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RESEARCH ARTICLE

Exploring constraint qualification-free optimality conditions for linear second-order cone programming

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ABSTRACT

Linear second-order cone programming (SOCP) deals with optimization problems characterized by a linear objective function and a feasible region defined by linear equalities and second-order cone constraints. These constraints involve the norm of a linear combination of variables, enabling the representation of a wide range of convex sets. The SOCP serves as a potent tool for addressing optimization challenges across engineering, finance, machine learning, and various other domains. In this paper, we introduce new optimality conditions tailored for SOCP problems. These conditions have the form of two optimality criteria that are obtained without the requirement of any constraint qualifications and are defined explicitly. The first criterion utilizes the concept of immobile indices of constraints. The second criterion, without relying explicitly on immobile indices, introduces a special finite vector set for assessing optimality. To demonstrate the effectiveness of these criteria, we present two illustrative examples highlighting their applicability. We compare the obtained criteria with other known optimality conditions and show the advantage of the former ones.

1. Introduction

A conic optimization problem is characterized by a constraint stipulating that the optimization variables must belong to a closed convex cone. Such problems encompass a wide spectrum of optimization problems and serve as a fundamental framework for addressing various real-world challenges. Conic problems form a broad and important class of optimization problems, since according to [1, 2], any convex optimization problem can be represented as a conic one. This universality underscores the essential significance of conic optimization in mathematical optimization theory. In recent years, conic optimization has attracted considerable attention due to its versatility and widespread applicability across diverse domains [3–5]. Among the most prominent and extensively studied subclasses of conic optimization problems are Linear Programming (LP) and convex Quadratic Programming (QP) problems. Another notable class of conic optimization problems is Semidefinite Programming (SDP), where the optimization is performed over the cone of positive semidefinite matrices. SDP has garnered significant interest owing to its utility in tackling a broad range of optimization tasks, including control theory, combinatorial optimization, and quantum information processing (see [6–8]).

Linear Second-Order Cone Programming (SOCP) deals with conic problems where the objective is to optimize a linear cost function over the intersection of an affine set and the product of the *second-order (Lorentz)* cones in a finite-dimensional vector space. The problems of LP, QP, and the quadratically constrained convex

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quadratic problems can be formulated as SOCP problems, which in turn, belong to a special class of SDP problems (see *e.g.* [9–11], and others).

The class of SOCP problems has been extensively studied in the past two decades due to its broad applicability across various fields of research, including engineering, finance, control theory, robust and combinatorial optimization. The literature dedicated to second-order problems is vast. For the applications, see, *e.g.* [10, 12, 13], and the references therein. As highlighted in [9], many of the SDP problems encountered in practical applications and considered in [7], can also be formulated as instances of SOCP problems, further emphasizing the significance and relevance of SOCP in optimization theory and practice.

Necessary and sufficient optimality conditions play an important role in optimization by providing a framework for identifying optimal solutions. By leveraging these conditions, researchers and practitioners can effectively discern the best possible outcome from the optimization process. Among the various types of optimality conditions, two prominent categories can be distinguished: the optimality conditions in *ordinary* (*punctual*) form as in, *e.g.*, [14–17] and *sequential* optimality conditions, see [18–20]. Additionally, other types of optimality conditions, such as those discussed in [21, 22], contribute to the comprehensive understanding of optimization processes and strategies.

To test ordinary optimality conditions for a primal feasible solution x^0, one has to find a finite vector y^0, which is a dual feasible solution, and check a finite number of equalities and inequalities constructed on the base of x^0 and y^0. When applying sequential optimality conditions to a feasible solution x^0, it is necessary to identify some sequences, $\{x^k\}$ and $\{y^k\}$, of vectors associated with the primal and dual variables, respectively, and check some conditions in the form of limits of functions built on the base of these sequences.

Optimality conditions are often formulated under certain additional conditions on the problem's constraints, known as *constraint qualifications* (CQ). Constraint qualifications are properties inherent in the analytical description of a feasible set ensuring that its structure around a given feasible point can be described by (first-order) approximations of the constraint functions (see *e.g.* [23]) and guarantee the Karush-Kuhn-Tucker (KKT) optimality conditions to hold at a local minimizer. The most widely used CQ for SOCP is the Slater condition (or *strict feasibility*) presupposing the existence of a feasible solution that belongs to the interior of the feasible set.

Constraint qualifications are particularly crucial for deriving primal and primal-dual characterizations of solutions in optimization and variational problems. They are essential for studying duality relations, conducting sensitivity and stability analysis, and justifying the convergence and evaluating the convergence rate of computational methods.

Many papers are dedicated to CQ conditions for different classes of optimization problems (see [14, 15, 18, 23–26], and others). One of the main challenges in this area is that for many conic problems in general and SOCP problems in particular, the CQs needed for formulation of optimality conditions may not hold (see, for example, [9, 16, 27], and the references therein). Therefore, it is very important to search for optimality conditions that do not rely on any CQ (referred to as *CQ-free optimality conditions*). Many research is dedicated to CQ-free optimality conditions for different classes of optimization problems (see [16, 19–21, 28, 29], and others). However, to the best of the authors' knowledge, no CQ-free optimality conditions in the ordinary form specifically designed for SOCP problems have been published to date.

In this paper, new CQ-free optimality conditions in the ordinary form are derived for SOCP problems. These conditions are formulated and proven in the form of two criteria. Illustrative examples demonstrate situations where classical conditions fail to test optimality, while the optimality criteria presented in the paper allow such a test.

The paper is structured as follows. In section 2, we formulate the problem and introduce the basic notation. In section 3, we introduce the set I_0 of special constraint indices referred to as *immobile*. Here the immobility of a constraint's index means that this constraint remains active for all feasible values of the problem's variables. We utilize the set of immobile indices to prove an optimality criterion for SOCP problems. This criterion does not use any additional conditions on the feasible set of the problem under consideration, making it an CQ-free optimality criterion. However, its application may be hindered by the requirement for information about the set I_0, which may not always be available. In the subsequent section 4, we present an alternative CQ-free optimality criterion wherein the set I_0 is not explicitly utilized. At the end of the section, we provide a short discussion on two different approaches to the CQ-free optimality conditions and on the properties of the approach proposed in the paper. Illustrative examples in section 5 highlight the new optimality conditions derived in the paper particularly in scenarios where the classical KKT optimality conditions fail to suffice. In section 6, motivated by the optimality criterion obtained by Gorokhovik in [21], for a more general class of convex problems and using the *lexicographical separations approach*, we formulate the optimality criteria for SOCP. We compare this criterion with that obtained in sections 3 and 4. The paper ends with some conclusions presented in section 7.

2. Problem's statement and basic notions

Consider a linear second-order cone programming problem in the form

$$\textbf{SOCP}: \qquad \max\, b^\top x \quad \text{s.t. } A_i x + c(i) \in \mathcal{SOC}(i), i \in I,$$

where $x \in \mathbb{R}^n$ is a vector of decision variables, $b \in \mathbb{R}^n$, $c(i) \in \mathbb{R}^{m_i+1}$, $A_i \in \mathbb{R}^{(m_i+1)\times n}$, $i \in I$, are given vectors and matrices; the sets

$$\mathcal{SOC}(i) := \{z = \begin{pmatrix} z_0 \\ z_* \end{pmatrix} \in \mathbb{R}^{m_i+1}, \quad z_0 \in \mathbb{R},\, z_* \in \mathbb{R}^{m_i} : ||z_*|| \le z_0\},\ i \in I,$$

are the second-order cones. Here $n \in \mathbb{N}$, $m_i \in \mathbb{N}$, $i \in I$; $||z_*|| = \sqrt{z_*^\top z_*}$, and the set $I \subset \mathbb{N}$ is supposed to be a finite index set.

Given $i \in I$, the second-order cone $\mathcal{SOC}(i)$ is convex, full-dimensional, *nice*, and consequently, is *facially exposed* (for definitions see *e.g.* [6]).

In what follows, for $i \in I$, we will suppose that a vector $z \in \mathcal{SOC}(i)$ has the form $z = (z_0, z_*^\top)^\top \in \mathbb{R}^{m_i+1}$, where $z_0 \in \mathbb{R}$, $z_* \in \mathbb{R}^{m_i}$.

Given $x \in \mathbb{R}^n$ and $i \in I$, denote

$$z(i, x) := A_i x + c(i).$$

For the problem (**SOCP**), the corresponding standard (Lagrangian) dual problem has the form

$$\textbf{SOCD}: \qquad \min \sum_{i\in I} c(i)^\top y(i) \quad \text{s.t. } \sum_{i\in I} A_i^\top y(i) = -b,\ \ y(i) \in \mathcal{SOC}(i),\ i \in I,$$

where vectors $y(i)$, $i \in I$, are the decision variables.

A vector $x \in \mathbb{R}^n$ is a *strictly feasible solution* in the problem (**SOCP**) if $z(i,x) \in \operatorname{int}\mathcal{SOC}(i)$ for all $i \in I$. A feasible solution of the problem (**SOCD**), consisting of vectors $y(i)$, $i \in I$, is called *strictly feasible* if $y(i) \in \operatorname{int}\mathcal{SOC}(i)$ for all $i \in I$. Here int$S$ stands for the interior of a set S.

Lemma 1. *[Weak duality, [9]] If $\bar{x}$ is feasible in the problem* (**SOCP**) *and $(\bar{y}(i), i \in I)$ is feasible in the dual problem* (**SOCD**)*, then the value of the objective function of* (**SOCP**) *evaluated at $\bar{x}$ is less than or equal to the value of the objective function of* (**SOCD**) *evaluated at $(\bar{y}(i), i \in I)$.*

Given a primal-dual pair of optimization problems (**P**) and (**D**), let val(**P**) and val(**D**) denote the optimal values of the cost functions of these problems. The difference $val(\mathbf{D}) - val(\mathbf{P})$ is called *the duality gap.*

From Lemma 1, it follows that for a pair of dual problems (**SOCP**) and (**SOCD**), the duality gap is non-negative. To guarantee that the duality gap is equal to zero, the problems should satisfy certain additional conditions.

The following theorems are proved in [9].

Theorem 1. *[Strong duality] If the second-order cone problems* (**SOCP**) *and* (**SOCD**) *have strictly feasible solutions, then they both have optimal solutions (are solvable) and* $val(\mathbf{SOCD}) - val(\mathbf{SOCP}) = 0$.

Theorem 2. *[KKT optimality conditions] Suppose that* (**SOCP**) *is strictly feasible (admits a strictly feasible solution). Then a feasible solution x^0 is optimal in this problem iff there exist vectors $y^0(i)$, $i \in I$, such that*

$$\sum_{i\in I} A_i^\top y^0(i) = -b, \qquad y^0(i) \in \mathcal{SOC}(i),\ y^0(i)^\top z(i, x^0) = 0\ \forall i \in I. \tag{1}$$

Without additional conditions (CQs) on the constraints of the problem (**SOCP**), the duality gap may be positive. In this case, the KKT optimality conditions may not be met (see [9,27], and the example below).

The aim of this study is to formulate and prove for the second-order cone problem (**SOCP**) new CQ-free optimality conditions in the ordinary form.

3. An optimality criterion for the primal second-order cone problem

Denote by X the set of feasible solutions of the problem (**SOCP**):

$$X := \{x \in \mathbb{R}^n : z(i, x) \in \mathcal{SOC}(i)\ \forall\, i \in I\}. \tag{2}$$

Notice that the set X is convex.

Suppose that $X \neq \emptyset$ and consider a subset of the index set I:

$$I_0 := \{i \in I :\ ||z_*(i, x)|| = z_0(i, x)\ \forall x \in X\}. \tag{3}$$

This subset plays an important role in our approach. It contains the indices of constraints that can be characterized as *always active* or *immobile* in the terminology of our previous papers (see *e.g.* [16, 30], and the references therein).

The constraints of the problem (**SOCP**) are said to satisfy the *Slater condition* if the problem admits a strictly feasible solution, *i.e.* there exists a vector $\bar{x} \in \mathbb{R}^n$ such that

$$z(i,\bar{x}) \in \operatorname{int} \mathcal{SOC}(i) \ \forall i \in I. \tag{4}$$

The Slater condition is one of CQs that guarantee the existence of KKT multipliers for a given optimal solution.

It is easy to show that conditions (4) are equivalent to the inequalities

$$||z_*(i,\bar{x})|| < z_0(i,\bar{x}) \ \forall i \in I. \tag{5}$$

Therefore, in terms of (3), one can see that the constraints of the problem (**SOCP**) satisfy the Slater condition iff $I_0 = \emptyset$. Hence, the emptiness of the set I_0 can be considered as a constraint qualification.

In what follows, we will use the following notation for $i \in I$:

$$\mathcal{R}_i := \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & -1 & .... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & -1 \end{pmatrix} \in \mathbb{R}^{(m_i+1)\times(m_i+1)};$$

$$\operatorname{int} \mathcal{SOC}(i) := \{z = \begin{pmatrix} z_0 \\ z_* \end{pmatrix} \in \mathbb{R}^{m_i+1} : \ ||z_*|| < z_0\},$$

$$\operatorname{bd}^+ \mathcal{SOC}(i) := \{z = \begin{pmatrix} z_0 \\ z_* \end{pmatrix} \in \mathbb{R}^{m_i+1} : \ ||z_*|| = z_0, \ z_0 > 0\}.$$

Then, for any $i \in I$, it holds

$$\mathcal{SOC}(i) = \operatorname{int} \mathcal{SOC}(i) \cup \operatorname{bd}^+ \mathcal{SOC}(i) \cup \{\mathbf{0}\}, \tag{6}$$

where $\mathbf{0}$ is the null vector in the corresponding real space $\mathbb{R}^{m_i+1}$.

Since $X \neq \emptyset$, then it is easy to show that there exists a vector $\tilde{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} ||z_*(i,\tilde{x})|| &< z_0(i,\tilde{x}) \ \forall i \in I \setminus I_0, \\ ||z_*(i,\tilde{x})|| &= z_0(i,\tilde{x}) \ \forall i \in I_0. \end{aligned} \tag{7}$$

A vector $\tilde{x}$ satisfying (7), is called a *minimally active feasible solution* of the problem (**SOCP**).

For $i \in I$, let $z \in \mathbb{R}^{m_i+1}$ and $y \in \mathbb{R}^{m_i+1}$ be complementary, *i.e.* satisfy the following *complementarity conditions*:

$$z^\top y = 0, \ z \in \mathcal{SOC}(i), \ y \in \mathcal{SOC}(i). \tag{8}$$

Then (see [9]) one of the next conditions takes a place:

a⁰) $z \in \operatorname{int} \mathcal{SOC}(i) \implies y = \mathbf{0}$;
b⁰) $z \in \operatorname{bd}^+ \mathcal{SOC}(i) \implies y = \alpha \mathcal{R}_i z, \ \alpha \geq 0$;
c⁰) $z = \mathbf{0} \implies \forall y \in \mathcal{SOC}(i)$.

Proposition 1. *Let $\tilde{x}$ be a minimally active feasible solution of the problem* (**SOCP**)*. Then for $i \in I_0$ and $x \in X$, there exists a corresponding number $\alpha_i = \alpha_i(x)$, such that*

$$z(i,x) = \alpha_i z(i,\tilde{x}), \ \alpha_i \geq 0. \tag{9}$$

Proof. Let $i \in I_0$ and $x \in X$. It follows from the convexity of the set X that $0.5(\tilde{x}+x) \in X$. From this inclusion and the definition of the index set I_0, one can conclude:

$$\begin{aligned} &||z_*(i,0.5(\tilde{x}+x))|| = z_0(i,0.5(\tilde{x}+x)), \\ &||z_*(i,\tilde{x})|| = z_0(i,\tilde{x}), \ ||z_*(i,x)|| = z_0(i,x). \end{aligned} \tag{10}$$

Consequently,

$$\begin{aligned} 0.5||z_*(i,\tilde{x}) + z_*(i,x)|| &= 0.5 z_0(i,\tilde{x}) + 0.5 z_0(i,x) \\ &= 0.5||z_*(i,\tilde{x})|| + 0.5||z_*(i,x)||. \end{aligned}$$

The equality $||z_*(i,\tilde{x}) + z_*(i,x)|| = ||z_*(i,\tilde{x})|| + ||z_*(i,x)||$ obtained above can be rewritten as follows: $(z_*(i,\tilde{x}) + z_*(i,x))^\top (z_*(i,\tilde{x}) + z_*(i,x)) = ||z_*(i,\tilde{x})||^2 + 2||z_*(i,\tilde{x})|| \cdot ||z_*(i,x)|| + ||z_*(i,x)||^2$, wherefrom we obtain $z_*(i,\tilde{x})^\top z_*(i,x) = ||z_*(i,\tilde{x})|| \cdot ||z_*(i,x)||$.

Taking into account the latter equality and the well-known relation $a^\top b = \cos(\varphi)||a|| \cdot ||b||$, where φ is the angle between the vectors a and b, we obtain that the cosine of the angle between vectors $z_*(i,x)$ and $z_*(i,\tilde{x})$ is equal to 1, and, hence, these vectors are collinear. This implies that

$$z_*(i,x) = \alpha_i z_*(i,\tilde{x}) \ \text{ with some } \alpha_i \geq 0. \tag{11}$$

It follows from (10) and (11) that

$$z_0(i,x) = ||z_*(i,x)|| = \alpha_i ||z_*(i,\tilde{x})|| = \alpha_i z_0(i,\tilde{x}).$$

The equality obtained, $z_0(i,x) = \alpha_i z_0(i,\tilde{x})$, together with (11) imply that relations (9) hold true for $i \in I_0$ and $x \in X$. □

Let us fix a minimally active feasible solution $\tilde{x}$ of the problem (**SOCP**) and denote

$$\gamma(i) := z(i,\tilde{x}) \ \forall i \in I_0.$$

Then it follows from Proposition 1 that for an immobile index $i \in I_0$ and for a feasible solution $x \in X$, the non-linear condition

$$z(i,x) \in \mathcal{SOC}(i) \iff ||z_*(i,x)|| \leq z_0(i,x)$$

can be replaced by $(m_i + 1)$ linear equalities $z(i,x) = \alpha_i z(i,\tilde{x})$ with one additional variable $\alpha_i \geq 0$. Based on this, it is easy to show that $X = \bar{X}$, where

$$\begin{aligned} \bar{X} := \{x \in \mathbb{R}^n : \ & z(i,x) \in \mathcal{SOC}(i) \ \forall i \in I \setminus I_0, \\ & z(i,x) = \alpha_i \gamma(i) \text{ with some } \alpha_i \geq 0 \ \forall i \in I_0\}. \end{aligned}$$

It follows from the considerations above that the problem (**SOCP**) is equivalent to the following one:

$$\mathbf{P}_*: \quad \max b^\top x$$
$$\text{s.t. } A_i x + c(i) = z(i),\ z(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_0;$$
$$A_i x + c(i) = \alpha_i \gamma(i),\ \alpha_i \geq 0\ \forall i \in I_0,$$

where the decision variables are vector $x \in \mathbb{R}^n$ and numbers α_i, $i \in I_0$.

Notice that in the problem ($\mathbf{P}_*$), there is a finite number of equality and inequality constraints

$$A_i x + c(i) = \alpha_i \gamma(i),\ \alpha_i \geq 0\ \forall i \in I_0,$$

that are linear w.r.t. $x \in \mathbb{R}^n$ and $\alpha_i \in \mathbb{R}$, $i \in I_0$. Moreover, there exists a feasible solution $\tilde{x}$ of the problem (**SOCP**) such that the feasible solution

$$\tilde{x},\ \tilde{\alpha}_i = 1,\ i \in I_0,\ \tilde{z}(i) = z(i, \tilde{x}),\ i \in I \setminus I_0,$$

of the problem ($\mathbf{P}_*$) satisfies the following strict inequalities:

$$\tilde{\alpha}_i > 0,\ i \in I_0,\ ||\tilde{z}_*(i)|| < \tilde{z}_0(i),\ i \in I \setminus I_0.$$

Hence, the constraints of this problem satisfy the generalized Slater condition (see [31]), and one can use the classical KKT optimality conditions for testing optimality of its feasible solution $(x^0,\ \alpha_i^0, i \in I_0)$.

Taking into account the equivalence of the problems (**SOCP**) and ($\mathbf{P}_*$), we obtain the following result.

Theorem 3. *[Optimality criterion 1] A feasible solution $x^0 \in X$ of the problem* (**SOCP**) *is optimal in this problem iff there exist vectors $y(i) \in \mathbb{R}^{m_i+1}$, $i \in I$, such that the following relations hold true:*

$$\sum_{i \in I} A_i^\top y(i) = -b,\ z(i, x^0)^\top y(i) = 0\ \forall i \in I; \quad (12)$$

$$y(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_0; \qquad y(i)^\top \gamma(i) \geq 0 \quad \forall i \in I_0. \quad (13)$$

Conditions (12), (13) are similar to the KKT conditions (1) but simpler than them. The difference is as follows: the conic conditions $y^0(i) \in \mathcal{SOC}(i)$, $i \in I_0$, in (1) are replaced by the linear ones $y(i)^\top \gamma(i) \geq 0$, $i \in I_0$, in (12), (13). Therefore, finding a solution to system (12), (13) is no more difficult than finding a solution to the KKT system (1).

It is evident that if $I_0 = \emptyset$, then conditions (12), (13) coincide with (1).

It should be noted here that the optimality criterion in the form of Theorem 3 does not use any additional conditions on the feasible set of the problem (**SOCP**) and is therefore an CQ-free optimality criterion. The only possible difficulty in its application is the need to know the set of immobile indices I_0.

In the next section, we will demonstrate an alternative CQ-free optimality criterion that does not explicitly rely on any knowledge of I_0.

4. An alternative CQ-free optimality criterion for the second-order cone programming

The optimality criterion presented in this section is based on the following idea used in literature for convex optimization problems (see, for example [22]).

For a given convex problem, at the first step, one attempts to obtain an *exact extended dual* problem (EEDP) explicitly formulated in terms of the data of the original primal problem (see [32–35]). The exact (strong) duality property entails that when the primal problem and its corresponding dual are consistent, their optimal values are equal, and the dual problem attains its optimal value.

The dual problem (EEDP) has an extended set of dual decision variables compared to the Lagrangian dual. Note that some regularization procedure is necessary to justify the exactness of this dual problem.

At the second step, taking into account the exactness of the extended dual problem (EEDP), attempts are made to formulate CQ-free optimality conditions for a feasible solution to the original primal problem using an optimal solution to this dual problem.

Below, we utilize this idea to derive an CQ-free optimality criterion for the problem (**SOCP**). Taking into account the specific nature of the problem under consideration, we are able to formulate the optimality conditions without an explicit representation of the corresponding exact extended dual problem. The regularization procedure associated with this formulation is implicitly embedded within the proof of the criterion.

It is worth noting that the KKT optimality conditions (see Theorem 2) are also based on a similar idea: these conditions are formulated using the set of vectors $y^0(i)$, $i \in I$, (the KKT multipliers for a given optimal solution) which, in fact, represents an optimal solution of the Lagrangian dual problem (**SOCD**). However, in the formulation of these conditions, this fact is not explicitly mentioned.

We commence by formally introducing a set of vectors that, in essence, constitutes a feasible solution of the exact extended dual problem.

Having fixed $i \in I$ and $k_0 \in \mathbb{N}$, $0 \le k_0 \le |I|$, consider the following set of vectors:

$$\{\pi(k,i) \in \mathbb{R}^{m_i+1},\ k = 0,1,\dots,k_0\}. \tag{14}$$

If $\pi(k,i) \not\equiv \mathbf{0}$ for all $k = 0,1,\dots,k_0$, denote

$$q_i = \min\{k : 0 \le k \le k_0,\ \pi(k,i) \ne \mathbf{0}\}.$$

We say that for a given $i \in I$, the set of vectors (14) satisfies Condition (A) if one of the following conditions is true:

A1) $\pi(k,i) \equiv \mathbf{0}$ for all $k = 0,1,\dots,k_0$;
A2) $\pi(q_i,i) \in \mathcal{SOC}(i)$, $\pi(k,i)^\top \mathcal{R}_i \pi(q_i,i) \ge 0$ for all $k = q_i+1,\dots,k_0$.

Here and in what follows, the set of indices $\{k = q, q+1,\dots,s\}$ is assumed to be empty if $s < q$.

Let us prove a technical proposition.

Proposition 2. *Suppose that $i \in I$ and that the set of vectors (14) satisfies Condition (A). Then for any $z \in \mathcal{SOC}(i)$, there exists $\bar\theta = \bar\theta(z) > 0$ such that*

$$\sum_{k=0}^{k_0} \theta^{k_0-k} z^\top \pi(k,i) \ge 0 \quad \forall \theta \ge \bar\theta. \tag{15}$$

Proof. If $\pi(k,i) \equiv \mathbf{0}$ for all $k = 0,1,\dots,k_0$, then inequalities (15) are trivially satisfied with any $\bar\theta > 0$.

Suppose that $\pi(k,i) \not\equiv \mathbf{0}$ for $k = 0,1,\dots,k_0$. In this case, we have

$$\sum_{k=0}^{k_0} \theta^{k_0-k} z^\top \pi(k,i) = \sum_{k=q_i}^{k_0} \theta^{k_0-k} z^\top \pi(k,i), \tag{16}$$

where $z^\top \pi(q_i,i) \ge 0$ since $z \in \mathcal{SOC}(i)$ and $\pi(q_i,i) \in \mathcal{SOC}(i)$.

If $z^\top \pi(q_i,i) > 0$, then evidently, the inequalities (15) hold true for a sufficiently large $\bar\theta > 0$.

Suppose that $z^\top \pi(q_i,i) = 0$. Since $z \in \mathcal{SOC}(i)$, we can distinguish the following three cases:

1) $z = \mathbf{0}$, 2) $z \in \operatorname{int} \mathcal{SOC}(i)$, and 3) $z \in \operatorname{bd}^+ \mathcal{SOC}(i)$.

In case 1), relations (15) are trivially satisfied with any $\bar\theta > 0$.

In case 2), the equality $z^\top \pi(q_i,i) = 0$ implies $\pi(q_i,i) = \mathbf{0}$ that contradicts the assumption $\pi(q_i,i) \ne \mathbf{0}$. Therefore, this case is impossible.

In case 3), the equality $z^\top \pi(q_i,i) = 0$ and the inequality $\pi(q_i,i) \ne \mathbf{0}$ imply $z = \alpha_i(z)\mathcal{R}_i \pi(q_i,i)$ with some $\alpha_i(z) > 0$. Hence, taking into account the latter relations and Condition A2), we obtain the following inequalities:

$$z^\top \pi(k,i) = \alpha_i(z) \pi(k,i)^\top \mathcal{R}_i \pi(q_i,i) \ge 0,$$

for all $k = q_i+1,\dots,k_0$. These inequalities together with equalities (16) and $z^\top \pi(q_i,i) = 0$, ensure that relations (15) are satisfied for any $\bar\theta > 0$. □

For a given $x \in X$, introduce an index set

$$I_a(x) := \{i \in I : ||z_*(i,x)|| = z_0(i,x)\}.$$

Theorem 4. *[Optimality Criterion 2] A vector $x^0 \in X$ is an optimal solution of the problem* (**SOCP**) *iff there exist an integer number k_0, $0 \le k_0 \le |I_a(x^0)|$, and the sets of vectors*

$$\{\pi(k,i) \in \mathbb{R}^{m_i+1}, k = 0,1,\dots,k_0\}, i \in I_a(x^0), \tag{17}$$

satisfying Condition (A) for all $i \in I_a(x^0)$, such that

$$\begin{aligned} &\sum_{i \in I_a(x^0)} A_i^\top \pi(k,i) = \mathbf{0} \ \forall k = 0,\dots,k_0-1; \\ &\sum_{i \in I_a(x^0)} A_i^\top \pi(k_0,i) = -b, \end{aligned} \tag{18}$$

and

$$\begin{aligned} &z(i,x^0)^\top \pi(k,i) = 0 \\ &\forall k = 0,1,\dots,k_0,\ \forall i \in I_a(x^0). \end{aligned} \tag{19}$$

Proof. *Sufficiency.* Suppose that there exists a set of vectors (17) satisfying Condition (A) and relations (18) and (19). Then it follows from (19) that

$$\begin{aligned} 0 &= \sum_{i \in I_a(x^0)} z(i,x^0)^\top \pi(k,i) \\ &= \sum_{i \in I_a(x^0)} [A_i x^0 + c(i)]^\top \pi(k,i) \\ &= \sum_{i \in I_a(x^0)} c(i)^\top \pi(k,i) + {x^0}^\top \sum_{i \in I_a(x^0)} A_i^\top \pi(k,i), \end{aligned}$$

for all $k = 0,1,\dots,k_0$. From these equalities and (18), we obtain

$$\begin{aligned} &\sum_{i \in I_a(x^0)} c(i)^\top \pi(k,i) = 0 \ \forall k = 0,\dots,k_0-1, \\ &\sum_{i \in I_a(x^0)} c(i)^\top \pi(k_0,i) = b^\top x^0. \end{aligned} \tag{20}$$

It follows from Proposition 2 that for any $x \in X$, there exists $\bar\theta = \bar\theta(x) > 0$ such that

$$\sum_{k=0}^{k_0} \bar\theta^{k_0-k} z(i,x)^\top \pi(k,i) \ge 0 \quad \forall i \in I_a(x^0). \tag{21}$$

For $x \in X$, taking into account (18) - (20), let us calculate

$$\begin{aligned} b^\top x &= -\sum_{i\in I_a(x^0)} x^\top A_i^\top \pi(k_0,i) \\ &= -\sum_{i\in I_a(x^0)} x^\top A_i^\top \sum_{k=0}^{k_0} \bar\theta^{k_0-k}\pi(k,i) \\ &= -\sum_{i\in I_a(x^0)} z(i,x)^\top \sum_{k=0}^{k_0} \bar\theta^{k_0-k}\pi(k,i) \\ &+ \sum_{i\in I_a(x^0)} c(i)^\top \sum_{k=0}^{k_0} \bar\theta^{k_0-k}\pi(k,i) \\ &= -\sum_{i\in I_a(x^0)} \sum_{k=0}^{k_0} \bar\theta^{k_0-k} z(i,x)^\top \pi(k,i) + b^\top x^0. \end{aligned}$$

These relations together with (21), permit one to conclude that $b^\top x \le b^\top x^0$ for all $x \in X$. Hence $x^0 \in X$ is an optimal solution of the problem (**SOCP**).

Necessity. Let $x^0 \in X$ be an optimal solution to the problem (**SOCP**). Let us construct a set of vectors (17) satisfying the Condition (A) and relations (18), (19). We will do this iteratively by performing the following iterations.

Iteration # 0. Consider the problem

$$\begin{aligned} \textbf{P-0}: \quad & \max \mu, \\ \text{s.t. } & A_i x + c(i) - e_0(i)\mu = z(i), \\ & z(i) \in \mathcal{SOC}(i)\ \forall i \in I, \end{aligned}$$

where $e_0(i) = (1, 0, \ldots, 0)^\top \in \mathbb{R}^{m_i+1}$, $i \in I$.

The constraints of this problem satisfy the Slater condition. In fact, for any $x \in X$, the vector $(x,\ \mu = -1,\ z(i),\ i \in I)^\top$ with $z(i) = (z_0(i) = z_0(i,x) + 1, z_*(i) = z_*(i,x)), i \in I$, is a feasible solution of the problem (**P-0**) satisfying the strict inequalities

$$||z_*(i)|| < z_0(i)\ \forall i \in I.$$

If this problem admits a feasible solution $(\bar x,\ \bar\mu,\ \bar z(i), i \in I)$ with $\bar\mu > 0$, then set $k_0 = 0$ and go to the Final Step.

Otherwise, for any $x \in X$, the vector

$$(x,\ \mu = 0,\ z(i) = z(i,x), i \in I) \qquad (22)$$

is an optimal solution of the problem (**P-0**). Since the constraints of this problem satisfy the Slater condition, applying the classical KKT optimality conditions to its optimal solution (22), we conclude that there exist vectors

$$\begin{aligned} & y^0(i) = \begin{pmatrix} y^0_0(i) \\ y^0_*(i) \end{pmatrix} \in \mathbb{R}^{m_i+1}, \\ & y^0_*(i) \in \mathbb{R}^{m_i},\ i \in I, \end{aligned} \qquad (23)$$

such that the following relations hold true for any $x \in X$:

$$\sum_{i\in I} A_i^\top y^0(i) = \mathbf{0},\ \sum_{i\in I} y^0_0(i) = 1, \qquad (24)$$

$$z(i,x)^\top y^0(i) = 0,\ y^0(i) \in \mathcal{SOC}(i)\ \forall i \in I. \quad (25)$$

Consider the index set

$$\Delta I_1 := \{i \in I : y^0_0(i) > 0\}.$$

It follows from (24) that $\Delta I_1 \neq \emptyset$. Let us show that

$$||z_*(i,x)|| = z_0(i,x)\ \forall i \in \Delta I_1,\ \forall x \in X, \qquad (26)$$

and consequently, the indices in ΔI_1 are immobile.

Suppose the contrary: there exist $i_0 \in \Delta I_1$ and $\bar x \in X$ such that $||z_*(i_0,\bar x)|| < z_0(i_0,\bar x)$. Then from the equality in (25) with $i = i_0$ and the conditions $z(i_0,\bar x) \in \mathcal{SOC}(i_0)$, $y^0(i_0) \in \mathcal{SOC}(i_0)$, we can conclude that $y^0(i_0) = \mathbf{0}$. But this contradicts the inequality $y^0_0(i_0) > 0$ that is fulfilled by construction. Hence equalities (26) are satisfied. Remind here that relations (24), (25) are valid for all $x \in X$.

Let us show that for all $i \in \Delta I_1$ and $x \in X$, the following is true:

$$\exists \alpha_i(x) \ge 0 \textit{ such that } z(i,x) = \alpha_i(x)\mathcal{R}_i y^0(i). \quad (27)$$

Let $i \in \Delta I_1$ and $x \in X$. If $y^0(i) \in \operatorname{int}\mathcal{SOC}(i)$, then it follows from the equality in (25) and the condition $z(i,x) \in \mathcal{SOC}(i)$, that $z(i,x) = \mathbf{0}$. Hence, in this case, relations (27) are satisfied with $\alpha_i = 0$. If $y^0(i) \in \operatorname{bd}^+\mathcal{SOC}(i)$, then it follows from (25) and the inclusion $z(i,x) \in \mathcal{SOC}(i)$, that $z(i,x) = \alpha_i(x)\mathcal{R}_i y^0(i)$ with some $\alpha_i(x) \ge 0$. Consequently, the equality in (27) holds true in this case as well. Taking into account that $y^0(i) \neq \mathbf{0}$ for $i \in \Delta I_1$, we conclude that relations (27) are proved.

It follows from (27) that for an immobile index $i \in \Delta I_1$ and for a feasible solution $x \in X$, the non-linear condition

$$z(i,x) \in \mathcal{SOC}(i) \iff ||z_*(i,x)|| \le z_0(i,x)$$

can be replaced by $(m_i + 1)$ linear equalities $z(i,x) = \alpha_i \mathcal{R}_i y^0(i)$ with one additional variable $\alpha_i \ge 0$. Based on this, it is easy to see that $X = X_0$, where

$$\begin{aligned} X_0 :=& \{x \in \mathbb{R}^n : z(i,x) \in \mathcal{SOC}(i), i \in I \setminus I_1; \\ & z(i,x) = \alpha_i \bar\gamma(i) \text{ with some } \alpha_i \ge 0,\ i \in I_1\}, \\ I_1 :=& \Delta I_1, \\ \bar\gamma(i) :=& \mathcal{R}_i y^0(i) \in \mathcal{SOC}(i)\ \forall i \in \Delta I_1. \end{aligned} \qquad (28)$$

In fact, if $x \in X_0$, then it is evident that $z(i,x) \in \mathcal{SOC}(i)$ for all $i \in I$. Hence, $x \in X$, and consequently, $X_0 \subset X$. Now suppose that $x \in X$. Then it follows from (27) and (28) that $x \in X_0$

and hence, $X \subset X_0$. The equality $X = X_0$ is proved.

The set of vectors (23) constructed above satisfies the conditions

$$\begin{aligned} &y^0(i) = \mathbf{0}\ \forall i \in I \setminus I_1;\\ &y^0(i) \in \mathcal{SOC}(i),\, y^0(i) \neq \mathbf{0}\ \forall i \in I_1. \end{aligned} \tag{29}$$

Go to the next Iteration #1 with the data (28).

Iteration # k ($k \geq 1$). At the beginning of this iteration, we have the following set and vectors:

$$I_k = \Delta I_1 \cup \cdots \cup \Delta I_k,\ \bar{\gamma}(i) = \mathcal{R}_i y^s(i) \in \mathcal{SOC}(i),$$
$$\bar{\gamma}_0(i) \neq 0,\ i \in \Delta I_{s+1},\ s = 0, 1, \ldots, k-1.$$

Consider the problem

$$\begin{aligned} \textbf{P-k}:\quad & \max \mu,\\ \text{s.t. } & A_i x + c(i) - e_0(i)\mu = z(i)\ \forall i \in I \setminus I_k,\\ & A_i x + c(i) = \alpha_i \bar{\gamma}(i)\ \forall i \in I_k,\\ & z(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_k, \alpha_i \geq 0\ \forall i \in I_k. \end{aligned}$$

The constraints of this problem satisfy the generalized Slater condition (see [31]).

If the problem (**P-k**) admits a feasible solution $(\bar{x},\ \bar{\mu},\ \bar{z}(i), i \in I \setminus I_k,\ \bar{\alpha}_i, i \in I_k)$ with $\bar{\mu} > 0$, then set $k_0 = k$ and go to the Final Step.

Otherwise, for any $x \in X$, the vector

$$\begin{aligned} &(x,\ \mu = 0,\ z(i,x), i \in I \setminus I_k;\\ &\alpha_i(x) = z_0(i,x)/\bar{\gamma}_0(i),\, i \in I_k) \end{aligned} \tag{30}$$

is an optimal solution to the problem (**P-k**). Taking into account that the constraints of this problem satisfy the generalized Slater condition and applying the KKT optimality conditions to its optimal solution (30), we conclude that there exist vectors

$$\begin{aligned} &y^k(i) = \begin{pmatrix} y_0^k(i) \\ y_*^k(i) \end{pmatrix} \in \mathbb{R}^{m_i+1},\\ &y_*^k(i) \in \mathbb{R}^{m_i},\ i \in I, \end{aligned} \tag{31}$$

such that the following relations hold true:

$$\sum_{i \in I} A_i^\top y^k(i) = \mathbf{0},\ \sum_{i \in I \setminus I_k} y_0^k(i) = 1, \tag{32}$$

$$\begin{aligned} &z(i,x)^\top y^k(i) = 0\ \forall i \in I;\\ &y^k(i) \in \mathcal{SOC}(i)\, \forall i \in I \setminus I_k;\\ &\bar{\gamma}(i)^\top y^k(i) \geq 0\, \forall i \in I_k. \end{aligned} \tag{33}$$

Consider the index set

$$\Delta I_{k+1} := \{i \in I \setminus I_k : y_0^k(i) > 0\}.$$

It follows from (32) that

$$\Delta I_{k+1} \neq \emptyset. \tag{34}$$

Similar to how it was done on the initial Iteration # 0, one can show that

$$||z_*(i,x)|| = z_0(i,x)\ \forall i \in \Delta I_{k+1},\ \forall x \in X, \tag{35}$$

$$\begin{aligned} &z(i,x) = \alpha_i(x)\mathcal{R}_i y^k(i),\\ &\alpha_i(x) \geq 0\ \forall i \in \Delta I_{k+1},\ \forall x \in X. \end{aligned} \tag{36}$$

Set

$$I_{k+1} = I_k \cup \Delta I_{k+1} = \Delta I_1 \cup \Delta I_2 \cup \cdots \cup \Delta I_{k+1},$$
$$\bar{\gamma}(i) = \mathcal{R}_i y^k(i),\ i \in\ \Delta I_{k+1}.$$

It follows from (36) that $X = X_k$, where

$$\begin{aligned} X_k := \{&x \in \mathbb{R}^n : z(i,x) \in \mathcal{SOC}(i), i \in I \setminus I_{k+1};\\ &z(i,x) = \alpha_i \bar{\gamma}(i) \text{ with some } \alpha_i \geq 0,\, i \in I_{k+1}\}. \end{aligned} \tag{37}$$

The set of vectors defined in (31)-(33), satisfies the following relations:

$$y^k(i) = \mathbf{0}\ \forall i \in I \setminus I_{k+1}, \tag{38}$$

$$y^k(i) \in \mathcal{SOC}(i),\ y_0^k(i) \neq 0\ \forall i \in \Delta I_{k+1}; \tag{39}$$

$$\begin{aligned} &y^k(i)^\top \mathcal{R}_i y^{s-1}(i) = y^k(i)^\top \bar{\gamma}(i) \geq 0\\ &\forall i \in \Delta I_s,\ s = 1, \ldots, k. \end{aligned} \tag{40}$$

Go to the next Iteration $\#(k+1)$ using the set I_{k+1} and vectors $\bar{\gamma}(i),\ i \in I_{k+1},\ y^s(i),\ i \in I,$ $s = 0, 1, \ldots, k$ found above.

Final Step. It follows from condition (34) that after a finite number of iterations, we will get to the Final Step with some k_0, $0 \leq k_0 \leq |I_0|$, where I_0 is the set of immobile indices of the constraints of the problem (**SOCP**) (see (3)).

From (26) and (35) we have:

$$I_{k_0} = \Delta I_1 \cup \cdots \cup \Delta I_{k_0} \subset I_0. \tag{41}$$

By construction, a number k_0 is such that for $k = k_0$, the problem (**P-k**) has a feasible solution

$$(\bar{x},\ \bar{\mu},\ \bar{z}(i),\, i \in I \setminus I_{k_0},\, \bar{\alpha}_i,\, i \in I_{k_0})$$

with $\bar{\mu} > 0$. Hence, $\bar{x} \in X_{k_0-1} = X$, where X_{k_0-1} is defined in (37) with $k = k_0 - 1$, and

$$||z_*(i,\bar{x})|| < z_0(i,\bar{x})\ \forall i \in I \setminus I_{k_0}. \tag{42}$$

Notice that for $k_0 = 0$, the set I_{k_0} is empty.

Taking into account (41) and (42), one can conclude that $I_{k_0} = I_0$.

Consider the following problem:

$$\begin{aligned} \textbf{P-R}:\quad & \max b^\top x,\\ \text{s.t. } & A_i x + c(i) = z(i),\ z(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_{k_0},\\ & A_i x + c(i) = \alpha_i \bar{\gamma}(i),\ \alpha_i \geq 0\ \forall i \in I_{k_0}. \end{aligned}$$

It follows from (42) that the constraints of this problem satisfy the generalized Slater condition. Since $X = X_{k_0-1}$, the optimality of the solution x^0 in the problem (**SOCP**) implies the optimality of the solution

$$\begin{aligned} &(x^0,\ z^0(i) = z(i,x^0),\, i \in I \setminus I_{k_0},\\ &\alpha_i^0 = z_0(i,x^0)/\bar{\gamma}_0(i),\, i \in I_{k_0}) \end{aligned}$$

in the problem (**P-R**). Applying the KKT optimality conditions to the problem (**P-R**) and its optimal solution, one can conclude that there exist vectors $y^{k_0}(i),\ i \in I$, such that

$$\begin{aligned} & y^{k_0}(i) \in \mathcal{SOC}(i) \quad \forall i \in I \setminus I_{k_0}; \\ & y^{k_0}(i)^\top \bar{\gamma}(i) = y^{k_0}(i)^\top \mathcal{R}_i y^{s-1}(i) \geq 0 \\ & \qquad\qquad \forall i \in \Delta I_s,\ \forall\, s = 1, \ldots, k_0, \\ & \sum_{i \in I} A_i^\top y^{k_0}(i) = -b, \\ & z(i, x^0)^\top y^{k_0}(i) = 0 \quad \forall i \in I. \end{aligned} \tag{43}$$

From the relations above, we get

$$y^{k_0}(i) = \mathbf{0}\ \forall i \in I \setminus I_a(x^0). \tag{44}$$

Notice that by construction, we have $I_{k_0} = I_0$, and, consequently, $I_0 \subset I_a(x^0)$ for all $x \in X$. Taking into account this inclusion, (44), and (38) (with $k = 0, \ldots, k_0 - 1$), we conclude that the vectors $y^k(i), i \in I,\ k = 0, 1, \ldots, k_0$, constructed here, satisfy the equalities

$$y^k(i) = \mathbf{0}\ \forall i \in I \setminus I_a(x^0),\ \forall k = 0, 1, \ldots, k_0.$$

It follows from the equalities above and relations (39), (40) (with $k = 0, \ldots, k_0 - 1$), together with (43) that the sets of vectors

$$\{\pi(k,i) = y^k(i),\ k = 0, \ldots, k_0\},\ \forall i \in I_a(x^0), \tag{45}$$

satisfy Condition (A) and relations (18)-(19). □

Remark 1. *In the theorem, it is affirmed that the integer k_0 is less than or equal to $|I_a(x^0)|$. In fact, the inequalities $k_0 \leq |I_0| \leq |I_a(x^0)|$ hold true and in the statement of the theorem, one can replace the inequality $k_0 \leq |I_a(x^0)|$ by a tighter estimate $k_0 \leq |I_0|$. However, we prefer to leave here the inequality $k_0 \leq |I_a(x^0)|$ since in a general case, one cannot expect to have any knowledge about the set I_0. Notice that if the set I_0 is known, one can use a more simple form of optimality conditions, namely Criterion 1.*

Considering the problems ($\mathbf{P}_*$) and (**P-R**), one can see that they are similar but at the same time there are some differences between them.

It was mentioned above that $I_{k_0} = I_0$. Let us introduce a subset

$$I_{00} = \{i \in I_0 :\ z_0(i, x) = 0\ \forall x \in X\}.$$

For $i \in I_0 \setminus I_{00}$, we have $\gamma(i) = \beta_i \bar{\gamma}(i)$ with $\beta_i = \gamma_0(i)/\bar{\gamma}_0(i) > 0$, *i.e.* the vectors $\gamma(i)$ and $\bar{\gamma}(i)$ coincide up to a positive nonzero factor.

For $i \in I_{00}$, we have $\gamma(i) = \mathbf{0}$ and $\bar{\gamma}(i) \neq \mathbf{0}$.

In the problem ($\mathbf{P}_*$), for $x \in X$, the corresponding variables $\alpha_i, i \in I_0 \setminus I_{00}$, are uniquely determined by the rule $\alpha_i = z_0(i, x)/\gamma_0(i),\ i \in I_0 \setminus I_{00}$, and we can choose any non-negative values for $\alpha_i,\ i \in I_{00}$.

In the problem (**P-R**), for $x \in X$, the formulas $\alpha_i = z_0(i, x)/\bar{\gamma}_0(i),\ i \in I_0$, uniquely define the corresponding variables $\alpha_i,\ i \in I_0$.

4.1. A short discussion

It was mentioned earlier that Criterion 2 proved in this section, is based on the utilization of an optimal solution to the exact extended dual problem (EEDP). In fact, the set (45) constitutes a part of an optimal solution

$$\{y^k(i),\ k = 0, \ldots, k_0\}, i \in I, \tag{46}$$

to the problem (EEDP). The vectors in (46) serve as a generalization of the vectors of KKT multipliers for a given optimal solution x^0. However, unlike the vectors of KKT multipliers, which may not exist for some problems, an optimal solution to the exact extended dual problem always exists provided that the optimal value of problem (**SOCP**) is finite.

It follows from the iterative nature of the proof of Theorem 4 that testing the optimality criterion is not much more difficult than checking the KKT system. In fact, to construct generalized multipliers (46), one has to test sequentially, for $k = 0, \ldots, k_0$, the classical KKT optimality conditions in the second-order programming problem (**P-k**) for the feasible solution ($\bar{x}$, $\mu = 0$, $z(i) = z(i, \bar{x})$, $i \in I$) with a fixed $\bar{x} \in X$, and one time in the second-order programming problem (**P-R**) for the feasible solution (x^0, $\alpha_i^0 = z_0(i, x^0)/\bar{\gamma}_0(i)$, $i \in I_0$).

Note here the following:

- The number k_0 satisfies the inequality $k_0 \leq |I_0|$ and hence, it is finite. One may expect the number k_0 to be less than $|I_0|$, since $|I_0| = \sum_{k=1}^{k_0} |\Delta I_k|$ and, as a rule, $|\Delta I_k| > 1$ for $k = 1, \ldots, k_0$.
- The constraints of all second-order problems (**P-k**), $k = 0, \ldots, k_0$, and the problem (**P-R**) satisfy the Slater condition.
- For $k = 1, \ldots, k_0$, the KKT system for the problem (**P-k**) is simpler than the KKT system for the problem (**P-(k-1)**), and the KKT system for the problem (**P-R**) is the simplest among them.

If $I_0 = \emptyset$, then $k_0 = 0$. It is easy to see that in this case, conditions (18), (19) coincide with the KKT conditions (1), where $y^0(i) = \pi(0, i)$ for $i \in I_a(x^0)$ and $y^0(i) = \mathbf{0}$ for $i \in I \setminus I_a(x^0)$. Hence the KKT conditions (1) are a particular case of conditions (18), (19) with $k_0 = 0$.

In case $I_0 \neq \emptyset$, conditions (18), (19) are more complex than the KKT conditions, since to test them, one has to find an extended dual optimal solution. But notice that the KKT conditions are useless if, for the problem under consideration, the dual gap is positive or/and the corresponding

Lagrangian dual problem has no solution. In such situations, the KKT conditions can never be satisfied.

In contrast to the KKT conditions, Criterion 2 can always recognize the optimality of a given feasible solution, as an optimal generalized dual solution exists and there is no duality gap. This represents the main and significant advantage of conditions (18), (19) compared to the KKT conditions.

As mentioned earlier, verifying sequential optimality conditions requires finding sequences of vectors $\{x^k\}$ and $\{y^k\}$ associated with primal and dual variables, and checking certain conditions in the form of limits of functions constructed on the basis of these sequences. It is important to note that if certain CQs are not satisfied, the sequence $\{y^k\}$ may become "irregular" (or not well-defined), since $||y^k|| \to \infty$ as $k \to \infty$. This irregularity may pose challenges in numerical methods for constructing such sequences and in verifying conditions in the form of limits.

In contrast, to test the optimality Criterion 2, one needs to find a finite set (46) of concrete vectors which are "well defined" and check a finite set of equality and inequality conditions.

One drawback of our approach is the requirement to know the set I_0 in order to apply the optimality Criterion 1. This can pose a challenge, as identifying this set may take additional effort or computational resources. However, it is worth noting that if we do know this set, our optimality conditions offer advantages over traditional KKT conditions, providing a practical framework for solving optimization problems.

The second drawback of our approach is that when applying the optimality Criterion 2, we need to construct an extended (generalized) vector of Lagrange multipliers. Despite this, the criterion offers the advantage of being CQ-free.

It is known that the violation of CQs can lead to difficulties in implementation of numerical methods of the primal-dual type using the classical KKT optimality conditions. This difficulty arises from the non-existence of classical Lagrange multipliers. It can be overcome by utilizing (iteratively and in an approximate form) of some CQ-free optimality conditions, in either sequential or ordinary form. Since the optimality conditions obtained in the paper are CQ-free, they can be used for this purpose as well as the CQ-free optimality conditions in sequential form as in [18–20] *et al.*

5. Examples

Example 1. Consider the problem (**SOCP**) with the following data: $n = 6$, $I = \{1, 2, 3\}$, $m_1 = 3$, $m_2 = 3$, $m_3 = 2$,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$c(1) = (0, 6, 0, 0)^\top$; $c(2) = (0, 4, 6, -2)^\top$;
$c(3) = (-4, -2, -2)^\top$, $b = (4, 2, -1, -3, 2, -5)^\top$.

Set $x^0 = (2, 1, -3, 0, 1, 1)^\top$ and calculate $z(i, x^0) = A_i x^0 + c(i)$, $i \in I$. In this example, we have:

$$z(1, x^0) = (1, 0, 1, 0)^\top, \; z(2, x^0) = (0, 0, 0, 0)^\top,$$
$$z(3, x^0) = (3, 3, 0)^\top.$$

Consequently, x^0 is a feasible solution of this problem and $I_a(x^0) = I$.

Set $k_0 = 1$, and consider the following vectors: $\pi(0, 1) = (1, 0, -1, 0)^\top$, $\pi(0, 2) = (1, 0, 0, 0)^\top$, $\pi(0, 3) = (0, 0, 0)^\top$, $\pi(1, 1) = (-2, 2, 2, 1)^\top$, $\pi(1, 2) = (3, -1, 1, -1)^\top$, $\pi(1, 3) = (3, -3, 0)^\top$.

It is easy to check that the vectors $\pi(k, i)$, $k = 0, 1$, satisfy Condition (A) for all $i \in I = I_a(x^0)$ and conditions (18), (19). Hence, according to Theorem 4 the vector x^0 is an optimal solution in the problem under consideration.

Now, suppose that in this example, the set I_0 is known: $I_0 = \{1, 2\}$. Using this information, let us test the optimality of the solution x^0 by applying Theorem 3.

Set $\tilde{x} = (1.0, 0.8, -3.4, -0.2, 1.8, 2.2)^\top$ and calculate

$$z(1, \tilde{x}) = (0.8, 0, 0.8, 0)^\top, \; z(2, \tilde{x}) = (0, 0, 0, 0)^\top,$$
$$z(3, \tilde{x}) = (3, 0.4, 0.8)^\top.$$

It is easy to see that the vector $\tilde{x}$ is a minimally active feasible solution and hence, we can choose $\gamma(i) = z(i, \tilde{x})$ for $i \in I_0$.

Set:

$$y(1) = (1, 2, -1, 1)^\top, y(2) = (-1, -1, 1, -1)^\top,$$
$$y(3) = (3, -3, 0)^\top.$$

It is easy to check that these vectors and x^0 satisfy conditions (12) and (13). Hence we have illustrated that the conditions of Theorem 3 are fulfilled as well.

Now, let us show that for the optimal solution x^0, the (classical) KKT optimality conditions formulated in Theorem 2, are not satisfied.

Suppose that in this example, for the optimal solution x^0, there exist vectors $y^0(i)$, $i \in I$, satisfying (1). Then it follows from the conditions

$$y^0(i) \in \mathcal{SOC}(i),\ z(i,x^0) \in \mathcal{SOC}(i),$$
$$y^0(i)^\top z(i,x^0) = 0 \text{ for } i=1 \text{ and } i=3$$

that $y^0(1) = (\alpha, 0, -\alpha, 0)^\top$, $y^0(3) = (\beta, -\beta, 0)^\top$ with some $\alpha \geq 0$ and $\beta \geq 0$.

This implies

$$A_1^\top y^0(1) = \mathbf{0},\ A_3^\top y^0(3) = \beta(-1, 0-1, 0, 0, 1)^\top.$$

Consequently,

$$\sum_{i\in I} A_i^\top y^0(i) = -b \iff$$
$$A_2^\top y^0(2) + \beta(-1, 0-1, 0, 0, 1)^\top = -b.$$

It is easy to check here that there are no $y^0(2) \in \mathbb{R}^4$ and β satisfying the latter linear system. Thus we have shown that there do not exist vectors $y^0(i)$, $i \in I$, satisfying (1).

Let us show that in this example the duality gap is zero. In fact, one can check directly that for all sufficiently small $\varepsilon > 0$, the vectors $y(1,\varepsilon) = (4\varepsilon + \frac{1}{\varepsilon}, 2 + \frac{3}{2}\varepsilon,\ -\frac{1}{\varepsilon}, 1)^\top$, $y(2,\varepsilon) = (10, -1 - \frac{5}{2}\varepsilon, 1+3\varepsilon, -1+2\varepsilon)^\top$, and $y(3,\varepsilon) = (3+\varepsilon, -3, \varepsilon)^\top$ satisfy the following conditions:

$$\sum_{i=1}^{3} A_i^\top y(i,\varepsilon) = -b,\ y(i,\varepsilon) \in \mathcal{SOC}(i)\ \forall i = 1,2,3;$$
$$\sum_{i=1}^{3} c^\top(i) y(i,\varepsilon) = 10 + 7\varepsilon.$$

Hence, these vectors form a feasible solution to the dual problem (**SOCD**) and the corresponding value of the dual cost function is equal to $10+7\varepsilon \geq b^\top x^0 = 10$. Consequently, in this example, we have the equality $val(\mathbf{SOCP}) = val(\mathbf{SOCD})$, but the dual problem has no optimal solution.

Thus in this example, despite the zero duality gap, the KKT optimality conditions do not allow to test the optimality of x^0.

Example 2. Now, we will analyze a problem (**SOCP**) with a positive duality gap. Let us consider a problem from subsection 2.2 in [27]. This problem can be formulated as problem (**SOCP**) with the following data:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$c(1) = (0,\, 0,\, -1)^\top$, $c(2) = (0,\, 0)^\top$, $b = (0,\, -1)^\top$, $I = \{1, 2\}$, $m_1 = 2$, $m_2 = 1$, $n = 2$.

It has been shown in [27] that vector $x^0 = (0.5,\, 1)^\top$ is an optimal solution to the primal problem, the corresponding Lagrangian dual problem also possesses an optimal solution, but a duality gap is positive and equals to 1. In this scenario, it becomes evident that the optimality of the given optimal solution can not be verified using the KKT optimality conditions. However, we will demonstrate that the optimality conditions derived in this paper, allow us to address this issue.

First, we will apply Theorem 3. In this example, $I_0 = \{1\}$ and $\tilde{x} = (1, 1)^\top$ is a minimally active feasible solution. Consequently, we obtain: $z(1,x^0) := A_1 x^0 + c(1) = (0.5,\, 0.5,\, 0)^\top$, $\gamma(1) := A_1\tilde{x} + c(1) = (1,\, 1,\, 0)^\top$, $z(2,x^0) := A_2 x^0 + c(2) = (0.5,\, 0.5)^\top$. One can easily verify that x^0 is a primal feasible solution, and it and the vectors $y(1) = (0,\, 0,\, 1)^\top$, $y(2) = (0,\, 0)^\top$ satisfy conditions (12), (13). Hence, due to Theorem 3 we conclude that, indeed, the vector x^0 is an optimal solution to the problem (**SOCP**) under consideration.

One can check that the conditions of Theorem 4 are satisfied with $\pi(0,1) = (1,\, -1,\, 0)^\top$, $\pi(1,1) = (0,\, 0,\, -1)^\top$, $\pi(0,2) = \pi(1,2) = (0,\, 0)^\top$.

6. Optimality conditions for SOCP based on a lexicographic approach

In paper [21], for convex programming problems in the form

$$\mathbf{CP}: \qquad \min f_0(x), \quad \text{s.t. } f_i(x) \leq 0,\ i \in I,$$

where $x \in \mathbb{R}^n$, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I \cup \{0\}$, are given convex functions, an optimality criterion was proposed based on another approach, namely the *lexicographical separations approach.*

Like the optimality criteria 1 and 2 proved in sections 3 and 4 for the problem (**SOCP**) (Theorems 3 and 4, respectively), this criterion does not require the fulfillment of any additional conditions for the constraints of the original problem. In this section, we will apply the optimality criterion from [21] to the problem (**SOCP**) and compare the result with the criteria proven in the previous sections.

It is evident that the problem (**SOCP**) can be formulated in the form (**CP**) with the following convex functions:

$$\begin{aligned} &f_0(x) := -b^\top x, \\ &f_i(x) := ||z_*(i,x)|| - z_0(i,x),\ i \in I, \end{aligned} \tag{47}$$

where, as before, $z(i,x) := A_i x + c(i) \in \mathbb{R}^{m_i+1}$, $z(i,x)^\top = (z_0(i,x), z_*^\top(i,x))$, $z_0(i,x) \in \mathbb{R}$, $z_*(i,x) \in \mathbb{R}^{m_i}$, $i \in I$.

Then the criterion from [21] can be reformulated as follows.

Theorem 5. *[Optimality criterion 3] A feasible solution x^0 of the problem* **(CP)** *with the functions defined by formula (47), is optimal if and only if there exist an integer number s, $0 \le s \le |I_a(x^0)|$, a vector $\lambda = (\lambda_i, i \in I)$, and an ordered partition*

$$\Delta I_0,\ \Delta I_1, \ldots, \Delta I_s, \tag{48}$$

of the index set I satisfying

(a) *the nonnegativity condition $\lambda_i \ge 0,\ i \in I$,*
(b) *the complementary slackness condition $\lambda_i f_i(x^0) = 0, i \in I$;*
(c) *the minimum conditions*

$$\sum_{i\in\Delta I_k} \lambda_i f_i(x^0) = \min_{x\in Q_k} \sum_{i\in\Delta I_k} \lambda_i f_i(x), \quad k = 0, 2, ..., s-1, \tag{49}$$

and

$$f_0(x^0) + \sum_{i\in\Delta I_s} \lambda_i f_i(x^0) = \min_{x\in Q_s} \big(f_0(x) + \sum_{i\in\Delta I_s} \lambda_i f_i(x)\big), \tag{50}$$

where $Q_0 = \mathbb{R}^n$ and

$$Q_{k+1} = \{x \in Q_k : \sum_{i\in\Delta I_k} \lambda_i f_i(x^0) = \sum_{i\in\Delta I_k} \lambda_i f_i(x)\},$$
$$k = 0, \ldots, s-1.$$

Notice that the functions $f_i(x), i \in I$, defined in (47) are convex but not smooth.

Let us compare the optimality criteria 2 and 3.

Criterion 3 looks simpler than Criterion 2, because it requires less input data for its testing. Indeed, in Criterion 3, we need to know the number s, the partition (48), and $|I|$-vector λ while in Criterion 2, we need to know the number k_0 and the set of vectors (17).

However, Criterion 2 is more constructive (since it is explicit) than Criterion 3. To apply Criterion 3, it is necessary to check whether the partition (48) and the $|I|$-vector λ satisfy conditions (49), (50). These conditions have an implicit form, since to check them, it is necessary to sequentially solve the optimization problems (49), (50) and construct (explicitly) their optimal solution sets Q_k, $k = 0, \ldots, s$. At the same time, to apply Criterion 2, one just needs to check whether the vectors in (17) satisfy conditions (18) and (19), which are **explicit** and can be easy verified.

Note that based on the explicit criterion 2, for the problem (**SOCP**), it is easy to formulate an *implicit* criterion, close in form to Criterion 3.

Theorem 6. *[Optimality criterion 4] A feasible solution $x^0 \in X$ is an optimal solution of the problem* **(SOCP)** *if and only if there exists an integer number s, $0 \le s \le |I_a(x^0)|$, a vector $\lambda = (\lambda_i, i \in I)$ and an ordered partition (48) of the index set I satisfying the following conditions:*

(a) $\lambda_i > 0,\ i \in \Delta I_k \ne \emptyset,\ k = 0, \ldots, s-1;$ $\lambda_i \ge 0,\ \lambda_i f_i(x^0) = 0,\ i \in \Delta I_s;$
(b) *the minimum conditions (49), (50), where*

$$Q_0 = \mathbb{R}^n,\ Q_{k+1} = \{x \in Q_k : f_i(x) = 0, i \in \Delta I_k\},$$
$$k = 0, \ldots, s-1.$$

The main difference between Theorems 5 and 6 is the way the sets Q_k, $k = 1, \ldots, s$, are defined.

7. Conclusions

Despite the fact that the second-order cone problems have been sufficiently studied, most of optimality conditions for these problems are formulated with some CQ. Constraint qualifications, while useful in many optimization problems, can impose restrictive assumptions on the problem structure and hinder the applicability of optimality conditions. By seeking optimality conditions that do not rely on such qualifications, researchers and practitioners can achieve a more robust and flexible framework for solving SOCPs.

The novelty of the paper consists in new optimality conditions for the second-order cone problems, namely Criteria 1 and 2. These optimality criteria are obtained using the approach based on the concept of immobile index set of the constraints of the problem and allow to detect optimality of a given feasible solution without any CQs. The absence of constraint qualifications in these criteria enhances the applicability of the theory to a broader range of optimization problems.

The findings presented in the paper enable us to conclude that the approach to optimality conditions, which is based on immobile indices and was developed in our earlier works, can be applied to the optimization of second-order cone problems.

It is worth mentioning here that there exist different formulations of exact dual problems. In the paper, we used one of them. Alternatively, it is possible to apply the same approach to other exact dual formulations and develop new optimality conditions that may have distinct properties and other ways of implementation. In the future, we will apply our approach to different classes of optimization problems.

In conclusion, it is important to recognize that all known optimality conditions for conic problems, in general, and SOCP problems, in particular, have their drawbacks and favorable properties. Nevertheless, by familiarizing oneself with a wide spectrum of optimality conditions, one can gain a more comprehensive understanding of the problem and its inherent characteristics. This empowers users to make informed decisions and select the most suitable method according to their specific requirements and preferences.

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RESEARCH ARTICLE

An accurate finite difference formula for the numerical solution of delay-dependent fractional optimal control problems

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ABSTRACT

Time-delay fractional optimal control problems (OCPs) are an important research area for developing effective control and optimization strategies to address complex phenomena occurring in various natural sciences, such as physics, chemistry, biology, and engineering. By considering fractional OCPs with time delays, we can design control strategies that take into account the system's history and optimize its behavior over a given time horizon. However, applying the Pontryagin principle of maximization to solve these problems leads to a boundary value problem (BVP) that includes delay and advance terms, making analytical solutions difficult and demanding. To address this issue, this paper presents a precise finite difference formula to solve the aforementioned advance-delay BVP numerically. The suggested approximate method's error analysis and convergence properties are provided, and several illustrative examples demonstrate the applicability, validity, and accuracy of the proposed approach. Simulation results confirm the proposed technique's advantages for the optimal control of delay fractional dynamical equations.

1. Introduction

Over the past few years, fractional calculus (FC), as a generalization of classical calculus, has attracted the attention of scientists and engineers for describing various types of physical phenomena [1]. In fact, this calculus is known as a powerful tool for the modelling of complex dynamical systems related to memory effects and non-locality [2]. The FC has some applications in epidemic modelling [3], finance [4], diffusion equations [5], outbreak control [6], quasi-synchronization [7], image diagnosis [8], chaos control [9], etc. Due to the difficulty of analytical solution for fractional dynamical systems, some efficient approximation approaches have been proposed for the numerical solution of various problems containing fractional-order operators, *e.g.*, differential equations [10], delay-dependent systems [11], etc.

Optimal control problems (OCPs) play a crucial role in determining the best strategies for controlling dynamic systems over time, with applications ranging from engineering and economics to biology and robotics [12–14]. A delay fractional OCP tries to find a control law for a delay fractional dynamical system by minimizing a cost functional in terms of the corresponding state and control variables [15]. The study of time-delay fractional OCPs is critical to develop efficient control and optimization strategies for addressing complex phenomena in various natural sciences, such as physics, chemistry, biology, and engineering.

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However, due to the high complexity of fractional OCPs with time-delay, it is extremely difficult to obtain their analytical solution [16]. To solve this issue, in the past decade, some numerical techniques have been developed including finite difference method [17, 18], Bernstein polynomials [19], Legendre polynomials [20, 21], linear programming technique [22], Lagrange polynomials [23], neural network [24], Taylor expansions [25], Chelyshkov wavelets [26], embedding process [27], and fractional orthogonal basis functions [28]. More recently, the paper [29] presented a collocation method for solving nonlinear delay fractional optimal control systems with constraints on the state and control variables. Another study [30] focused on time-optimal feedback control of nonlocal Hilfer fractional state-dependent delay inclusion with Clarke's subdifferential. The new work [31] also introduced Mittag-Leffler wavelets and their applications for solving fractional OCPs with and without delay.

The field of fractional OCPs with time delays presents a significant challenge due to the complexity introduced by considering both FC and time-delay terms simultaneously. While there is existing research on fractional OCPs and time-delay systems independently, the intersection of these two areas remains relatively unexplored. Current methods for solving delay-dependent fractional OCPs often face difficulties in providing accurate and efficient solutions due to the intricate nature of the boundary value problem (BVP) resulting from applying the Pontryagin maximum principle. Analytical solutions for such advance-delay BVPs are scarce, leading to a gap in the literature regarding effective numerical solution techniques tailored specifically for this challenging class of problems. Therefore, there is a pressing need for innovative approaches that can accurately and reliably address the unique characteristics of delay-dependent fractional OCPs, providing researchers and practitioners with appropriate tools for optimizing complex dynamical systems subjected to FC and time delays.

This research article addresses the above-mentioned critical research gap in the field of fractional OCPs with time delays. The study's significance lies in its focus on developing effective control and optimization strategies for complex phenomena present in various natural sciences and engineering, where FC and time delays play crucial roles. By introducing a precise finite difference formula to numerically solve advance-delay BVPs arising from applying the Pontryagin maximum principle, this research offers an innovative approach tailored specifically for this challenging class of problems. The study's novelty is evident in its unique contributions, including the development of a novel numerical solution technique for delay-dependent fractional OCPs, comprehensive error analysis and convergence properties of the proposed method, as well as illustrative examples demonstrating its applicability and accuracy. This research's potential impact is substantial, as it provides researchers and practitioners with appropriate tools for optimizing complex dynamical systems subjected to FC and time delays, ultimately advancing the state-of-the-art in this underexplored intersection of FC and time-delay systems.

2. Problem Statement

Consider the following fractional dynamical system with time-delay

$$\begin{cases} {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = A_1(\tau)z(\tau) + A_d(\tau)z(\tau - m) \\ \qquad\qquad +B_1(\tau)v(\tau), \quad \tau_0 \leq \tau \leq \tau_f, & \text{(1a)} \\ z(\tau) = \psi(\tau), \quad \tau_0 - m \leq \tau \leq \tau_0, & \text{(1b)} \end{cases}$$

in which $z \in \mathbb{R}^q$ is the state vector, and the symbol ${}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau)$ signifies the left Caputo fractional derivative [32]

$${}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = \frac{1}{\Gamma(1-\gamma)} \int_{\tau_0}^{\tau} (\tau - \xi)^{-\gamma} \frac{dz(\xi)}{d\xi} d\xi, \quad (2)$$

in which the derivative order is denoted by γ $(0 < \gamma \leq 1)$. Also, the parameter m is the state time-delay, $v \in \mathbb{R}^r$ is the control variable, and the coefficients $A_1(\tau)$, $A_d(\tau)$, and $B_1(\tau)$ are continuous-time matrix functions. Following the optimal control concept, it is desired to determine the control $v(\tau)$ minimizing the following performance index

$$J = \frac{1}{2} \int_{\tau_0}^{\tau_f} \left(z^T(\tau) Q z(\tau) + v^T(\tau) R v(\tau) \right) d\tau, \quad (3)$$

where the matrices $R \in \mathbb{R}^{r\times r}$ and $Q \in \mathbb{R}^{q\times q}$ are, respectively, assumed to be positive definite and positive semi-definite.

Theorem 1. *(Pontryagin conditions of optimality) Under the constraint given by the dynamical system* (1), *if* $(z(\tau), v(\tau))$ *is a minimizer of* (3), *then the costate vector* $y(\tau)$

exists such that the following conditions are satisfied:

- *the Hamiltonian system, for $\tau_0 \le \tau \le \tau_f$,*

$$\begin{cases} {}_{\tau_0}^{C}\mathbb{D}_{\tau}^{\gamma} z(\tau) = \dfrac{\partial \mathcal{H}}{\partial y(\tau)}, & \text{(4a)} \\ {}_{\tau}^{R}\mathbb{D}_{\tau_f}^{\gamma} y(\tau) = \dfrac{\partial \mathcal{H}}{\partial z(\tau)} + A_2(\tau) y(\tau+m), & \text{(4b)} \end{cases}$$

- *the stationary condition, for $\tau_0 \le \tau \le \tau_f$,*

$$\frac{\partial \mathcal{H}}{\partial v(\tau)} = 0, \tag{5}$$

- *and the transversality condition*

$$y(\tau)|_{\tau=\tau_f} = 0, \tag{6}$$

where ${}_{\tau}^{R}\mathbb{D}_{\tau_f}^{\gamma} y(\tau)$ $(0 < \gamma \le 1)$ is the γ-th order right Riemann-Liouville fractional derivative of $y(\tau)$ defined by [32]

$${}_{\tau}^{R}\mathbb{D}_{\tau_f}^{\gamma} y(\tau) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{d\tau} \int_{\tau}^{\tau_f} (\xi - \tau)^{-\gamma} y(\xi) d\xi, \tag{7}$$

$A_2(\tau) = A_d(\tau + m)\chi_{[\tau_0, \tau_f - m]}(\tau)$, and $\chi_{[a,b]}$ represents the characteristic function on the interval $[a, b]$. The function $\mathcal{H}$, called the Hamiltonian, has also the following form

$$\begin{aligned} \mathcal{H} := &0.5 \left(z^T(\tau) Q z(\tau) + v^T(\tau) R v(\tau) \right) \\ &+ y^T(\tau) \left(A_1(\tau) z(\tau) + A_d(\tau) z(\tau - m) \right. \\ &\left. + B_1(\tau) v(\tau) \right). \end{aligned} \tag{8}$$

Proof. First, we adjoin the dynamical constraint (1) to the performance index (3) by introducing the Lagrange multiplier $y(\tau) \in \mathbb{R}^q$, so the following augmented functional can be formed

$$J_a(v) = \int_{\tau_0}^{\tau_f} \left[\mathcal{H} - y^T(\tau)\, {}_{\tau_0}^{C}\mathbb{D}_{\tau}^{\gamma} z(\tau) \right] d\tau. \tag{9}$$

Let $\delta f(\tau)$ denote the variation of the function $f(\tau)$; then we take the variation of $J_a(v)$ as

$$\begin{aligned} \delta J_a(v) = \int_{\tau_0}^{\tau_f} \Bigg\{ &\left[\frac{\partial \mathcal{H}}{\partial z(\tau)} \right]^T \delta z(\tau) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial z(\tau - m)} \right]^T \delta z(\tau - m) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial y(\tau)} - {}_{\tau_0}^{C}\mathbb{D}_{\tau}^{\gamma} z(\tau) \right]^T \delta y(\tau) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial v(\tau)} \right]^T \delta v(\tau) \\ &- y^T(\tau)\, {}_{\tau_0}^{C}\mathbb{D}_{\tau}^{\gamma} \delta z(\tau) \Bigg\} d\tau. \end{aligned} \tag{10}$$

Next, it is easily derived that

$$\begin{aligned} &\int_{\tau_0}^{\tau_f} \left\{ \left[\frac{\partial \mathcal{H}}{\partial z(\tau - m)} \right]^T \delta z(\tau - m) \right\} dt \\ &= \int_{\tau_0}^{\tau_f} \left\{ y^T(\tau) A_d^T(\tau) \delta z(\tau - m) \right\} d\tau \\ &= \int_{m}^{\tau_f} (A_d(\tau) y(\tau))^T \delta z(\tau - m) d\tau \\ &= \int_{\tau_0}^{\tau_f} (A_2(\tau) y(\tau + m))^T \delta z(\tau) d\tau, \end{aligned} \tag{11}$$

where $A_2(\tau) = A_d(\tau + m)\chi_{[\tau_0, \tau_f - m]}(\tau)$, and $\chi_{[a,b]}$ denotes the characteristic function on the interval $[a, b]$. Furthermore, by using the fractional integration by parts [32] and taking into account the transversality condition (6), we have

$$\begin{aligned} &\int_{\tau_0}^{\tau_f} y^T(\tau)\, {}_{\tau_0}^{C}\mathbb{D}_{\tau}^{\gamma} \delta z(\tau) d\tau \\ &= \int_{\tau_0}^{\tau_f} \left({}_{\tau}^{R}\mathbb{D}_{\tau_f}^{\gamma} y(\tau) \right)^T \delta z(\tau) d\tau. \end{aligned} \tag{12}$$

From Eqs. (10), (11) and (12), we deduce

$$\begin{aligned} \delta J_a(v) = \int_{\tau_0}^{\tau_f} \Bigg\{ &\left[\frac{\partial \mathcal{H}}{\partial z(\tau)} + A_2(\tau) y(\tau + m) \right. \\ &\left. - {}_{\tau}^{R}\mathbb{D}_{\tau_f}^{\gamma} y(\tau) \right]^T \delta z(\tau) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial y(\tau)} - {}_{\tau_0}^{C}\mathbb{D}_{\tau}^{\gamma} z(\tau) \right]^T \delta y(\tau) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial v(\tau)} \right]^T \delta v(\tau) \Bigg\} d\tau. \end{aligned} \tag{13}$$

On an extremal v^*, we require that $\delta J_a(v^*) = 0$. Thus, in Eq. (13), each factor multiplying a variation has to be vanished. Since $z(\tau_0)$ is specified, it is concluded $\delta z(\tau_0) = 0$, but $\delta z(\tau_f)$ is not equal to 0; thus, it is required that $y(\tau_f) = 0$. Furthermore, the necessary conditions given by Eqs. (4) and (5) are achieved by setting to 0 the coefficients of $\delta z(\tau)$, $\delta y(\tau)$, and $\delta v(\tau)$ in Eq. (13). □

Applying the Pontryagin's optimality conditions given by Theorem 1 for the time-delay fractional OCP (1)-(3) leads to the following fractional advance-delay BVP

$$\begin{cases} {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = A_1(\tau)z(\tau) \\ +A_d(\tau)z(\tau-m) - S(\tau)y(\tau), & (14a) \\ \tau_0 \le \tau \le \tau_f, \\ \\ {}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} y(\tau) = Qz(\tau) + A_1^T(\tau)y(\tau) \\ +A_2(\tau)y(\tau+m), & (14b) \\ \tau_0 \le \tau \le \tau_f, \end{cases}$$

with the following conditions

$$\begin{cases} z(\tau) = \psi_1(\tau), \quad \tau_0 - m \le \tau \le \tau_0, & (15a) \\ y(t_f) = 0, & (15b) \end{cases}$$

where $y(\tau+m)$ is the advance term in time, $z(\tau-m)$ is the time-delay argument, and $S(\tau) = B_1(\tau)R^{-1}B_1^T(\tau)$. Moreover, the optimal control law has the following form

$$v^*(\tau) = -R^{-1}B_1^T(\tau)y(\tau), \quad \tau_0 \le \tau \le \tau_f. \quad (16)$$

The analytical solution of the fractional BVP (14)-(15), including the advance and the delay arguments, is not accessible. Thus, our main objective is to develop an effective approximate procedure to solve the above-mentioned BVP numerically.

3. Some Notations and Lemmas

The fractional derivatives in the senses of left Caputo and right Riemann-Liouville have previously been defined in (2) and (7), respectively. In the following, we give some more definitions and properties of Caputo and Riemann-Liouville fractional operators.

The left Riemann-Liouville fractional derivative of $z(\tau)$ is defined by [32]

$${}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{d\tau}\int_{\tau_0}^{\tau}(\tau-\xi)^{-\gamma}z(\xi)d\xi, \quad (17)$$

where $0 < \gamma \le 1$ denotes the fractional order.

Regarding the left and right fractional derivatives in the senses of Riemann-Liouville and Caputo, the following properties hold [32]

$$\begin{cases} {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = {}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) \\ \qquad - \dfrac{z(\tau_0)}{\Gamma(1-\gamma)}(\tau-\tau_0)^{-\gamma}, \\ \\ {}^{C}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} z(\tau) = {}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} z(\tau) \\ \qquad - \dfrac{z(\tau_f)}{\Gamma(1-\gamma)}(\tau_f-\tau)^{-\gamma}. \end{cases} \quad (18)$$

Definition 1. *In order to approximate the left and right Riemann-Liouville fractional derivatives, the shifted Grünwald-Letnikov (SGL) difference operators are defined as below [33]*

$$\Lambda^{\gamma}_{h,p} z(\tau) = \frac{1}{h^{\gamma}} \sum_{k=0}^{[\frac{\tau-\tau_0}{h}]+p} w_k^{(\gamma)} z(\tau-(k-p)h), \quad (19)$$

$$\Upsilon^{\gamma}_{h,p} z(\tau) = \frac{1}{h^{\gamma}} \sum_{k=0}^{[\frac{\tau_f-\tau}{h}]+p} w_k^{(\gamma)} z(\tau+(k-p)h), \quad (20)$$

where h is the time step size, p is an integer, and $w_k^{(\gamma)} = (-1)^k \begin{pmatrix} \gamma \\ k \end{pmatrix}$. Also, within the following power series, the coefficients $w_k^{(\gamma)}$ are satisfied

$$(1-x)^{\gamma} = \sum_{k=0}^{\infty} w_k^{(\gamma)} x^k, \quad (21)$$

so the following recursive formula computes them

$$w_0^{(\gamma)} = 1, \quad w_k^{(\gamma)} = (1 - \frac{\gamma+1}{k}) w_{k-1}^{(\gamma)}, \quad k \ge 1. \quad (22)$$

From (21) and (22), some important properties of the coefficients $w_k^{(\gamma)}$ can easily be deduced, as stated in the following lemma.

Lemma 1. *Let $0 < \gamma < 1$; then the coefficients $w_k^{(\gamma)}$, given by Eq. (22), satisfy the properties*

(1) $w_0^{(\gamma)} = 1, \quad w_1^{(\gamma)} = -\gamma, \quad w_k^{(\gamma)} < 0, \quad k \ge 2,$
(2) $-\sum_{k=1}^{n} w_k^{(\gamma)} < 1, \quad \forall\ n \ge 1,$
(3) $\sum_{k=0}^{\infty} w_k^{(\gamma)} = 0.$

Now, the space function $\mathcal{L}^j(\mathbb{R})$ is defined as

$$\mathcal{L}^j(\mathbb{R}) = \Big\{z : \int_{-\infty}^{\infty}(1+|\omega|)^j |\hat{z}(\omega)| d\omega < \infty; \ \hat{z} \text{ is the Fourier transform of } z\Big\}. \quad (23)$$

It is easy to show that for $0 < \gamma \le 1$, if $z \in \mathcal{L}^2(\mathbb{R})$, then $z \in \mathcal{L}^{1+\gamma}(\mathbb{R})$.

Lemma 2. *Let $z(\tau) \in C^j(\mathbb{R})$, $\frac{d^{j+1}z(\tau)}{d\tau^{j+1}} \in \mathcal{L}^1(\mathbb{R})$, $\frac{d^k z(\tau)}{d\tau^k}|_{\tau=\tau_0} = 0$ for $k = 0,1,2,\ldots,j$, and $0 < \gamma \le 1$; then*

$$\Lambda^{\gamma}_{h,p} z(\tau) = {}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) + \sum_{l=1}^{j-1} \omega_l(p)\ {}^{R}_{\tau_0}\mathbb{D}^{\gamma+l}_{\tau} z(\tau) h^l + \mathcal{O}(h^j), \quad (24)$$

in which $\omega_l(p)$ is the coefficient of the power series $\left(\frac{1-e^{-x}}{x}\right)^{\gamma} e^{px} - 1$; in particular,

$$\omega_1(p) = p - \frac{\gamma}{2}, \quad \omega_2(p) = \frac{\gamma}{24} + \frac{1}{2}(p - \frac{\gamma}{2})^2. \quad (25)$$

Proof. The proof of this lemma is easily followed from Theorem 1 in [34]. □

Using Lemma 2, we can formulate a third-order difference operator for the Riemann-Liouville fractional derivative (17), as given by the following definition.

Definition 2. *We define a weighted SGL difference operator for the Riemann-Liouville fractional derivative* (17) *as follows*

$$ {}^{R}_{\tau_0}\Delta^{\gamma}_{h} z(\tau) = \frac{2+\gamma}{2}\Lambda^{\gamma}_{h,0} z(\tau) - \frac{\gamma}{2}\Lambda^{\gamma}_{h,-1} z(\tau), \quad (26) $$

where the operator $\Lambda^{\gamma}_{h,p}$ has been given by (19).

Lemma 3. *Let $0 < \gamma \le 1$, and $z(\tau)$, its Fourier transform, and ${}^{R}_{\tau_0}\mathbb{D}^{\gamma+2}_{\tau} z(\tau)$ belong to $\mathcal{L}^1(\mathbb{R})$. Then for $\tau \in \mathbb{R}$*

$$ {}^{R}_{\tau_0}\Delta^{\gamma}_{h} z(\tau) = {}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) + \mathcal{O}(h^2), \quad (27) $$

uniformly as $h \to 0$, where the operator ${}^{R}_{\tau_0}\Delta^{\gamma}_{h} z(\tau)$ has been defined in (26).

Proof. Let $\mathcal{F}[z(\tau)](\omega) = \hat{z}(\omega) = \int \mathrm{e}^{-i\omega\xi} z(\xi) d\xi$ be the Fourier transform of $z(\tau)$, where $i = \sqrt{-1}$; thus, we have $\mathcal{F}[z(\tau - kh)](\omega) = \mathrm{e}^{-ik\omega h}\hat{z}(\omega)$. For each $\tau \in \mathbb{R}$, we also have $\mathcal{F}[{}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau)](\omega) = (i\omega)^{\gamma}\hat{z}(\omega)$. Applying the Fourier transform to the both sides of Eq. (26), for each $\tau \in \mathbb{R}$ we obtain

$$ \begin{aligned} &\mathcal{F}[{}^{R}_{\tau_0}\Delta^{\gamma}_{h} z(\tau)](\omega) \\ &= \frac{1}{h^{\gamma}}(1-\mathrm{e}^{-i\omega h})^{\gamma}\left(\frac{2+\gamma}{2} - \frac{\gamma}{2}\mathrm{e}^{-i\omega h}\right)\hat{z}(\omega) \\ &= \sigma_2(i\omega h)(i\omega)^{\gamma}\hat{z}(\omega), \end{aligned} \quad (28) $$

where

$$ \begin{aligned} \sigma_2(x) &= \left(\tfrac{1-\mathrm{e}^{-x}}{x}\right)^{\gamma}\left(\tfrac{2+\gamma}{2} - \tfrac{\gamma}{2}\mathrm{e}^{-x}\right) \\ &= 1 - \frac{\gamma}{24}(5+3\gamma)x^2 + \mathcal{O}(x^3). \end{aligned} \quad (29) $$

There exists a positive constant C_2 such that $|1 - \sigma_2(-ix)| \le C_2|x|^2$. Now, we apply the inverse Fourier transform; since $z(\tau) \in \mathcal{L}^{\gamma+2}(\mathbb{R})$, we derive

$$ \begin{aligned} &\left|{}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) - {}^{R}_{\tau_0}\Delta^{\gamma}_{h} z(\tau)\right| \\ &= \Big|\frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega\tau} \times \\ &\qquad (\mathcal{F}[{}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) - {}^{R}_{\tau_0}\Delta^{\gamma}_{h} z(\tau)](\omega)) d\omega\Big| \\ &= \Big|\frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega\tau} \times \\ &\qquad (1-\sigma_2(i\omega h))(i\omega)^{\gamma}\hat{z}(\omega) d\omega\Big| \\ &\le |h|^2 \frac{1}{2\pi i}\int_{-\infty}^{\infty} |\omega|^{2+\gamma}|\hat{z}(\omega)| d\omega \\ &\le C_2|h|^2 \frac{1}{2\pi i}\int_{-\infty}^{\infty} |1+\omega|^{2+\gamma}|\hat{z}(\omega)| d\omega \\ &\le \tilde{C}|h|^2, \end{aligned} \quad (30) $$

where $\tilde{C} = \frac{C_2}{2\pi i}\int_{-\infty}^{\infty} |1+\omega|^{2+\gamma}|\hat{z}(\omega)| d\omega$. □

Definition 3. *From* (26), *we can formally define the second-order weighted SGL difference (SGL2) operators as follows for the left and right Riemann-Liouville fractional derivatives*

$$ {}^{R}_{\tau_0}\Delta^{\gamma}_{h} z(\tau_n) = \frac{1}{h^{\gamma}}\sum_{k=0}^{n} g^{(\gamma)}_{k} z(\tau_n - kh), \quad (31) $$

$$ {}^{R}_{\tau_f}\Delta^{\gamma}_{h} z(\tau_n) = \frac{1}{h^{\gamma}}\sum_{k=0}^{n} g^{(\gamma)}_{k} z(\tau_n + kh), \quad (32) $$

where h is the time step size and

$$ \begin{cases} g^{(\gamma)}_{0} = \frac{2+\gamma}{2} w^{(\gamma)}_{0}, \\ g^{(\gamma)}_{k} = \frac{2+\gamma}{2} w^{(\gamma)}_{k} - \frac{\gamma}{2} w^{(\gamma)}_{k-1}, \quad k = 2, 3, \ldots. \end{cases} \quad (33) $$

Lemma 3 shows that the SGL2 operator (31) has the second-order of accuracy at every time level.

Remark 1. *Let $z(\tau_0) = 0$ and $0 < \gamma \le 1$; then by using integrating by parts, we have*

$$ \begin{aligned} {}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(h) &= \frac{1}{\Gamma(1-\gamma)}\int_{\tau_0}^{h} \frac{z'(\xi)}{(h-\xi)^{\gamma}} d\xi \\ &= \frac{z'(\tau_0)h^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{1}{\Gamma(2-\gamma)}\int_{\tau_0}^{h} \frac{z''(\xi)}{(h-\xi)^{\gamma-1}} d\xi. \end{aligned} \quad (34) $$

Therefore, if the function $z(\tau)$ has no derivative at $\tau = \tau_0$, then the SGL2 formula (31) *is of accuracy order $1-\gamma$. Moreover, the SGL2 formula is of accuracy order $2-\gamma$ if $z'(\tau_0) = 0$ and the second derivative of $z(\tau)$ does not exist at $\tau = \tau_0$.*

Now, we present the following properties for $\{g^{(\gamma)}_{k}\}$ by using Lemmas 1 and 3.

Lemma 4. *For $0 < \gamma \le 1$, the following properties are satisfied by the coefficients in* (33)*:*

(1) $g^{(\gamma)}_{0} = 1 + \frac{\gamma}{2}$, $g^{(\gamma)}_{1} = -\frac{\gamma(\gamma+3)}{2}$, $g^{(\gamma)}_{2} = \frac{\gamma(\gamma+3\gamma-2)}{4}$, $g^{(\gamma)}_{k} < 0$, $k \ge 3$,
(2) $-\sum_{k=1}^{n} g^{(\gamma)}_{k} < g^{(\gamma)}_{0}$, $\forall\ n \ge 2$,
(3) $\sum_{k=0}^{\infty} g^{(\gamma)}_{k} = 0$.

4. Numerical Method Formulation

Following the theoretical parts given in the previous section, here we formulate an accurate finite difference method to solve the fractional advance-delay BVP (14)-(15). To this end, first consider that the approximate values of $z(\tau_n)$ and $y(\tau_n)$ are denoted by z_n and y_n, respectively. Applying the SGL2 formulas (31) and (32) on the uniform grid points $\tau_n = \tau_0 + nh$ ($n = 0, 1, \ldots, N$) with $h = \frac{\tau_f - \tau_0}{N}$ as the time step size, a full discretization of the Pontryagin's conditions

(14)-(15) is formulated as follows

$$\begin{cases} {}_{\tau_0}^{R}\Delta_h^{\gamma} z_n = A_1(\tau_n) z_n + A_d(\tau_n)\hat{z}_n \\ \qquad -S(\tau_n) y_n,\ \tau_0 \le \tau \le \tau_f, & (35a) \\ {}_{\tau_f}^{R}\Delta_h^{\gamma} y_n = Q z_n + A_1^T(\tau_n) y_n \\ \qquad +A_2(\tau_n)\tilde{y}_n,\ \tau_0 \le \tau \le \tau_f, & (35b) \\ z_{-n} = \psi(\tau_{-n}),\ n = 0,1,2,\ldots, & (35c) \\ y_N = 0, & (35d) \end{cases}$$

where $\tau_{-n} = \tau_0 - nh$, and

$$\hat{z}_n = z(\tau_n - h) \approx \begin{cases} \psi(\tau_n - h), \\ \tau_n - h \le \tau_0, \\ \\ p_1(\tau_n - h; z_k, z_{k+1}), \\ \tau_0 < \tau_k \le \tau_n - h < \tau_{k+1}, \end{cases} \quad (36)$$

$$\tilde{y}_n = y(\tau_n + h) \approx \begin{cases} p_1(\tau_n + h; y_{i-1}, y_i), \\ \tau_{i-1} \le \tau_n + h < \tau_i, \\ \\ 0, \\ \tau_f \le \tau_n + h, \end{cases} \quad (37)$$

in which $0 \le i, k \le N-1$. Besides, the function p_1 is the linear interpolation polynomial

$$p_1(\xi; z_k, z_{k+1}) = \frac{\xi - \tau_k}{h} z_{k+1} + \frac{\tau_{k+1} - \xi}{h} z_k, \quad (38)$$

determined by the support points (τ_k, z_k) and (τ_{k+1}, z_{k+1}). Therefore, the value of the optimal control for $n = 0, 1, \ldots, N$ is approximated by

$$v_n^* = -R^{-1} B_1^T(\tau_n) y_n, \quad (39)$$

where v_n^* represents the numerical approximation of $v^*(\tau_n)$.

5. Numerical Examples

Here, we employ three numerical examples to show the effectiveness of the proposed finite difference technique. Comparative results are also given to verify the superiority of the suggested scheme over the other methodologies available in the literature.

Example 1. *As the first case, consider a delay fractional OCP in the form of minimizing*

$$J = \frac{1}{2}\int_0^2 \left(z^2(\tau) + v^2(\tau)\right) d\tau, \quad (40)$$

subject to

$$\begin{cases} {}_{0}^{C}\mathbb{D}_{\tau}^{\gamma} z(\tau) = \tau z(\tau - 1) + v(\tau), & 0 \le \tau \le 2, \\ z(\tau) = 1, & -1 \le \tau \le 0. \end{cases} \quad (41)$$

Solving the problem (40)-(41) for different values of γ, we portray, in Figure 1, the approximate state and control functions. Meanwhile, the performance index values $J = 1.0807, 1.0658, 1.0510$ were attained for $\gamma = 0.8, 0.9, 1$, respectively. As can be seen from Figure 1, the numerical approximation goes to the classic solution when γ tends to unity. Also, as depicted in Table 1, the cost functional values obtained by our proposed scheme is less than those previously achieved in [35] by using a linear programming (LP) control strategy. Thus, the given comparative discussion in this part verifies the efficiency of the suggested technique for solving the fractional OCP (40)-(41).

Table 1. Comparison of the approximate values for J (Example 1).

γ	Method	
	LP strategy [35]	Proposed technique
0.8	1.0807	1.0658
0.9	1.0658	1.0658
0.1	1.0514	1.0510

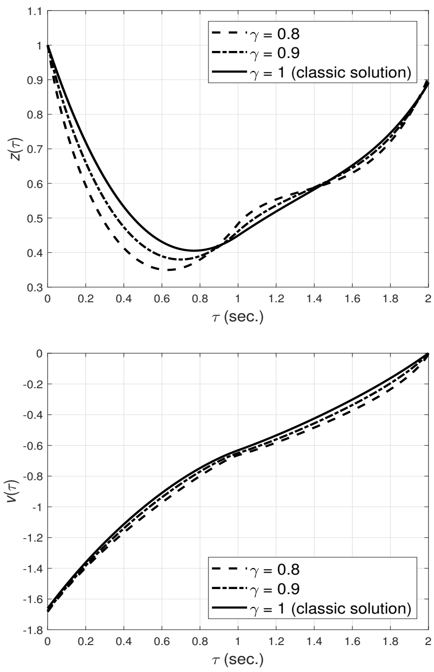

Figure 1. Simulation curves of $z(\tau)$ and $v(\tau)$ for Example 1.

Example 2. *Let us take into account, as the second example, the performance index*

$$J = \frac{1}{2}\int_0^1 \left\{ (z_1(\tau) + z_2(\tau))^2 + v^2(\tau) \right\} d\tau, \quad (42)$$

together with the delay fractional dynamical equations

$$\begin{cases} {}^C_0\mathbb{D}^\gamma_\tau z_1(\tau) = \tau z_1(\tau) + z_2(\tau - \frac{1}{4}),\ 0 \le \tau \le 1, \\ {}^C_0\mathbb{D}^\gamma_\tau z_2(\tau) = \tau^2 z_2(\tau) - 5z_1(\tau - \frac{1}{4}) \\ \quad -z_2(\tau - \frac{1}{4}) + v(\tau),\ 0 \le \tau \le 1, \end{cases} \tag{43}$$

and the initial conditions

$$\begin{bmatrix} z_1(\tau) \\ z_2(\tau) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad -\frac{1}{4} \le \tau \le 0. \tag{44}$$

We plot the state and control variables in Figure 2 for some values of γ. Also, the performance index values $J = 2.7999, 2.2393, 1.7548$ were obtained for $\gamma = 0.8, 0.9, 1$, respectively. Comparing the results with those reported in [35] shows a good agreement, a fact which confirms the efficiency of our proposed scheme to solve the delay fractional OCP (42)-(44). In addition, the classic solution is recovered by the fractional response in Figure 2 when γ goes to 1, a fact which is in line with the correctness of our numerical implementation.

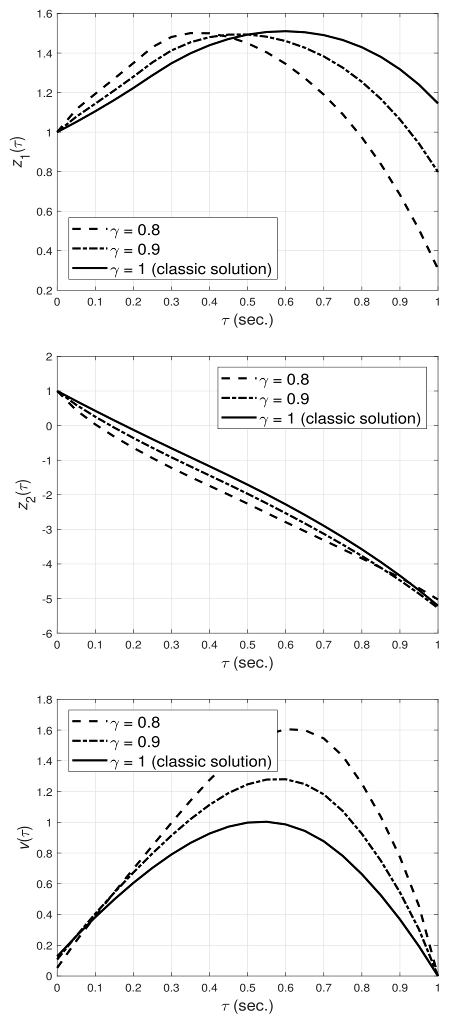

Figure 2. Simulation curves of $z_1(\tau)$, $z_2(\tau)$, and $v(\tau)$ for Example 2.

Example 3. *As a practical case, here we consider the minimization of*

$$J = \int_0^{t_f} \left(10^4 z_1^2(\tau) + v^2(\tau)\right) d\tau, \tag{45}$$

subject to the simplified fractional model

$$\begin{aligned} {}^C_0\mathbb{D}^\gamma_\tau z(\tau) &= \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & -2\xi\omega \end{bmatrix} z(\tau) \\ &+ \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z(\tau - 0.33) \\ &+ \begin{bmatrix} 0 \\ 0 \\ \omega^2 \end{bmatrix} v(\tau), \quad \tau \ge 0, \end{aligned} \tag{46}$$

which is connected to a wind tunnel at the NASA Langley Research Center. The vector $z(\tau)$ *represents* $z(\tau) = (z_1(\tau), z_2(\tau), z_3(\tau))$, *the parameters in the model* (46) *take the values* $\frac{1}{a} = 1.964$, $\xi = 0.8$, $\omega = 6$, *and* $k = -0.0117$, *and the initial conditions are considered as*

$$z(\tau) = \begin{bmatrix} -0.1 \\ 8.547 \\ 0 \end{bmatrix}, \quad -0.33 \le \tau \le 0. \tag{47}$$

Simulation curves of $z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$, and $v(\tau)$ for $\tau_f = 20$ and $\gamma = 0.8, 0.9, 1$ are shown in Figure 3. This figure confirms the convergence of the fractional response to the classic solution, given in [36], as γ goes to 1. Comparison of our numerical findings with those reported in [35] also shows that the new scheme is accurate and efficient to solve the delay fractional OCP (45)-(47).

6. Conclusion

In this study, we presented an approximate numerical solution for time-delay fractional OCPs using a novel finite difference formula. We began by formulating the optimality conditions as a system of fractional advance-delay BVPs and then applied our accurate finite difference method to solve these complex problems. The error analysis and convergence properties of the proposed method were discussed in detail, demonstrating its reliability and effectiveness. Through several illustrative examples and associated simulation results, we showed the accuracy, validity, and correctness of our approach. In particular, our third example, which is connected to a wind tunnel at the NASA Langley Research Center, served as a practical case demonstrating the applicability of our method to real-world problems in engineering

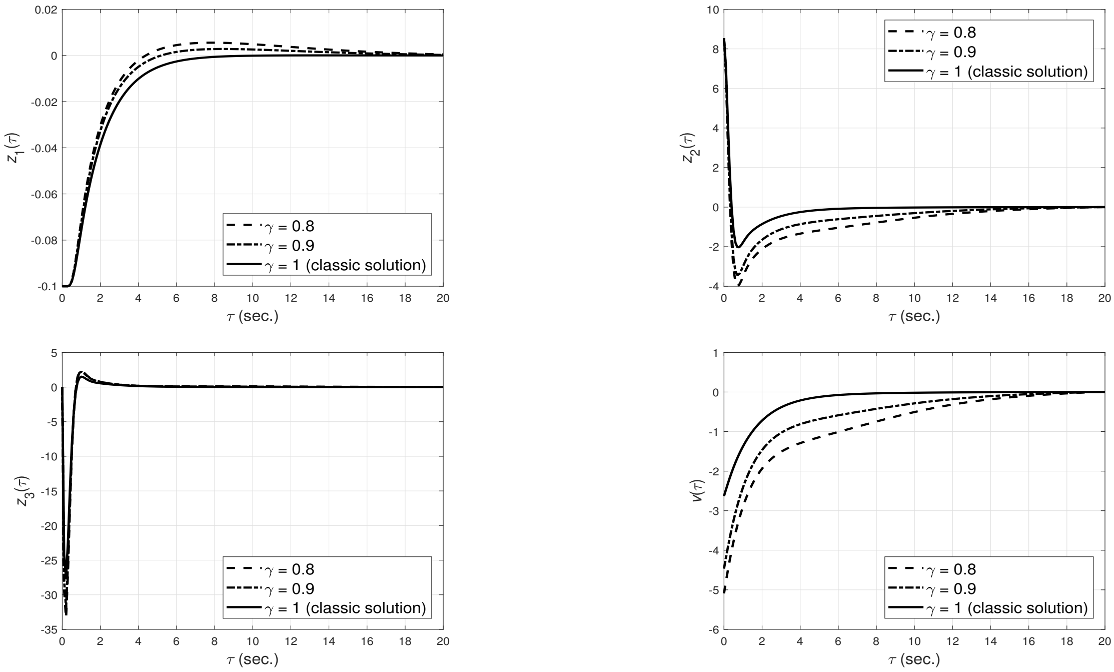

Figure 3. Simulation curves of $z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$, and $v(\tau)$ for Example 3.

and aerodynamics. Furthermore, comparative experiments highlighted the superiority of our new method over other approximation schemes developed in previous studies. These results not only validate the effectiveness of our approach but also emphasize its potential for addressing challenging problems in various natural sciences and engineering disciplines. Looking ahead, future perspectives of our work include exploring extensions of the proposed method to more complex systems and further practical applications. Future research directions may also involve further refining the algorithm, exploring additional applications across diverse scientific disciplines, and potentially integrating advanced computational techniques to enhance the method's efficiency.

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RESEARCH ARTICLE

Artificial bee colony algorithm for operating room scheduling problem with dedicated/flexible resources and cooperative operations

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ABSTRACT

In this study operating room scheduling (ORS) problem is addressed in multi-resource manner. In the addressed problem, besides operating rooms (ORs) and surgeons, the anesthesia team is also considered as an additional resource. The surgeon(s) who will perform the operation have already been assigned to the patients and is a dedicated resource. The assignment of the anesthesia team has been considered as a decision problem and a flexible resource. In this study, cooperative operations are also considered. A mixed integer linear programming (MILP) model is proposed for the problem. Since the problem is NP-hard, an artificial bee colony (ABC) algorithm is proposed for the problem. The solutions of the ABC are compared with the MILP model and random search.

1. Introduction

For many hospitals, operating rooms (ORs) are the costliest unit, but they are also the unit that makes the biggest contribution to the hospital's income. Therefore, the planning of ORs is important for hospitals [1]. Scheduling activities are important in the effective management of ORs. Patient assignment to ORs and determining the starting time of the operations becomes a complex problem due to additional resources [2]. In many hospitals, ORs are scheduled manually. As a result of the manual solutions of such a complex problem, ineffective schedules are created. By using optimization methods in the operating room scheduling (ORS) problem, it may be possible for the hospital management to serve more effectively to patients and managed the ORs efficiently [2].

ORS problems are an important problem that is studied frequently. Literature reviews on the ORS problem are reachable to related articles [3-8]. ORS problems can be classified according to various criteria. These criteria can be considered as the resources, resource types, scheduling period, objective functions, patient types, solution methods and additional features [9].

ORS problems are resource-constrained problems. The limited resources considered in ORS problems are surgeons, downstream beds [10], nurses, anesthesia team and equipment/tools. If the resources under consideration have been previously assigned to patients, they are classified as dedicated resources. If the assignment of resources is considered as a decision problem, it is classified as flexible resources [11].

According to the scheduling period, it is considered as a single/multi period. If scheduling is done for only one day, it is called a single period, if it is done for more than one day, it is called multi-period [12]. The scheduling of ORs is considered in two stages in hospitals. In the first stage, the patient's operation is assigned to a future date and it is long-term planning. The second stage is short-term planning, and it is the stage of determining the operation start times and assignment of ORs to patients on the relevant day. In short-term planning, only daily planning is done in hospitals [2].

Classification of the patients can be made as elective and emergency patients. In some studies, only elective patients are considered. Because in many hospitals, separate ORs are dedicated for emergency surgeries [2]. There are also studies that consider both elective and emergency patients [13]. In some studies, patients are prioritized according to the urgency of their surgery [14].

Many different objective functions are considered in ORS problems. There are multi-objective studies as well as studies that consider single objective function. Minimizing total cost, tardiness, overtime, idle time, waiting time, number of ORs, total completion time, maximum completion time (makespan), maximizing resource balancing [15], maximizing number of patients [16], service level are objective functions of the ORS problems [4].

*Corresponding author

Solution approaches of the ORS problems can be classified as exact and heuristic solutions. Since ORS problems are NP-hard problems, heuristic algorithms were proposed for solving large-sized problems [17]. Heuristic algorithms do not guarantee the best solution. Mathematical modeling [18], decomposition algorithms ([19] and [20]), branch and price, branch and cut [21], column generation [22] are exact solution methods that guarantee the best solution.

Real-life constraints should be taken into account as much as possible while defining the ORS problem. In other words, the problem should reflect the real-life problem as much as possible [23]. For this purpose, additional features are taken into account in many studies. In some studies, some parameters are considered fuzzy or stochastic [11]. Another feature that has been addressed is necessity of more than one surgeon in an operation [2]. Such operations are considered as cooperative operations. All employed surgeons must be available in order to perform the operation of the relevant patient. In some studies, making up of the team is considered [24]. In addition, the skill compatibility feature and the eligibilities on ORs and surgeons are considered. Not every patient can be assigned to every OR or surgeon with eligibility constraints [2].

The ORS literature was reviewed considering the classification of the problem. In most of the early studies on the subject, only surgeons and/or the ORs were considered as resources [25]. Fei et al. [25], proposed a column generation method for the solution of ORS problem. Fei et al. [26], proposed hybrid genetic algorithm (GA) for ORS problem. They considered multi- period feature. Vijayakumar et al. [27], considered nurses and equipment as additional resources. They proposed heuristic algorithms. Priorities of patients was taken into account. Agnetis et al. [19], proposed a decomposition algorithm for ORS problem. Fügener et al. [28], considered multiple downstream units for ORS problem. They proposed an exact solution method. Aringhieri et al. [29], proposed two-level heuristic algorithm for the ORS problem with downstream beds. Jebali et al. [30], used stochastic programming for ORS with downstream beds. They considered multi- period feature. Pariente et al. [31], proposed heuristic algorithm for ORS problem with objective function of maximizing service level. They considered priorities of patients. Wang et al. [32], considered nurses and anesthesiologist as additional resources for the solution of ORS problems. Constraint programming was used in the study. Heydari and Soudi [33], used stochastic programming for ORS problem. They considered downstream beds and elective/emergency patients. Vali- Siar et al. [12], considered nurses, anesthesiologist and downstream beds as additional resources. They proposed genetic algorithm (GA). Hamid et al. [24], considered downstream beds and equipment as additional resources for the ORS problem. NSGA II algorithm was proposed for the solution of the problem. Addis et al. [34], used robust optimization for multi- period ORS problem. Ahmed and Ali [35], used fuzzy TOPSIS and MILP model for the problem of ORS with objective functions of maximizing patient preferences and minimizing total cost. Coban [36], proposed a heuristic and an optimization model for the ORS problem with equipment. Khaniyev et al. [37], proposed heuristic algorithms for ORS problem. They considered uncertainty on parameters. Zhang et al. [11], used stochastic programming for the problem of ORS with downstream beds. Objective function of the problem is minimizing total cost. Britt et al. [38], considered multi- period ORS problem. Downstream beds and equipment were taken into account as additional resources. Roshanaei and Naderi [21], used benders decomposition algorithm for ORS problem. The objective function of the problem was maximization total scheduled surgical times. Park et al. [2], proposed a mathematical model for ORS problem with preferences and cooperative operations. Rachuba et al. [39], taken into account downstream beds for the problem of ORS. Simulation is used for the solution of the problem. Mazloumian et al. [18], proposed a robust multi- objective integer linear programming (MOILP) model for the solution of ORS problem with downstream beds. Azaiez et al. [40], proposed heuristic algorithm for ORS problem with makespan minimization. Makboul et al. [41], considered priorities of patients for ORS problem. Robust optimization was used for the problem. Oliveira et al. [42], considered anesthesiologist as an additional resource for ORS problem with multi- period. Integer linear programming (ILP) model was proposed for the problem. Lotfi and Behnamian [1], proposed multi-objective variable neighborhood search algorithm for multi- period ORS problem.

Heuristic algorithms have been proposed in very few of the studies in which additional resources such as nurses, downstream beds, and anesthesia team are taken into account. In many studies, only surgeons are taken as additional resources. In addition, there are studies that consider downstream beds as additional resources. There are few studies that consider the anesthesia team ([32],[12],[42]). Among these studies, Vali-Siar et al. [12] proposed a GA. Other studies used optimization or simulation methods.

In ORS problems, setup times have been neglected in many studies. However, in real life, the ORs are being prepared for the next operation when an operation is completed. Different equipment and tools are used in different operations. Some tools and equipment are mobile. After an operation is completed, setup must begin for the next operation immediately. During the setup phase, the cleaning of the OR, the transportation of the necessary tools, the sterilization of the used resources, the preparation of the surgeons, nurses and the anesthesia team are carried out [43]. Setup of an operation varies depending on the operation scheduled before it in the same OR. For example, when two operations using the same mobile devices are scheduled sequentially, the setup time may be shortened according to the sequential scheduling of operations using different mobile devices. In other words, setup times are sequence dependent [44]. There are few studies that consider sequence-dependent setup times in ORS problems. It was observed that additional resources considered in ORS problems such as the surgeons, beds and anesthesia team were neglected in many studies about ORS problem with setups [44]. Arnaout and Kulbashian [45], considered sequence dependent setup times in the ORS problem. Additional resources were not considered in the problem. The objective function was makespan minimization. Simulation was used for the problem. Arnout [46], proposed a heuristic algorithm for the solution of the ORS problem with sequence dependent setup times. Additional resources were not taken into account. Hamid et al. [43], used simulation for the ORS problem with sequence dependent setup times. Intensive care unit (ICU) beds were taken into account as an additional resource. The objective function is makespan minimization. Zhao and Li [47], considered sequence dependent setup times in the ORS problem. The use of additional resources was not taken into account in the study. They minimized the total cost. A nonlinear programming model and constraint programming used to solve the problem.

In this study, the problem is defined by considering a state hospital. Anesthesia teams are taken into account in the study. Anesthesia teams consist of specialist doctors, nurses and anesthesia technicians. An anesthesia team accompanies the patient during the operation. Assigning an anesthesia team to patients is an decision problem. In other words, the anesthesia team is a flexible resource. The relevant anesthesia team can serve only one patient at a time. Since there are limited number of anesthesia teams in hospitals, patient waiting occur if there is no team available. In addition, the case of more than one surgeon involvement in some operations is considered. Surgeons can only perform one operation at a time. The patient's operation may be start as long as the employed surgeon or surgeons are idle. Since the assignment of surgeon(s) to operations are predetermined, surgeons are considered as a dedicated resource. In addition, the setup time of the OR for the relevant patient varies depending on the previous operation in the same OR. In other words, operation setup times are sequence dependent. By solving the problem, the anesthesia team and OR are assigned to the patients and the order of the operation is determined. A MILP model and ABC algorithm are proposed for the problem. The proposed algorithm is compared with the MILP and random search.

According to the literature review, it was seen that sequence-dependent setup times were not addressed in many studies [48]. In addition, heuristic algorithm has not been proposed for the ORS problems, which took into account the sequence-dependent setup times and additional flexible/ dedicated resources. Literature is given in Appendix Table A1.

In this study an ORS problem is addressed that is not considered in the literature. Sequence dependent setup times, both flexible and dedicated resources are taken into account and a very complex operating room scheduling problem is addressed. In many studies that is proposed heuristic algorithm to similar problems, mathematical models are used to calculate objective function value of the solutions, due to complexity of the obtaining a feasible solution considering all resources. Collaboration with optimization model may be time consuming. In this study a heuristic algorithm is proposed to solve this complex problem. The unique value of the ABC algorithm is the decoding algorithm, calculation of objective function of the solutions, considering all flexible/dedicated resources.

With this study, a heuristic algorithm is proposed to a problem that is not considered before. The success of the proposed algorithm is demonstrated comparing the results of heuristic with MILP model results through small size problems. Only small size test problems are solvable in reasonable time (3600 seconds). For large size test problems, the ABC algorithm is compared with random search.

In the second section of the study, the problem definition and mathematical model are given. In the section third, heuristic algorithm is given. In the fourth section, test problems are derived and parameters of heuristic algorithms are determined. In addition, the success of the heuristic is demonstrated. The last section is the conclusion section.

2. Optimization model

The addressed problem is described in detail in this section. A MILP model has been proposed. The proposed model is applied to an example problem.

2.1. Problem definition

A state hospital was taken into account in defining the problem under consideration. In the study, operational (short time) scheduling activity was addressed. The assignment of OR to patients, the order of the operations, assignment of anesthesia team to operations are achieved by the solution of the addressed problem. In order to perform the operation of n number of patients, the patient must be assigned to an OR among m ORs. An operation of a patient may begin as long as the surgeon or surgeons who will perform the operation are available and an anesthesia team must be assigned to the operation of the patient. Each surgeon and anesthesia team can only operate on one patient at a time. Some operations may require more than one surgeon. If the surgeon or at least one of the surgeons who will perform the operation is in the operation of another patient or if there is no idle anesthesia team, patient waiting occur. Since both surgeons and the anesthesia team are taken into account, a multi-resource problem is defined. Since the surgeon(s) who will perform the operation of the patient is determined before the operation day, surgeons are a dedicated resource. The anesthesia team to be assigned to the patient is considered as a decision problem and is a flexible resource. Before starting the operation, OR must be prepared for the operation. Setup is done in the same OR immediately after the operation of the previous patient is completed. In the setup phase, the cleaning of the OR, sterilization and positioning of the necessary equipment and devices are conducted. The setup of the operations can be done simultaneously in different ORs. Setup times are sequence dependent.

Characteristics of the model:

- Two different type of resource is considered as flexible and dedicated resource. Surgeon(s) that perform each operation is predetermined and is a dedicated resource. The assignment of anesthesia team to operations is conducted by the MILP model and is a decision problem. The anesthesia teams are a flexible resource.
- Appropriate constraints have been added to the model so that each resource can only perform one operation at a time.
- Before the operation, setup of the operation is conducted.
- More than one surgeon may be involved in an operation.
- If at least one surgeon that will involve in an operation is in another operation at a time, there will be a waiting times of patients.
- If an anesthesia team is needed for different operations at the same time and there is no anesthesia team available, waiting times will be occurred.

Assumptions:

- The operation times and setup times are deterministic.
- The surgeon(s) that perform each operation are predetermined.
- The setup of an operation is conducted after the completion of the previous operation.
- Patients do not have anesthesia team preference.
- All patients have equal priority.
- The resource responsible for the setup is ignored.

2.2. MILP model

Sets and Indices

p, *l* and *k* show patient indices and *N={p,l,k| p=l=k=1,…,n}*
o shows OR index and *M={o| o=1,…,m}*
r shows position index and *N={r| r=1,…,n}*
d shows surgeon index and *U={d| d=1,…,u}*
g shows anesthesia team and *A={g| g=1,…,a}*

Parameters

t_p: Operation time of the patient p
SQ_p: Setup of OR for patient p that is scheduled on the first position
$ST_{p,l}$: Setup time of OR for patient l that is scheduled after patient p
B: Very big number

$$H_{p,d}: \begin{cases} 1, & If\ f\ surgeon\ d\ operates\ the\ patient\ p \\ 0, & Otherwise \end{cases}$$

Decision Variables

$$y_{p,r,o}: \begin{cases} 1, & If\ patient\ p\ is\ assigned\ to\ room\ o\ on\ position\ r \\ 0, & Otherwise \end{cases}$$

$$x_{p,g}: \begin{cases} 1, & If\ patient\ p\ is\ assigned\ to\ anesthesia\ team\ g \\ 0, & Otherwise \end{cases}$$

$f_{p,l}$:

$$\begin{cases} 1, If\ operation\ completion\ time\ of\ \ p\ is\ less\ than\\ \quad the\ operation\ start\ time\ of\ patient\ l \\ 0, If\ operation\ completion\ time\ of\ \ l\ is\ less\ than\\ \quad the\ operation\ start\ time\ of\ patient\ p \end{cases}$$

C_p: Operation completion time of patient p
W_p: Operation starting time of patient p
I_p: Waiting time of patient p
T_l: Setup completion time of patient l
C_{max}: Maximum completion time

Model

$$Min\ Z_1 = C_{max} \tag{1}$$

$$T_l + B(1 - y_{l,r,o}) \geq SQ_l \quad \forall\ l, r, o\ and\ r{=}1 \tag{2}$$

$$T_l - B(1 - y_{l,r,o}) \leq SQ_l \quad \forall\ l, r, o\ and\ r{=}1 \tag{3}$$

$$T_l + B(2 - y_{l,r,o} - y_{k,r-1,o}) \geq C_k + ST_{k,l} \quad \forall\ k, l, r, o, l{\neq}k, \ r > 1 \tag{4}$$

$$T_l - B(2 - y_{l,r,o} - y_{k,r-1,o}) \leq C_k + ST_{k,l} \quad \forall\ k, l, r, o, l{\neq}k, \ r > 1 \tag{5}$$

$$C_l = T_l + t_l + I_l \ \ \forall\ l \tag{6}$$

$$W_p = T_p + I_p \quad \forall p \tag{7}$$

$$C_l \leq W_p + Bf_{p,l} + B(2 - x_{l,g} - x_{p,g}) \quad \forall p, l, g\ and\ p < l \tag{8}$$

$$C_p \leq W_l + B(1 - f_{p,l}) + B(2 - x_{l,g} - x_{p,g}) \quad \forall p, l, g\ and\ p < l \tag{9}$$

$$C_l \leq W_p + Bf_{p,l} + B(2 - H_{p,d} - H_{l,d}) \quad \forall p, l, d\ and\ p < l \tag{10}$$

$$C_p \leq W_l + B(1 - f_{p,l}) + B(2 - H_{p,d} - H_{l,d}) \quad \forall p, l, d\ and\ p < l \tag{11}$$

$$\textstyle\sum_g x_{p,g} = 1 \quad \forall p \tag{12}$$

$$\textstyle\sum_p y_{p,r,o} \leq 1 \quad \forall\ r, o \tag{13}$$

$\sum_r \sum_o y_{p,r,o} = 1 \;\; \forall\, p$ (14)

$\sum_p y_{p,r,o} - \sum_l y_{l,r-1,o} \leq 0 \quad \forall\, r, o \text{ and } r > 1$ (15)

$C_{max} \geq C_p \qquad \forall\, p$ (16)

$y_{p,r,o}, x_{p,g}, f_{p,l} \in \{0,1\} \text{ and}$

$C_p, T_l, V_p, I_p, C_{max} \geq 0$ (17)

Constraint (1) minimizes makespan. Constraints (2-3) calculate the setup completion time of the patients that is scheduled on the first position of each OR. Constraints (4-5) calculate the setup completion time of the patients that is scheduled except for the first position of each OR. Constraint (6) calculates the operation completion time of the patients. Constraint (7) calculates the operation starting time of the patients. Constraints (8-9) prevent simultaneous operations on patients assigned to the same anesthesia team. Constraints (10-11) prevent simultaneous operations on patients assigned to the same surgeon(s). Constraint (12) ensures that an anesthesia team is assigned to each patient. Constraint (13) satisfied that maximum one patient can be assigned to a position of an OR. Constraint (14) provides that assignment of each patient to an OR. Constraint (15) allows patients to be assigned in sequence. Constraint (16) calculates C_{max}. Constraints (17) are sign constraints.

An example is given in Figure 1. Parameters of the problem is given in Appendix Table B1. Accordingly, patients 1,8, and 2 were assigned to OR 1, patients 4, 7 and 3 were assigned to OR 2, and patients 9, 5 and 6 were assigned to OR 3. First Anesthesia team was assigned to the 1st patient, and the 2nd Anesthesia team was assigned to the 4th patient. The anesthesia team assigned to patients is indicated in parentheses next to the patient number in the Figure 1. The anesthesia team assigned to other patients is given in the Figure 1. (Dx) denotes the required surgeon(s) for operation of the relevant patient. For example, for patient 6 the second surgeon (D2) employed for the operation. If the Figure 1 is examined, it is seen that the anesthesia teams and surgeon(s) are performed only one operation at the same time. The setups of operations can be done at the same time. The setup of the operations starts as soon as the previous operation is completed in the same OR. The objective function of the optimal solution is 814.

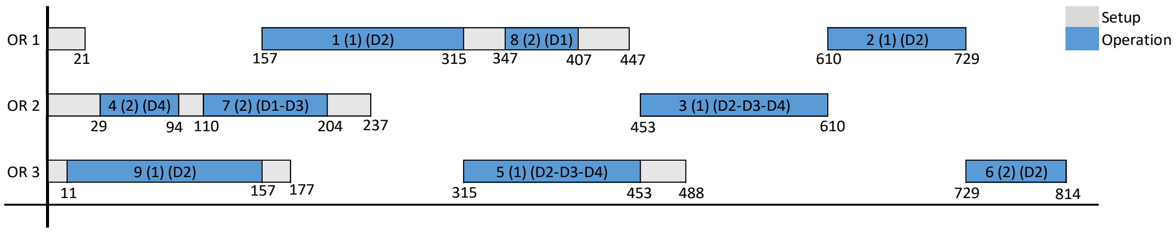

Figure 1. Gantt Chart of the optimal schedule

3. ABC algorithm

3.1. Steps of the algorithm

ABC algorithm was proposed in 2005 by Karaboğa [49]. ABC algorithm was designed by modeling the foraging behavior of bees. ABC algorithm is an algorithm based on swarm intelligence. The algorithm has 3 stages: employed bee stage, onlooker bee stage and scout bee stage. The algorithm makes intensification at the employed and onlooker bee stages. It makes diversification at the scout bee stage. At the end of the employed bee stage, the probability value of the solutions is calculated. Accordingly, the probability values of high-quality solutions are also high. Probability values are taken into account when choosing a solution at the onlooker bee stage. High quality solutions are more likely to be selected [50]. New solution is generated for selected solution by one of the insertion or swap methods. If the new solution produced is a better solution, the existing resource is replaced with the new solution, otherwise the I_i value of the relevant resource is increased by one. In the algorithm, bees are in a position to turn to higher quality resources. For resources whose I_i value is equal to the limit value, the scout bee stage is run and the related solution is replaced with a randomly derived solution. The steps continue until the predetermined number of iterations is achieved [51]. The ABC algorithm is given below [49,51].

```
Procedure: ABC algorithm
Input: Problem parameters, Iteration
number (T), Limit value, Population
size (2N)
Output: Optimal or near optimal
solution
Construct initial population with
size N randomly and calculate the
fitness (f(i)) of the each solution;
t←0;
While (t<T)
      Assume trial value of each
      resource 0;
      //Employed bee phase
      For i=1:N
            Match resource i with a
            resource randomly and
            generate a new resource
            by two point crossover
            and calculate fitness
            value of the new
            resource;
```

```
            If new resource better
            than resource i
                  Replace resource i
                  with the new
                  resource;
                  trial(i)←0;
            Else
                trial(i)←trial(i)+1;
            End
      End
      Determine the maximum fitness
      value as F;
      Calculate probability value of
      resources;
      Probability(i)←0.9(f(i)/F)+0.1;
      //Onlooker bee phase
      Assign each onlooker bee to a
      resource considering
      Probability values;
      For i=1:N
            Match resource i with a
            resource randomly and
            generate a new resource
            by two point crossover
            and calculate fitness
            value of the new
            resource;
            If new resource better
            than resource i
                  Replace resource i
                  with the new
                  resource;
                  trial(i)←0;
            Else
               trial(i)← trial(i)+1;
            End
      End
      Record the resource with best
      fitness value;
      //Scout bee phase
      Find the resource with maximum
      trial number as i*;
      If (trial(i*)>limit)
            Replace resource i with
            a random solution;
            trial(i*)←0;
      End
      t←t+1;
End
```

Fitness values of the solutions are calculated by Equation 18. Z(i) is the objective function value of the resource i.

$$f(i)=\begin{cases} \frac{1}{1+Z(i)} & if\ Z \geq 0 \\ 1+|Z(i)| & if\ Z < 0 \end{cases} \tag{18}$$

3.2. Representation of the solutions and decoding algorithm

The matrix of V_p^{Pop} is used to represent the solutions. Pop denotes the number of individuals in the population. The V_p^{Pop} matrix consists of the number of Pop rows and the number of n (number of patients) columns. The number of columns is equal to the number of patients and the number of rows is equal to the population size. $V_p^i \in$[1,n] and $V_p^i \neq V_l^i$. Each row constitutes of permutation representation of the patients. In other words, patients are ranked randomly in each row of the V_p^{Pop} matrix. The representation of the solutions is given in Figure 2. The assigned OR and the anesthesia team of patients are determined by the decoding algorithm. Therefore, this information is not included in the representation of the solutions.

1st Ind.	V_1^1	V_2^1	...	V_n^1	
2nd Ind.	V_1^2	V_2^2	...	V_n^2	$V_p^i \in$[1,n] and
⋮	⋮	⋮	⋮	⋮	$V_p^i \neq V_l^i\ \forall p,i,l$
popth Ind.	V_1^{pop}	V_2^{pop}	...	V_n^{pop}	

Figure 2. Representation of the solutions

Objective function of the number of Pop solutions are calculated by decoding algorithm. With the decoding algorithm, the patients are assigned to the ORs and the anesthesia team and order of the operations are determined. In addition, anesthesia teams and surgeon(s) operate only one operation at the same time. Some of the abbreviations used in the algorithm are given in the description of the MILP model. Newly defined abbreviations are given below.

OT_o : Operation completion time of the last patient that is assigned to OR o

o^*: The OR that the next patient will be assigned

g_p: The anesthesia team that is assigned to patient p

PA_g: Operation completion time of the patient that is assigned to anesthesia team g

p'_o: The patient that is last assigned to OR o

k_o: The number of patients that is assigned to OR o

seq_k^o: The patient that is scheduled the order of k in OR o

The operation times of the patient V_p^1 that is assigned to OR o' and the patient V_l^1 that is assigned to OR o'' is not overlap as long as the one of the following conditions is met.

Case 1: The operation completion time of patient V_p^1 is smaller than operation starting time of patient V_l^1. This situation is represented by Equation 19. Case 1 is shown in Figure 3.

$$C_{V_p^1} \leq W_{V_l^1} \tag{19}$$

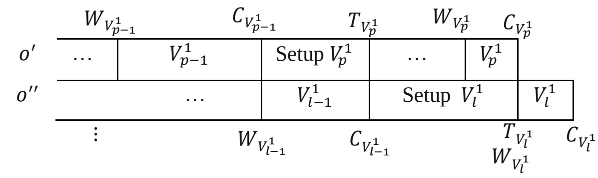

Figure 3. Case 1

In Figure 3, the operation time of V_p^1 and V_l^1 do not overlap. Because the operation completion time of patient V_p^1 is equal to the operation start time of patient V_l^1.

Case 2: The operation start time of patient V_p^1 is greater than the operation completion time of patient V_l^1. This situation is represented by Equation 20. Case 2 is shown in Figure 4.

$$W_{V_p^1} \geq C_{V_l^1} \tag{20}$$

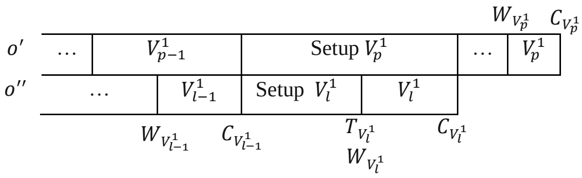

Figure 4. Case 2

In Figure 4, the operation time of V_p^1 and V_l^1 do not overlap. Because the operation starting time of patient V_p^1 is greater than the operation completion time of patient V_l^1.

By using the decoding algorithm, feasible solutions are obtained from each solution representation and the objective functions are calculated.

In the decoding algorithm, first of all, for each solution, the OT_o values, which shows the operation completion time of the patient who was last assigned to the OR o, are taken as 0. The first patient in each row is assigned to the first OR. In the first solution, the first patient is shown as patient V_1^1 and the OR to which it will be assigned is o^*. The setup of the first patient V_1^1 begins at time 0. The setup completion time of the patient V_1^1 ($T_{V_1^1}$) is calculated. Since patient V_1^1 is in the first order of the OR it is calculated as $T_{V_1^1} = SQ_{V_1^1}$. After the setup is completed, the operation starts and the operation start time is shown as $W_{V_1^1}$. The operation completion time ($C_{V_1^1}$) is calculated as $W_{V_1^1}+t_{V_1^1}$. The time that OR is used is recorded as an interval ($W_{V_1^1}-C_{V_1^1}$). The patient is randomly assigned to the $g_{V_1^1}^*$ anesthesia team. The operation time of the anesthesia team is taken as the interval ($W_{V_1^1} - PA_{V_1^1}$) and the value of $PA_{V_1^1}$ is equal to the value of $C_{V_1^1}$. Patient V_1^1 who was last assigned to OR o^* is recorded as p'_{o^*}. The next patient V_2^1 is assigned to the OR o^* that is the smallest setup completion time ($o^* \leftarrow arg \min\limits_{o}(OT_O + ST_{p'_o,V_2^1})$). $T_{V_2^1}$ value is calculated as ($OT_O + ST_{p'_o,V_2^1}$) or if no patient has been assigned to the relevant OR yet is calculated as ($OT_o + SQ_{V_2^1}$). First, after determining the o^* OR to which the V_2^1 patient will be assigned, the $T_{V_2^1}$ is calculated. The patient's operation completion time is calculated as $W_{V_2^1} + t_{V_2^1}$. If this value coincides with the operation times of other ORs, the surgeon(s) in the conflicting ORs and the surgeon(s) employed in the operation of patient V_2^1 are checked. If the same surgeon(s) is employed, the operation start time of the V_2^1 is postponed. If different surgeon(s) are employed, the patient V_2^1 is assigned a different anesthesia team than the patients with the overlap. If there is no free anesthesia team, the earliest completed anesthesia team is assigned to the patient. These steps are repeated for all patients. The decoding algorithm is given below.

Procedure: Decoding algorithm
Input: A solution (V_p^1), problem parameters
Output: Objective function of the solution
$OT_o \leftarrow 0$; $k_o \leftarrow 0$;
//The first patient V_1^1 is assigned to first OR and first //anesthesia team;
$o^* \leftarrow 1; T_{V_1^1} \leftarrow SQ_{V_1^1}; C_{V_1^1} \leftarrow SQ_{V_1^1} + t_{V_1^1};$
$g_{V_1^1}^* \leftarrow 1; PA_{g_{V_1^1}^*} \leftarrow C_{V_1^1}; OT_{o^*} \leftarrow C_{V_1^1}; p'_{o^*} \leftarrow V_1^1; k_{o^*} \leftarrow k_{o^*} + 1;$ $seq_k^{o^*} \leftarrow V_1^1;$ $W_{V_1^1} \leftarrow T_{V_1^1};$
For i=2:n
    $o^* \leftarrow \arg\min\limits_{o}(OT_o + ST_{p'_o,V_i^1})$;
    $T_{V_i^1} \leftarrow OT_{o^*} + ST_{p'_{o^*},V_i^1}$; $p'_{o^*} \leftarrow V_i^1$;
    $seq_k^{o^*} \leftarrow V_i^1$; $W_{V_i^1} \leftarrow T_{V_i^1}$;
    $x \leftarrow T_{V_i^1} + t_{V_i^1}$; z←0;
    //The operation starting time of patient V_i^1 is determined considering //the surgeons;
    While (j<=m)
        Δ←1;
        For l=1:k_j
            U←seq_l^j;
            If ($x \leq W_U$) or ($W_{V_i^1} \geq C_U$)
                //No overlap
            Else
                z←z+1;
                Overlap(z)=U;
                If ($H_{U,d} == H_{V_i^1,d}$)
                    $W_{V_i^1} \leftarrow C_U$;
                    $x \leftarrow W_{V_i^1} + t_{V_i^1}$;
                    j←1; Δ←0;
                End
            End
            If (Δ==0)
                Break
            End
        End
        If (Δ==1)
            j←j+1;
        End
    End
    z←0;
    //Assignment of anesthesia team and updating of operation starting //time considering anesthesia teams;
    For j=1:m

For l=1: k_j
U← seq_l^j;
If (x≤ W_U)or($W_{V_i^1} \geq C_U$)
//No overlap
Else
z←z+1;
Overlap(z)=U;
G\{ g_U^* };
End
End
End
If (G=={})
$g_{V_i^1}^* \leftarrow \arg\min_{g} PA_{g^*}$;
$W_{V_i^1} \leftarrow \max(PA_{g_{V_i^1}^*}, W_{V_i^1})$;
Else
$g_{V_i^1}^* \leftarrow \arg\max_{g \in \{G\}} PA_{g^*}$;
$W_{V_i^1} \leftarrow \max(PA_{g_{V_i^1}^*}, W_{V_i^1})$;
End
$C_{V_i^1} \leftarrow W_{V_i^1} + t_{V_i^1}$; $PA_{g_{V_i^1}^*} \leftarrow C_{V_i^1}$;
$OT_{o^*} \leftarrow C_{V_i^1}$; $k_{o^*} \leftarrow k_{o^*} + 1$;
End

4. Computational results

4.1. Parameters of the heuristic

Although in most of the studies on heuristic algorithms parameter levels are determined without an analytic method, in this study Taguchi experimental design (TED) method is used to determine the levels of the ABC algorithm parameters. The parameters of the ABC algorithm are N, T and limit value. Firstly, alternative parameter levels are determined through preliminary experiments and given in Table 1. L27 orthogonal array is chosen due to there are 3 parameters and 3 levels for each parameter. In TED method, signal-to-noise ratio (S/N) is used as a measure to determine the characteristics of engineering problems. To optimize the ABC algorithm parameters "the smaller, the better" performance criterion is used in TED method due to the addressed problem has a minimization objective function. The calculation of S/N is given in Equation 21. In Equation 21, n is the number of observations in each experiment and Y_i is the objective function of ABC algorithm with the related parameters. The optimal parameters are selected considering the highest S/N values. Minitab 16 for Windows (Minitab Inc.) is used to apply TED method to problem.

$$\frac{S}{N} = -10 \times \log\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right) \tag{21}$$

For the test problem with 7 ORs algorithm was run at the relevant parameter levels. The main effects plot for S/N ratios for the algorithm is given in Figure 5. In ABC algorithm, N level sets to 1000, T level is 100 and limit is 10.

Table 1. Parameter levels of the ABC algorithm

Parameters	Levels
N	500/750/1000
T	50/75/100
limit	5/7/10

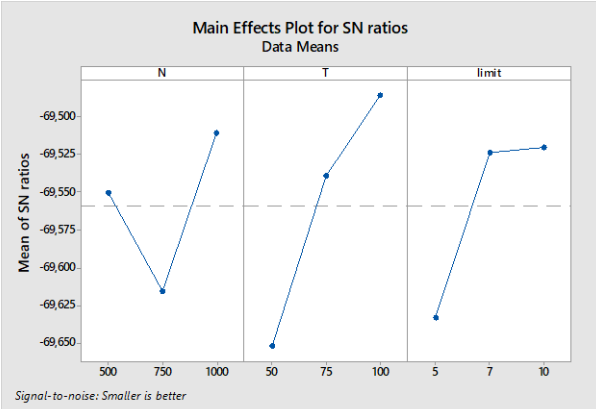

Figure 5. S/ N ratios of the algorithm

4.2. Comparisons

Properties of test problems are given in this section. The number of ORs (m) set to 3, 5, 7 or 10. The number of patients was taken as 3*m, 5*m, 7*m and 10*m. The number of surgeons was taken as 4, 7, 10 and 14, and the number of anesthesia team as 2, 3, 5 and 7. The parameter t_p were derived according to a uniform distribution in the U(40,170) range. Sequence-dependent setup times are derived in accordance with the uniform distribution in the range of U(20,50), U(10,40) or U(30,85). The $H_{p,d}$ parameter is derived so that 60% of the patients receive service from only one surgeon, 25% of the patients receive service from two surgeons and 15% from 3 surgeons. For each problem type two test problems are derived.

Test problems are run with the MILP model, ABC algorithm and random search. The results of random search also is an upper bound for the related test problem since for all test problems random search gave worse solution than ABC algorithm. In random search, random solutions are generated and the objective function of these solutions are calculated using the proposed decoding algorithm. The random search is run the same duration of ABC algorithm for the related test problem.

The time limit of the MILP model is 3600 seconds. The results are given in Table C1-C4 in Appendix section. Objective function values, CPU values and Error values obtained by using the relevant algorithm are given in the tables. Error value is calculated with Equation 22.

$$Error = \frac{(Solution\ of\ the\ algorithm - The\ obtained\ best\ solution)}{The\ obtained\ best\ solution} \tag{22}$$

The results with 3 OR are given in Table C1. Model gave optimal solutions for 7 test problems with number of 9 or 15 patients. Also, the heuristic algorithm found optimal solutions to these problems. ABC algorithm gave better results except six test problems. MILP model found no feasible solutions to number of 5 test problems with the number of 21 or 30 patients within 3600 seconds. For other test problems feasible solutions were found by MILP model. Accordingly, the ABC algorithm gave the better results for all test problems. The results with 5 ORs are given in Table C2. MILP model found feasible solutions to test problems with the number of 15 or 25 patients within time limit. MILP model could not find a solution to test problems with the number of 35 or 50 patients within time limits. Accordingly, the ABC algorithm also found better solutions for test problems with 5 ORs. The results of 7 ORs are given in Table C3. MILP model found feasible solutions to test problems with number of 21 patients. No feasible solutions were found other test problems by MILP model for the number of 7 ORs. The results of 10 ORs are given in Table C4. According to Table C4, MILP model found feasible solution to only one test problem. The ABC algorithm found better solutions than MILP model and random search.

5. Conclusions

ORs are one of the most important resources of hospitals. Therefore, effective scheduling of ORs has an important role in the effective management of the hospital. ORS problems are multi-resource problems. In this study, the ORS problem was defined by considering the anesthesia team as well as the surgeons. While surgeons are a dedicated resource, the anesthesia team is a flexible resource. In the ORS problem, sequence dependent setup times are taken into account. Although the ORS problem is an important problem, there are few studies that take into account the sequence-dependent setup times. A MILP model is proposed. ABC algorithm has been developed for large scale test problems. A heuristic algorithm is proposed for the first time to solve the ORS problem with multi-resource, sequence-dependent setup times. An algorithm has been developed to calculate the objective functions of the solutions. The proposed ABC algorithm is compared with MILP model. As a result, the ABC algorithm gave more successful results than MILP model. In future studies, the problem can be handled with multi- objective functions. Objective functions such as tardiness minimization, maximization of resource utilization may be considered besides makespan minimization. In multi- objective optimization problems, pareto optimal solutions are found. In multi- objective optimization, all obtained solutions are compared with each other to select non-dominated solutions in solution space that is increased the complexity of the problem. Different methods may be used such as Augmented ε- constraint method to obtain pareto optimal solutions to multi- objective optimization problems. Extracting Pareto optimal solutions from the solution space can significantly increase the running time of the heuristic algorithm. In this study, surgeons and anesthesia teams are considered as resources. In future studies, the resource conducts the setup and other resources such as machines used in operations may be taken into account. In this study all patients have same priority. In future studies patients may be prioritized. In this study, operations and setup times are considered deterministic. Stochastic parameters can be taken into account. Different heuristic algorithms may be proposed to solve the problem or exact solution methods may be used to solve the problem.

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Appendix

A. Related studies

Table A1. Related studies

Article	**Add. Res.**	**Res. Type**	**Patients**	**Obj. Funct.**	**Method**	**Add Pro.**
Zhang, 2021	d	a	a	Minimizing total cost	Column Gen. Based Heuristic, stoch. Prog.	a,b
Rachuba, 2022	d	b	a	Maximizing total number of patients	Chance const. Opt., Simulation	a,b
Park, 2021	a	a	a	Minimizing number of ORs and overtime	Mathematical model	c,e
Mazloumian, 2022	d	a	a	Minimizing total waiting Minimizing postponed Minimizing loss incurred	Robust MOILP Model	a,b
Lotfi, 2022	a	b	c	Minimizing total comp. time Minimizing makespan	Multi- obj. variable neigh. Search	b
Khaniyev, 2020	-	-	a	Minimizing waiting, idle and overtime	Heuristic algorithms	a
Hamid, 2019 a	a,d,e	a	a	Minimizing total cost Maximizing service level Maximizing consistecy score	NSGA II	f
Britt, 2021	a,d,e	b	a	Minimizing total cost Minimizing number of ORs	Hybrid heuristic alg.	a,b,c
Azaiez, 2022	d	b	a	Minimizing max. compl. time	MILP, Heuristics	b
Aringhieri, 2015	a,d	a	a	Minimizing total cost of waiting time	Two level heuristic, ILP model	
Addis, 2016	-	-	a	Minimizing total waiting time and tardiness	Robust optimization	a,b
Roshanaei, 2021	a	a	a	Maximizing total scheduled surgical times	Benders decomposition, MILP	
Wang, 2015	a,b,c, d	b	a	Minimizing makespan	MILP, Constraint prog.	f
Makboul, 2022	a,d	c	a	Maximizing score of surgeries	Robust opt.	a,b,f
Fei, 2010	a	a	a	Minimizing cost	Hybrid GA	b
Coban, 2020	e	a	a	Minimizing total cost	Heuristic, MILP model	
Pariente, 2015	a	b	a	Maximizing service level	Heuristics	f
Oliveira, 2022	a,c	c	a	Minimizing deviations	ILP model, simulation	b
Ahmed, 2020	a	b	a	Max. Patient preference Minimizing total cost	Fuzzy TOPSIS, MILP model	f
Agnetis, 2014	a	b	a	Maximizing total score	Decomposition	b
Jebali, 2015	d	b	a	Minimizing costs	Stoch. Prog.	a,b
Vijayakumar, 2013	a,b,e	a	a	Maximizing number of patients	Heuristic	b,f
Heydari, 2016	d	b	c	Minimizing makespan and overtime	Stochastic prog.	a,b
This study	**a,c**	**c**	**a**	**Minimizing max. compl. time**	**Heuristic,MILP**	**d,e**

Add. Resources: a: Surgeon, b: Nurse, c: Anesthesiologist, d: Downstream beds, e: Equipment/Tools

Additional Resource Type: a: Dedicated, b: Flexible, c: Hybrid

Patients: a: Elective, b: Emergency, c: Hybrid

Add. properties: a: Uncertainty on parameters, b: Multi- period, c: Preferences, d: Setup times, e: Cooperative operations, f: Priorities of patients

B. Parameters of example problem

The proposed MILP model was coded in the GAMS 24.0.2 program. Solved with CPLEX solver. For the first test problem, the MILP model was run. In Table A1 parameters of the problem are given. There are 9 patients, 3 ORs, 4 surgeons and 2 anesthesia teams.

Table B1. Parameters of SQ_p, $ST_{p,l}$ $H_{p,d}$ and t_p

p	t_p	SQ_p	$ST_{p,l}$									$H_{p,d}$			
			1	2	3	4	5	6	7	8	9	1	2	3	4
1	158	21	29	32	34	12	39	25	33	32	35	0	1	0	0
2	119	26	15	24	29	38	35	37	27	27	36	0	1	0	0
3	157	20	11	37	22	11	32	15	14	28	18	0	1	1	1
4	65	29	20	22	22	22	28	15	16	13	20	0	0	0	1
5	138	34	33	17	32	31	35	35	19	19	26	0	1	1	1
6	85	32	20	35	34	27	18	30	17	24	22	0	1	0	0
7	94	14	26	40	33	39	17	26	12	33	28	1	0	1	0
8	60	35	36	40	38	22	10	26	16	17	20	1	0	0	0
9	146	11	13	32	32	26	20	35	27	39	37	0	1	0	0

C. Solutions of test problems

Table C1. Solution of test problems with 3 ORs

n	ST	MILP Model			ABC			Random Search	
		Z	CPU	Error	Z	CPU	Error	Z	Error
9	U(10,40)	**814***	1882	0	**814**	6.59	0	**814**	0
9	U(10,40)	**630***	2804	0	**630**	5.61	0	632	0.003
9	U(20,50)	**759***	1620	0	**759**	4.91	0	**759**	0
9	U(20,50)	**742***	1874	0	**742**	4.64	0	758	0.022
9	U(30,85)	530	3600	0.017	**521**	6.13	0	**521**	0
9	U(30,85)	514	3600	0.024	**502**	5.45	0	**502**	0
15	U(10,40)	**980***	1235	0	**980**	9.39	0	995	0.015
15	U(10,40)	**1065***	2152	0	**1065**	7.78	0	1087	0.021
15	U(20,50)	1061	3600	0.002	**1059**	7.96	0	**1059**	0
15	U(20,50)	**878***	2252	0	**878**	9.28	0	**878**	0
15	U(30,85)	998	3600	0.034	**965**	7.74	0	989	0.025
15	U(30,85)	983	3600	0.005	**978**	7.59	0	1001	0.023
21	U(10,40)	1482	3600	0.086	**1365**	11.86	0	1385	0.015
21	U(10,40)	1281	3600	0.063	**1205**	14.79	0	1227	0.018
21	U(20,50)	1434	3600	0.075	**1334**	12.49	0	1337	0.002
21	U(20,50)	-	3600	-	**1166**	12.57	0	1200	0.029
21	U(30,85)	1698	3600	0.103	**1540**	12.71	0	1621	0.052
21	U(30,85)	-	3600	-	**1338**	13.65	0	1452	0.085
30	U(10,40)	1908	3600	0.181	**1616**	23.86	0	1663	0.029
30	U(10,40)	-	3600	-	**1500**	20.52	0	1559	0.039
30	U(20,50)	2277	3600	0.368	**1665**	16.98	0	1738	0.044
30	U(20,50)	-	3600	-	**1644**	17.58	0	1658	0.009
30	U(30,85)	2172	3600	0.225	**1773**	22.45	0	1868	0.054
30	U(30,85)	-	3600	-	**1916**	16	0	1927	0.006
* optimal solution		Average		0.062			0		0.02

Table C2. Solution of test problems with 5 ORs

n	ST(p,l)	MILP Model			ABC			Random Search	
		Z	CPU	Error	Z	CPU	Error	Z	Error
15	U(10,40)	680	3600	0.012	**672**	10.75	0	684	0.018
15	U(10,40)	579	3600	0.032	**561**	12.44	0	564	0.005
15	U(20,50)	580	3600	0.133	**512**	10.04	0	513	0.002
15	U(20,50)	**566**	3600	0	**566**	11.8	0	577	0.019
15	U(30,85)	793	3600	0.025	**774**	12.89	0	798	0.031
15	U(30,85)	731	3600	0.046	**699**	10.53	0	778	0.11
25	U(10,40)	1276	3600	0.44	**886**	23.33	0	920	0.038
25	U(10,40)	-	3600	-	**915**	17.49	0	923	0.009
25	U(20,50)	1359	3600	0.162	**1170**	24.35	0	1189	0.016
25	U(20,50)	1391	3600	0.218	**1142**	19.97	0	1184	0.037
25	U(30,85)	1930	3600	0.885	**1024**	21.69	0	1198	0.169
25	U(30,85)	-	3600	-	**1012**	23.56	0	1032	0.02
35	U(10,40)	-	3600	-	**1191**	23.3	0	1230	0.033
35	U(10,40)	-	3600	-	**1317**	33.58	0	1378	0.046
35	U(20,50)	-	3600	-	**1514**	27.9	0	1698	0.121
35	U(20,50)	-	3600	-	**1494**	29.1	0	1495	0.001
35	U(30,85)	-	3600	-	**1277**	31.86	0	1319	0.033
35	U(30,85)	-	3600	-	**1293**	30.96	0	1387	0.073
50	U(10,40)	-	3600	-	**2022**	40.71	0	2023	0
50	U(10,40)	-	3600	-	**1892**	35.37	0	2077	0.098
50	U(20,50)	-	3600	-	**2180**	41.71	0	2223	0.02
50	U(20,50)	-	3600	-	**1967**	39.75	0	2094	0.065
50	U(30,85)	-	3600	-	**1872**	38.53	0	1999	0.068
50	U(30,85)	-	3600	-	**1960**	41.46	0	2111	0.077
		Average		0.19			0		0.046

Table C3. Solution of test problems with 7 ORs

n	ST(p,l)	MILP Model			ABC			Random Search	
		Z	CPU	Error	Z	CPU	Error	Z	Error
21	U(10,40)	542	3600	0.146	**473**	17.27	0	497	0.051
21	U(10,40)	641	3600	0.009	**635**	14.56	0	645	0.016
21	U(20,50)	747	3600	0.201	**622**	16.83	0	629	0.011
21	U(20,50)	929	3600	0.078	**862**	16.33	0	927	0.075
21	U(30,85)	801	3600	0.004	**798**	19.92	0	801	0.004
21	U(30,85)	712	3600	0.029	**692**	14.11	0	785	0.134
35	U(10,40)	-	3600	-	**876**	31.26	0	985	0.124
35	U(10,40)	-	3600	-	**1022**	36.01	0	1108	0.084
35	U(20,50)	-	3600	-	**1163**	32.53	0	1189	0.022
35	U(20,50)	-	3600	-	**1064**	26.07	0	1089	0.023
35	U(30,85)	-	3600	-	**1193**	32.8	0	1198	0.004
35	U(30,85)	-	3600	-	**927**	28.63	0	1010	0.09
49	U(10,40)	-	3600	-	**1198**	46.04	0	1267	0.058
49	U(10,40)	-	3600	-	**1275**	41.35	0	1355	0.063
49	U(20,50)	-	3600	-	**1198**	38.9	0	1299	0.084
49	U(20,50)	-	3600	-	**1340**	42.65	0	1374	0.025

49	U(30,85)	-	3600	-	**1540**	39.15	0	1643	0.067
49	U(30,85)	-	3600	-	**1602**	40.86	0	1620	0.011
70	U(10,40)	-	3600	-	**1998**	65.94	0	2104	0.053
70	U(10,40)	-	3600	-	**1619**	68.37	0	1779	0.099
70	U(20,50)	-	3600	-	**1868**	66.32	0	1963	0.051
70	U(20,50)	-	3600	-	**1765**	63.92	0	1910	0.082
70	U(30,85)	-	3600	-	**1766**	68.58	0	1856	0.051
70	U(30,85)	-	3600	-	**1925**	65.78	0	2034	0.057
			Average	0.077			0		0.055

Table C4. Solution of test problems with 10 ORs

n	ST(p,l)	MILP Model			ABC			Random Search	
		Z	CPU	Error	Z	CPU	Error	Z	Error
30	U(10,40)	1101	3600	0.59	**691**	28.35	0	741	0.072
30	U(10,40)	-	3600	-	**780**	29.31	0	803	0.029
30	U(20,50)	-	3600	-	**735**	24.28	0	743	0.011
30	U(20,50)	-	3600	-	**733**	25.44	0	764	0.042
30	U(30,85)	-	3600	-	**639**	25.09	0	658	0.03
30	U(30,85)	-	3600	-	**826**	24.54	0	995	0.205
50	U(10,40)	-	3600	-	**1154**	50.48	0	1163	0.008
50	U(10,40)	-	3600	-	**985**	46.8	0	1051	0.067
50	U(20,50)	-	3600	-	**943**	51.19	0	1012	0.073
50	U(20,50)	-	3600	-	**1082**	53.68	0	1083	0.001
50	U(30,85)	-	3600	-	**1075**	51.1	0	1159	0.078
50	U(30,85)	-	3600	-	**1119**	46.44	0	1252	0.119
70	U(10,40)	-	3600	-	**1686**	75.05	0	1721	0.021
70	U(10,40)	-	3600	-	**1623**	75.81	0	1641	0.011
70	U(20,50)	-	3600	-	**1706**	73.99	0	1714	0.005
70	U(20,50)	-	3600	-	**1262**	80.06	0	1405	0.113
70	U(30,85)	-	3600	-	**1469**	81.6	0	1614	0.099
70	U(30,85)	-	3600	-	**1563**	73.56	0	1593	0.019
100	U(10,40)	-	3600	-	**2332**	112.7	0	2383	0.022
100	U(10,40)	-	3600	-	**2373**	106.3	0	2393	0.008
100	U(20,50)	-	3600	-	**2086**	135.8	0	2254	0.081
100	U(20,50)	-	3600	-	**1927**	137.1	0	2121	0.101
100	U(30,85)	-	3600	-	**2034**	146.9	0	2287	0.124
100	U(30,85)	-	3600	-	**2300**	155.2	0	2406	0.046
			Average	0.59			0		0.057

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RESEARCH ARTICLE

Existence and uniqueness study for partial neutral functional fractional differential equation under Caputo derivative

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ABSTRACT

The partial neutral functional fractional differential equation described by the fractional operator is considered in the present investigation. The used fractional operator is the Caputo derivative. In the present paper, the fractional resolvent operators have been defined and used to prove the existence of the unique solution of the fractional neutral differential equations. The fixed point theorem has been used in existence investigations. For an illustration of our results in this paper, an example has been provided as well.

1. Introduction

Modeling by taking into account the memory effect is the attraction of the fractional calculus. The concept of memory is not taken by the ordinary derivative, thus modeling with the ordinary derivative gives incomplete dynamics. The works related to fractional calculus continue to impress the mathematician communities. There exist now many papers addressing the application of fractional calculus, we cite the following paper which brings information on the application of this field of mathematics to biology [1–3], engineerings [4–7], physics and applications [8–12] and fluid modeling [13–16]. Modeling with the Caputo derivative is more adequate due to the inconvenience of the Riemann-Liouville fractional derivative. It is noticed that the Riemann-Liouville derivative of the constant function does not give zero, it is a serious inconvenience in the pratic because many initial conditions are constant or null. Due to this fact, we model in this paper using the Caputo derivative. The field of fractional calculus has attracted many authors due to the diversity and existence of many fractional operators. He has the Caputo derivative [17, 18], and the Riemann-Liouville derivative version of the fractional operator also exists, see more details in the paper [19]. We have the Atangana-Baleanu derivative which has two versions, the Caputo version and the Riemann-Liouville version. The Caputo-Fabrizio derivative exists but is with the exponential kernel [20]. Note that the Antangana-Baleanu derivative has as a kernel the Mittag-Leffler function as described in the paper [21]. There exist many other derivatives as conformable derivatives, Hilfer derivatives, and others, the difference between them is not significant, just the kernel change in many of them. In this

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paper, the application of fractional calculus to neutral fractional differential equations has been considered. The problem considered in this paper is represented by the following neutral fractional differential equation

$$\begin{cases} D_t^\alpha \left(x(t) - Bx(t-h)\right) = Ax(t) + Cx(t-h) \;\; t \geq 0 \\ x_0 = \varphi \in \mathcal{C}_X. \end{cases} \tag{1}$$

The literature review concerning the fractional neutral functional differential equations as described in the equation (1) or similar to the previous equation is very large. In [22], the authors addressed the neutral differential equation using Caputo derivative, and with the utilization of the resolvent operator, the authors also used fixed point theorem to prove the existence of the solution of the considered neutral differential equation. In [23], Wen et al provide the Complete controllability of nonlinear fractional neutral functional differential equations described by the Caputo derivative. In this paper, the authors provided an interesting example of a neutral fractional differential equation to illustrate their main results. In [24], Wang, et al. presented applying an iterative technique, sufficient conditions are obtained for the existence of the solution of the nonlinear neutral fractional integrodifferential equation described by the Riemann-Liouville derivatives of different fractional orders. In this paper, the existence has been proved without the resolvent operators. In [25], using the conformable derivative Li et al. provided the existence of the unique solution of the class of the fractional Integral neutral differential equations. In [26], the author proposed the investigation in a fractional context related to the existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integrodifferential equations. In [27], we can find the application of Krasnoselskii's fixed point theorem on periodicity and stability in neutral nonlinear differential equations. In integer versions many investigations have been made related to neutral differential equations in different types, the studies related to the existence, and the controllability are already made as well, see the following Ezzinbi et al papers investigations [28, 29]. In [30], Sene proposed a new fundamental result concerning the contribution of the resolvent operator for proving the existence of the unique solution of the fractional integrodifferential equation under the Caputo derivative. It is very important to model with the Caputo derivative or with integer derivative, it is also important to be sure that the investigations can be made on the considered fractional model. To make sure that the model is well defined in mathematics, it is important to prove the existence and uniqueness of the solution of the model using one known fixed point theorem. This paper's novelties can be summarized in different points. The first is to prove the existence and the uniqueness of the solution using resolvent operators. This problem is interesting because the resolvent operator in the fractional context is a new problem in the literature. The second problem is that the fractional neutral functional differential equations described by the Caputo derivative have been used. The last novel and interesting thing is that we used the fixed point theorem to prove our main results in this paper.

The present paper is organized in the following form. In Section 2, we recall the necessary tools for our investigation as the fractional operators and the fixed point theorem. In Section 3, we start with the main results concerning the existence of the solution of the fractional neutral differential equation using the resolvents operators. In Section 4, we illustrate our main results with an example to highlight our results. In Section 5 we finish with the conclusion and future direction of investigations.

2. Preliminaries

In this section we recall the preliminary definition necessary for our investigations. We begin with the fractional operator, we continue with the fractional resolvents necessary to define our solutions.

Some important results on the fixed point theorem can also be recalled because they will be used in our investigation, we mean Schauder's Fixed Point Theorem used in many papers in the literature.

Definition 1. *The Riemann-Liouville integral of order $\alpha > 0$ for a continuous function defined on $[0, 1]$ is given by:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{2}$$

with $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2. *If $f \in C^n([0, 1], \mathbb{R})$ and $n - 1 < \alpha \leq n$, then, the Caputo fractional derivative is* *given by:*

$$D^{\alpha}f(t) = I^{n-\alpha}\frac{d^n}{dt^n}(f(t)) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds. \quad (3)$$

Lemma 1. *Taking $n \in \mathbb{N}^*$ and $n-1<\alpha<n$, then the general solution of $D^{\alpha}y(t)=0$ is given by*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i \quad (4)$$

such that $c_i \in \mathbb{R}, i = 0,1,2,..,n-1$.

Lemma 2. *Taking $n \in \mathbb{N}^*$ and $n-1<\alpha<n$, then, we have*

$$I^{\alpha}D^{\alpha}y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i \quad (5)$$

such that $c_i \in \mathbb{R}, i = 0,1,2,..,n-1$.

Definition 3. *Let X be a Banach space. Then a map $T : X \longrightarrow X$ is called a contraction mapping on X if there exists $q \in [0,1)$ such that*

$$\|T(x)-T(y)\| \le q\|x-y\|$$

for all $x, y \in X$.

Theorem 1 (Banach's fixed point theorem)**.** *[31] Let Ω be a non-empty closed subset of a Banach space X. Then, any contraction mapping T of Ω into itself has a unique fixed point.*

Theorem 2 (Schauder's fixed point theorem)**.** *[31] Let X be a Banach space, and let $N : X \longrightarrow X$ be a completely continuous operator. If the set $E = \{y \in X : y = \lambda N y \text{ for some } \lambda \in (0,1)\}$ is bounded, then N has fixed points.*

3. Main results

In this section, we give the procedure to get the analytical solution of the neutral functional differential equation including the resolvent operator. The novelty of this section will be the use of the Laplace transform to get resolvent and to get the analytical solution. We start this procedure by applying the Laplace transform we get the following form

Let's give the solution of the fractional differential equation given by Eq. (1). The form is described in the following lemma, for the simplification we consider $f(t, x_t) = Ax(t) + Cx(t-h)$.

Lemma 3. *Let the neutral fractional differential equation described by the Caputo derivative, under the initial condition described in Eq. (1), the solution are described by the following form*

$$\begin{cases} x(t) = \varphi(0) - B\varphi(-h) + Bx(t-h) \\ + \int_0^t (t-s)^{\alpha-1}[f(s,x_s)]\,ds \quad t \ge 0 \\ x(t) = \varphi(t) \quad t \in [-h, 0]. \end{cases} \quad (6)$$

Proof. The procedure of the proof uses the application of the Riemann-Liouville integral to the equation Eq. (1). Using the lemma 2 in the equation 1, we get

$$x(t) - Bx(t-h) = \int_0^t (t-s)^{\alpha-1}[f(s,x_s)]\,ds + c_0, \quad (7)$$

where c_0 is a real constant. Using the initial condition of the equation (1), we obtain

$$\begin{cases} x(t) = \varphi(0) - B\varphi(-h) + Bx(t-h) \\ + \int_0^t (t-s)^{\alpha-1}[f(s,x_s)]\,ds \quad t \ge 0 \\ x(t) = \varphi(t) \quad t \in [-h, 0]. \end{cases}$$

□

The next problem will consist to rewrite the solution described in Eq. (6) the neutral fractional differential equation in terms of the resolvent operator. We make the following lemma.

Lemma 4. *We consider that Eq. (1) are hold and then we should have the following relationship described by the following form*

$$\begin{cases} x(t) = R_{\alpha}(t)[\varphi(0) - B\varphi(-h)] + Bx(t-h) \\ + \int_0^t (t-s)^{\alpha-1} S_{\alpha}(t-s)[AB]\,x(s-h)ds \\ + \int_0^t (t-s)^{\alpha-1} S_{\alpha}(t-s)[C]\,x(s-h)ds \quad t \ge 0 \\ x(t) = \varphi(t) \quad t \in [-h, 0] \end{cases}$$

where the resolvent operator in our context is defined by the following expressions for simplifications.

Proof. The proof, we apply the Laplace transform to the equation represented in Eq. (6) we get the series of transformations given in the forthcoming equations. We have that

$$\bar{x} = \frac{1}{q}[\varphi(0) - B\varphi(-h)] + B\bar{x}_h - q^{-\alpha}A\bar{x} + q^{-\alpha}C\bar{x}_h$$
$$= q^{\alpha-1}[q^{\alpha}I + A]^{-1}[\varphi(0) - B\varphi(-h)] + q^{\alpha}[q^{\alpha}I + A]^{-1}B\bar{x}_h + [q^{\alpha}I + A]^{-1}C\bar{x}_h \quad (8)$$
$$x(t) = \varphi(t) \in t \in [-h, 0] \quad (9)$$

For the rest of the proof, we suppose that

$$\mathcal{L}\{T_\alpha(t)\}(q) = [q^\alpha I + A]^{-1} \tag{10}$$

where the so-called in our present paper the fractional analytic semigroup $\{T_\alpha(t)\}_{t\geq 0}$, there is that there exist constant M such that $M = \sup_{t\in[0,+\infty[} |T_\alpha(t)| < \infty$ and for any $\alpha \in (0,1)$, we can found a constant C_α verifying the condition that $|A^\alpha T_\alpha(t)| \leq C_\alpha t^{-\alpha}$. Replacing Eq. (10) in Eq. (8), we get the following relationships

$$\begin{aligned}\bar{x} &= q^{\alpha-1}\int_0^\infty e^{-q^\alpha s} T_\alpha(s)[\varphi(0) - B\varphi(-h)]\,ds \\ &+ q^\alpha \int_0^\infty e^{-q^\alpha s} T_\alpha(s) B\bar{x}_h ds \\ &+ \int_0^\infty e^{-q^\alpha s} T_\alpha(s) C\bar{x}_h ds.\end{aligned} \tag{11}$$

Before beginning the simplification in the previous expression we suppose the following density of probability is well known in the literature of fractional calculus and can be found in, we have the following form

$$\begin{aligned}\varpi_\alpha(\theta) =& \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\alpha n-1} \\ &\times \frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha).\end{aligned} \tag{12}$$

The form of its Laplace transform can be represented by $\int_0^\infty e^{-q\theta}\varpi_\alpha(\theta)\,d\theta = e^{-q^\alpha}$, this relation will be replaced by its values in the forthcoming calculations. We now begin the simplification in Eq. (11), for the next calculations the sketch is inspired by the paper in the literature, we have to calculate the first form of Eq. (11) given in the following equation

$$\begin{aligned}& q^{\alpha-1}\int_0^\infty e^{-q^\alpha s} T_\alpha(s)[\varphi(0) - B\varphi(-h)]\,ds \\ &= \int_0^\infty \alpha (qt)^{\alpha-1} e^{-(qt)^\alpha} T_\alpha(t^\alpha)[\varphi(0) - B\varphi(-h)]\,dt \\ &= -\int_0^\infty \frac{1}{q}\frac{d}{dt}\left[e^{-(qt)^\alpha} T_\alpha(t^\alpha)[\varphi(0) - B\varphi(-h)]\right]dt\end{aligned}$$

we continue the variable change and we use the probability density described in Eq. (12), we get the following forms

$$\begin{aligned}& q^{\alpha-1}\int_0^\infty e^{-q^\alpha s} T_\alpha(s)[\varphi(0) - B\varphi(-h)]\,ds \\ &= -\int_0^\infty \frac{1}{q}\frac{d}{dt}\left[e^{-(qt)^\alpha} T_\alpha(t^\alpha)[\varphi(0) - B\varphi(-h)]\right]dt \\ &= \int_0^\infty\int_0^\infty \theta\varpi_\alpha(\theta)\left[e^{-qt\theta} T_\alpha(t^\alpha)[\varphi(0) - B\varphi(-h)]\right]d\theta dt \\ &= \int_0^\infty e^{-qt}\left(\int_0^\infty \varpi_\alpha(\theta)\left[T_\alpha\left(\frac{t^\alpha}{\theta^\alpha}\right)[\varphi(0) - B\varphi(-h)]\right]d\theta\right)dt.\end{aligned}$$

We take the second expression from Eq. (11) and continue the simplifications, we have the following relationships

$$\begin{aligned}& \int_0^\infty e^{-q^\alpha s} T_\alpha(s) C\bar{x}_h ds \\ &= \int_0^\infty\int_0^\infty \alpha t^{\alpha-1} e^{-(qt)^\alpha} T_\alpha(t^\alpha) C\bar{x}_h(s)\,ds dt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\varpi_\alpha(\theta) e^{-(qt\theta)} T_\alpha(t^\alpha) t^{\alpha-1} C\bar{x}_h(s)\,d\theta ds dt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\varpi_\alpha(\theta) e^{-qt} T_\alpha\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha} C\bar{x}_h(s)\,d\theta ds dt \\ &= \int_0^\infty e^{-qt}\left(\alpha\int_0^\infty\int_0^\infty \varpi_\alpha(\theta) T_\alpha\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^\alpha}{\theta^\alpha} C\bar{x}_h(s)\,d\theta ds\right)dt.\end{aligned}$$

We take the last calculation for giving a more simple form of Eq. (11), the formula which we will simplify is given by the following relationships

$$\begin{aligned}& q^\alpha\int_0^\infty e^{-q^\alpha s} T_\alpha(s) B\bar{x}_h ds \\ &= \int_0^\infty\int_0^\infty \alpha q^\alpha t^{\alpha-1} e^{-(qt)^\alpha} T_\alpha(t^\alpha) B\bar{x}_h ds dt \\ &= \int_0^\infty\left[\int_0^\infty -T_\alpha(t^\alpha) B\bar{x}_h ds\right] de^{-(qt)^\alpha}\end{aligned} \tag{13}$$

Applying the integration by parts, and introducing the function described in Eq. (12), according to the calculations, we arrive at the following calculation for Eq. (13), that is

$$\begin{aligned}& q^\alpha\int_0^\infty e^{-q^\alpha s} T_\alpha(s) B\bar{x}_h ds = \int_0^\infty e^{-qt} Bx(t-h)\,dt \\ &+ \int_0^\infty e^{-qt}\left[q\int_0^t\int_0^\infty \varpi_\alpha(\theta) A T_\alpha\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right) C\frac{(t-s)^\alpha}{\theta^\alpha} d\theta ds\right]dt\end{aligned} \tag{14}$$

We now try to compute the inverse of the Laplace transform by inverting Eq. (11) by considering the simplified form described in the previous equation, we get the following form as the final expression

$$\begin{aligned} x(t) &= \int_0^\infty \phi_\alpha T_\alpha(t^\alpha\theta)[\varphi(0) - B\varphi(-h)]\,d\theta + Bx(t-h) \\ &+ q\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\phi_\alpha AB(s-h) T_\alpha\theta(t-s)^\alpha d\theta ds \\ &+ q\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\phi_\alpha T_\alpha\theta(t-s)^\alpha Cx(s-h)\,d\theta ds,\end{aligned}$$

where $\phi_{\alpha}(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi\left(\theta^{-\frac{1}{\alpha}}\right)$. With the previous representation of the solution we can now define our resolvents operators which we will consider to continue our investigation, the resolvents are represented as

$$R_{\alpha}(t)x = \int_0^{\infty}\phi_{\alpha}(\theta)T_{\alpha}(t^{\alpha}\theta)xd\theta, \tag{15}$$

and

$$S_{\alpha}(t)x = q\int_0^{\infty}\theta\phi_{\alpha}(\theta)T_{\alpha}(t^{\alpha}\theta)xd\theta. \tag{16}$$

Using Eq. (15) and Eq. (16) we get the solutions represented in the Lemma 4 when the following condition is respected $t \geq 0$. The second form of the solution is that $x(t) = \varphi(t),\ \ t \in [-h, 0]$. We end the proof of our lemma. □

Lemma 5. *The resolvents operators $R_{\alpha}(t)$ and $S_{\alpha}(t)$ are strongly continuous, and furthermore verify the relationships that they are bounded operators and satisfies the conditions that*

$$\begin{cases} \|R_{\alpha}(t)x\| \leq M\|x\| \\ and \\ \|S_{\alpha}(t)x\| \leq \frac{M\alpha}{\Gamma(1+\alpha)}\|x\| \end{cases} \tag{17}$$

where M is constant.

Now, we are ready to prove the existence of the mild solution of the neutral fractional differential equation defined in Eq. (1). We make a certain number of assumptions necessary in our investigations.

(A1) The resolvent operators $R_{\alpha}(t)$ and $S_{\alpha}(t)$ are compact operators for every $t \geq 0$.

(A2) The function $Cx(t-h)$ is messurable, continuous and satifies the condition that there exists $q \in (0,1)$ and $m \in L^{1/q}([0,T], \mathbb{R}^+)$, we have that $|Cx(t-h)| \leq m(t)\rho(\|x_t\|)$ for all $x \in C$ and furtermore almost all $t \in [0,T]$.

(A3) Let for the function $Bx(t-h)$ and we have existence of a constant $\beta \in (0,1)$ and two constant k and k_1 satisfying the condition that $Bx(t-h) \in D\left(A^{\beta}\right)$ and for $x, y \in C$ and $t \in [0,a]$ we have $\left\|A^{\beta}Bx - A^{\beta}By\right\| \leq k\|x-y\|$ and $\left\|A^{\beta}Bx\right\| \leq k_1(\|x_t\|+1)$.

For the main results of our present paper, we make the following theorem. This theorem proves the existence of the mild solution. In our investigation, we use Schauder Fixed Point Theorem, which is more appropriate for this study.

Theorem 3. *Under the hypotheses (A1), (A2) and (A3) the problem (1) has at least one mild solution.*

Proof. We begin by proving the boundedness of some mathematical expressions. Let the function that

$$\begin{aligned} &\left\|\int_0^t (t-s)^{\alpha-1}S_{\alpha}(t-s)ABx(s-h)ds\right\| \\ &\leq \int_0^t \left|(t-s)^{\alpha-1}A^{1-\beta}S_{\alpha}(t-s)A^{\beta}Bx(s-h)\right|ds \\ &\leq \int_0^t (t-s)^{\alpha-1}\left|A^{1-\beta}S_{\alpha}(t-s)A^{\beta}Bx(s-h)\right|ds \end{aligned}$$

We use the assumption described by (A3) and the statement posed in lemma 5, the next established results are well known in fundamental mathematics as the Lebesgue integrability of the function into the integration, we get the following relationships

$$\begin{aligned} &\left\|\int_0^t (t-s)^{\alpha-1}S_{\alpha}(t-s)ABx(s-h)ds\right\| \\ &\leq \int_0^t (t-s)^{\alpha-1}\left|A^{1-\beta}S_{\alpha}(t-s)A^{\beta}Bx(s-h)\right|ds \\ &= \int_0^t (t-s)^{\alpha-1}\frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)(t-s)^{\alpha(1-\beta)}}k_1(\|x_t\|+1)ds \\ &= \frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}k_1(\|x_t\|+1)\int_0^t (t-s)^{\alpha-1}ds \\ &= \frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}k_1(\|x_t\|+1)T^{\alpha\beta} \end{aligned} \tag{18}$$

As in the previous bound we also continue the simplification by trying to find a bound for the next integration, we have the following relationship

$$\begin{aligned} &\left\|\int_0^t (t-s)^{\alpha-1}S_{\alpha}(t-s)Cx(s-h)ds\right\| \\ &\leq \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t \left|(t-s)^{\alpha-1}Cx(s-h)\right|ds. \end{aligned}$$

Applying Holder inequality and using the assumption described in (A2), we get the following relationships

$$\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) Cx(s-h)ds\right\|$$
$$\leq \frac{M\alpha}{\Gamma(1+\alpha)}\left[\int_0^\infty (t-s)^{\frac{\alpha-1}{1-\beta}} ds\right]^{1-\beta} \|m\|_{L^{1/\beta}[0,t]}\,\rho(\|x\|)$$
$$\leq \frac{M\alpha}{\Gamma(1+\alpha)} \frac{\alpha M N a^{\left(1+\frac{\alpha-1}{1-\beta}\right)(1-\beta)} \rho(\|x\|)}{\Gamma(1+\alpha)\left(1+\frac{\alpha-1}{1-\beta}\right)^{1-\beta}}.$$

These two previous relationships will help us in the application of the fixed point theorem which we want to illustrate. The application of the fixed point need to defined an operator as the following form $\Phi : B_r \longrightarrow C([-h,a], X)$ such that

$$\Phi x(t) = R_\alpha(t)[\varphi(0) - B\varphi(-h)] + Bx(t-h)$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[AB]\,x(s-h)ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[C]\,x(s-h)ds$$
$$respect \quad to \quad t \geq 0 \tag{19}$$
$$\Phi x(t) = \varphi(t) \in t \in [-h,0] \tag{20}$$

Note that the set B_r is defined as all $x_t \in B_r$ satisfy the condition that $\|x_t\| \leq r$. The proof should de be divided into three parts, In the first part the operator Φ maps to itself. The first step is denoted by step1. We have the following.

Step 1: Let us prove that Φ maps to itself. We suppose that the following relationships will play important role in the proof-by-contradiction process, we have that

$$\lim_{r\to\infty}\left(\frac{\theta(r)}{r} + \frac{K}{r}\int_0^\infty \rho(s)\,ds\right) < 1. \tag{21}$$

We begin the proof by contradiction by applying the norm used in our paper to the function defined in Eq. (19), we also assume that all the assumptions have been verified as well, we have the following form

$$\|\Phi x(t)\| \leq M\|\varphi\| + M\left|A^{-\beta}\right| k_1(\|\varphi\|+1)$$
$$+ \left|A^{-\beta}\right| k_1(\|x_t\|+1)$$
$$+ \frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)} k_1(\|x_t\|+1)T^{\alpha\beta}$$
$$+ \frac{M\alpha}{\Gamma(1+\alpha)} \frac{\alpha M N a^{\left(1+\frac{\alpha-1}{1-\beta}\right)(1-\beta)}}{\Gamma(1+\alpha)\left(1+\frac{\alpha-1}{1-\beta}\right)^{1-\beta}} \int_0^t \rho(s)\,ds$$

Let's consider that the function and the constant that

$$\theta(\|x_t\|) = \left|A^{-\beta}\right| k_1\|x_t\| + \frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)} k_1\|x_t\| T^{\alpha\beta} \tag{22}$$

and

$$K = \frac{M\alpha}{\Gamma(1+\alpha)} \frac{\alpha M N a^{\left(1+\frac{\alpha-1}{1-\beta}\right)(1-\beta)}}{\Gamma(1+\alpha)\left(1+\frac{\alpha-1}{1-\beta}\right)^{1-\beta}} \tag{23}$$

And then the previous equation can be written in the form that

$$\|\Phi x(t)\| \leq M\|\varphi\| + M\left|A^{-\beta}\right| k_1(\|\varphi\|+1)$$
$$+ \left|A^{-\beta}\right| k_1 + \frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)} k_1 T^{\alpha\beta}$$
$$+ \theta(\|x_t\|) + K\int_0^t \rho(s)\,ds$$

Let that for each strictly positive constant r, there exist exist $x \in B_r$, such that $\phi x \notin B_r$. For simplification in the calculations, we add a further constant notation that is

$$M_1 = M\|\varphi\| + M\left|A^{-\beta}\right| k_1(\|\varphi\|+1) + \left|A^{-\beta}\right| k_1 \tag{24}$$

The previous assumption can be written mathematically by the condition described in the following form

$$r < \|\phi x(t)\| \leq M_1 + \theta(r) + K\int_0^t \rho(s)\,ds \tag{25}$$

The next step consists to divide the previous Eq. (25) by our constant r, we get the following relationships

$$1 < \|\phi x(t)\| \leq \frac{M_1}{r} + \frac{\theta(r)}{r} + \frac{K}{r}\int_0^t \rho(s)\,ds \tag{26}$$

Applying the limit respect to r at infinity, we get the following relationship which will contradict our preliminary assumptions, we have that

$$1 < \|\phi x(t)\| \leq \liminf_{r\to\infty}\left[\frac{\theta(r)}{r} + \frac{K}{r}\int_0^t \rho(s)\,ds\right] \tag{27}$$

We notice that Eq. (27) is in contradiction with the assumption reported in Eq. (21). This means that $\|\Phi x(t)\| \leq r$, we conclude that Φ maps to itself.

Step 2: In the second step, we will prove that the operator $\Phi : B_r \longrightarrow B_r$ is continuously using the classical method for proving the continuity. We set that $\{x^n\} \subseteq B_r$ respect to the property that $x^n \to x$ on the set B_r. In our present context using the assumption that (A) and the fact that $x_t^n \to x_t$, we have in particular the following intermediary condition that $Cx^n(t-h) \to Cx(t-h)$ a:e $t \in [0,T]$ when $n \to \infty$. Furthermore with the assumption (A), we have in particular that $\|Cx^n(s-h) - Cx(s-h)\| \leq m(s)\|\rho(x^n(s)) - \rho(x(s))\|$. We notice with the previous condition that, the fact that the function ρ is Lipschitz continuous implies the convergence to zero of the previous relationship. In addition, using the classical dominated convergence theorem, we get the following transformation and convergence, which are

$$\begin{aligned}
&\|\Phi x^n(t) - \Phi x(t)\| \leq \|Bx^n(t-h) - Bx(t-h)\|_X \\
&+ \int_0^t (t-s)^{\alpha-1}|S_\alpha(t-s)[ABx^n(s-h) - ABx(s-h)]|\,ds \\
&+ \int_0^t (t-s)^{\alpha-1}|S_\alpha(t-s)[Cx^n(s-h) - Cx(s-h)]|\,ds \\
&\leq (k+1))A^{-\beta}|B|\|x^n(t) - x(t)\|_X \\
&+ \int_0^t (t-s)^{\alpha-1}|S_\alpha(t-s)[ABx^n(s-h) - ABx(s-h)]|\,ds \\
&+ \int_0^t (t-s)^{\alpha-1}|S_\alpha(t-s)[Cx^n(s-h) - Cx(s-h)]|\,ds
\end{aligned}$$

Using the previously established results and the Lipchitz property in the assumption (A), and adding the condition in Lemma 5, we have the following form

$$\begin{aligned}
&\|\Phi x^n(t) - \Phi x(t)\| \leq (k+1))A^{-\beta}|B|\|x^n(t) - x(t)\|_X \\
&+ \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}|[ABx^n(s-h) - ABx(s-h)]|\,ds \\
&+ \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}|[Cx^n(s-h) - Cx(s-h)]|\,ds
\end{aligned} \tag{28}$$

We observe by using Eq. (28) that $\|\Phi x^n(t) - \Phi x(t)\| \to 0$ as $n \to \infty$ where it follows that the continuity of the operator Φ. The next section will be consecrated to prove that the set $\{\phi x : x \in B_r\}$ is relatively compact.

Step 3: As recalled at the end of the previous step in this part we try to prove that the set described by $\{\phi x : x \in B_r\}$ is relatively compact. Let $x \in B_r$ and $t_1 \leq t_2 \leq T$. We use two sub-operators, we have the following form

$$\phi_a x = R_\alpha(t)[\varphi(0) - B\varphi(-h)] + Bx(t-h) \tag{29}$$

We have the following expressions by applying the norm used in our space

$$\begin{aligned}
&\|\Phi x(t_2) - \Phi x(t_1)\| \\
&\leq \|(R_\alpha(t_2) - R_\alpha(t_1))[\varphi(0) - B\varphi(-h)]\|_X \\
&+ \|Bx(t_2-h) - Bx(t_1-h)\|_X \\
&\leq \|(R_\alpha(t_2) - R_\alpha(t_1))\|_X[\varphi(0) - B\varphi(-h)] \\
&+ \|Bx(t_2-h) - Bx(t_1-h)\|_X
\end{aligned}$$

We first use that the resolvent operator R_α is strongly continuous, and then we get that when $t_2 \to t_1$ thus $\|\Phi x(t_2) - \Phi x(t_1)\| \to 0$. We set the second operator as the following form

$$\phi_b x = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[AB]\,x(s-h)ds \tag{30}$$

Let $x \in B_r$ and $t_1 \leq t_2 \leq T$, to evaluate the convergence as in the previous section, we have the following relationships

$$\begin{aligned}
&\|\Phi_b x(t_2) - \Phi_b x(t_1)\| \\
&= \|\int_0^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]\,x(s-h)ds \\
&- \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_1-s)[AB]\,x(s-h)ds\| \\
&\leq \|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]\,x(s-h)ds\| \\
&+ \|\int_0^{t_1} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]\,x(s-h)ds \\
&- \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_2-s)[AB]\,x(s-h)ds\| \\
&+ \|\int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_2-s)[AB]\,x(s-h)ds \\
&- \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_1-s)[AB]\,x(s-h)ds\| \\
&\leq \|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]\,x(s-h)ds\| \\
&+ \|\int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] S_\alpha(t_2-s)[AB]\,x(s-h)ds\| \\
&+ \|\int_0^{t_1} (t_1-s)^{\alpha-1}[S_\alpha(t_2-s) - S_\alpha(t_1-s)][AB]\,x(s-h)ds\|
\end{aligned} \tag{31}$$

The previous relation in Eq. (31) can be rewritten in terms of three integral denotes here by the following form

$$\|\Phi_b x(t_2) - \Phi_b x(t_1)\| \leq I_1 + I_2 + I_3, \tag{32}$$

we have the following relationships for the simplification of our expressions, we have that

$$I_1 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) [AB] x(s-h) ds \right\| \tag{33}$$

$$I_2 = \left\| \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \times S_\alpha (t_2 - s) [AB] x(s-h) ds \right\| \tag{34}$$

$$I_3 = \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [S_\alpha (t_2 - s) - S_\alpha (t_1 - s)] \times [AB] x(s-h) ds \right\| \tag{35}$$

We now proceed to the calculations of the expressions represented in Eq. (33), Eq. (34) and Eq. (35), we have the following calculations

$$\begin{aligned}
I_1 &= \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) [AB] x(s-h) ds \right\| \\
&\leq \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left| A^{1-\beta} S_\alpha (t_2 - s) A^\beta B x(s-h) \right| ds \\
&= \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \frac{\alpha \Gamma (1+\beta) C_{1-\beta}}{\Gamma (1+\alpha\beta) (t-s)^{\alpha(1-\beta)}} k_1(\|x_t\| + 1) ds \\
&= \frac{\alpha \Gamma (1+\beta) C_{1-\beta}}{\Gamma (1+\alpha\beta)} k_1(\|x_t\| + 1) \int_{t_1}^{t_2} (t_2 - s)^{\alpha\beta-1} ds \\
&= \frac{\Gamma (1+\beta) C_{1-\beta}}{\Gamma (1+\alpha\beta)} k_1(\|x_t\| + 1) (t_2 - t_1)^{\alpha\beta}
\end{aligned} \tag{36}$$

We continue with the second expression represented by the variable I_2 in Eq. (34), we have the following calculations

$$\begin{aligned}
I_2 &= \left\| \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] S_\alpha (t_2 - s) [AB] x(s-h) ds \right\| \\
&\leq \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] \left| A^{1-\beta} S_\alpha (t_2 - s) A^\beta B x(s-h) \right| ds \\
&\leq \frac{\alpha M C_{1-\beta}}{\Gamma (1+\alpha)} \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] k_1(\|x_t\| + 1) ds \\
&\leq \frac{\alpha M C_{1-\beta}}{\Gamma (1+\alpha)} k_1(\|x_t\| + 1) \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] ds \\
&\leq \frac{\alpha M C_{1-\beta}}{\Gamma (1+\alpha)} k_1(\|x_t\| + 1) \left[|(t_2 - t_1)^\alpha| \right]
\end{aligned}$$

where we are assumed that $\left| A^{1-\beta} \right| \leq C_{1-\beta}$. We continue with the third integral, we have the following bound

$$\begin{aligned}
I_3 &= \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [S_\alpha (t_2 - s) - S_\alpha (t_1 - s)] [AB] x(s-h) ds \right\| \\
&\leq \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\left| A^{1-\beta} S_\alpha (t_2 - s) - A^{1-\beta} S_\alpha (t_1 - s) \right| \right] \left| \left[A^\beta B \right] x(s-h) \right| ds \\
&\leq \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\left| A^{1-\beta} S_\alpha (t_2 - s) - A^{1-\beta} S_\alpha (t_1 - s) \right| \right] k_1(\|x_t\| + 1) ds \\
&\leq \frac{t_1^\alpha k_1(\|x_t\| + 1)}{\alpha} \sup_{s \in [0, t_1]} \left[\left| A^{1-\beta} S_\alpha (t_2 - s) - A^{1-\beta} S_\alpha (t_1 - s) \right| \right]
\end{aligned} \tag{37}$$

Note that from the continuity of the resolvent operator S_α, follows also the continuity of the operator $A^{1-\beta} S_\alpha$. Then from Eq.(36) to Eq.(38), we observe that $t_2 \to t_1$ thus $\|\Phi_b x(t_2) - \Phi_b x(t_1)\|_X \to 0$. We finish this sub-section with the term represented by

$$\phi_c x = \int_0^t (t-s)^{\alpha-1} S_\alpha (t-s) [C] x(s-h) ds \tag{38}$$

Let $x \in B_r$ and $t_1 \leq t_2 \leq T$, to evaluate the convergence as in the previous section, we have the following relationships and referring to the previous section we have the following relationships

$$\|\Phi_c x(t_2) - \Phi_c x(t_1)\|_X \leq I_1 + I_2 + I_3 \tag{39}$$

where

$$I_1 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) C x(s-h) ds \right\| \tag{40}$$

$$I_2 = \left\| \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \times S_\alpha (t_2 - s) C x(s-h) ds \right\| \tag{41}$$

$$I_3 = \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [S_\alpha (t_2 - s) - S_\alpha (t_1 - s)] \times C x(s-h) ds \right\| \tag{42}$$

We do the same as the previous sub-section but here the Holder inequality is used many times, we begin with the expression represented by

$$\begin{aligned}
I_1 &= \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) C x(s-h) ds \right\| \\
&\leq \frac{\alpha M \rho (\|x_t\|)}{\Gamma (1+\alpha)} \left[\int_{t_1}^{t_2} (t_2 - s)^{\frac{1-\alpha}{1-\eta}} ds \right]^{1-\eta} \|m\|_{L^{1/\eta}[t_1, t_2]} .
\end{aligned} \tag{43}$$

For simplification in the rest of the calculutaion we take that $L_1 = \|m\|_{L^{1/\eta}[t_1,t_2]}$ and $\kappa = \frac{1-\alpha}{1-\eta}$, and then we get the following relationship

$$I_1 \leq \frac{\alpha M \rho\left(\|x_t\|\right) L_1 \left(t_2 - t_1\right)^{(1+\kappa)(1-\eta)}}{\Gamma\left(1+\alpha\right)\left(1+\kappa\right)^{1-\eta}}.$$

We now continue with the expression represented by the I_2 in Eq. (34), here also the Holder inequality is used for the simplification of the upbound, we have the following relationships

$$\begin{aligned} I_2 &= \left\| \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \right. \\ &\left. \times S_\alpha\left(t_2 - s\right) C x(s-h) ds \right\| \\ &\leq \tfrac{\alpha M \rho(\|x_t\|)}{\Gamma(1+\alpha)} \left[\int_0^{t_1} \left(t_1 - s\right)^{\frac{1-\alpha}{1-\eta}} - \left(t_2 - s\right)^{\frac{1-\alpha}{1-\eta}} ds \right]^{1-\eta} \\ &\times \|m\|_{L^{1/\eta}[t_1,t_2]} \\ &= \tfrac{\alpha M L_1 \rho(\|x_t\|)}{\Gamma(1+\alpha)(1+\kappa)^{1-\eta}} \left[t_1^{1+\kappa} - t_2^{1+\kappa} + \left(t_2 - t_1\right)^{1+\kappa} \right]^{1-\eta} \\ &\leq \tfrac{\alpha M L_1 \rho(\|x_t\|)}{\Gamma(1+\alpha)(1+\kappa)^{1-\eta}} \left(t_2 - t_1\right)^{(1+\kappa)(1-\eta)} \end{aligned} \tag{44}$$

We finish by repeating the same calculations with the expression described in I_3 at Eq. (35). We have to do the following results after the application of the Holder,

$$\begin{aligned} I_1 &= \left\| \int_0^{t_1} \left(t_1 - s\right)^{\alpha-1} \left[S_\alpha\left(t_2 - s\right) - S_\alpha\left(t_1 - s\right) \right] \right. \\ &\left. \times C x(s-h) ds \right\| \\ &\leq \tfrac{L_1 \rho(\|x_t\|) t_1^{(1+\kappa)(1-\eta)}}{(1+\kappa)^{1-\eta}} \\ &\times \sup_{s\in[0,t_1]} \left[S_\alpha\left(t_2 - s\right) - S_\alpha\left(t_1 - s\right) \right]. \end{aligned} \tag{45}$$

The first remark is that the resolvent operator S_α, follows also the continuity of the operator $A^{1-\beta} S_\alpha$. Then from Eq.(43) to Eq.(45), we observe that $t_2 \to t_1$ thus $\|\Phi_c x(t_2) - \Phi_c x(t_1)\|_X \to 0$. That ends the proof of the third step by concluding that the $\{\phi x : x \in B_r\}$ is relatively compact. □

4. Illustrative example

In this section we add an illustrative example to illustrate the findings of our paper, we take the partial neutral functional fractional differential equation under Caputo derivative described by the form that

$$D^\alpha \left[x\left(t, z\right) - \int_0^\pi g(z,y) x_t\left(\theta, y\right) dy \right] = \frac{\partial^2 x\left(t, z\right)}{\partial z^2} + f\left(t, x_t\right) \tag{46}$$

$$x\left(t, 0\right) = x\left(t, \pi\right) = 0, \quad 0 < t \leq 1, \tag{47}$$

$$x\left(\theta, z\right) = \phi\left(\theta, z\right), \quad -r \leq \theta \leq 0, \tag{48}$$

where the function g is an continuous function and measurable, $x_t\left(\theta, z\right) = x\left(t+\theta, z\right)$, $\phi\left(\theta, z\right)$ is also assumed to be continuous and the function f is specified later in the example. The next section will be to write the previous equation in terms of Eq. (1) representing our mean result. The second step will be to verify all the assumptions considered in this paper.

For the rest we suppose that $X = L^2\left([0,\pi]\right)$. We define an operator $A : D\left(A\right) \subset X \to X$ such that $Av = v''$ where the considered domain is defined by the set

$$\begin{aligned} D\left(A\right) = \{ v \in X : v; v' \text{are absolutely continuous;} \\ v'' \in X : v(0) = v(\pi) = 0 \}. \end{aligned} \tag{49}$$

Thus and the operator defined by A generates a compact semigroup $T(t)$ in X and it is having some properties summarized as the following properties. We have that $T\left(t\right) v = \sum_{n=1}^{\infty} e^{n^2 t} \left(v, e_n\right) e_n$ where $v \in X$. The second properties is that for each $v \in X$, we have that $A^{-1/2} v = \sum_{n=1}^{\infty} \frac{1}{n} \left(v, e_n\right) e_n$. The third properties is that the operator $A^{1/2}$ can be obtained by the form that $A^{1/2} v = \sum_{n=1}^{\infty} n \left(v, e_n\right) e_n$ where where the set $D\left(A^{1/2}\right) = \{ v \in X : \sum_{n=1}^{\infty} n \left(v, e_n\right) e_n \in X \}$. Note that in the previous part we works with $e_n\left(z\right) = \sqrt{\frac{2}{\pi}} \sin\left(nz\right)$ where $0 \leq z \leq \pi$. It is not hard to see that the family $\{e_n\}$ with $n = 1, 2, 3, ...$ represent an orthonormal base for our set X. Let consider that $Bx\left(t-h\right)\left(z\right) = \int_0^\pi g(z,y) x_t\left(\theta, y\right) dy$. We assume that g is continuously differentiable and satisfies the condition that $b(t,.,0) = b(t,.,\pi) = 0$. Let the function f is Lipschitz continuous according to the following properties that $\|f\left(t, \xi_1\right) - f\left(t, \xi_2\right)\| \leq a \|\xi_1 - \xi_1\|$ where $\xi_1, \xi_2 \in R$. Finally, the fractional differential equation represented by Caputo derivative of order $\alpha = 0.5$ can be presented as the form

$$D_t^{\alpha}\left[x\left(t\right)-Bx\left(t-h\right)\right]$$
$$=Ax\left(t\right)+f\left(t,x_t\right)\quad t\geq 0 \tag{50}$$
$$x_0=\varphi\in\mathcal{C}_X \tag{51}$$

where $f\left(t,x_t\right)=\frac{1}{t^{1/3}}\sin x_t$. we can see that the function f is Lipchitz continuous, and it is trivial to see the assumption (A2) is satisfied. The verification of the last assumption assumption (A3), the sketch of the proof can be found in, we proceed as the following, let that $\int_0^{\pi}\int_0^{\pi}g^2(z,y)dydz<\infty$, furthermore we consider that $\frac{\partial}{\partial z}g(z,y)$ is measurable too and satisfying the conditions that $g(0,y)=g(\pi,y)=0$ and $\delta=\left(\int_0^{\pi}\int_0^{\pi}\left[\frac{\partial}{\partial z}g(z,y)\right]^2dydz\right)^{\frac{1}{2}}<\infty$. For simplification in our calculations we let that $\int_0^{\pi}g(z,y)v(z)dy=U_hv\left(z\right)$. Using the fact that $\int_0^{\pi}\int_0^{\pi}g^2(z,y)dydz<\infty$ generate that U_h is bounded into the set X and we have that $U_hv\in D\left(A^{1/2}\right)$ and $\left\|A^{1/2}U_h\right\|<\infty$. The second assumption $\delta=\left(\int_0^{\pi}\int_0^{\pi}\left[\frac{\partial}{\partial z}g(z,y)\right]^2dydz\right)^{\frac{1}{2}}<\infty$ and using the definition of e_n, we get the following relationship

$$\begin{aligned}(U_h(v),e_n)&=\int_0^{\pi}e_n\left(\int_0^{\pi}g(z,y)x_t\left(\theta,y\right)dy\right)dz\\&=\frac{1}{n}\sqrt{\frac{2}{\pi}}\left(\int_0^{\pi}\frac{\partial}{\partial z}g(z,y)x_tdy,\cos nz\right)\end{aligned} \tag{52}$$

Thus $\left\|\int_0^{\pi}\frac{\partial}{\partial z}g(z,y)v(z)dy\right\|\leq\delta$ and then we get $\left\|A^{1/2}U_h\left(v\right)\right\|\leq\delta$, wchich generate the satisfaction of the assumption $(A3)$. The present results can be compared with the results in the same direction in [25, 32, 33]. The difference is the used resolvent operators. The nature of the resolvent depends on the used problem and the used fractional operators. It is important to see that the resolvent operators defined in this paper for existence depend on the order of the fractional operator. But in general, the works are in good agreement.

5. Conclusion

In this paper, we have focussed on the existence of the unique solution of the partial neutral functional fractional differential equation described by the Caputo derivative. The novelties of this work were the use of the fractional resolvents operators to arrive to prove existence via fixed point theorem. The present investigation can be made with the other operators by in our idea it should depend on new fractional resolvent operators. Their definitions should differ according to the fractional operators. This idea can be an open problem for future investigations.

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RESEARCH ARTICLE

Intuitionistic fuzzy eigenvalue problem

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ABSTRACT

The purpose of this paper is the study of the eigenvalues of the second order fuzzy boundary value problem (FBVP). By using the (α-β)-level set of intuitionistic fuzzy numbers and Zadeh's extension principle, the FBVP is solved with the proposed method. Furthermore, a numerical example is illustrated and the advantages of the proposed approach are compared with other well-known methods such as the solutions based on the generalized Hukuhara derivative.

1. Introduction

Consider the following FBVP

$$u'' = -\lambda u, \quad t \in [a,b] \tag{1}$$

which satisfy the conditions

$$\widehat{a^i}_1 u(a) = \widehat{a^i}_2 u'(a) \tag{2}$$

$$\widehat{b^i}_1 u(b) = \widehat{b^i}_2 u'(b) \tag{3}$$

where $\widehat{a^i}_1, \widehat{a^i}_2, \widehat{b^i}_1, \widehat{b^i}_2$ intuitionistic fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a^i}_1$ and $\widehat{a^i}_2$ and at least one of the numbers $\widehat{b^i}_1$ and $\widehat{b^i}_2$ are nonzero.

The subject of fuzzy differential equations (FDEs) was first introduced by Kaleva [1] and Seikkala [2] and has been expended and studied by many researchers for the purpose of modeling problems in science and engineering [3–6]. Most practical problems require the solution of an FDE satisfying fuzzy initial or boundary conditions., so a fuzzy initial value problem (IVP) or boundary value problem (BVP) should be solved. There are several approaches to solve fuzzy problems such as the Hukuhara derivative or Seikkala derivative, the differential inclusion and the derivative based on the Zadeh's extension principle which is widely used for FDEs [7–16].

Puri and Dan introduced the H-derivative [17], and later it was further explored by Kaleva [1] and Seikkala [2]. But in some cases the H-derivative method has a disadvantage that a fuzzy differential equation may have only solutions with nondecreasing lengths of the diameter of the level sets [1, 18]. This disadvantage was solved by Hüllermeier [19], who interpreted a FDE as a family of differential inclusions. Another approach to solve fuzzy problem has been proposed, including Zadeh's extension principle expanding the ordinary differential equations to the fuzzy cases [20]. Then the arithmetic operations are considered to be operations on fuzzy numbers [21].

An effective concept of the differentiability of fuzzy-valued functions is given as the strongly generalized differentiability concept (gh-differentiability) which was first introduced by Bede et al [22]. The fuzzy solutions with gh-differentiability have some not an interval solutions which are associated with the existence of switch points [23]. In addition, Gasilov et al. argued that the solutions obtained by the method of

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Khastan and Nieto [7] are difficult to evaluate, because the solutions to the four different problems may not reflect the nature of the phenomenon being studied [9].

Recently, intutionistic fuzzy set theory (IFST) has become very popular. It is used in various industries, robotics, in audiovisual systems etc. Therefore, many researchers have dedicated their time to the development of IFST. Atanassov [24] generalized the concept of fuzzy set theory by intuitionistic fuzzy set (IFS) which is an extension of fuzzy set introduced by Zadeh [25]. The degree of acceptance in fuzzy sets is only considered, otherwise IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than one [26, 27]. The concept of intuitionistic fuzzy metric space has been introduced Melliani et al. [28] and differential and partial differential equations have been discussed under intuitionistic fuzzy environment.

On the other hand, Melliani et al. [10] gave the the existence and uniqueness theorem of a solution to the intuitionistic FDE. Numerical solution of intuitionistic FDE by Runge-Kutta method has been introduced with intuitionistic treatment in [29] and by Euler method has been discussed by Nirmala and Chenthur Pandian based on the $\alpha-$level [30].

In literature, although there are many approaches to solve the FDEs, there are only few papers such as [11–14, 31] in which the eigenvalues and the eigenfunctions of the FBVP are examined by using different methods such as H-differentiabilty, gH-differentiability and the Zadeh's extension principle.

The main aim of this research is to find eigenvalues of FBVP under the intuitionistic Zadeh's extension principle [32].

In this work, the solutions of the intuitionistic fuzzy eigenvalue problem are studied. The rest of this study is organized as follows, In Section 2, consists of basic definitions related to intuitionistic fuzzy set theory. In Section 3, intuitionistic fuzzy problem and a numerical example is given. Conclusion of the paper is in section 4.

2. Preliminaries

Before proceeding to the solution method, the notations and definitions that will be used throughout the paper are given. To denote an intuitionistic fuzzy number, a bar of the form $\widehat{\ }^{i}$ is placed over a letter. Also, $\widehat{u^i}(t)$ is written for intuitionistic fuzzy-valued functions defined over the real numbers.

Definition 1. *[26] Let $A \subseteq X$ and let $\mu_A(t) : X \to [0,1]$, $\zeta_A(t) : X \to [0,1]$ be two functions such that $0 \le \mu_A(t) + \zeta_A(t) \le 1$. The set*

$$\widehat{A^i} = \{(t, \mu_A(t), \zeta_A(t)) : t \in X, \; \mu_A(t), \zeta_A(t) : X \to [0,1]\}$$

is called an intuitionistic fuzzy set of X.

Here $\mu_A(t)$ is called membership function and $\zeta_A(t)$ is called non-membership function and the set of all intuitionistic fuzzy sets of X will be denoted by $IF(X)$.

Definition 2. *[26] Let $\widehat{A^i} \in IF(X)$. The set*

$$A(\alpha,\beta) = \{t \in X : \alpha,\ \beta \in [0,1] \ ; \; \mu_A(t) \ge \alpha,\ \zeta_A(t) \le \beta,\ 0 \le \alpha+\beta \le 1\}$$

is called the (α,β)-level of the intuitionistic fuzzy set $\widehat{A^i}$.

Theorem 1. *[26] Let $\widehat{A^i} \in IF(X)$. Then $A(\alpha,\beta) = A(\alpha) \cap A^*(\beta)$ holds.Here $A(\alpha)$ is $\alpha-$level set and $A^*(\beta)$ is $\beta-$level set.*

Definition 3. *[26] An intuitionistic fuzzy set $\widehat{A^i} \in IF(R^n)$ satisfying the following properties is called an intuitionistic fuzzy number in R^n*

1) *$\widehat{A^i}$ is a normal set, i.e., $\exists\ t_0 \in R^n$ such that $\mu_A(t_0) = 1$ and $v_A(t_0) = 0$,*

2) *$A(0)$ and $A^*(1)$ are bounded sets in R^n,*

3) *$\mu_A : R^n \to [0,1]$ is an upper semi-continuous function, i.e.,*

$\forall\ k \in [0,1]$, $(\{t \in A : \mu_A(t) < k\})$ is an open set.

4) *$\zeta_A : R^n \to [0,1]$ is a lower semi-continuous function, i.e.,*

$\forall\ k \in [0,1](\{t \in A : \zeta_A(t) > k\})$ is an open set.

5) *The membership function $\mu_A(t)$ is quasi-concave, i.e.,*

$\forall\ n \in [0,1]$, $\forall\ x, y \in R^n$

$$\mu_A(nt + (1-n)x) \ge min(\mu_A(t), \mu_A(x)),$$

6) *The non-membership function $\zeta_A(t)$ is quasi-convex; i.e.,*

$\forall\ n \in [0,1]$, $\forall x, y \in R^n$

$$\zeta_A(nt + (1-n)x) \le max(\zeta_A(t), \zeta_A(x)).$$

The set of all intuitionistic fuzzy numbers of R^n will be denoted by $IFN(R^n)$.

Definition 4. *[10] A triangular intuitionistic fuzzy number (TIFN) $\widehat{A^i} \in IF(R^n)$ is defined with the following membership and non-membership functions:*

$$\mu_A(t) = \begin{cases} \frac{t-a_1}{a_2-a_1};\ a_1 \le t \le a_2 \\ \frac{a_2-t}{a_3-a_2};\ a_2 \le t \le a_3 \\ 0;\ otherwise \end{cases}$$

and

$$\zeta_A(t)=\begin{cases}\frac{a_2-t}{a_2-a_1^*};\ a_1^*\le t\le a_2\\ \frac{t-a_2}{a_3^*-a_2};\ a_2\le t\le a_3^*\\ 1;\ otherwise\end{cases}.$$

Here $a_1^*\le a_1\le a_2\le a_3\le a_3^*$ *and it is denoted by* $\widehat{A^i}=(a_1,a_2,a_3;a_1^*,a_2,a_3^*)$.

Remark 1. *[33] Let* $\widehat{A^i}\in IFN(R)$*. Then* $\left[\widehat{A}\right]^\alpha$ *and* $\left[\widehat{A^*}\right]^\beta$ *are closed and bounded intervals such that*

$$\left[\widehat{A}\right]^\alpha=\left[A_\alpha^-,A_\alpha^+\right]=\left[(a_2-a_1)\,\alpha+a_1,a_3-(a_3-a_2)\,\alpha\right]$$

and

$$\left[\widehat{A^*}\right]^\alpha=\left[a_2-(a_2-a_1^*)\,\alpha,(a_3^*-a_2)\,\alpha+a_2\right].$$

Definition 5. *[32] Let X and Y be two sets and $g:X\to Y$ be a function. Let $\widehat{A^i}$ be an intuitionistic fuzzy set in X. Then $f(\widehat{A^i})$ is an intuitionistic fuzzy set in Y such that for every $y\in Y$*

$$\mu_{g\left(\widehat{A^i}\right)}(y)=\begin{cases}\sup\{\mu_A(x):g(x)=y\};y\in g(x)\\ 0;\qquad y\notin f(x),\end{cases}$$

and

$$\zeta_{g\left(\widehat{A^i}\right)}(y)=\begin{cases}\inf\{\zeta_A(x):g(x)=y\};y\in g(x)\\ 1;\qquad y\notin g(x),\end{cases}$$

Definition 6. *[33] The function*

$$\theta(x)=\begin{cases}1,&x\ge0\\0,&x<0\end{cases}$$

is called the Heaviside step function.

3. Numerical Method for the FBVP

Here, the eigenvalues and the fuzzy eigenfunctions of the intuitionistic fuzzy problem (1)-(3) are investigated. Then, similar to the method applied by Titchmarsh [34], we will use the solutions of (1) that satisfy the fuzzy initial conditions instead of the fuzzy boundary conditions. To solve intuitionistic fuzzy IVPs, the method created by Akin and Bayeğ is used [33]. To do this, firstly the crisp IVP will be solved.

Then, the solution of intuitionistic FIVPs will be obtained from classical solutions using the intuitionistic Zadeh's extension principle. The fuzzy solutions do not require the analysis of existence of switching endpoints of α and β levels, because Heaviside (step) function will be applied during the interval operations on α and β levels.

Now, let the linear and homogeneous differential equation (1) be considered separately with intuitionistic fuzzy boundary conditions (2) and (3),respectively.

$$\begin{cases}\chi''+\lambda\chi=0\\ \chi(a)=\widehat{a^i}_2,\ \chi'(a)=\widehat{a^i}_1\end{cases}\tag{4}$$

and

$$\begin{cases}\Psi''+\lambda\Psi=0\\ \Psi(b)=\widehat{b^i}_2,\ \Psi'(b)=\widehat{b^i}_1.\end{cases}\tag{5}$$

where $\widehat{a^i}_1,\widehat{a^i}_2,\widehat{b^i}_1,\widehat{b^i}_2$ intuitionistic triangular fuzzy numbers, $\lambda>0$.

Theorem 2. *[33] Let $\widehat{\chi^i}$ (t) and $\widehat{\Psi^i}$ (t) be the solution of the intuitionistic IVP in (4) and (5) obtained by intuitionistic Zadeh's extension principle. Let α and β levels of $\widehat{\chi^i}$ (t) and $\widehat{\Psi^i}$ (t), $\widehat{a_k^i}$ and $\widehat{b_k^i}$ $(k=1,2)$ be given by $[\chi_\alpha^-(t,\lambda),\chi_\alpha^+(t,\lambda)]$, $[\Psi_\alpha^-(t,\lambda),\Psi_\alpha^+(t,\lambda)]$ and $\left[(\chi^*)_\alpha^-(t,\lambda),(\chi^*)_\alpha^+(t,\lambda)\right]$, $\left[(\Psi^*)_\alpha^-(t,\lambda),(\Psi^*)_\alpha^+(t,\lambda)\right]$; $\left[(a_k)_\alpha^-,(a_k)_\alpha^+\right]$, $\left[(b_k)_\alpha^-,(b_k)_\alpha^+\right]$ and $\left[(a_k^*)_\alpha^-,(a_k^*)_\alpha^+\right]$, $\left[(b_k^*)_\alpha^-,(b_k^*)_\alpha^+\right]$, respectively. Then the α and β levels of the solution can be determined as follows:*

$$\begin{cases}\chi_\alpha^-=\sum_{k=1}^2\left[(a_k)_\alpha^+-\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\\ \chi_\alpha^+=\sum_{k=1}^2\left[(a_k)_\alpha^-+\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\\ (\chi^*)_\beta^-=\sum_{k=1}^2\left[(a_k^*)_\alpha^+-\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\\ (\chi^*)_\beta^+=\sum_{k=1}^2\left[(a_k^*)_\alpha^-+\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\end{cases}$$

and

$$\begin{cases}\Psi_\alpha^-=\sum_{k=1}^2\left[(a_k)_\alpha^+-\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\\ \Psi_\alpha^+=\sum_{k=1}^2\left[(a_k)_\alpha^-+\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\\ (\Psi^*)_\beta^-=\sum_{k=1}^2\left[(a_k^*)_\alpha^+-\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\\ (\Psi^*)_\beta^+=\sum_{k=1}^2\left[(a_k^*)_\alpha^-+\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\end{cases}$$

Here $K_{1k}(t)$ and $K_{2k}(t)$ are Heaviside function.

$$\chi_\alpha^-=\sum_{k=1}^2\left[(a_k)_\alpha^+-\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

$$\chi_\alpha^+=\sum_{k=1}^2\left[(a_k)_\alpha^-+\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

$$(\chi^*)_\alpha^-=\sum_{k=1}^2\left[(a_k^*)_\alpha^+-\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

$$(\chi^*)_\alpha^+=\sum_{k=1}^2\left[(a_k^*)_\alpha^-+\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

First, let us look for the solution to the problem in Equation (4). Then, by performing similar operations, we find the solution to the problem (5). First of all we solve the following crisp IVP related to the fuzzy IVP in Eq. (4) and then apply intuitionistic Zadeh's Extension Principle to the solution [33]:

$$\begin{cases} \chi'' + \lambda\chi = 0 \\ \chi(a) = a_2, \ \chi'(a) = a_1 \end{cases} \quad (6)$$

where a_1, a_2 and λ are real numbers. The general solution of the differential equation (6) can be written as:

$$\chi_\lambda(t) = C_1\chi_1(t) + C_2\chi_2(t), \quad (7)$$

where C_1 and C_2 are arbitrary constants; $\chi_1(t)$ and $\chi_1(t)$ are linearly independent functions satisfying the Eq. (6).

Let us substitute the initial conditions to find the coefficients C_1 and C_2 in equation Eq. (7). Therefore, the following system of equations is obtained:

$$\begin{cases} C_1\chi_1(a) + C_2\chi_2(a) = a_2 \\ C_1\chi_1'(a) + C_2\chi_2'(a) = a_1 \end{cases} \quad (8)$$

In Eq.(8) C and B are unknown coefficients and the following notations are used for convenience.

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix};$$

$$w_{11} = \chi_1(a), \ w_{12} = \chi_2(a), \ w_{21} = \chi_1'(a), \ w_{22} = \chi_2'(a);$$

$$\overrightarrow{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \ \overrightarrow{a} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}.$$

According to these notations, (8) is written in the matrix form:

$$W\overrightarrow{C} = \overrightarrow{a}.$$

Using Cramer's method, C_1 and C_2 are obtained as follows:

$$C_J = \frac{|W_1|}{|W|} - \frac{|W_2|}{|W|}.$$

Here

$$|W| = \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} = w_{11}w_{22} - w_{21}w_{12},$$

$$|W_1| = \begin{vmatrix} a_2 & w_{12} \\ a_1 & w_{22} \end{vmatrix} = a_2w_{22} - a_1w_{12},$$

$$|W_2| = \begin{vmatrix} w_{11} & a_2 \\ w_{21} & a_1 \end{vmatrix} = a_1w_{11} - a_2w_{21}.$$

Thus, C_1 and C_2 can be rewritten as

$$C_1 = \frac{|W_1|}{|W|} = \frac{a_2w_{22} - a_1w_{12}}{|W|},$$

$$C_2 = \frac{|W_2|}{|W|} = \frac{a_1w_{11} - a_2w_{21}}{|W|}.$$

C_1 and C_2 can be rewritten as follows to simplify the above results, respectively

$$C_1 = a_2f_{22} - a_1f_{12},$$
$$C_2 = a_1f_{11} - a_2f_{21}$$

where $f_{ij} = \frac{w_{ij}}{|W|}; i, j = 1, 2.$

From the results for C_1 and C_2, the classical solution of the given crisp IVP can be derived as follows:

$$\begin{aligned} \chi_\lambda(t) &= C_1\chi_1(t) + C_2\chi_2(t), \\ &= (a_2f_{22} - a_1f_{12})\chi_1(t) \\ &\quad + (a_1f_{11} - a_2f_{21})\chi_2(t). \end{aligned}$$

This solution can also be written as:

$$\begin{aligned} \chi_\lambda(t) = a_2(f_{22}\chi_1(t) - f_{21}\chi_2(t)) \\ + a_1(f_{11}\chi_2(t) - f_{12}\chi_1(t)). \end{aligned}$$

Next the following notations are used for the sake of its comprehension:

$$\begin{aligned} K_{11}(t) &= f_{22}\chi_1(t) - f_{21}\chi_2(t), \\ K_{12}(t) &= f_{11}\chi_2(t) - f_{12}\chi_1(t). \end{aligned} \quad (9)$$

Thus the solution of the crisp IVP (6) can be written as:

$$\chi_\lambda(t) = a_2K_{11}(t) + a_1K_{12}(t). \quad (10)$$

It is easy to see that the solution in Eq. (10) is linearly dependent only on the initial values. Now, Zadeh's extension principle is applied to the intuitionistic fuzzy sets and the solution of the fuzzy IVP as follows:

$$\widehat{\chi^i}_\lambda(t) = \widehat{a^i}_2K_{11}(t) + \widehat{a^i}_1K_{12}(t) \quad (11)$$

In terms of α and β levels of the intuitionistic fuzzy numbers it is obtained that

$$\begin{cases} \left[\chi_\alpha^-(t,\lambda), \chi_\alpha^+(t,\lambda)\right] = \sum_{k=1}^{2}\left[(a_k)_\alpha^-, (a_k)_\alpha^+\right]K_{1k}(t) \\ \left[(\chi^*)_\alpha^-(t,\beta), (\chi^*)_\alpha^+(t,\beta)\right] = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^-, (a_k^*)_\alpha^+\right]K_{1k}(t) \end{cases}$$

where $\chi_\alpha^-(t,\lambda)$, $(a_k)_\alpha^-$, ; $(\chi^*)_\alpha^-(t,\lambda)$, $(a_k^*)_\alpha^-$ are lower bounds for $\alpha-$levels and $\beta-$levels, respectively and $\chi_\alpha^+(t,\lambda)$, $(a_k)_\alpha^+$, ; $(\chi^*)_\alpha^+(t,\lambda)$, $(a_k^*)_\alpha^+$ are upper bounds for $\alpha-$levels and $\beta-$levels, respectively

Using the Heaviside function and interval arithmetic the α and β levels of the solution $\widehat{\chi^i}_\lambda(t)$ can be written as follows:

$$\begin{cases} \chi_\alpha^- = \sum_{k=1}^{2}\left[(a_k)_\alpha^+ - \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t) \\ \chi_\alpha^+ = \sum_{k=1}^{2}\left[(a_k)_\alpha^- + \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t) \\ (\chi^*)_\alpha^- = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^+ - \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t) \\ (\chi^*)_\alpha^+ = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^- + \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t) \end{cases} \quad (12)$$

For $\left[\widehat{\Psi^i}_\lambda(t)\right]^\alpha$, a solution is found for the problem (5) by doing similar operations. So the solution of the crisp IVP $\Psi_\lambda(t)$ can be written as:

$$\Psi_\lambda(t) = b_2 K_{21}(t) + b_1 K_{22}(t). \quad (13)$$

Then Zadeh's extension principle is applied and the solution of the fuzzy IVP as follows:

$$\widehat{\Psi^i}_\lambda(t) = \widehat{a^i}_2 K_{21}(t) + \widehat{a^i}_1 K_{22}(t). \quad (14)$$

By taking $\alpha-$levels and $\beta-$levels, into account in the solution (5) and using the Heaviside function, the solution $\widehat{\Psi^i}_\lambda(t)$ can be written as follows:

$$\begin{cases} \Psi_\alpha^- = \sum_{k=1}^{2}\left[(a_k)_\alpha^+ - \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \\ \Psi_\alpha^+ = \sum_{k=1}^{2}\left[(a_k)_\alpha^- + \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \\ (\Psi^*)_\alpha^- = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^+ - \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \\ (\Psi^*)_\alpha^+ = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^- + \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \end{cases} \quad (15)$$

Because the eigenvalues of the problem (1)-(3) if and only if consist of the zeros of function $W(\chi, \Psi)(t, \lambda)$ in [34], Wronskian function is found from the classical solutions (10) and (13) for classic eigenvalue λ as follows :

$$W(\chi, \Psi)(t, \lambda) = \chi_\lambda(t)\Psi'_\lambda(t) - \chi'_\lambda(t)\Psi_\lambda(t). \quad (16)$$

Now we give the following numerical example to demonstrate the proposed method.

Example 1. *Consider the intuitionistic fuzzy boundary value problem*

$$-u'' = \lambda u \quad (17)$$
$$\widehat{2^i}u(0) = \widehat{1^i}u'(0) \quad (18)$$
$$\widehat{4^i}u(1) = \widehat{3^i}u'(1) \quad (19)$$

where $\widehat{1^i} = (0, 1, 2; -1, 1, 3)$, $\widehat{2^i} = (1, 2, 3; 0, 2, 4)$, $\widehat{3^i} = (2, 3, 4; 1, 3, 5)$, $\widehat{4^i} = (3, 4, 5; 2, 4, 6)$ *intuitionistic triangular fuzzy numbers and* $\lambda = p^2$, $p > 0$. *From problem* (17)-(19) *, we get two intuitionistic FIVPs as follows:*

$$\chi'' + p^2\chi = 0, \quad \chi(0) = \widehat{1^i}, \quad \chi'(0) = \widehat{2^i} \quad (20)$$

and

$$\Psi'' + p^2\Psi = 0, \quad \Psi(1) = \widehat{3^i}, \quad \Psi'(1) = \widehat{4^i}. \quad (21)$$

Let us first solve the crisp IVP:

$$\chi'' + p^2\chi = 0, \quad \chi(0) = 1, \quad \chi'(0) = 2.$$

By solving the differential equation in the crisp IVP, the general crisp solution is obtained as:

$$\chi(t, \lambda) = C_1 \cos(pt) + C_2 \sin(pt).$$

The functions $K_{11}(t)$ *and* $K_{12}(t)$ *are obtained as follows:*

$$\begin{aligned} K_{11}(t) &= \cos(pt) \\ K_{12}(t) &= \frac{1}{p}\sin(pt). \end{aligned} \quad (22)$$

Thus the solution of the crisp IVP can be written using (22) *as:*

$$\begin{aligned} \chi(t, \lambda) &= a_2 K_{11}(t) + a_1 K_{12}(t) \\ &= \frac{2}{p}\sin(pt) + \cos(pt) \end{aligned} \quad (23)$$

Similarly, the solution $\Psi(t, \lambda)$ *is written as:*

$$\Psi(t, \lambda) = \frac{4}{p}\sin(pt - p) + 3\cos(pt - p). \quad (24)$$

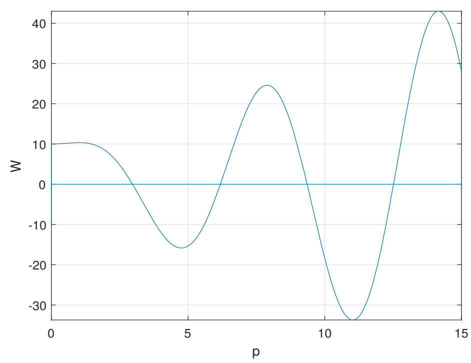

Figure 1. The function $W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (4-6)\cos(p)$.

Then, Wronskian functions can be gotten from Eq. (16) *as:*

$$\begin{aligned} W(\lambda) &= W(\chi, \Psi)(t, \lambda) \\ &= \left(3p + \frac{8}{p}\right)\sin(p) + (-2)\cos(p). \end{aligned}$$

The classic eigenvalues of problem (17)-(19) *consist of the zeros of the* $W(\lambda)$ *functions. For this reason, an infinite number of eigenvalues satisfying the equation* $W(\lambda) = 0$ *can be obtained by calculating p values in Matlab programme in Figure 1.*

Table 1. Eigenvalues of the fuzzy problem.

	p_n	λ_n
$n=1$	*3.30241*	*10.90581*
$n=2$	*6.38091*	*40.71581*
$n=3$	*9.49291*	*90.11511*
$n=4$	*12.61831*	*159.22151*
$n=5$	*15.74981*	*248.05621*
$n \approx$	$n\pi$	$(n\pi)^2$

The first five eigenvalues are found numerically and then the approximation of the remaining eigenvalues is written in tabl e 1.

From (12) *and* (15) *$\alpha-$levels and $\beta-$levels of the solutions $\widehat{\chi^i}_\lambda(t)$ and $\widehat{\Psi^i}_\lambda(t)$, respectively can be found as follows:*

$$\begin{aligned}
\chi_\alpha^-(t,\lambda) &= [2-\alpha-2(1-\alpha)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[3-\alpha-2(1-\alpha)\theta(K_{12}(t))]K_{12}(t),\\
\chi_\alpha^+(t,\lambda) &= [\alpha+2(1-\alpha)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[\alpha+1+2(1-\alpha)\theta(K_{12}(t))]K_{12}(t),\\
(\chi^*)_\alpha^-(t,\beta) &= [2\beta+1-(4\beta)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[2+2\beta-(4\beta)\theta(K_{12}(t))]K_{12}(t),\\
(\chi^*)_\alpha^+(t,\beta) &= [1-2\beta+(4\beta)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[2-2\beta+(4\beta)\theta(K_{12}(t))]K_{12}(t).
\end{aligned}$$

and

$$\begin{aligned}
\Psi_\alpha^- &= [4-\alpha-2(1-\alpha)\theta(K_{21}(t))]K_{21}(t)\\
&\quad+[5-\alpha-2(1-\alpha)\theta(K_{22}(t))]K_{22}(t),\\
\Psi_\alpha^+ &= [2+\alpha+2(1-\alpha)\theta(K_{21}(t))]K_{21}(t)\\
&\quad+[3+\alpha+2(1-\alpha)\theta(K_{22}(t))]K_{22}(t),\\
(\Psi^*)_\alpha^- &= [3+2\beta-(4\beta)\theta(K_{21}(t))]K_{21}(t)\\
&\quad+[4+2\beta-(4\beta)\theta(K_{22}(t))]K_{22}(t),\\
(\Psi^*)_\alpha^+ &= [3-2\beta+(4\beta)\theta(K_{11}(t))]K_{21}(t)\\
&\quad+[4-2\beta+(4\beta)\theta(K_{22}(t))]K_{22}(t).
\end{aligned}$$

where $\theta(t)$ is the Heaviside function, $K_{11}(t) = \cos(pt)$, $K_{12}(t) = \frac{1}{p}\sin(pt)$, $K_{21}(t) = \cos(pt-p)$ *and* $K_{22}(t) = \frac{1}{p}\sin(pt-p)$.

In particular, $p_1 = 3.30241$ *in Table 1 and substitute* (25) *and* (25) *are selected. The α and β levels of the solutions $\widehat{\chi^i}_{p_1}(t)$ and $\widehat{\Psi^i}_{p_1}(t)$ are given in Figures 2, 3 and Figures 4, 5.*

Consider the FBVP given as in (17)-(19), using gh-differentiabilty by converting the FDE into a family of systems of classical differential equation [35]. Now we have that the graphical representation of the endpoint functions χ_α^-, χ_α^+ in Figure 6 and Ψ_α^-, Ψ_α^+ in Figure 7 obtained of (1,1)-system for every $\alpha \in [0,1]$. *In Figure 6 and 7, it is seen that the $\widehat{\chi}$ and $\widehat{\Psi}$ functions do not fulfil the fuzzy solution properties duo to the existence of switching points in the entire interval* $[0, 3.5]$.

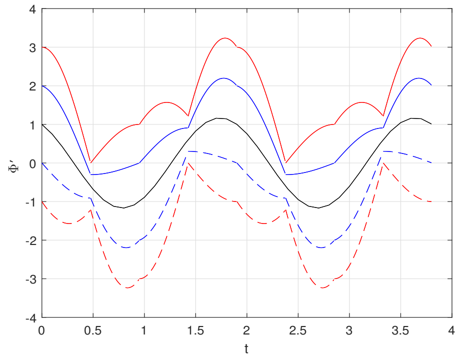

Figure 2. The $\widehat{\chi^i}_\lambda(t)$ solution in Example 1. The black line represents the reel solution. The red and blue lines represent upper solution for $\beta = 1$ and $\alpha = 0$, respectively and the dashed red and blue lines represent lower solution for $\beta = 1$ and $\alpha = 0$, respectively

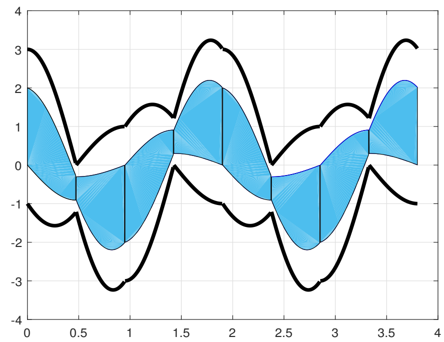

Figure 3. The blue region of the intersection of fuzzy solution $[\chi]^\alpha$ and $[\chi^*]^\alpha$ of the intuitionistic fuzzy solution in Example 1

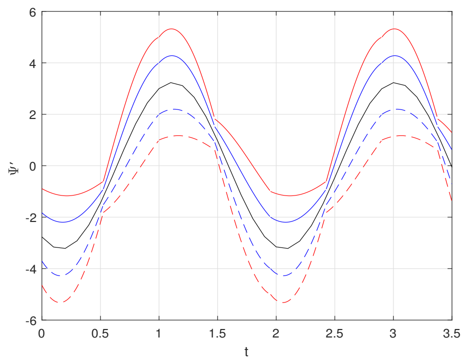

Figure 4. The $\widehat{\Psi^i}_\lambda(t)$ solution in Example 1. The black line represents the crisp solution. The red and blue lines represent upper solution for $\beta = 1$ and $\alpha = 0$, respectively and the dashed red and blue lines represent lower solution for $\beta = 1$ and $\alpha = 0$, respectively

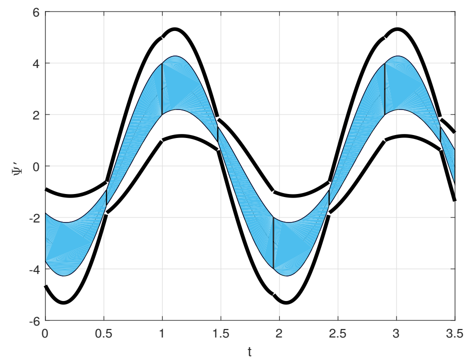

Figure 5. The blue region of the intersection of fuzzy solution $[\psi]^{\alpha}$ and $[\psi^{*}]^{\alpha}$ of the intuitionistic fuzzy solution in Example 1

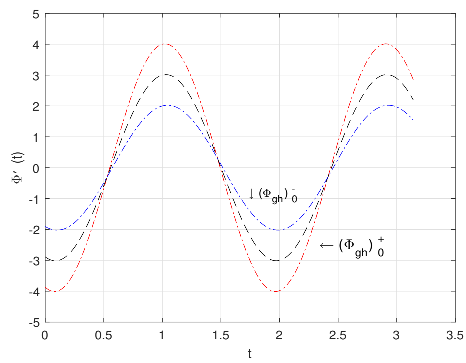

Figure 6. The χ solution of the (1,1)-system related to (17)-(19) in the sense of gH-derivative.The blue line and the red line represent respectively the left and right end-points of the 0-level of the solution the black line represent the reel solution in Example 1

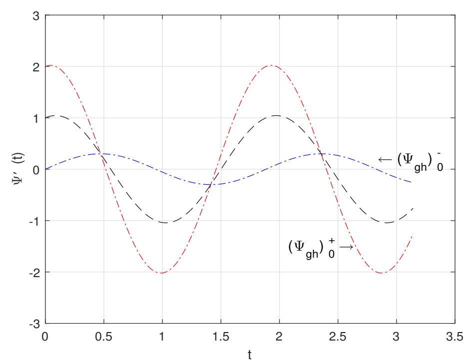

Figure 7. The ψ solution of (1,1)-system related to (17)-(19) in the sense of gH-derivative.The blue line and the red line represent respectively the left and right end-points of the 0-level of the solution the black line represent the reel solution for Example 1

4. Conclusion

The main contribution of this article is the study of intuitionistic fuzzy eigenvalue problem with boundary values given by intuitionistic fuzzy numbers. The eigenvalues of the fuzzy problem are found mainly on the idea of the intuitionistic Zadeh's extension principle. To do this the method proposed in Theorem 2 is used. Then one of the obtained eigenvalues is arbitrarily selected and substituted in the fuzzy solutions to obtain the intuitionistic fuzzy eigenfunctions $\widehat{\chi^{i}}_{\lambda}(t)$ and $\widehat{\Psi^{i}}_{\lambda}(t)$ which are shown in Figures 2, 3, 4 and 5. To prevent switch-points as illustrated in Figure 6 and in Figure 7, Heaviside function is used during the interval operations on α and β-levels.

The approach using the gH-derivative is equivalent to the study of some systems of classical differential equations, which can lead to an additional study of switching points as shown in Figures 6 and 7. Moreover from this approach, the sign of the solution is considered itself and the signs of its first and second derivatives.

By using the method in this paper, fuzzy eigenfunctions are obtained without dealing with these unfavourable situations.

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RESEARCH ARTICLE

Further refinements and inequalities of Fejér's type via GA-convexity

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ABSTRACT

In this study, we introduce some new mappings in connection with Hermite-Hadamard and Fejér type integral inequalities which have been proved using the GA-convex functions. As a consequence, we obtain certain new inequalities of the Fejér type that provide refinements of the Hermite-Hadamard and Fejér type integral inequalities that have already been obtained.

1. Introduction

For convex functions the following double inequality has great significance in literature and is known as Hermite-Hadamard's inequality [1,2]:

Let $\tau : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$, be a convex function, then

$$\tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\tau(\nu)d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}. \quad (1)$$

The inequality (1) holds in reversed direction if τ is concave.
Fejér [3], established the following double inequality as a weighted generalization of (1):

$$\tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\int_{\varkappa_1}^{\varkappa_2} r(\nu)d\nu \leq \int_{\varkappa_1}^{\varkappa_2}\tau(\nu)r(\nu)d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}(\nu)d\nu, \quad (2)$$

where $\tau : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$ is any convex function and $r : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\nu = \frac{\varkappa_1+\varkappa_2}{2}$.

These inequalities have many extensions and generalizations, see [4]- [50]. Dragomir *et al.* [7], obtained the refinement of the first inequality in (1). Yang and Hong [42], obtained the following Hermite-Hadamard-type inequality which is a refinement of the second inequality in (1). Tseng *et al.* [35], established the Fejér-type inequalities that refined 2. Yang and Tseng [42] and

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Tseng *et al.* [35] established the Fejér-type inequalities which are weighted generalizations of results from [7] and [42]. Dragomir *et al.* [12] provided further Hermite-Hadamard-type inequality related to (1) that refine the second inequality in (1). Tseng et al. [36, 37], obtained some very fascinating results related to Fejér's result (2) which are weighted generalizations of a result proven in [12]. Tseng *et al.* [38] considered the following mappings defined over an interval $[0,1]$ and discussed important results that characterize the properties of the those mappings and also proved Fejér-type inequalities that provide refinements of the Hermite-Hadamard's (1) and Fejér's inequality (2):

$$G(\alpha) := \frac{1}{2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)\right],$$

$$Q(\alpha) := \frac{1}{2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\varkappa_2\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\varkappa_1\right)\right],$$

$$H(\alpha) := \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\tau\left(\alpha\nu + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)d\nu,$$

$$H_r(\alpha) := \int_{\varkappa_1}^{\varkappa_2}\tau\left(\alpha\nu + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)r(\nu)\,d\nu,$$

$$I(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\frac{\varkappa_1+\nu}{2} + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right) + \tau\left(\alpha\frac{\varkappa_2+\nu}{2} + (1-\alpha)\frac{\varkappa_1+\varkappa_2}{2}\right)\right]r(\nu)\,d\nu,$$

$$P(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_1 + \left(\frac{1-\alpha}{2}\right)\nu\right) + \tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_2 + \left(\frac{1-\alpha}{2}\right)\nu\right)\right]d\nu,$$

$$P_r(\alpha) := \frac{1}{2(\varkappa_2-\varkappa_1)} \times\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_1 + \left(\frac{1-\alpha}{2}\right)\nu\right)r\left(\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\left(\frac{1+\alpha}{2}\right)\varkappa_2 + \left(\frac{1-\alpha}{2}\right)\nu\right)r\left(\frac{\nu+\varkappa_2}{2}\right)\right]d\nu,$$

$$N(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\nu+\varkappa_2}{2}\right)\right]r(\nu)\,d\nu,$$

$$L(\alpha) := \frac{1}{2(\varkappa_2-\varkappa_1)} \times\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\nu\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\nu\right)\right]d\nu,$$

$$L_r(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\nu\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\nu\right)\right]r(\nu)\,d\nu$$

and

$$S_r(\alpha) := \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\alpha\varkappa_1 + (1-\alpha)\frac{\nu+\varkappa_2}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\varkappa_1+\nu}{2}\right) + \tau\left(\alpha\varkappa_2 + (1-\alpha)\frac{\nu+\varkappa_2}{2}\right)\right]r(\nu)\,d\nu,$$

where $\tau : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ is a convex function and $r : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\nu = \frac{\varkappa_1+\varkappa_2}{2}$.

Remark 1. *It should be noted that* $H = H_r = I$, $P = P_r = N$ *and* $L = L_r = S_r$ *on* $[0,1]$ *as* $r(\nu) = \frac{1}{\varkappa_2-\varkappa_1}$, $\nu \in [\varkappa_1, \varkappa_2]$.

Tseng *et al.* [38], proved new Fejér-type inequalities related to the mappings G, Q, H_r, P_r, I, N, L_r and S_r defined above. These results generalize known results obtained in connection to the Hermite-Hadamard inequality and therefore are useful in obtaining various results for means for a given convex function τ and particular weight function r.

Here we point out few important findings from Tseng *et al.* [35, 39] that were used to prove results from [38].

Lemma 1. *[35] Let* $\tau : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ *be a convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 + \nu_2 = \kappa_1 + \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

The assumptions in Lemma 1 can be weakened as in the following lemma:

Lemma 2. *[39] Let* $\tau : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ *be a convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \kappa_2 \le \varkappa_2$ *and* $\varkappa_1 \le \kappa_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 + \nu_2 = \kappa_1 + \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

Lemma 3. *[39] Let* τ, G, Q *be defined as above. Then* Q *is symmetric about* $\frac{1}{2}$, Q *is decreasing on* $\left[0, \frac{1}{2}\right]$ *and increasing on* $\left[\frac{1}{2}, 1\right]$,

$$G(2\alpha) \le Q(\alpha), \quad \alpha \in \left[0, \frac{1}{4}\right],$$

$$G(2\alpha) \geq Q(\alpha), \quad \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right],$$

$$G(2(1-\alpha)) \geq Q(\alpha), \quad \alpha \in \left[\frac{1}{2}, \frac{3}{4}\right]$$

and

$$G(2(1-\alpha)) \leq Q(\alpha), \quad \alpha \in \left[\frac{3}{4}, 1\right].$$

Here we cite two important results form Tseng *et al.* [38].

Theorem 1. *[38] Let τ, r, H, P_r, L_r and S_r be defined as above. Then*

(i) *The inequality*

$$\begin{aligned}
&\int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu \\
&\quad \leq 2\left[\int_{\varkappa_1}^{\frac{3\varkappa_1+\varkappa_2}{4}} \tau(\nu) r(2\nu - \varkappa_1) d\nu \right. \\
&\qquad \left. + \int_{\frac{\varkappa_1+3\varkappa_2}{4}}^{\varkappa_2} \tau(\nu) r(\varkappa_2 - 2\nu) d\nu\right] \\
&\quad \leq \int_0^1 P_r(\alpha) d\alpha \leq \frac{1}{2}\left[\int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu \right. \\
&\qquad \left. + \frac{\tau(\varkappa_1) + \tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu\right] \qquad (3)
\end{aligned}$$

holds.

(ii) *The inequalities*

$$\begin{aligned}
L_r(\alpha) &\leq P_r(\alpha) \leq (1-\alpha) \int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu \\
&\quad + \alpha \cdot \frac{\tau(\varkappa_1) + \tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \\
&\quad \leq \frac{\tau(\varkappa_1) + \tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq N(\alpha) - G(\alpha) \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \\
&\leq \frac{\tau(\varkappa_1) + \tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu - N(\alpha) \qquad (5)
\end{aligned}$$

hold for all $\alpha \in [0,1]$.

(iii) *If τ is differentiable on $[\varkappa_1, \varkappa_2]$, then we have the inequalities*

$$\begin{aligned}
0 &\leq \alpha\left[\frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \tau(\nu) d\nu \right. \\
&\quad \left. - \tau\left(\frac{\varkappa_1 + \varkappa_2}{2}\right)\right] \inf_{\nu \in [\varkappa_1, \varkappa_2]} r(\nu) \\
&\quad \leq P_r(\alpha) - \int_{\varkappa_1}^{\varkappa_2} \tau(\nu) r(\nu) d\nu, \qquad (6)
\end{aligned}$$

$$\begin{aligned}
0 &\leq P_r(\alpha) - \tau\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \\
&\leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \\
&\qquad \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \qquad (7)
\end{aligned}$$

$$\begin{aligned}
0 &\leq L_r(\alpha) - H_r(\alpha) \\
&\leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \\
&\qquad \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
0 &\leq P_r(\alpha) - L_r(\alpha) \\
&\leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \\
&\qquad \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \qquad (9)
\end{aligned}$$

$$\begin{aligned}
0 &\leq P_r(\alpha) - H_r(\alpha) \\
&\leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \\
&\qquad \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \qquad (10)
\end{aligned}$$

$$\begin{aligned}
0 &\leq N(\alpha) - I(\alpha) \\
&\leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \\
&\qquad \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq S_r(\alpha) - I(\alpha) \\
&\leq \frac{(\varkappa_2 - \varkappa_1)\left(\tau'(\varkappa_2) - \tau'(\varkappa_1)\right)}{4} \\
&\qquad \times \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \qquad (12)
\end{aligned}$$

hold for all $\alpha \in [0,1]$.

Theorem 2. *[38] Let τ, r, G, Q, H_r, P_r and S_r be defined as above. Then*

(i) *The inequalities*

$$\begin{aligned}
H_r(\alpha) &\leq Q(\alpha) \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu \\
&\leq \frac{\tau(\varkappa_1) + \tau(\varkappa_2)}{2} \int_{\varkappa_1}^{\varkappa_2} r(\nu) d\nu, \\
&\qquad \alpha \in \left[0, \frac{1}{3}\right] \qquad (13)
\end{aligned}$$

and

$$\tau\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu \le Q(\alpha)\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu \le P_r(\alpha), \quad \alpha\in\left[\frac{1}{3},1\right] \quad (14)$$

hold.

(ii) *The inequality*

$$0 \le S_r(\alpha) - G(\alpha)\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu \le \frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+Q(\alpha)\right]\times\int_{\varkappa_1}^{\varkappa_2} r(\nu)\,d\nu - S_r(\alpha) \quad (15)$$

hold for all $\alpha\in[0,1]$.

Convex functions are a fundamental concept in mathematics, and geometrically-arithmetically convex functions, or GA-convex functions, represent an exciting generalization of this concept that offers new insights and applications.

Definition 1. *[7] Suppose* $I\subseteq(0,\infty)$ *is an interval of positive real numbers. A function* $\tau: I\to\mathbb{R}$ *is considered to be GA-convex, if*

$$\tau\left(\nu^{\alpha}\kappa^{1-\alpha}\right) \le \alpha\tau(\nu)+(1-\alpha)\tau(\kappa) \quad (16)$$

for all $\nu,\kappa\in I$ *and* $\alpha\in[0,1]$. *A function* $\tau: I\to\mathbb{R}$ *is GA-concave if the inequality in (16) reversed.*

We have gathered crucial information regarding GA-convex and convex functions, which we will utilize to demonstrate our main findings.

Theorem 3. *[7] If* $[\varkappa_1,\varkappa_2]\subset(0,\infty)$ *and the function* $\mathcal{G}:[\ln\varkappa_1,\ln\varkappa_2]\to\mathbb{R}$ *is convex (concave) on* $[\ln\varkappa_1,\ln\varkappa_2]$, *then the function* $\tau:[\varkappa_1,\varkappa_2]\to\mathbb{R}, \tau(\alpha)=\mathcal{G}(\ln\alpha)$ *is GA-convex (concave) on* $[\varkappa_1,\varkappa_2]$.

Remark 2. *It is obvious from Theorem 3 that if* $\tau:[\varkappa_1,\varkappa_2]\to\mathbb{R}$ *is GA-convex on* $[\varkappa_1,\varkappa_2]\subset(0,\infty)$, *then* $\tau\circ\exp$ *is convex on* $[\ln\varkappa_1,\ln\varkappa_2]$. *It follows that* $\tau\circ\exp$ *has finite lateral derivatives on* $(\ln\varkappa_1,\ln\varkappa_2)$ *and by gradient inequality for convex functions, we have*

$$\tau\circ\exp(\nu)-\tau\circ\exp(\kappa)(\nu-\kappa)\ge\varphi(\exp\kappa)\exp(\kappa),$$

where $\varphi(\exp\kappa)\in\left[\tau'_-(\exp\kappa),\tau'_+(\exp\kappa)\right]$ *for any* $\nu,\kappa\in(\ln\varkappa_1,\ln\varkappa_2)$.

The following inequality of Hermite-Hadamard type for *GA*-convex functions holds (see [31] for an extension for GA h-convex functions):

Theorem 4. *[31] Let* $\tau: I\subseteq(0,\infty)\to\mathbb{R}$ *be a GA-convex function and* $\varkappa_1,\varkappa_2\in I$ *with* $\varkappa_1<\varkappa_2$. *If* $\tau\in L([\varkappa_1,\varkappa_2])$, *then the following inequalities hold:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\le\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_2}^{\varkappa_1}\frac{\tau(\nu)}{\nu}d\nu \le\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}. \quad (17)$$

The notion of geometrically symmetric functions was introduced in [25].

Definition 2. *[25] A function* $r:[\varkappa_1,\varkappa_2]\subseteq(0,\infty)\to\mathbb{R}$ *is geometrically symmetric with respect to* $(0,\infty)$, *if*

$$r(\nu)=r\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)$$

holds for all $\nu\in[\varkappa_1,\varkappa_2]$.

Fejér type inequalities using GA-convex functions using geometrically symmetric functions were presented in Latif *et al.* [25].

Theorem 5. *[25] Let* $\tau: I\subseteq(0,\infty)\to\mathbb{R}$ *be a GA-convex function and* $\varkappa_1,\varkappa_2\in I$ *with* $\varkappa_1<\varkappa_2$. *If* $\tau\in L([\varkappa_1,\varkappa_2])$ *and* $r:[\varkappa_1,\varkappa_2]\subseteq(0,\infty)\to\mathbb{R}$ *is nonnegative, integrable and geometrically symmetric with respect to* $\sqrt{\varkappa_1\varkappa_2}$, *then*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_2}^{\varkappa_1}\frac{r(\nu)}{\nu}d\nu\le\int_{\varkappa_2}^{\varkappa_1}\frac{\tau(\nu)r(\nu)}{\nu}d\nu \le\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_2}^{\varkappa_1}\frac{r(\nu)}{\nu}d\nu. \quad (18)$$

Suppose that $\tau: I\subseteq(0,\infty)\to\mathbb{R}$ is GA-convex on I and $\varkappa_1,\varkappa_2\in I$, let $\mathcal{H},\mathcal{F}\ ,\varkappa_{\in},\mathcal{I}_r:[0,1]\to\mathbb{R}$ be defined by

$$\mathcal{H}(\alpha):=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{1}{\nu}\tau\left(\nu^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)d\nu,$$

$$\mathcal{F}(\alpha):=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2}\frac{1}{\nu\kappa}\tau\left(\nu^{\alpha}\kappa^{1-\alpha}\right)d\nu d\kappa,$$

$$\mathcal{P}(\alpha):=\frac{1}{2(\ln\varkappa_2-\ln\varkappa_1)}\times\int_{\varkappa_1}^{\varkappa_2}\frac{1}{\nu}\left[\tau\left(\varkappa_2^{\frac{1+\alpha}{2}}\nu^{\frac{1-\alpha}{2}}\right)+\tau\left(\varkappa_1^{\frac{1+\alpha}{2}}\nu^{\frac{1-\alpha}{2}}\right)\right]d\nu$$

and

$$\mathcal{I}_r(\alpha):=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\left(\sqrt{\varkappa_1\nu}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\left(\sqrt{\nu\varkappa_2}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu,$$

where $r:[\varkappa_1,\varkappa_2]\subseteq(0,\infty)\to\mathbb{R}$ is nonnegative, integrable and geometrically symmetric with respect to $\sqrt{\varkappa_1\varkappa_2}$.

Latif *et al.* [21] obtained the following refinements for the inequalities (17):

Theorem 6. *[21] Let the function $\tau : I \subseteq (0, \infty) \to \mathbb{R}$ be as above. Then*

(i) *$\mathcal{H}$ is GA-convex on $[0, 1]$.*

(ii) *We have*

$$\inf_{\alpha\in[0,1]} \mathcal{H}(\alpha) = \mathcal{H}(0) = \tau\left(\sqrt{\varkappa_1\varkappa_2}\right)$$

and

$$\sup_{\alpha\in[0,1]} \mathcal{H}(\alpha) = \mathcal{H}(1) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu)}{\nu} d\nu.$$

(iii) *$\mathcal{H}$ increases monotonically on $[0, 1]$.*

The following theorem holds:

Theorem 7. *[21] Let $\tau : [\varkappa_1, \varkappa_2] \subseteq (0, \infty) \to \mathbb{R}$ be as above. Then*

(i) *$\mathcal{F}(\alpha + \frac{1}{2}) = \mathcal{F}(\frac{1}{2} - \alpha)$ for all α in $[0, \frac{1}{2}]$.*

(ii) *$\mathcal{F}$ is GA-convex on $[0, 1]$.*

(iii) *We have*

$$\sup_{\alpha\in[0,1]} \mathcal{F}(\alpha) = \mathcal{F}(0) = \mathcal{F}(1) = \frac{1}{(\ln\varkappa_2 - \ln\varkappa_1)^2}\int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu}\tau(\nu)\, d\nu$$

and

$$\inf_{\alpha\in[0,1]} \mathcal{F}(\alpha) = \mathcal{F}\left(\frac{1}{2}\right) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu\kappa}\tau\left(\sqrt{\nu\kappa}\right) d\nu d\kappa.$$

(iv) *The inequality*

$$\tau\left(\sqrt{\nu\kappa}\right) \le \mathcal{F}\left(\frac{1}{2}\right)$$

is valid.

(v) *$\mathcal{F}$ decreases monotonically on $[0, \frac{1}{2}]$ and increases monotonically on $\left[\frac{1}{2}, 1\right]$.*

(vi) *The inequality $\mathcal{H}(\alpha) \le \mathcal{F}(\alpha)$ holds true for all $\alpha \in [0, 1]$.*

Theorem 8. *[21] Let $\mathcal{P} : [0, 1] \to \mathbb{R}$ and $\tau : [\varkappa_1, \varkappa_2] \subset (0, \infty) \to \mathbb{R}$ be defined as above. Then*

(i) *$\mathcal{P}$ is GA-convex on $(0, 1]$.*

(ii) *The following hold:*

$$\inf_{\alpha\in[0,1]} \mathcal{P}(\alpha) = \mathcal{P}(0) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \frac{\tau(\nu)}{\nu} d\nu$$

and

$$\sup_{\alpha\in[0,1]} \mathcal{P}(\alpha) = \mathcal{P}(1) = \frac{\tau(\varkappa_1) + \tau(\varkappa_2)}{2}.$$

(iii) *$\mathcal{P}$ increases monotonically on $[0, 1]$.*

Theorem 9. *[27] Let $\tau, r, \mathcal{I}_r$ be defined as above. Then $\mathcal{I}_r$ is GA-convex, increasing on $[0, 1]$ and for all $\alpha \in [0, 1]$, we have the following Fejér type inequalities:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2} \frac{r(\nu)}{\nu} d\nu \le \mathcal{I}_r(0) \le \mathcal{I}_r(\alpha) \le \mathcal{I}_r(1) = \frac{1}{2}\int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\sqrt{\varkappa_1\nu}\right) + \tau\left(\sqrt{\nu\varkappa_2}\right)\right] \frac{r(\nu)}{\nu} d\nu. \tag{19}$$

Mathematical inequalities are very useful tools in establishing a number of important results in various branches of mathematical and physical sciences. Later on, mathematicians observed that the convexity plays an important role to prove novel results in theory of inequalities. Moreover, over the past three decades researchers are trying to generalize the classical convexity notion and convex functions so that they can prove new generalized and novel results in the field of mathematical inequalities that can also serve as refinements of previously proven results.

Indeed there are several generalizations of the classical convexity and convex functions in classical sense but one of them is know as the geometrically-arithmetically convexity (GA-covexity) and geometrically-arithmetically convex functions (GA-covex functions). Using the notion of GA-covexity, Noor et al. [31] and Latif et al. [25] proved results of Hermite-Hadamard and Féjer type.

The study explores a recent research study that builds upon previous work and offers fresh insights. Using their knowledge of studies conducted in [10–15, 17, 18, 34–39, 41–45], we define new mappings pertaining to two specific inequalities, namely (17) and (18). We then utilize these mappings to establish new Féjer type inequalities for GA-convex functions, employing innovative techniques and a variant of Lemma 4 for GA-convex functions to achieve results that refine (17) and (18) that are variants of inequalities given in Theorems 1 and Theorem 2. The researchers also highlight the implications of their findings and suggest future research directions, underscoring their commitment to advancing the field and making meaningful contributions.

2. Main Results

Let $\tau : [\varkappa_1, \varkappa_2] \subset (0, \infty) \to \mathbb{R}$ be a GA-convex mapping and let $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}_r$, $\mathcal{I}_r$, $\mathcal{P}$, $\mathcal{P}_r$, $\mathcal{N}$, $\mathcal{S}_r : [0, 1] \to \mathbb{R}$ be defined by

$$\mathcal{G}(\alpha) := \frac{1}{2}\left[\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\right],$$

$$\mathcal{Q}(\alpha) := \frac{1}{2}\left[\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right) + \tau\left(\varkappa_2\varkappa_1^{1-\alpha}\right)\right],$$

$$\mathcal{H}(\alpha) := \frac{1}{\ln \varkappa_2 - \ln \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \frac{1}{\nu} \tau\left(\nu^{\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{1-\alpha}\right) d\nu,$$

$$\mathcal{H}_r(\alpha) := \int_{\varkappa_1}^{\varkappa_2} \tau\left(\nu^{\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{1-\alpha}\right) \frac{r(\nu)}{\nu} d\nu,$$

$$\mathcal{P}_r(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\frac{1+\alpha}{2}} \nu^{\frac{1-\alpha}{2}}\right) \frac{r\left(\sqrt{\varkappa_1 \nu}\right)}{\nu} + \tau\left(\varkappa_2^{\frac{1+\alpha}{2}} \nu^{\frac{1-\alpha}{2}}\right) \frac{r\left(\sqrt{\nu \varkappa_2}\right)}{\nu^2}\right] d\nu,$$

$$\mathcal{N}(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \left(\sqrt{\varkappa_1 \nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \left(\sqrt{\nu \varkappa_2}\right)^{1-\alpha}\right)\right] \frac{r(\nu)}{\nu} d\nu,$$

$$\mathcal{L}(\alpha) := \frac{1}{2(\ln \varkappa_2 - \ln \varkappa_1)} \times \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \nu^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \nu^{1-\alpha}\right)\right] \frac{d\nu}{\nu},$$

$$\mathcal{L}_r(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \nu^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \nu^{1-\alpha}\right)\right] \frac{r(\nu)}{\nu} d\nu$$

and

$$\mathcal{S}_r(\alpha) := \frac{1}{2} \int_{\varkappa_1}^{\varkappa_2} \left[\tau\left(\varkappa_1^{\alpha} \left(\sqrt{\varkappa_1 \nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha} \left(\sqrt{\nu \varkappa_2}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \left(\sqrt{\varkappa_1 \nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha} \left(\sqrt{\nu \varkappa_2}\right)^{1-\alpha}\right)\right] \frac{r(\nu)}{\nu} d\nu.$$

Remark 3. *It should be noted that* $\mathcal{H} = \mathcal{H}_r = \mathcal{I}_r$, $\mathcal{P} = \mathcal{P}_r = \mathcal{N}$ *and* $\mathcal{L} = \mathcal{L}_r = \mathcal{S}_r$ *on* $[0,1]$ *when we take* $r(\nu) = \frac{1}{\ln \varkappa_2 - \ln \varkappa_1}$, $\nu \in [\varkappa_1, \varkappa_2]$.

In order to obtain the results of this section the author proved the following important lemma:

Lemma 4. *[27] Let* $\tau : [\varkappa_1, \varkappa_2] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 \nu_2 = \kappa_1 \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

The assumptions in Lemma 4 can be weakened as in the following lemma:

Lemma 5. *Let* $\tau : [\varkappa_1, \varkappa_2] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function and let* $\varkappa_1 \le \kappa_1 \le \nu_1 \le \kappa_2 \le \varkappa_2$ *and* $\varkappa_1 \le \kappa_1 \le \nu_2 \le \kappa_2 \le \varkappa_2$ *with* $\nu_1 \nu_2 = \kappa_1 \kappa_2$. *Then*

$$\tau(\nu_1) + \tau(\nu_2) \le \tau(\kappa_1) + \tau(\kappa_2).$$

We also need the following new lemma to prove our main results.

Lemma 6. *Let* τ, $\mathcal{G}$, $\mathcal{Q}$ *be defined as above. Then* $\mathcal{Q}$ *is symmetric about* $\frac{1}{2}$, Q *is decreasing on* $\left[0, \frac{1}{2}\right]$ *and increasing on* $\left[\frac{1}{2}, 1\right]$. *Moreover the following inequalities hold:*

$$\mathcal{G}(2\alpha) \le \mathcal{Q}(\alpha), \quad \alpha \in \left[0, \frac{1}{4}\right], \tag{20}$$

$$\mathcal{G}(2\alpha) \ge \mathcal{Q}(\alpha), \quad \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right], \tag{21}$$

$$\mathcal{G}(2(1-\alpha)) \ge \mathcal{Q}(\alpha), \quad \alpha \in \left[\frac{1}{2}, \frac{3}{4}\right] \tag{22}$$

and

$$\mathcal{G}(2(1-\alpha)) \le \mathcal{Q}(\alpha), \quad \alpha \in \left[\frac{3}{4}, 1\right]. \tag{23}$$

Proof. The GA-convexity of $\mathcal{Q}(\alpha)$ on $(0,1]$ follows from the GA-convexity of τ on $[\varkappa_1, \varkappa_2]$. It is clear that $\mathcal{Q}(\alpha)$ is symmetric about $\frac{1}{2}$. Let $0 < \alpha_1 < \alpha_2 \le \frac{1}{2} \le \alpha_3 < \alpha_4 \le 1$, then according to Lemma 4, the following inequalities hold:
The inequality

$$\tau\left(\varkappa_2^{\alpha_2} \varkappa_1^{1-\alpha_2}\right) + \tau\left(\varkappa_1^{\alpha_2} \varkappa_2^{1-\alpha_2}\right) \le \tau\left(\varkappa_2^{\alpha_1} \varkappa_1^{1-\alpha_1}\right) + \tau\left(\varkappa_1^{\alpha_1} \varkappa_2^{1-\alpha_1}\right)$$

holds for $\nu_1 = \varkappa_2^{\alpha_2} \varkappa_1^{1-\alpha_2}$, $\nu_2 = \varkappa_1^{\alpha_2} \varkappa_2^{1-\alpha_2}$, $\kappa_1 = \varkappa_2^{\alpha_1} \varkappa_1^{1-\alpha_1}$, $\kappa_2 = \varkappa_1^{\alpha_1} \varkappa_2^{1-\alpha_1}$.
The inequality

$$\tau\left(\varkappa_2^{\alpha_3} \varkappa_1^{1-\alpha_3}\right) + \tau\left(\varkappa_1^{\alpha_3} \varkappa_2^{1-\alpha_3}\right) \le \tau\left(\varkappa_2^{\alpha_4} \varkappa_1^{1-\alpha_4}\right) + \tau\left(\varkappa_1^{\alpha_4} \varkappa_2^{1-\alpha_4}\right)$$

holds for $\nu_1 = \varkappa_2^{\alpha_3} \varkappa_1^{1-\alpha_3}$, $\nu_2 = \varkappa_1^{\alpha_3} \varkappa_2^{1-\alpha_3}$, $\kappa_1 = \varkappa_2^{\alpha_4} \varkappa_1^{1-\alpha_4}$, $\kappa_2 = \varkappa_1^{\alpha_4} \varkappa_2^{1-\alpha_4}$.
Thus, $\mathcal{Q}$ is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$.
Now, we consider the following two cases:
Case 1. $\alpha \in \left[0, \frac{1}{4}\right]$
By choosing $\nu_1 = \varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$, $\nu_2 = \varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$, $\kappa_1 = \varkappa_2^{\alpha} \varkappa_1^{1-\alpha}$, $\kappa_2 = \varkappa_1^{\alpha} \varkappa_2^{1-\alpha}$ in Lemma 4, we get

$$\tau\left(\varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right) + \tau\left(\varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right) \le \tau\left(\varkappa_2^{\alpha} \varkappa_1^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha} \varkappa_2^{1-\alpha}\right)$$

for all $\alpha \in \left[0, \frac{1}{4}\right]$.
Case 2. $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$
By choosing $\nu_1 = \varkappa_2^{\alpha} \varkappa_1^{1-\alpha}$, $\nu_2 = \varkappa_1^{\alpha} \varkappa_2^{1-\alpha}$, $\kappa_1 = \varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$, $\kappa_2 = \varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}$ in Lemma 4, we get

$$\tau\left(\varkappa_2^{\alpha} \varkappa_1^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha} \varkappa_2^{1-\alpha}\right) \le \tau\left(\varkappa_1^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right) + \tau\left(\varkappa_2^{2\alpha} \left(\sqrt{\varkappa_1 \varkappa_2}\right)^{2\alpha-1}\right)$$

for all $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$.
Thus (20) and (21) are established. Using the symmetricity of $\mathcal{Q}$, (22) and (23) follow from (20) and (21), respectively. □

The author skillfully utilized Lemma 4 to obtain refined versions of Fejér type inequalities (18).

These refined inequalities not only extend the mappings related to (18), but also provide valuable insights into their properties. Overall, the author's work represents an important contribution to the field of inequalities.

Theorem 10. *[28] Let τ, $\mathcal{H}_r$, $\mathcal{P}_r$ and r be defined as above. Then*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu=\mathcal{H}_r(0)\le\mathcal{H}_r(\alpha)\le\mathcal{H}_r(1)=\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)\,r(\nu)}{\nu}d\nu=\mathcal{P}_r(0)\le\mathcal{P}_r(\alpha)\le\mathcal{P}_r(1)=\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (24)$$

Theorem 11. *[27] Let τ, $\mathcal{I}_r$, $\mathcal{N}$ and r be defined as above. Then*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\mathcal{I}_r(0)\le\mathcal{I}_r(\alpha)\le\mathcal{I}_r(1)=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu=\mathcal{N}(0)\le\mathcal{N}(\alpha)\le\mathcal{N}(1)=\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (25)$$

Corollary 1. *[27] Let τ, r be defined as above. Then, we have*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\frac{\tau\left(\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}\right)+\tau\left(\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu\le\frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)+\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\right]\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (26)$$

Theorem 12. *[29] Let τ, r, $\mathcal{G}$, $\mathcal{S}_r$, $\mathcal{L}_r$ be defined as above. Then, we have the following results:*

(i) *$\mathcal{L}_r$ is GA-convex on $(0,1]$.*

(ii) *The following inequalities hold for all $\alpha\in[0,1]$:*

$$\mathcal{H}_r(\alpha)\le\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\mathcal{L}_r(\alpha)\le(1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)\,r(\nu)}{\nu}d\nu+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (27)$$

$$\mathcal{S}_r(1-\alpha)\le\mathcal{L}_r(\alpha) \quad (28)$$

and

$$\frac{\mathcal{S}_r(\alpha)+\mathcal{S}_r(1-\alpha)}{2}\le\mathcal{L}_r(\alpha). \quad (29)$$

(iii) *The following bound holds true:*

$$\sup_{\alpha\in[0,1]}\mathcal{L}_r(\alpha)=\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (30)$$

Theorem 13. *[29] Let τ, r, $\mathcal{G}$, $\mathcal{I}_r$, $\mathcal{S}_r$ be defined as above. Then, we have the following results:*

(i) *$\mathcal{S}_r$ is convex on $[0,1]$.*

(ii) *The following inequalities hold for all $\alpha\in[0,1]$:*

$$\mathcal{I}_r(\alpha)\le\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\mathcal{S}_r(\alpha)\le(1-\alpha)\cdot\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\le\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (31)$$

$$\mathcal{I}_r(1-\alpha)\le\mathcal{S}_r(\alpha) \quad (32)$$

and

$$\frac{\mathcal{I}_r(\alpha)+\mathcal{I}_r(1-\alpha)}{2}\le\mathcal{S}_r(\alpha). \quad (33)$$

(iii) *The following identity holds true:*

$$\sup_{\alpha\in[0,1]}\mathcal{S}_r(\alpha)=\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (34)$$

Now, we can prove a new variant of Theorem 1 for GA-convex functions.

Theorem 14. *Let τ, r, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{P}_r$, $\mathcal{L}_r$ and $\mathcal{S}_r$ be defined as above. Then*

(i) *The inequalities*

$$\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu \le 2\left[\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\tau(\nu)\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu + \int_{\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}}^{\varkappa_2}\tau(\nu)\frac{r\left(\frac{\nu^2}{\varkappa_2}\right)}{\nu}d\nu\right] \le \int_0^1 \mathcal{P}_r(\alpha)\,d\alpha \le \frac{1}{2}\left[\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\right] \quad (35)$$

hold.

(ii) *The inequalities*

$$\mathcal{L}_r(\alpha) \le \mathcal{P}_r(\alpha) \le (1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (36)$$

and

$$0 \le \mathcal{N}(\alpha) - \mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu - \mathcal{N}(\alpha) \quad (37)$$

hold for all $\alpha \in [0,1]$.

(iii) *If* τ *is differentiable on* $[\varkappa_1,\varkappa_2]$, *then we have the following inequalities:*

$$0 \le \alpha\left[\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu - \tau(\sqrt{\varkappa_1\varkappa_2})\right] \times \inf_{\nu\in[\varkappa_1,\varkappa_2]} r(\nu) \le \mathcal{P}_r(\alpha) - \int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu, \quad (38)$$

$$0 \le \mathcal{P}_r(\alpha) - \tau(\sqrt{\varkappa_1\varkappa_2})\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (39)$$

$$0 \le \mathcal{L}_r(\alpha) - \mathcal{H}_r(\alpha) \le \frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (40)$$

$$0 \le \mathcal{P}_r(\alpha) - \mathcal{L}_r(\alpha) \le \frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (41)$$

$$0 \le \mathcal{P}_r(\alpha) - \mathcal{H}_r(\alpha) \le \frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (42)$$

$$0 \le \mathcal{N}(\alpha) - \mathcal{I}_r(\alpha) \le \frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (43)$$

and

$$0 \le \mathcal{S}_r(\alpha) - \mathcal{I}_r(\alpha) \le \frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4} \times \int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (44)$$

hold for all $\alpha \in [0,1]$.

Proof. (i) By using integration techniques and the assumptions on r, we get the following identities:

$$\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\int_0^{\frac{1}{2}}\left[\tau(\nu)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right] \times \frac{r(\nu)}{\nu}d\alpha d\nu, \quad (45)$$

$$2\left[\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\tau(\nu)\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu + \int_{\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}}^{\varkappa_2}\tau(\nu)\frac{r\left(\frac{\nu^2}{\varkappa_2}\right)}{\nu}d\nu\right] = 2\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\left[\tau(\nu)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu = 2\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\int_0^{\frac{1}{2}}\left[\tau(\sqrt{\varkappa_1\nu})+\tau\left(\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}\right)\right] \times \frac{r(\nu)}{\nu}d\alpha d\nu, \quad (46)$$

$$\int_0^1 \mathcal{P}_r(\alpha)\,d\alpha = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \int_0^1 \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu$$
$$= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\alpha d\nu$$
$$= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau\left(\varkappa_1^{1-\alpha}\nu^{\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^1 \left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\right)\right]\frac{r(\nu)}{\nu}d\alpha d\nu \tag{47}$$

and

$$\frac{1}{2}\left[\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\right] = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau(\varkappa_1)+\tau(\nu)\right]\frac{r(\nu)}{\nu}d\alpha d\nu + \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \int_0^{\frac{1}{2}} \left[\tau(\varkappa_2)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\frac{r(\nu)}{\nu}d\alpha d\nu. \tag{48}$$

According to Lemma 4, the following inequalities hold for all $\alpha \in \left[0, \frac{1}{2}\right]$ and $\nu \in \left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$:

The inequality

$$\tau(\nu) + \tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right) \le \tau\left(\sqrt{\varkappa_1\nu}\right) + \tau\left(\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}\right) \tag{49}$$

holds with the choices $\nu_1 = \nu$, $\nu_2 = \frac{\varkappa_1\varkappa_2}{\nu}$, $\kappa_1 = \sqrt{\varkappa_1\nu}$ and $\kappa_2 = \sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}$.

The inequality

$$\tau\left(\sqrt{\varkappa_1\nu}\right) \le \frac{1}{2}\left[\tau\left(\varkappa_1^{1-\alpha}\nu^{\alpha}\right) + \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right] \tag{50}$$

holds with the choices $\nu_1 = \nu_2 = \sqrt{\varkappa_1\nu}$, $\kappa_1 = \varkappa_1^{1-\alpha}\nu^{\alpha}$ and $\kappa_2 = \varkappa_1^{\alpha}\nu^{1-\alpha}$.

The inequality

$$\tau\left(\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}\right) \le \frac{1}{2}\left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\right)\right] \tag{51}$$

holds with the choices $\nu_1 = \nu_2 = \sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}$, $\kappa_1 = \varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$ and $\kappa_2 = \varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}$.

The inequality

$$\frac{1}{2}\left[\tau\left(\varkappa_1^{1-\alpha}\nu^{\alpha}\right) + \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right] \le \frac{\tau(\varkappa_1)+\tau(\nu)}{2} \tag{52}$$

holds with the choices $\nu_1 = \varkappa_1^{1-\alpha}\nu^{\alpha}$, $\nu_2 = \left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)$, $\kappa_1 = \varkappa_1$ and $\kappa_2 = \nu$.

The inequality

$$\frac{1}{2}\left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\right)\right] \le \frac{\tau(\varkappa_2)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)}{2} \tag{53}$$

holds with the choices $\nu_1 = \varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\nu_2 = \varkappa_2^{1-\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}$, $\kappa_1 = \frac{\varkappa_1\varkappa_2}{\nu}$ and $\kappa_2 = \varkappa_2$.

Multiplying the inequalities (49)-(53) by $\frac{r(\nu)}{\nu}$ and integrating them over α on $\left[0, \frac{1}{2}\right]$ and over ν on $\left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$ and using identities (45)-(48), we derive (35).

(ii) Using substitution rules for integration and the assumptions on r, we have the following identities:

$$\mathcal{P}_r(\alpha) = \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu$$
$$= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right] \times \frac{r(\nu)}{\nu}d\nu \tag{54}$$

and

$$\mathcal{L}_r(\alpha) = \frac{1}{2}\left[\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu^2}d\nu\right] + \frac{1}{2}\left[\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2} \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\frac{r(\nu)}{\nu}d\nu\right] = \frac{1}{2}\mathcal{P}_r(\alpha) + \frac{1}{2}\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \left[\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu \tag{55}$$

for all $\alpha \in [0, 1]$.

By choosing $\nu_1 = \varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\nu_2 = \varkappa_2^{\alpha}\nu^{1-\alpha}$, $\kappa_1 = \varkappa_1^{\alpha}\nu^{1-\alpha}$, $\kappa_2 = \varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$ in Lemma 5, we observe that the inequality:

$$\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right) \le \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) \quad (56)$$

holds for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$.
Multiplying the inequality (56) by $\frac{r(\nu)}{\nu}$, integrating both sides over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ and using identities (54) and (55), we derive the first inequality of (36). The second and third inequalities of (36) can be obtained by the GA-convexity of τ and (18).
Using substitution rules for integration and the hypothesis of r, we have the following identity:

$$\mathcal{N}(\alpha)=\frac{1}{2}\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\frac{\varkappa_1\varkappa_2^2}{\nu}}\right)^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu$$

$$=\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\times\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu=\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)\right.$$

$$\left.+\tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu \quad (57)$$

for all $\alpha\in[0,1]$.
By Lemma 4, the following inequalities hold for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right]$:
The inequality

$$\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right) \le \tau(\varkappa_1)+\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \quad (58)$$

holds for $\nu_1=\varkappa_1^{\alpha}\nu^{1-\alpha}$, $\nu_2=\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2$ and $\kappa_2=\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.
The inequality

$$\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) \le \tau(\varkappa_2)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \quad (59)$$

holds for $\nu_1=\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}$, $\nu_2=\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2$ and $\kappa_2=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.
Multiplying the inequalities (58)-(59) by $\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}$ and integrating them over ν on $\left[\varkappa_1,\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right]$ and using (57), we have

$$\mathcal{N}(\alpha)\le\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{G}(\alpha)\right]\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (60)$$

for all $\alpha\in[0,1]$. The second inequality in (37) is a consequence of (60).
Applying Lemma 4, we observe that the inequality:

$$\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \le \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) \quad (61)$$

holds for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ when $\nu_1=\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\nu}\right)$, $\nu_2=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$, $\kappa_1=\varkappa_1^{\alpha}\nu^{1-\alpha}$ and $\kappa_2=^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$.

Multiplying the inequalities (61) by $\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}$ and integrating them over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ and using the first part of the identity (57), we get (37).
(iii) Integrating by parts, we have

$$\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\frac{1}{\nu}\left(\ln\varkappa_1-\ln\nu\right)\times\left[\nu\tau'(\nu)-\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]d\nu =\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu-\tau\left(\sqrt{\varkappa_1\varkappa_2}\right). \quad (62)$$

Using substitution rules for integration, we have the following identity:

$$\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\times\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\frac{1}{\nu}\left[\tau(\nu)+\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]d\nu. \quad (63)$$

Since $\tau:[\varkappa_1,\varkappa_2]\to\mathbb{R}$ is harmonic convex on $[\varkappa_1,\varkappa_2]$, hence $g:[\ln\varkappa_1,\ln\varkappa_2]$ defined by $g(\nu):=\tau\circ\exp(\nu)$ is convex on $[\ln\varkappa_1,\ln\varkappa_2]$.
Using the convexity of g and the fact that $r(\nu)\ge 0$ on $[\ln\varkappa_1,\ln\varkappa_2]$, the inequality

$$\begin{aligned}&[g(\alpha\ln\varkappa_1+(1-\alpha)\nu)-g(\nu)]\,r(\ln\nu)\\&+[g(\alpha\ln\varkappa_2+(1-\alpha)(\ln\varkappa_1+\ln\varkappa_2-\nu))\\&-g(\ln\varkappa_1+\ln\varkappa_2-\nu)]\,r(\ln\nu)\\&\geq\alpha(\ln\varkappa_1-\nu)\,g'(\nu)\,r(\ln\nu)\\&+\alpha(\nu-\ln\varkappa_1)\,g'(\ln\varkappa_1+\ln\varkappa_2-\nu)\,r(\ln\nu)\\&=\alpha(\nu-\ln\varkappa_1)\\&\times\left[g'(\ln\varkappa_1+\ln\varkappa_2-\nu)-g'(\nu)\right]r(\ln\nu)\qquad(64)\end{aligned}$$

holds for all $\alpha\in[0,1]$ and $\nu\in\left[\ln\varkappa_1,\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right]$.

The inequality (64) can be re-written as

$$\begin{aligned}&\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)-\tau(\nu)\right]\frac{r(\nu)}{\nu}\\&+\left[\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)-\tau\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\frac{r(\nu)}{\nu}\\&\geq\nu\alpha(\ln\nu-\ln\varkappa_1)\,\tau'(\nu)\frac{r(\nu)}{\nu}\\&-\alpha(\ln\nu-\ln\varkappa_1)\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\frac{r(\nu)}{\nu}\\&=\alpha(\ln\nu-\ln\varkappa_1)\\&\times\left[\nu\tau'(\nu)-\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\frac{r(\nu)}{\nu}\\&\geq\alpha(\ln\nu-\ln\varkappa_1)\\&\times\left[\nu\tau'(\nu)-\frac{\varkappa_1\varkappa_2}{\nu}\tau'\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)\right]\\&\times\frac{1}{\nu}\inf_{\nu\in[\varkappa_1,\varkappa_2]}r(\nu)\qquad(65)\end{aligned}$$

for all $\alpha\in[0,1]$ and $\nu\in\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$.
Integrating the above inequality over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$, multiplying both sides by $\frac{1}{\ln\varkappa_2-\ln\varkappa_1}$ and using (17), (54), (63) and (65), we derive (38). We also observe that

$$\begin{aligned}&\frac{g(\ln\varkappa_1)-g\left(\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)}{2}\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&\leq\frac{1}{2}\left(\ln\varkappa_1-\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)\\&\times g'(\ln\varkappa_1)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&=\left(\frac{\ln\varkappa_1-\ln\varkappa_2}{4}\right)g'(\ln\varkappa_1)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\qquad(66)\end{aligned}$$

and

$$\begin{aligned}&\frac{g(\ln\varkappa_2)-g\left(\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)}{2}\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&\leq\frac{1}{2}\left(\ln\varkappa_2-\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)\\&\times g'(\ln\varkappa_2)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&=\left(\frac{\ln\varkappa_2-\ln\varkappa_1}{4}\right)g'(\ln\varkappa_2)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu.\qquad(67)\end{aligned}$$

Adding (66) and (67), we get

$$\begin{aligned}&\frac{g(\ln\varkappa_1)+g(\ln\varkappa_2)}{2}\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&-g\left(\frac{\ln\varkappa_1+\ln\varkappa_2}{2}\right)\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu\\&\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(g'(\ln\varkappa_2)-g'(\ln\varkappa_1)\right)}{4}\\&\times\int_{\ln\varkappa_1}^{\ln\varkappa_2}r(\ln\nu)\,d\nu.\qquad(68)\end{aligned}$$

The inequality (68) is equivalent to

$$\begin{aligned}&\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\&-\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\&\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4}\\&\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu.\qquad(69)\end{aligned}$$

Finally, inequalities (39)-(44) follow from inequalities (24), (25), (27), (31), (36) and (69). □

Corollary 2. *If $r(\nu)=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}$, $\nu\in[\varkappa_1,\varkappa_2]$, then Hermite-Hadamard-type inequalities, that are obvious consequences of Theorem 14, are given as follows:*

(i) *The inequalities*

$$\begin{aligned}&\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\\&\leq\frac{2}{\ln\varkappa_2-\ln\varkappa_1}\end{aligned}$$

$$\times\left[\int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}}\frac{\tau(\nu)}{\nu}d\nu+\int_{\varkappa_1^{\frac{1}{4}}\varkappa_2^{\frac{3}{4}}}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu\right]$$
$$\leq\int_0^1\mathcal{P}(\alpha)\,d\alpha$$
$$\leq\frac{1}{2}\left[\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu+\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\right] \quad (70)$$

hold.

(ii) *The inequalities*

$$\mathcal{L}(\alpha)\leq\mathcal{P}(\alpha)\leq\frac{1-\alpha}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \quad (71)$$

and

$$0\leq\mathcal{P}(\alpha)-\mathcal{G}(\alpha)\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}-\mathcal{P}(\alpha) \quad (72)$$

hold for all $\alpha\in[0,1]$.

(iii) *If* τ *is differentiable on* $[\varkappa_1,\varkappa_2]$, *then we have the inequalities:*

$$0\leq\alpha\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\left[\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu-\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\right]\leq\mathcal{P}(\alpha)-\frac{1}{\ln\varkappa_2-\ln\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu, \quad (73)$$

$$0\leq\mathcal{P}(\alpha)-\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4}, \quad (74)$$

$$0\leq\mathcal{L}(\alpha)-\mathcal{H}(\alpha)\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4}, \quad (75)$$

$$0\leq\mathcal{P}(\alpha)-\mathcal{L}(\alpha)\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4} \quad (76)$$

and

$$0\leq\mathcal{P}(\alpha)-\mathcal{H}(\alpha)\leq\frac{(\ln\varkappa_2-\ln\varkappa_1)\left(\varkappa_2\tau'(\varkappa_2)-\varkappa_1\tau'(\varkappa_1)\right)}{4}, \quad (77)$$

hold for all $\alpha\in[0,1]$.

Remark 4. *The inequality (35) gives a new refinement of the Fejér's inequality (18).*

Remark 5. *The inequality (36) refines the Fejér-type inequality (27).*

In the next theorem, we point out some inequalities for the functions $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{S}_r$ considered above.

Theorem 15. *Let* τ, r, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{S}_r$ *be defined as above. Then the following Fejér type inequalities hold true:*

(i) *The inequalities*

$$\mathcal{H}_r(\alpha)\leq\mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (78)$$

hold for $\alpha\in\left[0,\frac{1}{3}\right]$ *and*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\mathcal{P}_r(\alpha), \quad (79)$$

hold for $\alpha\in\left[\frac{1}{3},1\right]$.

(ii) *The inequalities*

$$0\leq\mathcal{S}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{Q}(\alpha)\right]\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu+\mathcal{S}_r(\alpha), \quad (80)$$

hold for all $\alpha\in[0,1]$.

Proof. (i) Here we consider the following two cases:

Case 1. $\alpha\in\left[0,\frac{1}{3}\right]$.

Using substitution rules for integration and the hypothesis of r, we have the following identity:

$$\mathcal{H}_r(\alpha)=\int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}}\left[\tau\left(\nu^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)\right]\frac{r(\nu)}{\nu}d\nu. \quad (81)$$

We observe that the following inequality is a result of application of Lemma 4:

The inequality

$$\tau\left(\nu^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \le \tau\left(\varkappa_1^{1-\alpha}\varkappa_2^{\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right) \quad (82)$$

holds for $\nu_1 = \nu^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$, $\nu_2 = \left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$, $\kappa_1 = \varkappa_1^{1-\alpha}\varkappa_2^{\alpha}$, $\kappa_2 = \varkappa_1^{\alpha}\varkappa_2^{1-\alpha}$ in Lemma 4, where $\alpha \in \left[0, \frac{1}{3}\right]$ and $\nu \in \left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$.

Multiplying the inequality (82) by $\frac{r(\nu)}{\nu}$, integrating both sides over ν on $\left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$ and using identity (81), we derive the first inequality of (78). From Lemma 6, we get that

$$\sup_{\alpha\in\left[0,\frac{1}{2}\right]} \mathcal{Q}(\alpha) = \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}.$$

Thus the second inequality in (78) is established.

Case 2. $\alpha \in \left[\frac{1}{3}, 1\right]$.

By choosing $\nu_1 = \varkappa_1^{\alpha}\varkappa_2^{1-\alpha}$, $\nu_2 = \varkappa_1^{1-\alpha}\varkappa_2^{\alpha}$, $\kappa_1 = \varkappa_1^{\alpha}\nu^{1-\alpha}$, $\kappa_2 = \varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$ in Lemma 6, where $\alpha \in \left[\frac{1}{3}, 1\right]$ and $\nu \in \left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$, we get

$$\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right)+\tau\left(\varkappa_1^{1-\alpha}\varkappa_2^{\alpha}\right) \le \tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right). \quad (83)$$

Multiplying the inequality (83) by $\frac{r(\nu)}{\nu}$, integrating both sides over ν on $\left[\varkappa_1, \sqrt{\varkappa_1\varkappa_2}\right]$ and using identity (54), we derive the second inequality of (79). From Lemma 6, we get that

$$\inf_{\alpha\in\left[\frac{1}{2},1\right]} \mathcal{Q}(\alpha) = \tau\left(\sqrt{\varkappa_1\varkappa_2}\right).$$

Thus the first inequality in (79) is also achieved.

(ii) Using substitution rules for integration and the hypothesis of r, we have the following identity:

$$\begin{aligned}
2\mathcal{S}_r &= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\right] \times \frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu + \int_{\sqrt{\varkappa_1\varkappa_2}}^{\varkappa_2}\left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right) +\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_2}\right)}{\nu}d\nu \\
&= \int_{\varkappa_1}^{\sqrt{\varkappa_1\varkappa_2}} \left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) +\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu \\
&= \int_{\varkappa_1}^{\varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}} \left[\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right) + \tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right) \right. \\
&\quad \left. +\tau\left(\varkappa_2^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right) +\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)\right]\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}d\nu. \quad (84)
\end{aligned}$$

By using Lemma 4, we observe that the following inequality holds for all $\alpha \in [0,1]$ and $\nu \in \left[\varkappa_1, \varkappa_1^{\frac{3}{4}}\varkappa_2^{\frac{1}{4}}\right]$:

The inequality

$$\tau\left(\varkappa_1^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right) \le \tau(\varkappa_1)+\tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \quad (85)$$

holds for $\nu_1 = \varkappa_1^{\alpha}\nu^{1-\alpha}$, $\nu_2 = \varkappa_1^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}$, $\kappa_1 = \varkappa_1$ and $\kappa_2 = \varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.

The inequality

$$\tau\left(\varkappa_1^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right) \le \tau\left(\varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\varkappa_1^{\alpha}\varkappa_2^{1-\alpha}\right) \quad (86)$$

holds for $\nu_1 =, \varkappa_1^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}$, $\nu_2 = \varkappa_1^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\kappa_1 = \varkappa_1^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$ and $\kappa_2 = \varkappa_1^{\alpha}\varkappa_2^{1-\alpha}$.

The inequality

$$\tau\left(\varkappa_2^{\alpha}\nu^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}\right)$$
$$\leq\tau\left(\varkappa_2^{\alpha}\varkappa_1^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right) \quad (87)$$

holds for $\nu_1=\varkappa_2^{\alpha}\nu^{1-\alpha}$, $\nu_2=\varkappa_2^{\alpha}\left(\frac{\sqrt{\varkappa_1^3\varkappa_2}}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2^{\alpha}\varkappa_1^{1-\alpha}$ and $\kappa_2=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$.
The inequality

$$\tau\left(\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}\right)$$
$$\leq\tau\left(\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}\right)+\tau\left(\varkappa_2\right) \quad (88)$$

holds for $\nu_1=\varkappa_2^{\alpha}\left(\nu\sqrt{\frac{\varkappa_2}{\varkappa_1}}\right)^{1-\alpha}$, $\nu_2=\varkappa_2^{\alpha}\left(\frac{\varkappa_1\varkappa_2}{\nu}\right)^{1-\alpha}$, $\kappa_1=\varkappa_2^{\alpha}\left(\sqrt{\varkappa_1\varkappa_2}\right)^{1-\alpha}$ and $\kappa_2=\varkappa_2$.
Multiplying the inequalities (85)-(88) by $\frac{r\left(\frac{\nu^2}{\varkappa_1}\right)}{\nu}$ and integrating them over ν on $\left[\varkappa_1,\sqrt{\varkappa_1\varkappa_2}\right]$ and using identity (84), we get

$$2\mathcal{S}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{Q}(\alpha)\right]$$
$$\times\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (89)$$

for all $\alpha\in[0,1]$. Using (31) and (89), we derive (80). □

Corollary 3. *Let $r(\nu)=\frac{1}{\ln\varkappa_2-\ln\varkappa_1}$, $\nu\in[\varkappa_1,\varkappa_2]$ in Theorem 15. Then $\mathcal{I}_r(\alpha)=\mathcal{H}(\alpha)$, $\alpha\in[0,1]$ and therefore we observe that:*

(i) *The inequalities*

$$\mathcal{H}(\alpha)\leq\mathcal{Q}(\alpha)\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}, \quad (90)$$

hold for $\alpha\in\left[0,\frac{1}{3}\right]$ and

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\leq\mathcal{Q}(\alpha)\leq\mathcal{P}(\alpha), \quad (91)$$

hold for $\alpha\in\left[\frac{1}{3},1\right]$.

(ii) *The inequalities*

$$0\leq\mathcal{L}(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{1}{2}\left[\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}+\mathcal{Q}(\alpha)\right]$$
$$+\mathcal{L}(\alpha), \quad (92)$$

hold for all $\alpha\in[0,1]$.

The following Fejér-type inequalities can be deduced from Theorems 5, 10, 12, 13, 14, 15, Corollary 1 and Lemma 6 and we omit their proofs.

Theorem 16. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{I}_r$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha\in[0,1]$:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{H}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{S}_r(\alpha)\leq(1-\alpha)$$
$$\times\int_{\varkappa_1}^{\varkappa_2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu$$
$$+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (93)$$

and

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{I}_r(\alpha)\leq\mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{L}_r(\alpha)\leq\mathcal{P}_r(\alpha)$$
$$\leq(1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu$$
$$+\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (94)$$

Theorem 17. *Let τ, r, $\mathcal{H}_r$, $\mathcal{G}$, $\mathcal{I}_r$, $\mathcal{Q}$ be defined as above. Then, the following inequalities hold for all $\alpha\in\left[0,\frac{1}{4}\right]$:*

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\leq\mathcal{H}_r(\alpha)$$
$$\leq\mathcal{H}_r(2\alpha)\leq\mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu$$
$$\leq\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (95)$$

and

$$\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{I}_r(\alpha) \le \mathcal{I}_r(2\alpha) \le \mathcal{I}_r(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (96)$$

Theorem 18. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha \in \left[\frac{1}{4}, \frac{1}{3}\right]$:*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &\le \mathcal{H}_r(\alpha) \le \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu^2}d\nu \le \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{L}_r(2\alpha)\\ &\le \mathcal{P}_r(2\alpha) \le (1-2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (97)\end{aligned}$$

and

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &\le \mathcal{H}_r(\alpha) \le \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{S}_r(2\alpha) \le (1-2\alpha)\\ &\times\int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (98)\end{aligned}$$

Theorem 19. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right]$:*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &\le \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{L}_r(2\alpha)\\ &\le \mathcal{P}_r(2\alpha) \le (1-2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu, \quad (99)\end{aligned}$$

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &\le \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{S}_r(2\alpha) \le (1-2\alpha)\\ &\times\int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau\left(\sqrt{\varkappa_1\nu}\right)+\tau\left(\sqrt{\nu\varkappa_2}\right)\right]\frac{r(\nu)}{\nu}d\nu + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (100)\end{aligned}$$

and

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu &\le \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{G}(2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{P}_r(\alpha) \le \mathcal{P}_r(2\alpha)\\ &\le (1-2\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (101)\end{aligned}$$

Theorem 20. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha \in \left[\frac{1}{2}, \frac{2}{3}\right]$:*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\\ \le \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \le \mathcal{G}(2(1-\alpha))\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu\end{aligned}$$

$$\leq \mathcal{L}_r(2(1-\alpha)) \leq \mathcal{P}_r(2(1-\alpha)) \leq (2\alpha-1)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + 2(1-\alpha)\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (102)$$

and

$$\tau(\sqrt{\varkappa_1\varkappa_2})\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{G}(2(1-\alpha))\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{S}_r(2(1-\alpha)) \leq (2\alpha-1) \times \int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau(\sqrt{\varkappa_1\nu})+\tau(\sqrt{\nu\varkappa_2})\right]\frac{r(\nu)}{\nu}d\nu + 2(1-\alpha)\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (103)$$

Theorem 21. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha \in \left[\frac{2}{3}, \frac{3}{4}\right]$:*

$$\tau(\sqrt{\varkappa_1\varkappa_2})\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{G}(2(1-\alpha))\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{L}_r(\alpha) \leq \mathcal{P}_r(\alpha) \leq (1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (104)$$

and

$$\tau(\sqrt{\varkappa_1\varkappa_2})\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{G}(2(1-\alpha))\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{G}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{S}_r(\alpha) \leq (1-\alpha) \times \int_{\varkappa_1}^{\varkappa_2}\frac{1}{2}\left[\tau(\sqrt{\varkappa_1\nu})+\tau(\sqrt{\nu\varkappa_2})\right]\frac{r(\nu)}{\nu}d\nu \leq \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (105)$$

Theorem 22. *Let τ, r, $\mathcal{H}_r$, $\mathcal{P}_r$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{L}_r$, $\mathcal{S}_r$ be defined as above. Then, the following inequalities hold for all $\alpha \in \left[\frac{3}{4}, 1\right]$:*

$$\tau(\sqrt{\varkappa_1\varkappa_2})\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{H}_r(2(1-\alpha)) \leq \mathcal{G}(2(1-\alpha))\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{P}_r(\alpha) \leq (1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \quad (106)$$

and

$$\tau(\sqrt{\varkappa_1\varkappa_2})\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{I}_r(2(1-\alpha)) \leq \mathcal{G}(2(1-\alpha))\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \mathcal{Q}(\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu^2}d\nu \leq \mathcal{P}_r(\alpha) \leq (1-\alpha)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)r(\nu)}{\nu}d\nu + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2}\int_{\varkappa_1}^{\varkappa_2}\frac{r(\nu)}{\nu}d\nu. \quad (107)$$

Corollary 4. *Let τ, $\mathcal{Q}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{P}$, $\mathcal{L}$ be defined as above and $r(\nu) = \frac{1}{\ln\varkappa_2 - \ln\varkappa_1}$, then we have:*

(i) *The inequalities*

$$\tau(\sqrt{\varkappa_1\varkappa_2}) \leq \mathcal{H}(\alpha) \leq \mathcal{H}(2\alpha) \leq \mathcal{G}(2\alpha) \leq \mathcal{Q}(\alpha) \leq \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \quad (108)$$

hold for all $\alpha \in \left[0, \frac{1}{4}\right]$.

(ii) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{H}(\alpha) \le \mathcal{Q}(\alpha) \\ &\le \mathcal{G}(2\alpha) \le \mathcal{L}(2\alpha) \le \mathcal{P}(2\alpha) \\ &\le \left(\frac{1-2\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu \\ &\quad + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (109)\end{aligned}$$

hold for all $\alpha\in\left[\frac{1}{4},\frac{1}{3}\right]$.

(iii) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{G}(2\alpha) \\ &\le \mathcal{L}(2\alpha) \le \mathcal{P}(2\alpha) \\ &\le \left(\frac{1-2\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu \\ &\quad + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (110)\end{aligned}$$

and

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{P}(\alpha) \\ &\le \mathcal{P}(2\alpha) \le \left(\frac{1-2\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu \\ &\quad + 2\alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (111)\end{aligned}$$

hold for all $\alpha\in\left[\frac{1}{3},\frac{1}{2}\right]$.

(iv) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{G}(2(1-\alpha)) \\ &\le \mathcal{L}(2(1-\alpha)) \le \mathcal{P}(2(1-\alpha)) \\ &\le \left(\frac{2\alpha-1}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu \\ &\quad + 2(1-\alpha)\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (112)\end{aligned}$$

hold for all $\alpha\in\left[\frac{1}{2},\frac{2}{3}\right]$.

(v) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{Q}(\alpha) \le \mathcal{G}(2(1-\alpha)) \\ &\le \mathcal{G}(\alpha) \le \mathcal{L}(\alpha) \le \mathcal{P}(\alpha) \\ &\le \left(\frac{1-\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu \\ &\quad + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (113)\end{aligned}$$

hold for all $\alpha\in\left[\frac{2}{3},\frac{3}{4}\right]$.

(vi) *The inequalities*

$$\begin{aligned}\tau\left(\sqrt{\varkappa_1\varkappa_2}\right) &\le \mathcal{H}(2(1-\alpha)) \\ &\le \mathcal{G}(2(1-\alpha)) \le \mathcal{Q}(\alpha) \le \mathcal{P}(\alpha) \\ &\le \left(\frac{1-\alpha}{\ln\varkappa_2-\ln\varkappa_1}\right)\int_{\varkappa_1}^{\varkappa_2}\frac{\tau(\nu)}{\nu}d\nu \\ &\quad + \alpha\cdot\frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \\ &\le \frac{\tau(\varkappa_1)+\tau(\varkappa_2)}{2} \qquad (114)\end{aligned}$$

hold for all $\alpha\in\left[\frac{3}{4},1\right]$.

3. Conclusions

Overall, this paper aimed to introduce some new mappings in connection with Hermite-Hadamard and Fejér type integral inequalities which have been proved using the GA-convex functions. As a consequence, we obtained certain new inequalities of the Fejér type that provided refinements of the Hermite-Hadamard and Fejér type integral inequalities that have already been obtained. We believe that these new techniques will be important tools for interested researcher for investigating various variational problems for different types of convexities. We hope that this research can motivate the researchers to demonstrate new results for functions of two or more variables by considering the GA-convexity and coordinated GA-convex functions on a rectangle from a plane.

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RESEARCH ARTICLE

Design optimal neural network based on new LM training algorithm for solving 3D - PDEs

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ABSTRACT

In this article, we design an optimal neural network based on new LM training algorithm. The traditional algorithm of LM required high memory, storage and computational overhead because of it required the updated of Hessian approximations in each iteration. The suggested design implemented to converts the original problem into a minimization problem using feed forward type to solve non-linear 3D - PDEs. Also, optimal design is obtained by computing the parameters of learning with highly precise. Examples are provided to portray the efficiency and applicability of this technique. Comparisons with other designs are also conducted to demonstrate the accuracy of the proposed design.

1. Introduction

Partial differential equations (PDEs) based mathematical models can be used to describe a wide variety of physical issues. The PDEs govern a wide range of physical, chemical, and biological events [1, 2]. A mathematical model is a condensed, mathematically stated depiction of physical reality. Nonlinear PDEs are also crucial for study in a wide range of domains, including hydrodynamics, engineering, quantum field theory, optics, plasma physics, etc [3–5]. Since they frequently do not have exact solutions, numerical techniques are used to approximate them.

In addition, many researchers have been solve nonlinear PDEs by using homotopy analysis method (HAM) [6], Homotopy perturbation method (HPM) [7, 8], Variational Iteration method (VIM) [9], and Adomain decomposition methods (ADM) [10–17]. Moreover, a number of methods, including numerical approach used to solve different type of PDEs for more details see [18–22], iterations, differential, and Laplace transformation approaches, have been utilized to numerically and analytically solve comparable types of the wave-like and also heat-like problems. It is important to use a suitable method for solving any equation or problem. In recent years some authors used neural networks as an important method to solve many of real-world problems because of their specification. Some authors used ANNs for solving different types of differential equations such that Oraibi et. al. [23] first gave the concept of solving ordinary differential equations using a neural network by formulating a trial solution of the differential equation. The authors tested the applicability and accuracy of their developed method not only for ordinary differential equations but also for systems of coupled differential equations. Further, the authors compared their results with the results obtained by using other numerical methods, and reported

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that developed ANN method is superior in terms of memory requirements and accuracy.

Several attempts have been made to solve different types of differential equations using feed-forward neural networks. Hussein and Mohammed [24] reported a hybrid method by combining optimization techniques with neural networks to solve high-order ordinary differential equations. In a related work, Tawfiq and Hussein [25] introduced a novel method for solving boundary value problems using artificial neural networks. They also implemented the method for irregular domain boundaries with Dirichlet as well as Neumann boundary conditions and used for processing face recognition. Tawfiq [26] solved initial and boundary value problems using a single-layer finite element neural network and investigated the accuracy of the method for nonlinear forward and inverse problem, and also for a system of ordinary differential equations. Salih and Tawfiq [27] presented a functionally weighted neural network (FWNN) a new class of artificial neural networks incorporating an infinite number of nodes and showed that their new network has superior extrapolation capability over other networks then used to solve Troesch's problem. Hussien et. al. [28] proposed an artificial neural networks-based deep neural network and dropout to solve time dependent differential equations. The authors showed that artificial neural network-based deep is very well approximating dynamic systems represented by time-dependent differential equations. Ali and Tawfiq [29] in their paper used artificial neural networks to approximate the solution of unsteady state confined aquifer problem. The authors used linear and non-linear terms in different types of unsteady sate differential equations to illustrate the accuracy of the method. Ali et. al. [30] proposed feed forward neural network design for solving nonlinear second order, eigenvalue problem for partial differential equation. They presented example to show speed, accuracy and effectiveness of applying neural network technique and found their results more precise than other numerical methods. The proposed neural network based on new modification of BFGS update algorithm. Gupta and Batra [31] developed a vectorized algorithm and impleme-nted it in Python code using a deep artificial neural network to solve the system of ordinary differential equations. Further, to show the effectiveness of the proposed method he compared his results with the fourth-order Runge-Kutta method and showed the high accuracy of his proposed method. Hussien and Dhannoon [28] presented a meshless parameter estimation method for solving a system of partial differential equations using an artificial neural network. The authors demonstrated that the deep learning ANN-based approach is very effective in solving differential equations in reasonable computing times. They illustrated their method for linear and non-linear partial differential equations with Dirichlet and Neumann boundary conditions for both regular and irregular boundaries. Khamas et. al. [32] design suitable neural network to solve singular initial and boundary value problems. The proposed design used to determine the effect hookah smoking on health with different types of tobacco. Tawfiq et. al. [33] in their paper discussed pitfalls for solving differential equations with neural networks. They considered examples and counter-examples for numerical tests to substantiate their findings. ANNs have a lot of advantages including high learning ability, adaptiveness, parallel processing, fault-tolerance, error computation, and machine training making this method the preferred choice to solve ordinary, partial and singular differential equations with initial or/and boundary conditions [34]. The researchers used different design of ANNs depending on type of problems; number of given data or samples. While, the ANN reliability has been assessed in this research. The new approach of training based on the LM training algorithm has been proposed. The objective function for this research include the minimizing.

This article has been consisting as follows: In next section, define and gives a background of the ANNs. In section 3, LM training algorithm is presented. In section 4, modification for LM training algorithm will be given. In section 5, 3D equation Linear & non PDE presented then we design optimal ANN for solving this equation with implementation and discussions for the result will be given. Finally, the conclusions are given in section 6.

2. Neural networks

A neural network is a structure of parallel processing for distributing information in the form of connected layers consist of a set of nodes called neurons (also are called processing elements) is the basic processor in ANNs, along with directed line segments between them called links (also are called connections). All nodes can be taken any number of arrival connections and can have any number of coming out connections, but the signs must be the same [31]. In effect, all nodes have a one coming out connection that can branch out to form multiple output connections, each of which carries the same sign. Each node possesses a transfer (activation) function which can use input signs, and which produces the node's output sign. Generally, ANNs have been generalizations of mathematical models of human brain, based on the processing of information occurs at many connections nodes; signs are passed between nodes over connection links which has an associated weight; each node applies an transfer function to its weighted input net to determine its sign of output.

Thus for a given input vector x, the input to this neuron is $W_j^T x$. We assume that each of the hidden neurons has identical transfer function σ, but that bias bj. So the output from the j-th hidden neuron is $\sigma(W_j^T x + b_j)$.

Now we denote the weight connecting the j^{th} hidden node to the output by $\mho_j$. The output function $g(x)$ of the ANN is therefore [35]:

$$g\left(x\right) = \sum_{j=1}^{k} \mho j \sigma (W_j^T x + bj) \tag{1}$$

Note that σ must be sigmoidal functions, so we choice suitable σ herein defined as [32]:

$$\sigma\left(n_i\right) = \frac{2}{e^{-2ni} + 1} - 1 \tag{2}$$

Then, the ANN input-output equation is:

$$\hat{Y} = \Phi(x^T W^T + b^T) \mho^T$$

where $W \epsilon R^{n \times r}; \mho \epsilon R^{1 \times n}$ and $b \epsilon R^{n \times 1}$ are the adjustable input weights, output weights and bias respectively.

The structure of interconnections ANN can be classified to different classes of ANNs architecture such feed forward neural network (FFNN): organized of nodes are in the form of layers and arrival input from the previous layer then feed their output to the next layer, in a strictly the data goes from the input node to the output node as feed-forward way i.e., forward loops. Feedback neural network (FBNN): all possible connections are allowed between layers and neurons. The data transfer in the network as back loops. Herein we choose FFNN.

3. LM Training algorithm

Here's a simplified mathematical breakdown of the "trainlm" algorithm:

(1) Initialization:
- Initialize weights (W) and biases (b) randomly.
- Set the learning rate ($\eta = 0.001$) for the Levenberg-Marquardt algorithm.

(2) Forward Propagation:
- For each input sample xi:
- Calculate the weighted sum and apply the activation function for each neuron in the hidden layer:

$$a_{ij} = \sum_{k=1}^{n} w_{ijk} x_{ik} + b_{ij} \tag{3}$$

$$u_{ij} = \sigma(a_{ij})$$

- Propagate the activations to the output layer using a similar process:

$$a_{ik} = \sum_{j=1}^{m} w_{ijk} a_{ij} + b_{ik} \tag{4}$$

$$u_{ik} = \sigma(a_{ik})$$

(3) Calculate Error:
Compute the error (E_i) between predicted (neural) output (u_{ik}) and target (exact) output ($\dot{u}_{ik}$)

$$E_i = \frac{1}{2} \sum_{k=1}^{K} \left(\dot{u}_{ik} - u_{ik}\right)^2 \tag{5}$$

(4) Backpropagation:
- Compute the gradient of the error with respect to weights and biases in the output layer:

$$g_{ik} = -(\dot{u}_{ik} - u_{ik})\ \acute{\sigma}(a_{ik}) \tag{6}$$

$$\frac{\partial E_i}{\partial w_{ijk}} = g_{ik} a_{ij}$$

$$\frac{\partial E_i}{\partial b_{ik}} = g_{ik}$$

- Propagate the error gradient back to the hidden layer and compute gradients there

$$g_{ij} = \acute{\sigma}(a_{ij}) \sum_{k=1}^{K} w_{ijk} g_{ik} \tag{7}$$

$$\frac{\partial E_i}{\partial w_{ijk}} = g_{ij} x_{ik}$$

$$\frac{\partial E_i}{\partial b_{ij}} = g_{ij}$$

(5) Update Weights and Biases Using Levenberg-Marquardt:
- The Update the weights and biases using the Levenberg-Marquardt update rule:

$$w_{ijk}^{(t+1)} = w_{ijk}^{(t)} - \eta\ \rho$$

$$w_{ijk}^{(t+1)} = w_{ijk}^{(t)} - \left(J^T J + \lambda I\right)^{-1} J_k^T \; e \qquad (8)$$

$$b_{ijk}^{(t+1)} = b_{ijk}^{(t)} - \eta \; \rho \qquad (9)$$

Where, ρ is search direction.

(6) Repeat:
- Iterate through the dataset multiple times, adjusting weights and biases after each iteration.
- Stop when the error converges or a predefined number of iterations is reached.

Based on its speed, the algorithm seems to be the most efficient way to train feedforward neural networks of moderate size (with up to several hundred weights). Additionally, it has a streamlined implementation in MATLAB software, as the matrix equation solution is built-in. These attributes make it particularly effective in a MATLAB environment [28].

4. Suggested modification for LM training algorithm

In this section we will present suggested modified for LM training algorithm denoted by MLM as follow:

Algorithm 1.
Step 1: *Given point* $x_0 \in R^n$ *and constants* d_0, d_1, d_2, μ_0 *and* m *such that* $\mu_0 > m > 0$; $0 < d_0 < d_1 < d_2 < 1.\sigma \in (0, 2], \theta \in [0, 1]$ *Let* $k = 0$.
Step 2: *If* $\left\|J_k^T \; E_k\right\| < \epsilon$, *then stop. otherwise Solve*

$$\lambda_k = \mu_k \left(\frac{\theta \left\|E_k\right\|^\sigma}{1 + \left\|E_k\right\|^\sigma}\right) + \frac{(1-\theta)\left\|J_k^T E_k\right\|^\sigma}{1 + \left\|J_k^T E_k\right\|^\sigma} \qquad (10)$$

Step 2. Compute the search direction p_k

$$p_k = \left(J_k^T \; J_k + \lambda_k I\right)^{-1} J_k^T \; E_k. \qquad (11)$$

Step 3*: Calculate* $r_k = {}^{Ared_k}/{}_{Pred_k}$, *where* $Ared_k$ *is an actual reduction which equal to:*

$$Ared_k = \left\|E_k\right\|^2 - \left\|E\left(x_k + p_k\right)\right\|^2 \qquad (12)$$

and $Pred_k$ *is a predicted reduction which equal to:*

$$Pred_k = \left\|E_k\right\|^2 - \left\|E_k + J_k p_k\right\|^2 \qquad (13)$$

set

$$x_{k+1} = \begin{cases} x_k + p_k & if \; r_k \geq d_0 \\ x_k & otherwise \end{cases}$$

Step 4*: Choose* μ_{k+1} *as*

$$\mu_{k+1} = \begin{cases} 4\mu_k & if \; r_k < d_1 \\ \mu_k & if \; r_k \in [d_1, d_2] \\ \max\left\{\frac{\mu_k}{4}, m\right\} & if r_k > d_2 \end{cases}$$

Step 5*: Take* $k := k + 1$ *and go to Step 2.*

5. Design optimal ANN to solve 3D-differential equations

In this section we suggest optimal design ANN to solve 3D- PDEs. The optimum based on suitable choice of number of neurons in the hidden layer depending on trial and error. That is design ANN requires fully interconnection three layers; 1st layer is input layer consist 4 neurons in the input layer $(x, y, z\&t)$; 3^{rd} layer is output layer consist one neuron with linsig. transfer function which represents the solution of the network and 2nd layer is hidden layer with tanhsig. transfer function consist 9 neurons in 1st trial then 10 neurons in 2^{nd} trial then 13 neurons in 3^{rd} trial and 15 neurons in 4^{th} trial. So, we comparing between the number of neurons in hidden layer in the training ANN, for solving non- linear PDE we see that in case solving the linear equation when the number of neurans large (15 neurons) that make a good design for ANN to solve it according to time 00:00:08 with performance 4.7370e-07 and best epoch 726, see Figures 2, 3, 4 and 5. But Figure 1, illustrat the implementation and accuracy of suggested design in different values of time t. Whereas, in nonlinear equation the lower number of neurons (9 nodes) in the hidden layer give the better value according to time 00:00:00 with performance of the network solution $u_{net}\left(x, y, z, t; \theta\right)$ is **8.7805e-30** and best epoch 8, see Figures 7, 8, 9 and 10. The preformance of the network solution $u_{net}\left(x, y, z, t; \theta\right)$ is **8.7470e-10** which is best from archticher of ANN with one hidden layer. But Figure 6, illustrat the implementation and accuracy of suggested design in different values of time t. While in the case solving nonlinear equation take long time 00:02:21 in ANN consist 9 nodes in 1^{st} hidden layers and 3 nodes in 2nd hidden layers in 1000 epoch and the value of preformance is 1.8910e-11. However, this value is not good when comparing with one hidden layer network. In other words the best archticher is one hidden layer ANN with 9 nodes in hidden layer since it is sufficient to give good result for solving nonlinear equation.

Training suggested ANN by back propagation rule and using unconstrain optimazation methods new LM algorithm. For every input data x, y, z and t, the process from input layer to the hidden layer described as follows:

$$n_i = \sum_{i=1}^{9} \left(\; W_{x_i} x + W_{y_i} y + W_{z_i} z + W_{t_i} t\right) + b_1$$

where W_{x_i}, Wy_i, W_{z_i} and Wt_i are the weights interrelate of the inputs x, y, z and t to the hidden layer respectively, and b1 is the biases of hidden layer. Hence, it is activated by the log. sig. function as Eq.(2). The next step is the process of the interrelate of the hidden layer to the output layer which is based on the following formula:

$$h_i = \sum_{j=1}^{9} \mho_{ij}\sigma(n_i) + b_2 \tag{14}$$

where $\mho_{ij}$ are the weights of the hidden layer with output, and b2 is the biases. When Eq.(6) became to output layer, it turned into the form

$$u_{net}(x, y, z, t; \theta) = \sum_{j=1}^{9} \mho_i\sigma(h_i)$$

where $\mho_j$ are the weights of the hidden layers to the output layers.

Then, it is also easy to express the k-th derivatives of $u_{net}(x, y, z, t; \theta)$ in terms:

$$\frac{\partial^k u_{net}(x, y, z, t; \theta)}{\partial x^k} = \sum_{j=1}^{n} \frac{\partial^k \mho_j f(h_2)}{\partial x^k} \tag{15}$$

$$\frac{\partial^k u_{net}(x, y, z, t; \theta)}{\partial y^k} = \sum_{j=1}^{n} \frac{\partial^k \mho_j f(h_2)}{\partial y^k} \tag{16}$$

$$\frac{\partial^k u_{net}(x, y, z, t; \theta)}{\partial z^k} = \sum_{j=1}^{n} \frac{\partial^k \mho_j f(h_2)}{\partial z^k}$$

$$\frac{\partial^k u_{net}(x, y, z, t; \theta)}{\partial t^k} = \sum_{j=1}^{n} \frac{\partial^k \mho_j f(h_2)}{\partial t^k} \tag{17}$$

For $k = 1, \ldots, n$.

The mean square error (mse) will be computed to check the accuracy of the approximate solutions that obtained in these cases for different values of the epochs. Moreover, illustrates the target of output in each case and the behavior of gradient in the validation case at epoch 1000. Target values of training is 70, validation 15 and testing 15. The learning rate $(\eta) = 0.001$.

Example 1. *Consider the 2^{nd} order, 3D linear homogeneous hyperbolic PDE :*

$$u(x, y, z, t) = u_{xx} + u_{yy} + u_{zz} + u_t \ \textit{for}\ 0 < x, y \ \textit{and}\ z < 1$$

IC: $u(x, y, z, 0) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$
BCs:$u(0, y, z, t) = 0, u(1, y, z, t) = 0, u(x, 0, z, t) = 0, u(x, 1, z, t) = 0,\ u(x, y, 0, t) = 0, u(x, y, 1, t) = 0,\ u(x, y, 0, t) = 0,\ u(x, y, 1, t) = 0$
The exact solution [19] is $u(x, y, z, t) = \sin(\pi x)\sin(\pi y)\sin(\pi z)e^{xyzt}$.

We solve this equation by suggested design of ANN and implemented in MATLAB vol. 2023a, after training suggested ANN we see below the result of the equation at different time in Figures 1-7 and the value of neural network Table 1 with using sigmoidal functions as in eq.2 between the first and the hidden layer while between the hidden and last layer purlin function. Figure 8 show the performances of ANN, Figures 9-12 explain the performances of test, validation & training, Figure 13 show the valued of gradient, Mu & validation, finally in Figure 14 explain the errors between exact & suggested solution.

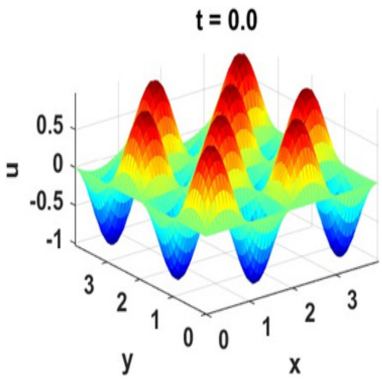

Figure 1. Results of suggested design for zero time of Example 1.

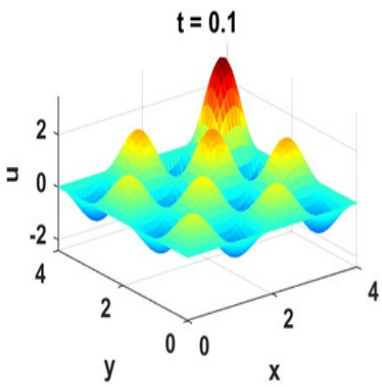

Figure 2. Results of suggested design for time 0.1 of Example 1.

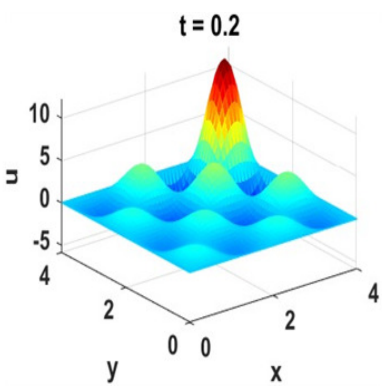

Figure 3. Results of suggested design for Example 1when time t= 0.2.

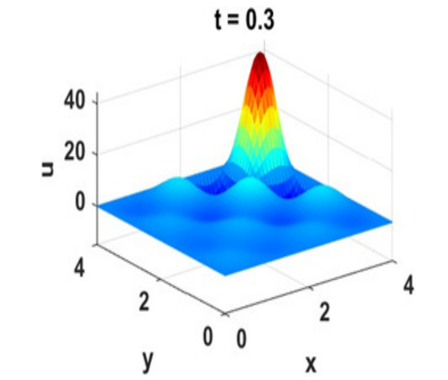

Figure 4. Results of suggested design when time t= 0.3 for Example 1.

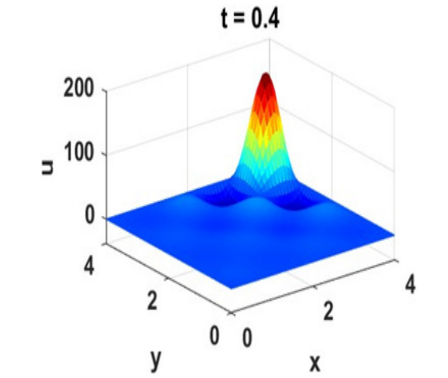

Figure 5. Results of suggested design when time t= 0.4 for Example 1.

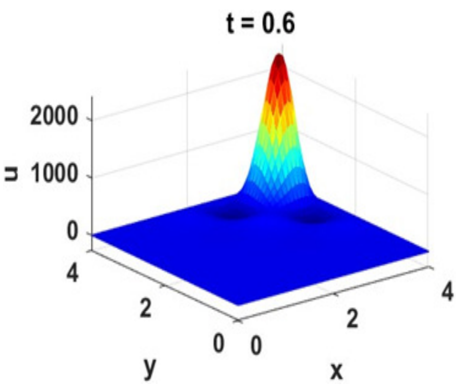

Figure 6. Results of suggested design when time t= 0.6 for Example 1.

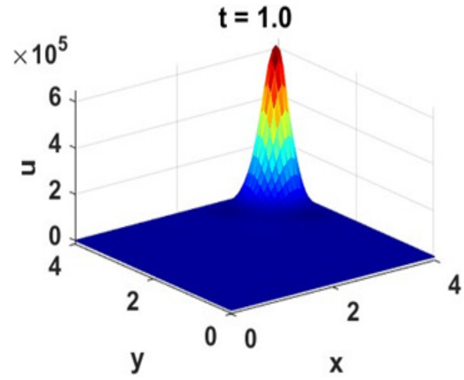

Figure 7. Results of suggested design when time t= 1 for Example 1.

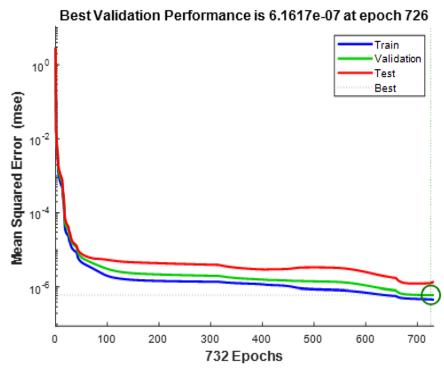

Figure 8. Comparison of Performances of ANN for Example 1, between train, test & validation in case 15 neurons in hidden layer.

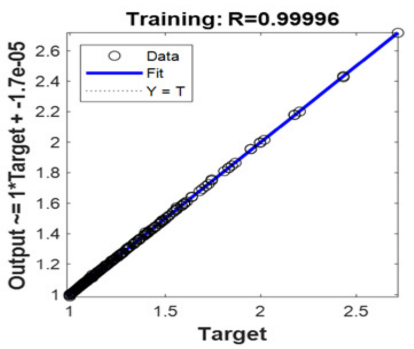

Figure 9. Performances of training for Example 1, in case 15 neurons in hidden layer.

Table 1. Results of suggested design in different cases for Example 1

No.	Layer & Nodes	Best epoch	Time	Best-perf.	Best-Vperf.	Best-tperf.	Gradient	lr.
1	9	1000	00:00:08	7.5071e-06	7.4696e-06	8.5628e-06	0.000104	0.001
2	10	538	00:00:06	6.6823e-06	6.7689e-06	6.7437e-06	0.000121	0.001
3	13	314	00:00:05	1.5217e-06	1.5500e-06	1.2775e-06	0.000241	0.001
4	15	726	00:00:08	4.7370e-07	6.1617e-07	1.2674e-06	0.00012	0.001
5	[9 3]	1000	00:00:11	4.7538e-08	5.2877e-08	4.8976e-08	0.00031	0.001
6	[9 9]	1000	00:00:17	6.4486e-09	6.4733e-09	6.3662e-09	1.98e-06	0.001
7	[9 19]	1000	00:00:33	8.7470e-10	9.0352e-10	1.2115e-09	9.35e-06	0.001

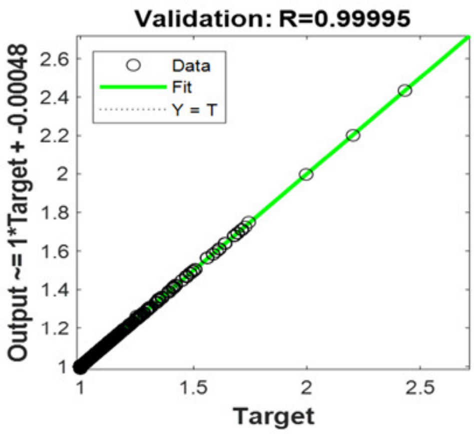

Figure 10. Performances of validation for Example 1, in case 15 neurons in hidden layer.

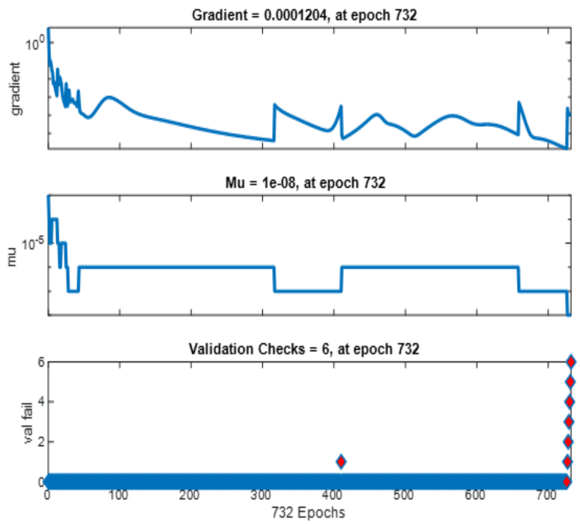

Figure 13. Gradient, Mu & validation for Example 1, in case 15 neurons in hidden layer.

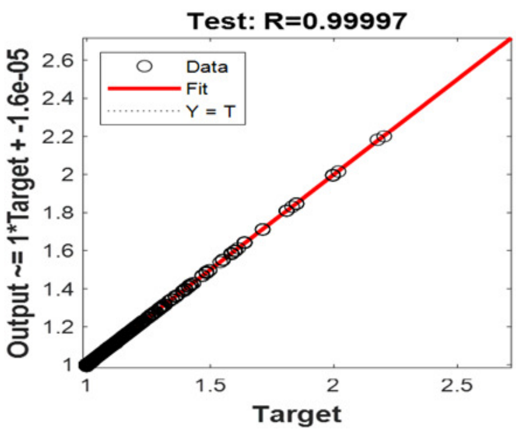

Figure 11. Performances of test for Example 1, in case 15 neurons in hidden layer.

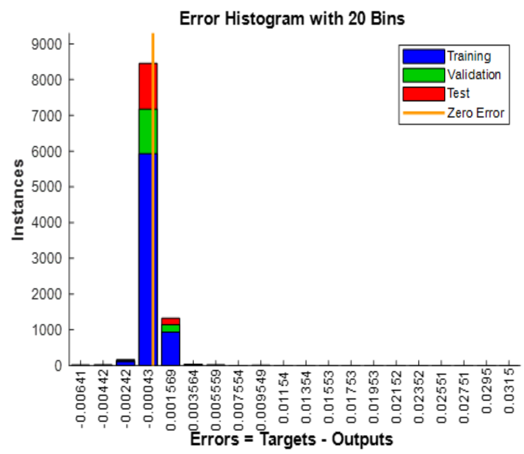

Figure 14. Errors between exact & suggested solution for Example 1, in case 15 neurons in hidden layer .

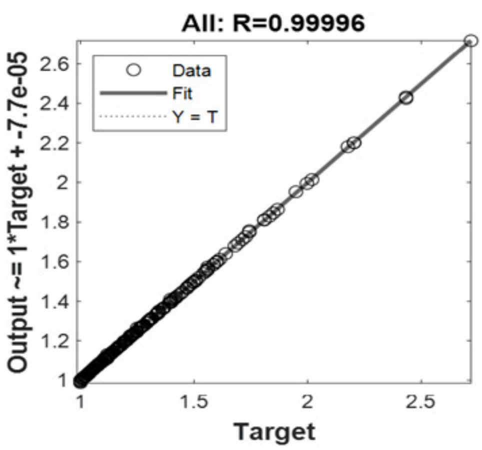

Figure 12. Comparison between exact & ANN results for Example 1, in case 15 neurons in hidden layer .

Example 2. *Consider the following 4^{th} order 3D nonlinear Jimbo-miwa equation*

$$u_{xxxy} + 3u_{xy}u_x + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} = 0$$

With ICs: $u_y(x, y, z, 0) = \frac{9}{2}\ sech^2\left(\frac{3}{2}(x + y + z)\right)$
Exact solution in [33, 34]: $u(x, y, z, t) = 3\tanh\left(\frac{3}{2}(x + y + z - 3t)\right)$

We solve that equation by suggested ANN and implemented in MATLAB vol. 2023a suggested design consist three layers: 1^{st} layer (input layer) consist of 4 nodes represent $\{x, y, z\ \&\ t\}$. In the hidden layer, we take the different case depending on number of neurons and 3rd layer (output layer) gives the solution of the network. Other design illustrate the results for different values of

Table 2. The value of parameters for suggested ANN in different cases for Example 2.

No.	Layer & Nodes	Best epoch	Time	Best-perf.	Best-Vperf.	Best-tperf.	Gradient	lr.
1	9	8	00:00:00	8.7805e-30	8.6926e-30	8.8637e-30	5.24e-15	0.001
2	10	11	00:00:09	3.2411e-31	3.2205e-31	3.2382e-31	1.41e-15	0.001
3	13	8	00:00:14	7.4300e-28	7.4141e-28	7.3858e-28	5.59e-14	0.001
4	15	8	00:00:22	1.0665e-23	1.0648e-23	1.0641e-23	3.06e-11	0.001
5	[9 3]	1000	00:02:21	1.8910e-11	1.8696e-11	1.8787e-11	7.22e-07	0.001
6	[9 9]	1000	00:03:35	3.863 6e-09	3.8756e-09	3.8627e-09	2.98e-06	0.001
7	[9 19]	1000	00:06:03	3.3635e-09	3.3775e-09	3.3723e-09	1.84e-06	0.001

time see Figures 15-21 and the value of parameters given in Table 2. Figure 22 shows the performances of ANN. Figures 23-26 illustrate the performances of training, test & validation case. Figure 27 illustrate the value of gradient, Mu & validation, finally in Figure 28, the errors between exact and neural solution in each case are presented.

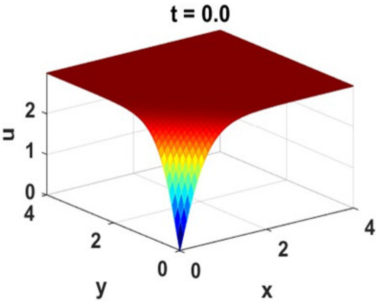

Figure 15. Results of suggested design for zero time of Example 2.

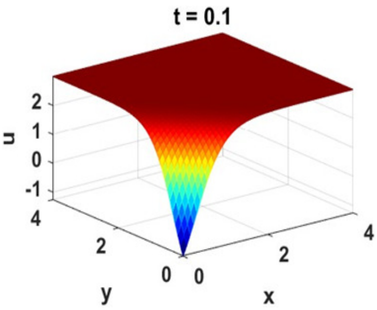

Figure 16. Results of suggested design when time t = 0.1 for Example 2.

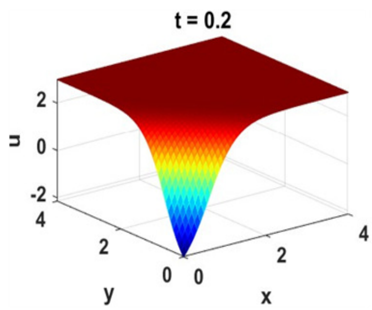

Figure 17. Results of suggested design when time t= 0.2 of Example 2.

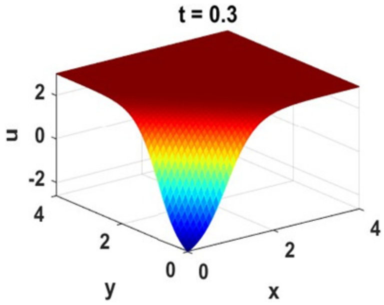

Figure 18. Results of suggested design when time t= 0.3 for Example 2.

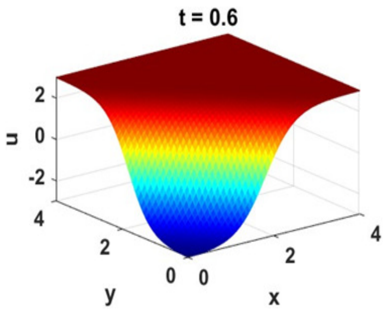

Figure 19. Results of suggested design when time t= 0.6 for Example 2.

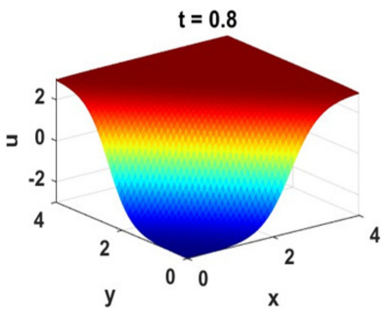

Figure 20. Results of suggested design when time t= 0.8 for Example 2.

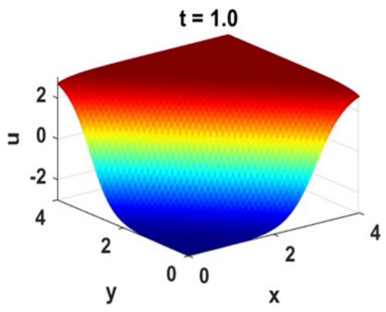

Figure 21. Results of suggested design when time t= 1 for Example 2.

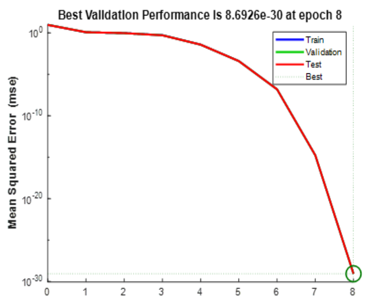

Figure 22. Performances of ANN for Example 2, in the case of 9 neurons in hidden layer.

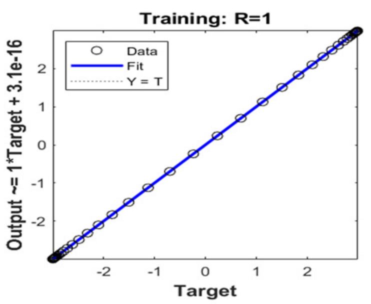

Figure 23. Performances of training ANN for Example 2, in the case 9 neurons in hidden layer.

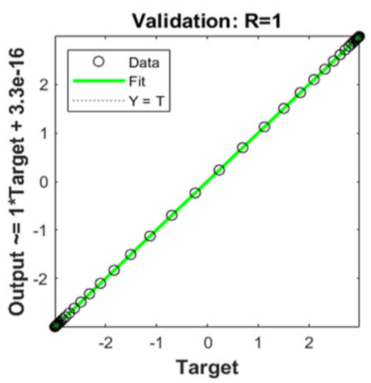

Figure 24. Performances of validation for Example 2, in the case 9 neurons in hidden layer.

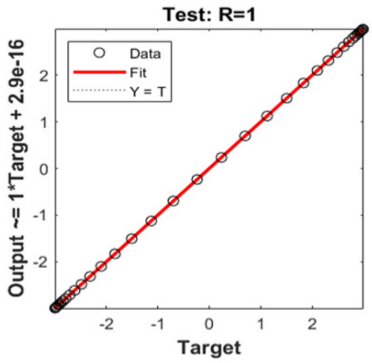

Figure 25. Performances of test for Example 2, in the case 9 neurons in hidden layer.

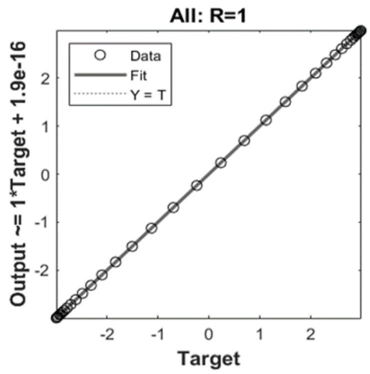

Figure 26. Comparison between exact & ANN result for 2, in the case 9 neurons in hidden layer.

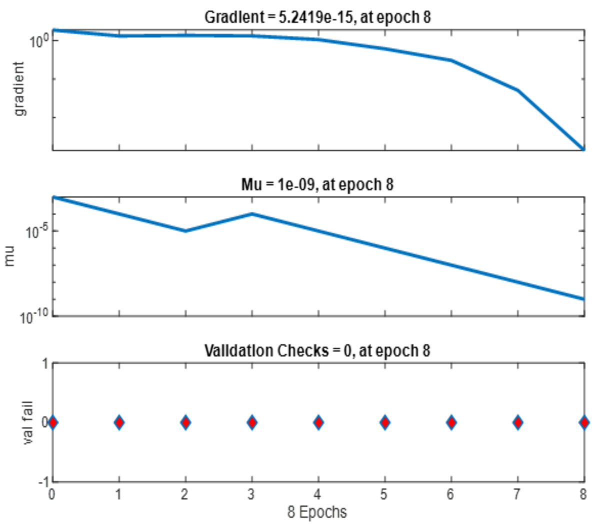

Figure 27. Gradient, Mu & validation for Example 2, in case 9 neurons in hidden layer.

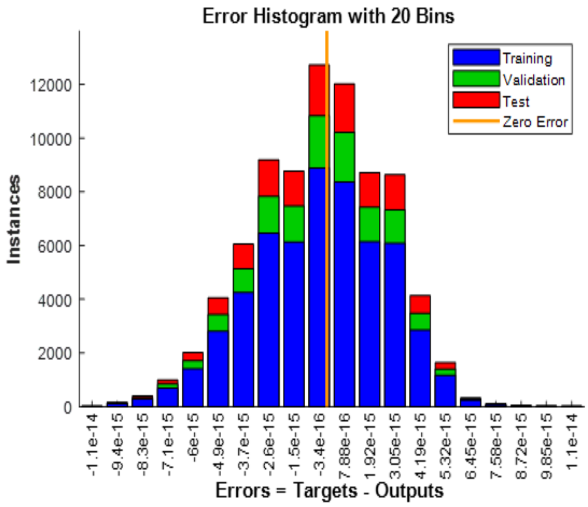

Figure 28. Errors between exact & neural solution for Example 2, in the case 9 neurons in hidden layer.

6. Conclusion

In this article, we suggest ANNs with different architecture based on number of layers and number of nodes in each layers. Suggested design trained by unconstrained optimization especially new LM training algorithm then used to solve 3D linear and nonlinear differential equations. The comparison between different design depending on the number of nodes in hidden layer has been presented. We see that in Example 1 (linear case) when the number of nodes large (15 neurons) we get good results and represent optimal design for ANN to solve this type of equations according to the time 00:00:08 with performance 4.7370e-07 and best epoch 726, whereas in Example 2, (nonlinear case) we see that the lower number of neurons (9 nodes) in the hidden layer gives the better results according to the time 00:00:00 with performance of the network solution $u_{net}(x, y, z, t; \theta)$ is 8.7805e-30 and best epoch 8. Also in the case of two hidden layers, the best archticher of ANN that gives good result in linear case is as [9 19] nodes, the best epoch is 1000 with long time 00:00:33 when comparing with one hidden layer but the preformance of the network solution $u_{net}(x, y, z, t; \theta)$ is 8.7470e-10 is best comparing with one hidden layer this means that the two hidden layer with the [9 19] nodes gives better result for linear case. While in the nonlinear case take long time 00:02:21 in [9 3] nodes with the best epoch is 1000 and the value of preformance is 1.8910e-11. However, that results is not good when we comparing with one hidden layer. This means that the one hidden layer ANN with 9 nodes in hidden layer is sufficient to get good result for solving nonlinear problems. Also, we conclude that many important article used original LM for training ANN such [38–41] can be resolve by training with new LM training algorithm to get best results

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RESEARCH ARTICLE

Analysis of COVID-19 epidemic with intervention impacts by a fractional operator

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ABSTRACT

This study introduces an innovative fractional methodology for analyzing the dynamics of COVID-19 outbreak, examining the impact of intervention strategies like lockdown, quarantine, and isolation on disease transmission. The analysis incorporates the Caputo fractional derivative to grasp long-term memory effects and non-local behavior in the advancement of the infection. Emphasis is placed on assessing the boundedness and non-negativity of the solutions. Additionally, the Lipschitz and Banach contraction theorem are utilized to validate the existence and uniqueness of the solution. We determine the basic reproduction number associated with the model utilizing the next generation matrix technique. Subsequently, by employing the normalized sensitivity index, we perform a sensitivity analysis of the basic reproduction number to effectively identify the controlling parameters of the model. To validate our theoretical findings, numerical simulations are conducted for various fractional order values, utilizing a two-step Lagrange interpolation technique. Furthermore, the numerical algorithms of the model are represented graphically to illustrate the effectiveness of the proposed methodology and to analyze the effect of arbitrary order derivatives on disease dynamics.

1. Introduction

In the realm of infectious diseases, mathematical modeling stands as a pivotal tool, offering insights into the spread and control mechanisms. The foundations of this discipline were laid in 1927 by Kermack and Mc Kendric, who introduced a fundamental compartment model for complex epidemic studies in epidemiology [1]. In the contemporary world, heightened attention has been directed towards research on an array of epidemic diseases like HIV, Malaria, Dengue, HBV, posing significant challenges in containment and prevention of disease within the human population. As the world grapples with these pre-existing health concerns, a new, unprecedented threat emerged on the horizon in late 2019, named COVID-19, originating in Wuhan, China. This novel coronavirus rapidly escalated into a pandemic, challenging our understanding of disease transmission and intervention strategies. Although, the exact origins of the virus remains elusive, it is believed to have originated from animals and potentially transmitted to humans through intermediaries such as SARS-CoV and MERS-CoV. COVID-19 manifests with a range of symptoms, from the typical fever, dry cough, and fatigue to severe respiratory distress with some cases being asymptomatic. During the pandemic, individuals infected with the coronavirus could

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spread it even without displaying the symptoms, constituting an incubation period of ranging from 2 to 14 days [2]. In the absence of a specific treatment for the first year of its emergence, non-pharmaceutical interventions took precedence such as isolation, mask-wearing, sanitization, and stringent restrictions on public gatherings. Governments around the world imposed lockdowns, constituting one of the largest quarantines in history, to curb the virus's spread. Consequently, understanding the role of different intervention strategies in transmission control remains a vital research focus. Several compartmental models analyzing the effect of various intervention strategies for COVID-19 have been proposed. In the study conducted by [3], a model was introduced to analyze the COVID-19 outbreak in China (Shanxi province). The researchers investigated the impact of the city lockdown date on the ultimate case count. They discovered that an earlier lockdown in the city could significantly reduce the number of infectious cases. Another study by [4], focused on the COVID-19 pandemic in the U.S.A, analyzing the impacts of non-pharmaceutical strategies. Additionally, [5] formulated a mathematical model to analyze the spread of COVID-19 in India. Their findings highlighted the significance of strict isolation measures for susceptible individuals, which could effectively bring down the rate of contact between susceptible and infected persons.

Nowadays, Fractional calculus is emerging as a vital branch of mathematics, extending traditional calculus by including integrals and derivatives with non-integer orders, enabling a more nuanced analysis of epidemic dynamics, originating from Leibniz's inquiry in 1695 [6]. Over the past three decades, researchers have delved into a range of fractional derivatives, such as Riemann-Liouville, Caputo, Caputo-Fabrizio, Atangana-Baleanu and more, captivated by their usage in diverse domains, including science, biology, economics, and engineering. Unlike traditional integer-order models focusing solely on the current state, fractional order models incorporate memory and hereditary effects, integrating past information to make more accurate epidemic predictions. Current advancements in epidemiological research emphasize the significance of utilizing models incorporating fractional order derivatives. A study investigated the behavior of HCV (Hepatitis C virus) disease, employing a mathematical model incorporating differential equations (DEs) of fractional-order. This model accounted for two crucial transmission components: interactions between the virus and cells, and the rate at which infected cells are cured, as presented in [7]. Also, in a study [8] researchers investigated the dynamics of COVID-19 transmission in Ethiopia, emphasizing on different age classes of infected population. The researchers employed Chebyshev polynomials to transform a fractional system into a set of algebraic equations. Additionally, [9] introduced an epidemic model of fractional order, integrating the classical Atangana-Baleanu-Caputo operator and Caputo operator, to investigate COVID-19 transmission. Considering these instances, it becomes apparent that employing fractional order derivatives in modeling real-life situations produces more precise outcomes than integer order scenarios. This statement finds support by a multitude of research investigations [10–16] in the field. In particular, Caputo fractional derivative (CFD) has found widespread application in various epidemic models, underscoring its utility. This significance is particularly evident when dealing with constant functions, as the Caputo derivative of such functions yields zero. The Caputo operator plays a pivotal role in solving ordinary differential equations, involving a subsequent fractional integral to achieve the desired order of fractional derivative. Notably, the Caputo fractional differential equation allows for the inclusion of local initial conditions in the model derivation process. Numerous researchers have successfully employed the Caputo operator to model diverse real-life scenarios, as evidenced by the literature [17–21].

Consequently, we emphasize the continued application of the Caputo operator in our current work, building upon the successful endeavors of previous researchers. This study investigates the dynamics of COVID-19 model considering the effect of intervention strategies introduced by [22]. By utilizing the CFD, our objective is to grasp the memory effect and non-local behavior essential for understanding the dynamics of COVID-19 infection. The choice of CFD lies in its capability to incorporate local primary conditions and enhance the accuracy of the model. The paper is structured in the described manner: Section 2 delves into fundamental mathematical concepts essential for the subsequent discussions. Section 3 describe the formulation and examination of the extension of COVID-19 model utilizing the CFD. In Section 4, we explore the non-negativity and boundedness of the model, accompanied by an exploration of the existence and uniqueness of solution for the given model. Section 5 determines the basic reproduction number and conducts a sensitivity analysis concerning each parameter. Section 6, presents a numerical simulation employing a two-step Lagrange interpolation method to validate the theoretical findings. Section 7, showcases the results and discussion. Finally, in Section 8, we draw conclusions from the entire study.

2. Preliminaries

Within this part, we will define some basic notations and definitions related to fractional calculus, that will be extensively utilized in this paper.

Definition 1. *Let $\phi : (0,\infty) \to \mathbb{R}$ be a function, then the Riemann-Liouville fractional integral operator [6] with order $\alpha > 0$ is expressed as:*

$$ {}^{c}{}_{0}\mathfrak{I}_{t}^{\alpha}\, \phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\phi(s)}{(t-s)^{\alpha-1}}\, ds; \quad t \geq 0, \quad (1)$$

here, $\Gamma(.)$ referred as a well-known Gamma function.

Definition 2. *Let $\phi : (0,\infty) \to \mathbb{R}$ be a function, then the CFD [6] with order $\alpha > 0$ is represented as*

$$ {}^{c}{}_{0}\mathfrak{D}_{t}^{\alpha}\, \phi(t) = \begin{cases} \frac{1}{\Gamma(\mathfrak{n}-\alpha)} \int_0^t \frac{\phi^{\mathfrak{n}}(s)}{(t-s)^{\alpha+1-\mathfrak{n}}}\, ds; \\ \qquad \alpha \in (\mathfrak{n}-1, \mathfrak{n}), \\ D_t^{\mathfrak{n}}\phi(t)\,; \qquad \alpha = \mathfrak{n}, \end{cases} \quad (2)$$

where, $t \geq 0$ and $\mathfrak{n}$ is any positive integer. When $\alpha \in (0,1)$,

$$ {}^{c}{}_{0}\mathfrak{D}_{t}^{\alpha}\, \phi(t) = \frac{1}{\Gamma(\mathfrak{n}-\alpha)} \int_0^t \frac{\phi'(s)}{(t-s)^{\alpha}}\, ds. \quad (3)$$

Also, the corresponding fractional integral with order $(\alpha > 0)$ is described as

$$ {}^{c}{}_{0}\mathfrak{I}_{t}^{\alpha}\, \phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\phi(s)}{(t-s)^{\alpha-1}}\, ds; \ \Re(\alpha) > 0. \quad (4)$$

Definition 3. *The Laplace transform(LT) [9] of the CFD with order $\alpha > 0$ is expressed as:*

$$ \mathscr{L}\left[{}^{c}{}_{0}\mathfrak{D}_{t}^{\alpha}\, \phi(t)\right](\mathfrak{s}) = \mathfrak{s}^{\alpha}\mathscr{L}[\phi(t)] - \sum_{m=0}^{\mathfrak{n}-1} \phi^{(m)}(0)\mathfrak{s}^{\alpha-m-1}, \quad (5)$$

where, $\alpha \in (\mathfrak{n}-1, \mathfrak{n}]$ and $\mathfrak{n} \in \mathbb{N}$.

Definition 4. *The Mittag-Leffler function [23] characterized by two parameters is expressed as*

$$ E_{\mathrm{a,b}}(\mathscr{S}) = \sum_{r=0}^{\infty} \frac{\mathscr{S}^r}{\Gamma(r\,\mathrm{a} + \mathrm{b})}, \quad (6)$$

where, $\mathrm{a,b} > 0$ and also, $E_{\mathrm{a},1}(\mathscr{S}) = E_{\mathrm{a}}(\mathscr{S})$. The LT of one parameter Mittag-Leffler function can be expressed as follows:

$$ \begin{aligned} \mathscr{L}[1 - E_{\mathrm{a}}(-kt^{\mathrm{a}})] &= \frac{k}{\mathfrak{s}(\mathfrak{s}^{\mathrm{a}} + k)}, \\ \mathscr{L}[E_{\alpha}(-kt^{\mathrm{a}})] &= \frac{\mathfrak{s}^{\mathrm{a}}}{\mathfrak{s}(\mathfrak{s}^{\mathrm{a}} + k)}. \end{aligned} \quad (7)$$

3. Formulation of Mathematical Model

Within this part, we develop a fractional-order epidemic model by applying the CFD operator to the classical integer-order model of COVID-19, as described in [22]. The COVID-19 integer-order model is defined by the given set of nonlinear ordinary DEs:

$$ \begin{aligned} \frac{dS(t)}{dt} &= (1-\rho)\Omega - \beta S(A+I) - (\mu+\lambda)S + \zeta Q_1, \\ \frac{dQ_1(t)}{dt} &= \rho\Omega - \sigma\beta Q_1(A+I) + \lambda S - (\mu+\zeta)Q_1, \\ \frac{dA(t)}{dt} &= \beta S(A+I) + \sigma\beta Q_1(A+I) \\ &\qquad - (q_1+q_2+\mu)A, \\ \frac{dQ_2(t)}{dt} &= q_1 A - (q_3+q_4+\mu)Q_2, \\ \frac{dI(t)}{dt} &= q_3 Q_2 + q_2 A + (\delta+\mu+\gamma)I, \\ \frac{dT(t)}{dt} &= \gamma I - (\mu+\eta)T, \\ \frac{dR(t)}{dt} &= q_4 Q_2 + \eta T - \mu R, \end{aligned} \quad (8)$$

with initial conditions

$$ \begin{aligned} &S(0) = S_0 > 0, Q_1(0) = Q_{1,0} \geq 0, A(0) = A_0 \geq 0, \\ &Q_2(0) = Q_{2,0} \geq 0, I(0) = I_0 \geq 0, \\ &T(0) = T_0 \geq 0, R(0) = R_0 \geq 0. \end{aligned} \quad (9)$$

Here, the entire population $\mathbb{P}(t)$ is segmented to seven sub-population compartments, say $S(t)$, $A(t), Q_1(t), Q_2(t), T(t), I(t)$, and $R(t)$ where the total population is sum of these compartments as:

$$ \begin{aligned} \mathbb{P}(t) = S(t) + Q_1(t) + A(t) + Q_2(t) + I(t) \\ + T(t) + R(t). \end{aligned} \quad (10)$$

When an individual is in good health but can contract the infection is susceptible (S), Susceptible individuals under quarantine due to lockdown measures are comprising in (Q_1), those in the community who exhibit no symptoms yet are in incubation period are categorized as Asymptomatic (A), those asymptomatic individuals who are self-quarantined (Q_2), those individuals who are seriously ill (I), those individuals who are isolated for treatment (T) and recovered population (R). The parameters mentioned in the model (8) are thoroughly defined and their corresponding values are presented in Table 1. In system (8) individuals in (Q_1) compartment, representing susceptible people under quarantine due to lockdown, interact to infected people with a reduced rate compare to individuals in the susceptible (S) compartment. This concept is governed by multiplying a scaling factor σ with the contact rate β, where $0 \leq \sigma \leq 1$ and $1-\sigma$ represents the effectiveness of lockdown i.e., $\sigma = 0$ describe the scenario of complete lockdown and $\sigma = 1$ describe the situation of no lockdown.

The above classical-integer order model of COVID-19 (8)-(9) is expanded into a fractional order system with an order α ($0 < \alpha \leq 1$). As, the model represented by equations (8) can be expressed in integral form as:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \int_0^t \kappa(t-s)[(1-\rho)\Omega - \beta S(A+I) \\
&\qquad - (\mu+\lambda)S + \zeta Q_1]ds, \\
\frac{dQ_1(t)}{dt} &= \int_0^t \kappa(t-s)[\rho\Omega - \sigma\beta Q_1(A+I) + \lambda S \\
&\qquad - (\mu+\zeta)Q_1]ds, \\
\frac{dA(t)}{dt} &= \int_0^t \kappa(t-s)[\beta S(A+I) + \sigma\beta Q_1(A+I) \\
&\qquad - (q_1+q_2+\mu)A]ds, \\
\frac{dQ_2(t)}{dt} &= \int_0^t \kappa(t-s)[q_1 A - (q_3+q_4+\mu)Q_2]ds, \\
\frac{dI(t)}{dt} &= \int_0^t \kappa(t-s)[q_3 Q_2 + q_2 A \\
&\qquad + (\delta+\gamma+\mu)I]ds, \\
\frac{dT(t)}{dt} &= \int_0^t \kappa(t-s)[\gamma I - (\eta+\mu)T]ds, \\
\frac{dR(t)}{dt} &= \int_0^t \kappa(t-s)[q_4 Q_2 + \eta T - \mu R]ds.
\end{aligned} \tag{11}
$$

In this context, $\kappa(t-s)$ represents the kernel function. On employing the power law of the kernel function as described in [24], we obtain:

$$
\kappa(t-s) = \frac{1}{\Gamma(\alpha-1)}(t-s)^{\alpha-2}. \tag{12}
$$

Now, on replacing the value of kernel from equation (12) into equation (11) and subsequently using the CFD with order $\alpha - 1$, we obtain:

$$
\begin{aligned}
{}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\left[\frac{dS(t)}{dt}\right] &= {}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\, {}^{c}_{0}\mathfrak{I}_t^{\alpha-1}[(1-\rho)\Omega \\
&\qquad - \beta S(A+I) - (\mu+\lambda)S + \zeta Q_1], \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\left[\frac{dQ_1(t)}{dt}\right] &= {}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\, {}^{c}_{0}\mathfrak{I}_t^{\alpha-1}[\rho\Omega \\
&\qquad - \sigma\beta Q_1(A+I) + \lambda S - (\mu+\zeta)Q_1], \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\left[\frac{dA(t)}{dt}\right] &= {}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\, {}^{c}_{0}\mathfrak{I}_t^{\alpha-1}[\beta S(A+I) \\
&\qquad + \sigma\beta Q_1(A+I) - (q_1+q_2+\mu)A], \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\left[\frac{dQ_2(t)}{dt}\right] &= {}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\, {}^{c}_{0}\mathfrak{I}_t^{\alpha-1}[q_1 A \\
&\qquad - (q_3+q_4+\mu)Q_2], \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\left[\frac{dI(t)}{dt}\right] &= {}^{c}_{0}\mathfrak{D}_t^{\alpha-1c}\, {}_{0}\mathfrak{I}_t^{\alpha-1}[q_3 Q_2 + q_2 A \\
&\qquad + (\delta+\gamma+\mu)I], \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\left[\frac{dT(t)}{dt}\right] &= {}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\, {}^{c}_{0}\mathfrak{I}_t^{\alpha-1}[\gamma I - (\eta+\mu)T], \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\left[\frac{dR(t)}{dt}\right] &= {}^{c}_{0}\mathfrak{D}_t^{\alpha-1}\, {}^{c}_{0}\mathfrak{I}_t^{\alpha-1}[q_4 Q_2 + \eta T - \mu R].
\end{aligned} \tag{13}
$$

Since, ${}^{c}_{0}\mathfrak{D}_t^{\alpha-1}$, ${}^{c}_{0}\mathfrak{I}_t^{\alpha-1}$ are inverse operators to each other. Therefore, the COVID-19 model with fractional order of α ($0 < \alpha \leq 1$) is formulated as:

$$
\begin{aligned}
{}^{c}_{0}\mathfrak{D}_t^{\alpha}\, S(t) &= (1-\rho)\Omega - \beta S(A+I) \\
&\qquad - (\mu+\lambda)S + \zeta Q_1, \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha}\, Q_1(t) &= \rho\Omega - \sigma\beta Q_1(A+I) + \lambda S - (\mu+\zeta)Q_1, \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha}\, A(t) &= \beta S(A+I) + \sigma\beta Q_1(A+I) \\
&\qquad - (q_1+q_2+\mu)A, \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha}\, Q_2(t) &= q_1 A - (q_3+q_4+\mu)Q_2, \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha}\, I(t) &= q_3 Q_2 + q_2 A + (\delta+\gamma+\mu)I, \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha}\, T(t) &= \gamma I - (\eta+\mu)T, \\
{}^{c}_{0}\mathfrak{D}_t^{\alpha}\, R(t) &= q_4 Q_2 + \eta T - \mu R,
\end{aligned} \tag{14}
$$

In the fractional order systems, maintaining dimensional consistency plays a pivotal role, ensuring that the units of measurement on both sides of the equations align smoothly. To achieve this consistency, a practical approach involves adjusting the parameters on the right-hand side of the equations, typically by raising their power to α, as discussed in [25–27]. In this context, our proposed fractional-order model takes the following form:

$$
\begin{aligned}
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, S(t) &= (1-\rho^{\alpha})\Omega^{\alpha} - \beta^{\alpha} S(A+I) \\
&\quad - (\mu^{\alpha}+\lambda^{\alpha})S + \zeta^{\alpha} Q_1, \\
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, Q_1(t) &= \rho^{\alpha}\Omega^{\alpha} - \sigma^{\alpha}\beta^{\alpha} Q_1(A+I) + \lambda^{\alpha} S \\
&\quad - (\mu^{\alpha}+\zeta^{\alpha})Q_1, \\
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, A(t) &= \beta^{\alpha} S(A+I) + \sigma^{\alpha}\beta^{\alpha} Q_1(A+I) \\
&\quad - (q_1^{\alpha}+q_2^{\alpha}+\mu^{\alpha})A, \\
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, Q_2(t) &= q_1^{\alpha} A - (q_3^{\alpha}+q_4^{\alpha}+\mu^{\alpha})Q_2, \\
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, I(t) &= q_3^{\alpha} Q_2 + q_2^{\alpha} A + (\delta^{\alpha}+\gamma^{\alpha}+\mu^{\alpha})I, \\
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, T(t) &= \gamma^{\alpha} I - (\eta^{\alpha}+\mu^{\alpha})T, \\
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, R(t) &= q_4^{\alpha} Q_2 + \eta^{\alpha} T - \mu^{\alpha} R,
\end{aligned} \tag{15}
$$

with the initial conditions:

$$
\begin{aligned}
&S(0) = S_0 > 0,\; Q_1(0) = Q_{1,0} \geq 0, A(0) = A_0 \geq 0, \\
&Q_2(0) = Q_{2,0} \geq 0, I(0) = I_0 \geq 0, \\
&T(0) = T_0 \geq 0, R(0) = R_0 \geq 0.
\end{aligned} \tag{16}
$$

4. Analytical Study of the Model

In this segment, we discuss certain key properties for the COVID-19 fractional order model(15).

4.1. Non-negativity and boundedness

To prove the positivity of solutions for fractional order model (15), we first discuss the subsequent lemma.

Lemma 1. *(Generalized Mean Value Theorem [28]). Let* $\phi(t)$ *is continuous on interval* [a, b] *and* ${}^{c}_{0}\mathfrak{D}^{\alpha}_{t} \in C(\mathrm{a},\mathrm{b}]$ *with* $0 < \alpha \leq 1$*, then*

$$
\phi(t) = \phi(\mathrm{a}) + \frac{1}{\Gamma(\alpha)}({}_{0}D^{\alpha}_{t}\phi)(\mathfrak{z})(t-\mathrm{a})^{\alpha}, \tag{17}
$$

where, $\mathrm{a} \leq \mathfrak{z} \leq t, \forall\, t \in (\mathrm{a},\mathrm{b}]$.

Thus, if ${}_{0}D^{\alpha}_{t}\phi(t) \geq 0$, $\forall\, t \in (\mathrm{a},\mathrm{b})$*, then* ϕ *is a non-decreasing function and if* ${}_{0}D^{\alpha}_{t}\phi(t) \leq 0$, $\forall\, t \in (\mathrm{a},\mathrm{b})$*, then* ϕ *is a non-increasing function.*

Theorem 1. *(Positivity). All solutions of the system* (15)-(16) *are non-negative and are remains in*

$$
\mathbb{R}^7_+ = \{\mathscr{Q}(t); \mathscr{Q}(t) = (S(t), Q_1(t), A(t), Q_2(t), I(t), T(t), R(t)) \in \mathbb{R}^7, \mathscr{Q}(t) \geq 0\}.
$$

Proof. We will prove the non-negativity of solutions for our system (15) by using the Lemma 1. Since,

$$
\begin{aligned}
&{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, S|_{S=0} = (1-\rho^{\alpha})\Omega^{\alpha} + \zeta^{\alpha} Q_1 \geq 0, \\
&{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, Q_1|_{Q_1=0} = \rho^{\alpha}\Omega^{\alpha} + \lambda^{\alpha} S \geq 0, \\
&{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, A|_{A=0} = \beta^{\alpha} S I + \sigma^{\alpha}\beta^{\alpha} Q_1 I \geq 0, \\
&{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, Q_2|_{Q_2=0} = q_1^{\alpha} A \geq 0, \\
&{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, I|_{I=0} = q_3^{\alpha} Q_2 + q_2^{\alpha} A \geq 0, \\
&{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, T|_{T=0} = \gamma^{\alpha} I \geq 0, \\
&{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, R|_{R=0} = q_4^{\alpha} Q_2 + \eta^{\alpha} T \geq 0.
\end{aligned} \tag{18}
$$

As, a result $\forall\, t > 0$, the solutions of the system remain positive and they will remain within $\mathbb{R}^7_+$. Also, the vector field consistently directs towards $\mathbb{R}^7_+$ on each hyperplane encompassing the non-negativity orthant. □

Theorem 2. *(Boundedness). All solutions of the system* (15)-(16) *starting in* $\mathbb{R}^7_+$ *is bounded.*

Proof. To establish the theorem, we derive the subsequent result from equations (15) as follows:

$$
\begin{aligned}
{}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, \mathbb{P}(t) &= {}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, S(t) + {}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, Q_1(t) + {}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, A(t) \\
&\quad + {}^{c}_{0}\mathfrak{D}^{\alpha}_{t} Q_2(t) + {}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, I(t) \\
&\quad + {}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, T(t) + {}^{c}_{0}\mathfrak{D}^{\alpha}_{t}\, R(t), \\
&= \Omega^{\alpha} - \mu^{\alpha}\mathbb{P}(t) - \delta^{\alpha} I, \\
&\leq \Omega^{\alpha} - \mu^{\alpha}\mathbb{P}(t).
\end{aligned}
$$

Utilizing the LT of CFD, as discussed in Definition 3, on the above equation, result in

$$
\begin{aligned}
&\mathfrak{s}^{\alpha}\mathscr{L}\big[\mathbb{P}(t)\big] - \mathfrak{s}^{\alpha-1}\,\mathbb{P}(0) \leq \frac{\Omega^{\alpha}}{\mathfrak{s}} - \mu^{\alpha}\mathscr{L}\big[\mathbb{P}(t)\big], \\
&\mathscr{L}\big[\mathbb{P}(t)\big]\big[\mathfrak{s}^{\alpha}+\mu^{\alpha}\big] \leq \frac{\Omega^{\alpha}}{\mathfrak{s}} + \mathfrak{s}^{\alpha-1}\mathbb{P}(0), \\
&\mathscr{L}\big[\mathbb{P}(t)\big] \leq \frac{\Omega^{\alpha}}{\mathfrak{s}(\mathfrak{s}^{\alpha}+\mu^{\alpha})} + \frac{\mathfrak{s}^{\alpha-1}}{\mathfrak{s}^{\alpha}+\mu^{\alpha}}\mathbb{P}(0), \\
&\mathbb{P}(t) \leq \frac{\Omega^{\alpha}}{\mu^{\alpha}}\mathscr{L}^{-1}\big[\frac{\mu^{\alpha}}{\mathfrak{s}(\mathfrak{s}^{\alpha}+\mu^{\alpha})}\big] + \mathbb{P}(0)\mathscr{L}^{-1}\big[\frac{\mathfrak{s}^{\alpha-1}}{\mathfrak{s}^{\alpha}+\mu^{\alpha}}\big].
\end{aligned}
$$

By using the Definition 4, we get

$$
\begin{aligned}
\mathbb{P}(t) &\leq \frac{\Omega^{\alpha}}{\mu^{\alpha}}\big[1 - E_{\alpha}(-\mu^{\alpha}t^{\alpha})\big] + \mathbb{P}(0)\big[E_{\alpha}(-\mu^{\alpha}t^{\alpha})\big] \\
&\leq \frac{\Omega^{\alpha}}{\mu^{\alpha}} - \big(\frac{\Omega^{\alpha}}{\mu^{\alpha}} - \mathbb{P}(0)\big)E_{\alpha}(-\mu^{\alpha}t^{\alpha}) \\
&\leq \frac{\Omega^{\alpha}}{\mu^{\alpha}} - c\,E_{\alpha}(-\mu^{\alpha}t^{\alpha}), \text{where } c = \frac{\Omega^{\alpha}}{\mu^{\alpha}} - \mathbb{P}(0).
\end{aligned}
$$

This indicates that $0 \leq \mathbb{P}(t) \leq \frac{\Omega^{\alpha}}{\mu^{\alpha}}$, as $t \to \infty$. Therefore, as a consequence the total population and the sub populations all are bounded. Thus,

every solution of the model (15)-(16) starts in region $\mathbb{R}^7_+$ and remains in the region:
$\mathfrak{A} = \{(S, Q_1, A, Q_2, I, T, R) \in \mathbb{R}^7_+ : S + Q_1 + A + Q_2 + I + T + R \leq \dfrac{\Omega^\alpha}{\mu^\alpha}\}$.

□

4.2. Existence and uniqueness of solution

We discuss the existence and uniqueness of the solution for the CFD model (15) by utilizing the Banach fixed point theory [29] in this segment.

Let $\mathsf{B}(\mathscr{J})$ denote a Banach space consisting of continuous real-valued functions defined on the interval $\mathscr{J} = [0, b]$, with the norm specified as:

$$\|(S, Q_1, A, Q_2, I, T, R)\| = \|S\| + \|Q_1\| + \|A\| + \|Q_2\| + \|I\| + \|T\| + \|R\|,$$

where,

$$\|S\| = \underset{t\in\mathscr{J}}{Sup}|S(t)|,\ \|Q_1\| = \underset{t\in\mathscr{J}}{Sup}|Q_1(t)|,$$
$$\|A\| = \underset{t\in\mathscr{J}}{Sup}|A(t)|,\ \|Q_2\| = \underset{t\in\mathscr{J}}{Sup}|Q_2(t)|,$$
$$\|I\| = \underset{t\in\mathscr{J}}{Sup}|I(t)|,\ \|T\| = \underset{t\in\mathscr{J}}{Sup}|T(t)|,$$
$$\|R\| = \underset{t\in\mathscr{J}}{Sup}|R(t)|.$$

Now, consider the DE,

$$\begin{aligned} &{}^C_0\mathfrak{D}^\alpha_t \mathscr{Q}(t) = \mathscr{G}(t, \mathscr{Q}(t))\ ;\ t \in \mathscr{J}, 0 < \alpha \leq 1, \\ &\mathscr{Q}(0) = \mathscr{Q}_0 \geq 0, \end{aligned} \tag{19}$$

where,

$$\mathscr{Q}(t) = (S(t), Q_1(t), A(t), Q_2(t), I(t), T(t), R(t))',$$
$$\mathscr{Q}(0) = (S_0, Q_{1,0}, A_0, Q_{2,0}, I_0, T_0, R_0)',$$
$$\mathscr{G}(t, \mathscr{Q}(t)) = (\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4, \mathscr{G}_5, \mathscr{G}_6, \mathscr{G}_7)',$$

and

$$\begin{aligned} \mathscr{G}_1(t, \mathscr{Q}(t)) &= \Omega^\alpha(1 - \rho^\alpha) - \beta^\alpha S(A + I) \\ &\quad - (\mu^\alpha + \lambda^\alpha)S + \zeta^\alpha Q_1, \\ \mathscr{G}_2(t, \mathscr{Q}(t)) &= \rho^\alpha\Omega^\alpha - \sigma^\alpha\beta^\alpha Q_1(A + I) + \lambda^\alpha S \\ &\quad - (\mu^\alpha + \zeta^\alpha)Q_1, \\ \mathscr{G}_3(t, \mathscr{Q}(t)) &= \beta^\alpha S(A + I) + \sigma^\alpha\beta^\alpha Q_1(A + I) \\ &\quad - (q_1^\alpha + q_2^\alpha + \mu^\alpha)A, \\ \mathscr{G}_4(t, \mathscr{Q}(t)) &= q_1^\alpha A - (q_3^\alpha + q_4^\alpha + \mu^\alpha)Q_2, \\ \mathscr{G}_5(t, \mathscr{Q}(t)) &= q_3^\alpha Q_2 + q_2^\alpha A + (\delta^\alpha + \mu^\alpha + \gamma^\alpha)I, \\ \mathscr{G}_6(t, \mathscr{Q}(t)) &= \gamma^\alpha I - (\eta^\alpha + \mu^\alpha)T, \\ \mathscr{G}_7(t, \mathscr{Q}(t)) &= q_4^\alpha Q_2 + \eta^\alpha T - \mu^\alpha R. \end{aligned}$$

Theorem 3. *All the kernels $\mathscr{G}_j$, where $j = 1, 2, 3, \cdots, 7$ fulfills the Lipschitz condition within the Banach space $\mathsf{B}(\mathscr{J})$.*

Proof. Consider, $\mathscr{Q}(t)$, $\overline{\mathscr{Q}(t)}$ be two functions, then

$$\begin{aligned} &\|\mathscr{G}_1(t, \mathscr{Q}(t)) - \mathscr{G}_1(t, \overline{\mathscr{Q}(t)})\| \\ &= \|(1 - \rho^\alpha)\Omega^\alpha - \beta^\alpha S(A + I) - (\mu^\alpha + \lambda^\alpha)S \\ &\qquad + \zeta^\alpha Q_1 - (1 - \rho^\alpha)\Omega^\alpha + \beta^\alpha\overline{S}(A + I) \\ &\qquad + (\mu^\alpha + \lambda^\alpha)\overline{S} - \zeta^\alpha Q_1\| \\ &= \|-\beta^\alpha(A + I)(S - \overline{S}) - (\mu^\alpha + \lambda^\alpha)(S - \overline{S})\| \\ &\leq |K_1|\,\|S - \overline{S}\|, \end{aligned}$$

where, $K_1 = -(\beta^\alpha(d_3 + d_5) + \mu^\alpha + \lambda^\alpha)$ and $\|A\| \leq d_3,\ \|I\| \leq d_5$.

$$\begin{aligned} &||\mathscr{G}_2(t, \mathscr{Q}(t)) - \mathscr{G}_2(t, \overline{\mathscr{Q}(t)})|| \\ &= ||\rho^\alpha\Omega^\alpha - \sigma^\alpha\beta^\alpha Q_1(A + I) + \lambda^\alpha S \\ &\qquad - (\mu^\alpha + \zeta^\alpha)Q_1 - \rho^\alpha\Omega^\alpha + \sigma^\alpha\beta^\alpha\overline{Q_1}(A + I) \\ &\qquad - \lambda^\alpha S + (\mu^\alpha + \zeta^\alpha)\overline{Q_1}|| \\ &= || - \sigma^\alpha\beta^\alpha(A + I)(Q_1 - \overline{Q_1}) \\ &\qquad - (\mu^\alpha + \zeta^\alpha)(Q_1 - \overline{Q_1})|| \\ &= || - (\sigma^\alpha\beta^\alpha(A + I) + (\mu^\alpha + \zeta^\alpha))(Q_1 - \overline{Q_1})|| \\ &\leq |K_2|\,||Q_1 - \overline{Q_1}||, \end{aligned}$$

where, $K_2 = -(\sigma^\alpha\beta^\alpha(d_3 + d_5) + (\mu^\alpha + \zeta^\alpha))$ and $||A|| \leq d_3,\ ||I|| \leq d_5$.

$$\begin{aligned} &||\mathscr{G}_3(t, \mathscr{Q}(t)) - \mathscr{G}_3(t, \overline{\mathscr{Q}(t)})|| \\ &= ||\beta^\alpha S(A + I) + \sigma^\alpha\beta^\alpha Q_1(A + I) \\ &\qquad - (q_1^\alpha + q_2^\alpha + \mu^\alpha)A - \beta^\alpha S(\overline{A} + I) \\ &\qquad - \sigma^\alpha\beta^\alpha Q_1(\overline{A} + I) + (q_1^\alpha + q_2^\alpha + \mu^\alpha)\overline{A}|| \\ &= ||\beta^\alpha S(A - \overline{A}) + \sigma^\alpha\beta^\alpha Q_1(A - \overline{A}) \\ &\qquad - (q_1^\alpha + q_2^\alpha + \mu^\alpha)(A - \overline{A})|| \\ &\leq |\beta^\alpha\,||S|| + \sigma^\alpha\beta^\alpha\,||Q_1|| + (q_1^\alpha + q_2^\alpha + \mu^\alpha)|\,||A - \overline{A}|| \\ &\leq |K_3|\,||A - \overline{A}||, \end{aligned}$$

where, $K_3 = (\beta^\alpha d_1 + \sigma^\alpha\beta^\alpha d_2 + q_1^\alpha + q_2^\alpha + \mu^\alpha)$ and $||S|| \leq d_1,\ ||Q_1|| \leq d_2$.

$$\begin{aligned} &||\mathscr{G}_4(t, \mathscr{Q}(t)) - \mathscr{G}_4(t, \overline{\mathscr{Q}(t)})|| \\ &= ||q_1^\alpha A - (q_3^\alpha + q_4^\alpha + \mu^\alpha)Q_2 - q_1^\alpha A \\ &\qquad + (q_3^\alpha + q_4^\alpha + \mu^\alpha)\overline{Q_2}|| \\ &= || - (q_3^\alpha + q_4^\alpha + \mu^\alpha)(Q_2 - \overline{Q_2})|| \\ &\leq |K_4|\,||Q_2 - \overline{Q_2}||, \end{aligned}$$

where, $K_4 = -(q_3^\alpha + q_4^\alpha + \mu^\alpha)$.

$$\begin{aligned}
&||\mathscr{G}_5(t,\mathscr{Q}(t)) - \mathscr{G}_5(t,\overline{\mathscr{Q}(t)})|| \\
&= ||q_3^\alpha Q_2 + q_2^\alpha A + (\delta^\alpha + \mu^\alpha + \gamma^\alpha)I - q_3^\alpha Q_2 - q_2^\alpha A \\
&\quad - (\delta^\alpha + \mu^\alpha + \gamma^\alpha)\overline{I}|| \\
&= || - (\delta^\alpha + \mu^\alpha + \gamma^\alpha)(I - \overline{I})|| \\
&\leq |K_5|\; ||I - \overline{I}||,
\end{aligned}$$

where, $K_5 = -(\delta^\alpha + \mu^\alpha + \gamma^\alpha)$.

$$\begin{aligned}
&||\mathscr{G}_6(t,\mathscr{Q}(t)) - \mathscr{G}_6(t,\overline{\mathscr{Q}(t)})|| \\
&= ||\gamma^\alpha I - (\eta^\alpha + \mu^\alpha)T - \gamma^\alpha I + (\eta^\alpha + \mu^\alpha)\overline{T}|| \\
&= || - (\zeta^\alpha + \mu^\alpha)(T - \overline{T})|| \\
&\leq |K_6|\, ||T - \overline{T}||,
\end{aligned}$$

where, $K_6 = -(\zeta^\alpha + \mu^\alpha)$.

$$\begin{aligned}
&||\mathscr{G}_7(t,\mathscr{Q}(t)) - \mathscr{G}_7(t,\overline{\mathscr{Q}(t)})|| \\
&= ||q_4^\alpha Q_2 + \eta^\alpha T - \mu^\alpha R - q_4^\alpha Q_2 - \eta^\alpha T - \mu^\alpha \overline{R}|| \\
&= || - \mu^\alpha (R - \overline{R})|| \\
&\leq |K_7|\, ||R - \overline{R}||,
\end{aligned}$$

where, $K_7 = -\mu^\alpha$.

After adding all the aforementioned equations, we get

$$\begin{aligned}
&\|\mathscr{G}(t,\mathscr{Q}(t)) - \mathscr{G}(t,\overline{\mathscr{Q}(t)})\| \\
&\leq \|\mathscr{G}_1(t,\mathscr{Q}(t)) - \mathscr{G}_1(t,\overline{\mathscr{Q}(t)})\| \\
&\qquad + \|\mathscr{G}_2(t,\mathscr{Q}(t)) - \mathscr{G}_2(t,\overline{\mathscr{Q}(t)})\| \\
&\qquad + \|\mathscr{G}_3(t,\mathscr{Q}(t)) - \mathscr{G}_3(t,\overline{\mathscr{Q}(t)})\| \\
&\qquad + \|\mathscr{G}_4(t,\mathscr{Q}(t)) - \mathscr{G}_4(t,\overline{\mathscr{Q}(t)})\| \\
&\qquad + \|\mathscr{G}_5(t,\mathscr{Q}(t)) - \mathscr{G}_5(t,\overline{\mathscr{Q}(t)})\| \\
&\qquad + \|\mathscr{G}_6(t,\mathscr{Q}(t)) - \mathscr{G}_6(t,\overline{\mathscr{Q}(t)})\| \\
&\qquad + \|\mathscr{G}_7(t,\mathscr{Q}(t)) - \mathscr{G}_7(t,\overline{\mathscr{Q}(t)})\| \\
&\leq |K_1|\,\|S - \overline{S}\| + |K_2|\,\|Q_1 - \overline{Q_1}\| + |K_3|\,\|A - \overline{A}\| \\
&\quad + |K_4|\,\|Q_2 - \overline{Q_2}\| + |K_5|\;\|I - \overline{I}\| \\
&\quad + |K_6|\,\|T - \overline{T}\| + |K_7|\,\|R - \overline{R}\| \\
&\leq K\|\mathscr{Q}(t) - \overline{\mathscr{Q}(t)}\|,
\end{aligned}$$

where, $K = Max\{|K_i|; i = 1,2,3,\cdots,7\}$ is the Lipschitz constant of the kernel $\mathscr{G}(t,\mathscr{Q}(t))$. Hence, $\mathscr{G}(t,\mathscr{Q}(t))$ satisfies the Lipschitz condition. □

Theorem 4. *If* $\dfrac{K}{\Gamma(\alpha+1)} \leq 1$*, then the model* (15) *possesses a unique solution.*

Proof. Consider, $\Psi : \mathsf{B} \to \mathsf{B}$ be a linear map represented by,

$$\Psi(\mathscr{Q}(t)) = \mathscr{Q}_0(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}} \times \mathscr{G}(s,\mathscr{Q}(s))ds,$$

and, $\mathscr{Q}(t), \overline{\mathscr{Q}(t)} \in \mathsf{B}$ then, we have

$$\begin{aligned}
&\|\Psi(\mathscr{Q}(t)) - \Psi(\overline{\mathscr{Q}(t)})\| \\
&= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(\mathscr{G}(s,\mathscr{Q}(s)) - \mathscr{G}(s,\overline{\mathscr{Q}(s)}))ds \right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| \mathscr{G}(s,\mathscr{Q}(s)) - \mathscr{G}(s,\overline{\mathscr{Q}(s)}) \right\| ds \\
&\leq \frac{K\,\|\mathscr{Q}(s) - \overline{\mathscr{Q}(s)}\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds \\
&\leq \frac{Kt^\alpha}{\alpha\Gamma(\alpha)}\, \|\mathscr{Q}(s) - \overline{\mathscr{Q}(s)}\|.
\end{aligned}$$

Thus, Ψ is a contraction, if $\dfrac{K}{\Gamma(\alpha+1)} \leq 1$.

Hence, from Banach contraction principle, the fractional order system (15) possesses a unique solution. □

5. The Reproduction Number and it's Sensitivity Analysis

5.1. Reproduction number

Epidemiologically, the basic reproduction number often denoted as $\mathbf{R}_0$ indicates the average count of new infections originating from one infected individual within a vulnerable population throughout their infectious period. It is a fundamental concept used to measure the potential for disease transmission in a population. If $\mathbf{R}_0 < 1$ then eventually disease will die out from population and if $\mathbf{R}_0 > 1$, the disease will persist and potentially lead to an outbreak. To calculate $\mathbf{R}_0$ we first determine the Disease Free Equilibrium point (DFE) denoted by $(\mathscr{E}_0^*)$. Since, Equilibrium points represent the solutions to equation describing the system, at which the variable experiences zero rate of change. Specifically, the disease free equilibrium (DFE) signifies a state where the disease does not persist within the population. By setting

$$\begin{aligned}
&{}^{c}{}_{0}\mathfrak{D}_t^\alpha\, S = {}^{c}{}_{0}\mathfrak{D}_t^\alpha\, Q_1 = {}^{c}{}_{0}\mathfrak{D}_t^\alpha\, A = {}^{c}{}_{0}\mathfrak{D}_t^\alpha\, Q_2 \\
&= {}^{c}{}_{0}\mathfrak{D}_t^\alpha\, I = {}^{c}{}_{0}\mathfrak{D}_t^\alpha\, T = {}^{c}{}_{0}\mathfrak{D}_t^\alpha\, R = 0,
\end{aligned}$$

we calculate the equilibrium points based on the system. Now, applying the necessary conditions involves setting all infectious compartments of the model to zero i.e. $A = Q_2 = I = T = R = 0$.

We obtained the DFE point of the model as follows:

$$\mathcal{E}_0^* = (S_0^*, Q_{1,0}^*, A_0^*, Q_{2,0}^*, I_0^*, T_0^*, R_0^*) = \Bigg(\frac{\Omega^\alpha(\mu^\alpha(1-\rho^\alpha)+\zeta^\alpha)}{\mu^\alpha(\lambda^\alpha+\mu^\alpha+\zeta^\alpha)}, \frac{\Omega^\alpha(\mu^\alpha\rho^\alpha+\lambda^\alpha)}{\mu^\alpha(\lambda^\alpha+\mu^\alpha+\zeta^\alpha)}, 0,0,0,0,0\Bigg).$$

We then apply the next-generation matrix method [30, 31] to evaluate the $\mathbf{R}_0$ of the model (15). This involves determining the spectral radius of the next generation matrix $(\mathcal{F}\mathcal{V}^{-1})$, in which $\mathcal{F}$ represent the Jacobian of matrix $\mathscr{F}$ (transmission compartment, signifying the appearance of new infections) and $\mathcal{V}$ express the Jacobian of matrix $\mathscr{V}$ (transition compartment) at the DFE point:

$$\mathcal{F} = \begin{bmatrix} \beta^\alpha(S_0^*+\sigma^\alpha Q_0^*) & 0 & \beta^\alpha(S_0^*+\sigma^\alpha Q_0^*) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{V} = \begin{bmatrix} b_2 & 0 & 0 & 0 \\ -q_1^\alpha & b_3 & 0 & 0 \\ -q_2^\alpha & -q_3^\alpha & b_4 & 0 \\ 0 & 0 & -\gamma^\alpha & b_5 \end{bmatrix}$$

$$\mathbf{R}_0 = \varrho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\beta^\alpha\Omega^\alpha[\sigma^\alpha(\rho^\alpha\mu^\alpha+\lambda^\alpha)+(\zeta^\alpha+\mu^\alpha(1-\rho^\alpha))]}{\mu^\alpha\, b_1\, b_3\, b_2^2} \times [b_3 b_2 + q_2^\alpha b_2 + q_3^\alpha q_1^\alpha], \tag{20}$$

where, $b_1 = \mu^\alpha + \zeta^\alpha + \lambda^\alpha$, $b_2 = q_1^\alpha + q_2^\alpha + \mu^\alpha$, $b_3 = q_3^\alpha + q_4^\alpha + \mu^\alpha$, $b_4 = \delta^\alpha + \gamma^\alpha + \mu^\alpha$ and $b_5 = \eta^\alpha + \mu^\alpha$.

5.2. Sensitivity analysis

Sensitivity analysis is crucial for assessing the robustness of model predictions and understanding how the output variable changes concerning variations in input parameters. Within this part, we delve into the sensitivity analysis of $\mathbf{R}_0$ and the model parameters by utilizing the Normalized Sensitivity Index as discussed in [32]. This method identifies the most influential parameter for $\mathbf{R}_0$ and their impacts on disease transmission. The normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. as discussed in [33]. Specifically, for the $\mathbf{R}_0$ concerning the parameter $\mathfrak{p}$, it is calculated as:

$$\wp_{\mathfrak{p}}^{\mathbf{R}_0} = \frac{\partial \mathbf{R}_0}{\partial\, \mathfrak{p}} \times \frac{\mathfrak{p}}{\mathbf{R}_0}. \tag{21}$$

Where, the sensitivity index of $\mathbf{R}_0$ w.r.t parameter $\mathfrak{p}$ is positive, if $\mathbf{R}_0$ increases concerning $\mathfrak{p}$ and negative if $\mathbf{R}_0$ decreases concerning $\mathfrak{p}$.

Table 1. Parameter description and their corresponding values sourced from the relevant literature [22].

Parameters	Biological meaning	Values
Ω	Recruitment rate of susceptible peoples	0.0000421
ρ	Fraction of individuals under quarantine due to the implemented lockdown	0.5
λ	Transmission rate at which Susceptible people moving to Quarantine class(Q_1)	0.5
β	Rate of transmission of infection between individuals	0.07
μ	Mortality rate	0.0000421
ζ	Transmission rate of Quarantine people moving to Susceptible class	0.0715
σ	efficacy factor of lockdown	0.5
q_1	Rate by which Asymptomatic individual move into self-Quarantine class Q_1	0.2
q_2	Rate by which Asymptomatic individual showing the symptoms	0.1428
q_3	Rate at which Self-Quarantine people enters into Infected class	0.21
q_4	Rate by which self-Quarantine people recovers	0.08
γ	Rate by which infected individuals are treated	0.11
η	Rate by which infected people are recovered with medical treatment	0.0917
δ	Disease induced death rate	0.05

Table 2. Sensitivity indices of R_0.

parameters	indices
Ω	+1
ρ	-3.27412e-05
β	+1
μ	-1.00031
ζ	+0.0972096
σ	+0.777734
q_1	-0.25982
q_2	-0.09752
q_3	+0.08925
q_4	-0.0892
δ	-0.200737
γ	-0.4416
λ	-0.0972096

However, calculating the sensitivity indices of $\mathbf{R}_0$ explicitly in terms of the model's parameters proves challenging due to the intricate nature of $\mathbf{R}_0$. Consequently, we assess the sensitivity indices using the values of parameters provided in Table 1. We obtained sensitivity indices for

$\mathbf{R}_0$ concerning the thirteen distinct parameters in the model that are displayed in Table 2. Additionally, a visual representation of these numerical sensitivity indices is provided in Figure 1. According to the computed sensitivity indices, a 10% increment in the recruitment rate (Ω), lockdown efficacy factor (σ), and the transmission rate (β) results in a 10%, 7.7%, and 10% increase in the value of $\mathbf{R}_0$, respectively. On the contrary,

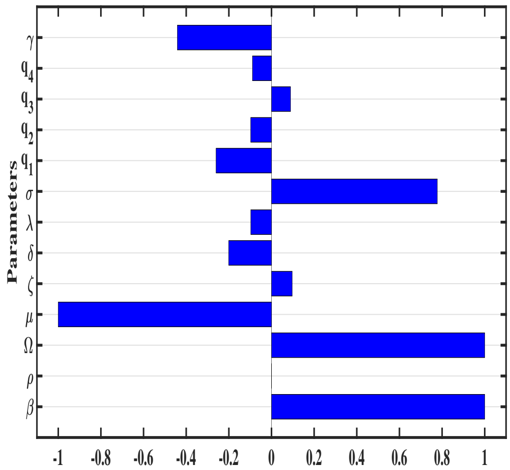

Figure 1. Sensitivity of the $\mathbf{R}_0$ concerning all thirteen parameters.

when it comes to parameters such as the natural death rate (μ), the treatment rate (γ), the rate at which symptomatic individuals enter self-quarantine (q_1) and disease-induced death rate (δ), an increase of 10% in their values results in a decrease of $\mathbf{R}_0$ by 4.4%, 2.5%, 2.1%, and 10% respectively.

Therefore, the findings indicate that a 10% rise in the transmission rate β and recruitment rate Ω, significantly increases $\mathbf{R}_0$, with a notable impact. Additionally, the lockdown scaling factor σ also demonstrates a substantial effect on $\mathbf{R}_0$. While, the remaining parameters exhibit low perturbation, exerting minimal influence on $\mathbf{R}_0$. This analysis is depicted in Figure 1, illustrating the high sensitivity of the transmission rate, and the significant impact of the lockdown scaling factor on $\mathbf{R}_0$.

6. Numerical Algorithm

We utilize numerical technique to approximate the solutions for nonlinear ordinary and partial differential equations that cannot be resolved through standard analytical techniques. In this study, the numerical approach is based on the two-step Lagrange interpolation approach, as detailed in [34–36] to address the fractional order COVID-19 model (15).

From equation (19), we have

$$ {}^{C}_{0}\mathfrak{D}^{\alpha}_{t}\mathscr{Q}(t) = \mathscr{G}(t, \mathscr{Q}(t)) \ , \ t \in [0, b] \, , 0 < \alpha \leq 1, \\ \mathscr{Q}(0) = \mathscr{Q}_0, \tag{22} $$

and its solution is

$$ \mathscr{Q}(t) = \mathscr{Q}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{G}(s, \mathscr{Q}(s))\, ds. \tag{23} $$

Let, $h = \dfrac{T}{\mathfrak{n}}$, $t_\vartheta = \vartheta h$, $\vartheta = 0, 1, 2\cdots, \mathfrak{n} \in \mathbb{Z}^+$, then at point $t = t_{\vartheta+1}$, equation (23) becomes

$$ \mathscr{Q}(t_{\vartheta+1}) = \mathscr{Q}(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{\vartheta+1}} (t_{\vartheta+1} - s)^{\alpha-1} \times \mathscr{G}(s, \mathscr{Q}(s))\, ds, $$

which can be expressed as,

$$ \mathscr{Q}(t_{\vartheta+1}) = \mathscr{Q}(0) + \frac{1}{\Gamma(\alpha)} \sum_{\varsigma=0}^{\vartheta} \int_{t_\varsigma}^{t_{\varsigma+1}} (t_{\vartheta+1} - s)^{\alpha-1} \times \mathscr{G}(s, \mathscr{Q}(s)) ds. \tag{24} $$

By approximating the function $\mathscr{G}(s, \mathscr{Q}(s))$ over interval $[t_\varsigma, t_{\varsigma+1}]$ by using the Lagrange polynomial,

$$ \mathscr{G}(s, \mathscr{Q}(s)) = \frac{s - t_{\varsigma-1}}{t_\varsigma - t_{\varsigma-1}} \mathscr{G}(t_\varsigma, \mathscr{Q}(t_\varsigma)) - \frac{s - t_\varsigma}{t_\varsigma - t_{\varsigma-1}} \mathscr{G}(t_{\varsigma-1}, \mathscr{Q}(t_{\varsigma-1})). \tag{25} $$

Using equation (25) in (24) and then simplifying the integral, we get

$$ \mathscr{Q}_{\vartheta+1} = \mathscr{Q}(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}(t_\varsigma, \mathscr{Q}(t_\varsigma)) \\ \left((2 + \vartheta - \varsigma - \alpha) (1 + \vartheta - \varsigma)^\alpha - (\vartheta - \varsigma)^\alpha (2 + \vartheta - \varsigma + 2\alpha) \right) \big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}(t_{\varsigma-1}, \mathscr{Q}(t_{\varsigma-1})) \\ \left((1 + \vartheta - \varsigma + \alpha) (\vartheta - \varsigma)^\alpha - (1 + \vartheta - \varsigma)^{\alpha+1} \right) \big]. \tag{26} $$

Using the aforementioned scheme (26) for numerical solution of our proposed model (15), we get

$$S_{\vartheta+1} = S(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_1\left(t_\varsigma, S(t_\varsigma)\right) \big((2+\vartheta-\varsigma-\alpha)\left(1+\vartheta-\varsigma\right)^\alpha - (\vartheta-\varsigma)^\alpha(2+\vartheta - \varsigma + 2\alpha)\big)\big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_1(t_{\varsigma-1}, S(t_{\varsigma-1})) \big((1+\vartheta-\varsigma+\alpha)(\vartheta-\varsigma)^\alpha - (1+\vartheta-\varsigma)^{\alpha+1}\big)\big], \tag{27}$$

$$Q_{1,\vartheta+1} = Q_1(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_2\left(t_\varsigma, Q_1(t_\varsigma)\right) \big((2+\vartheta-\varsigma-\alpha)(1+\vartheta-\varsigma)^\alpha - (\vartheta-\varsigma)^\alpha(2+\vartheta - \varsigma + 2\alpha)\big)\big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_2(t_{\varsigma-1}, Q_1(t_{\varsigma-1})) \big((1+\vartheta-\varsigma+\alpha)\left(\vartheta-\varsigma\right)^\alpha - (1+\vartheta-\varsigma)^{\alpha+1}\big)\big], \tag{28}$$

$$A_{\vartheta+1} = A(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_3\left(t_\varsigma, A(t_\varsigma)\right) \big((2+\vartheta-\varsigma-\alpha)(1+\vartheta-\varsigma)^\alpha - (\vartheta-\varsigma)^\alpha(2+\vartheta - \varsigma + 2\alpha)\big)\big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_3\left(t_{\varsigma-1}, A(t_{\varsigma-1})\right) \big((1+\vartheta-\varsigma+\alpha)(\vartheta-\varsigma)^\alpha - (1+\vartheta-\varsigma)^{\alpha+1}\big)\big], \tag{29}$$

$$Q_{2,\vartheta+1} = Q_2(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_4\left(t_\varsigma, Q_2(t_\varsigma)\right) \big((2+\vartheta-\varsigma-\alpha)(1+\vartheta-\varsigma)^\alpha - (\vartheta-\varsigma)^\alpha(2+\vartheta - \varsigma + 2\alpha)\big)\big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_4\left(t_{\varsigma-1}, Q_2(t_{\varsigma-1})\right) \big((1+\vartheta-\varsigma+\alpha)\left(\vartheta-\varsigma\right)^\alpha - (1+\vartheta-\varsigma)^{\alpha+1}\big)\big], \tag{30}$$

$$I_{\vartheta+1} = I(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_5(t_\varsigma, I(t_\varsigma)) \big((2+\vartheta-\varsigma-\alpha)(1+\vartheta-\varsigma)^\alpha - (\vartheta-\varsigma)^\alpha(2+\vartheta - \varsigma + 2\alpha)\big)\big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_5\left(t_{\varsigma-1}, I(t_{\varsigma-1})\right) \big((1+\vartheta-\varsigma+\alpha)(\vartheta-\varsigma)^\alpha - (1+\vartheta-\varsigma)^{\alpha+1}\big)\big], \tag{31}$$

$$T_{\vartheta+1} = T(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_6(t_\varsigma, T(t_\varsigma)) \big((2+\vartheta-\varsigma-\alpha)\left(1+\vartheta-\varsigma\right)^\alpha - (\vartheta-\varsigma)^\alpha(2+\vartheta - \varsigma + 2\alpha)\big)\big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_6(t_{\varsigma-1}, T(t_{\varsigma-1})) \big((1+\vartheta-\varsigma+\alpha)(\vartheta-\varsigma)^\alpha - (1+\vartheta-\varsigma)^{\alpha+1}\big)\big], \tag{32}$$

$$R_{\vartheta+1} = R(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_7(t_\varsigma, R(t_\varsigma)) \big((2+\vartheta-\varsigma-\alpha)\left(1+\vartheta-\varsigma\right)^\alpha - (\vartheta-\varsigma)^\alpha(2+\vartheta - \varsigma + 2\alpha)\big)\big] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{\varsigma=0}^{\vartheta} \big[\mathscr{G}_7(t_{\varsigma-1}, R(t_{\varsigma-1})) \big((1+\vartheta-\varsigma+\alpha)\left(\vartheta-\varsigma\right)^\alpha - (1+\vartheta-\varsigma)^{\alpha+1}\big)\big]. \tag{33}$$

7. Results and Discussion

We utilized the numerical method outlined in preceding subsection, and employed baseline values for parameters (as detailed in Table 1) and the initial conditions of the model from pertinent literature [22]. The initial conditions were specified as follows:

$$S(0) = 0.69 \times 10^9,\ Q_1(0) = 0.7 \times 10^9,\ A(0) = 3800,\ Q_2(0) = 800,\ I(0) = 601,\ T(0) = 825,\ R(0) = 566. \tag{34}$$

To illustrate the dynamics of the formulated COVID-19 model (15), we provide graphical visualizations in Figures 2, 3, 4 and 5. These visualizations enable us to analyze the influence of the CFD on the dynamics of population by altering key model parameters and exploring different values of fractional order. We used MATLAB software for simulating numerical results, and our discussed numerical approach provided approximate solutions, which are visually depicted in the referenced figures. Figure 2 displays the population dynamics of the discussed model, utilizing the CFD within a time sequence framework, measured in weeks. In Figure 2, the behaviors of $I(t)$, $Q_2(t)$, $T(t)$, and $R(t)$ are portrayed for fractional order values $\alpha = 0.80,\ 0.85,\ 0.90,\ 0.95$ and 1.

Figure 2a demonstrates that infection increase and decrease rapidly as the fractional order rises.

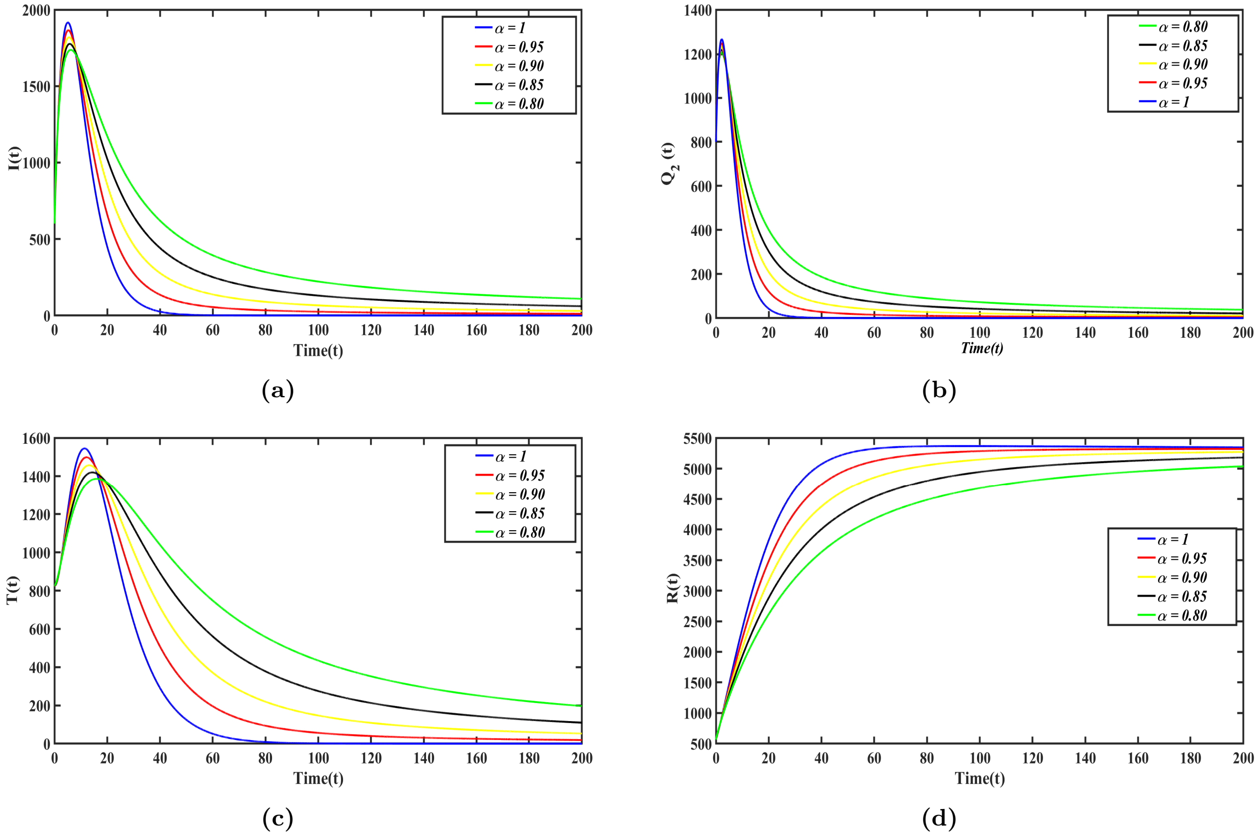

Figure 2. Solution behavior of I(t), $Q_2(t)$, T(t), R(t).

Similar patterns can be observed in Figures 2b and 2c, respectively. During this critical period, medical treatment plays a pivotal role in controlling infections, ensuring suitable care for individuals and facilitating their recovery, as indicated in Figure 2d. The recovered population increases over time, with variations observed for different fractional order values. It is noteworthy that, as α approach to 1, the fractional order model solution converges toward the solution obtained from the conventional integer-order model.The convergence becomes faster as the fractional order α approaches one. This behavior can be attributed to fractional order derivatives retaining the population dynamics of previous time instants, which effectively slows down the rate of reaching stability.

Figures 3, 4 and 5 illustrate the impact of highly sensitive parameters such as β (transmission rate), σ (lockdown scaling factor), and γ (rate of exposure to treatment class) on $\mathbf{R}_0$ and simultaneously explores the impact of the transmission rate, lockdown scaling factor and recovery rate on the presented model. We investigate how these governing factors influence the dynamics and behavior of the system. The strategies for managing the spread of the disease primarily revolve around minimizing the transmission of the covid-19 infection from individuals who are infected to those who are susceptible, and enhancing the rate of recovery. These measures are crucial in managing and preventing the continued dissemination of the disease. On the left side of the figures, pattern of the ($\mathbf{R}_0$) is displayed, while the right side illustrates the behavior of the infected population for distinct values of the specified parameters.

Figure 3, illustrates the dynamical behavior of $\mathbf{R}_0$ and COVID-19-infected individuals under various transmission rates (β), while the remaining parameters remain the same as in Table 1, with considering a fractional order $\alpha = 1$.

It reveals that β leads to a rapid and substantial increase in $\mathbf{R}_0$, and as its value escalates from 0.10 to 0.50, result in a corresponding rise in the infection. Figure 4, illustrates the dynamical behavior of $\mathbf{R}_0$ and COVID-19-infected individuals under various transmission rates (γ), while the remaining parameters remain the same as in Table 1, with considering a fractional order $\alpha = 1$. The variation of $\mathbf{R}_0$ concerning γ demonstrates an inverse relation. Increasing the value of γ significantly reduces the cases of infected

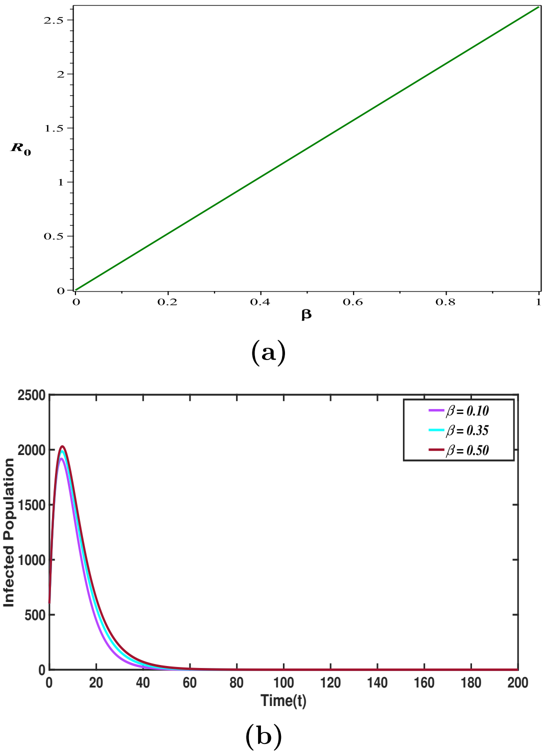

Figure 3. **(a)** Variation of $\mathbf{R}_0$ with β. **(b)** Variation of infected population.

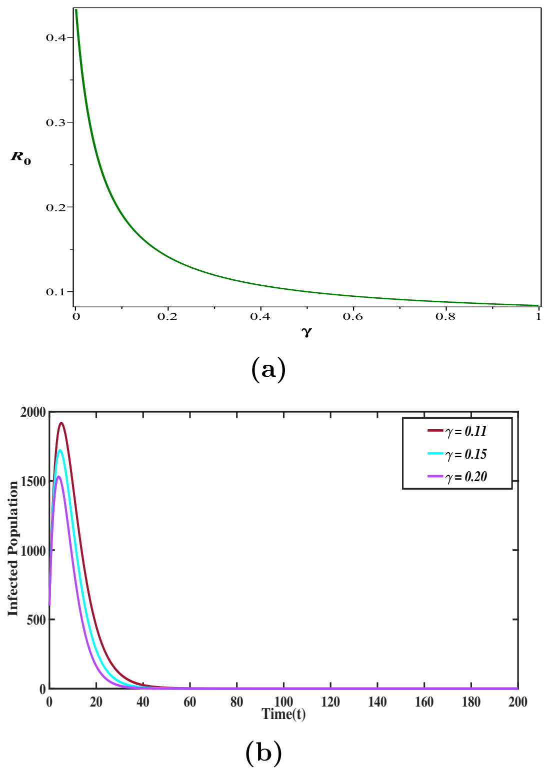

Figure 4. **(a)** Variation of $\mathbf{R}_0$ with γ. **(b)** Variation of Infected Population.

individuals, as depicted in Figure 4. Additionally, Figure 5 illustrates the dynamical behavior of $\mathbf{R}_0$ and COVID-19-infected individuals under various transmission rates (β), while the remaining parameters remain the same as in Table 1, with considering a fractional order $\alpha = 1$, where, $\sigma = 0$ corresponds to a state of complete lockdown, $\sigma = 0.5$ to a partial lockdown and, $\sigma = 1$ to a no lockdown scenario. It depicts the impact of the parameter σ on the $\mathbf{R}_0$ and on the infected population, ranging from 0 to 1. It is evident that without imposing a lockdown, infection levels would inevitably rise.

8. Conclusions

In our study, we investigated the mathematical model involving CFD to determine the transmission dynamics of COVID-19. Our analysis included fundamental assessments of the formulated model, ensuring boundedness and non-negativity within the feasible region. These analyses ensure that the model offers valuable and realistic perspective into the dynamics of COVID-19 outbreak. With addition to this, we established the existence and uniqueness of proposed model solutions with the help of Banach fixed point theorem. We computed the basic reproduction number $\mathbf{R}_0$ by employing the next-generation matrix technique, serving as

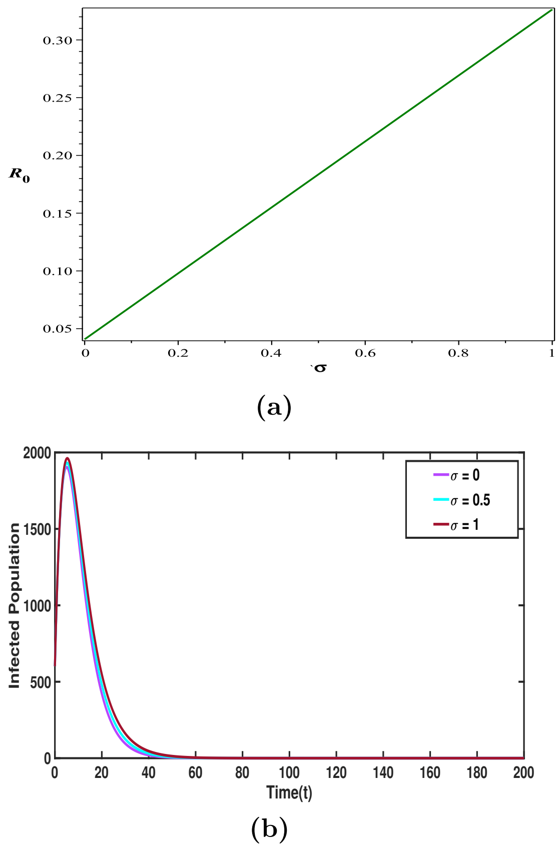

Figure 5. **(a)** Variation of $\mathbf{R}_0$ with σ. **(b)** Variation of Infected Population.

a threshold parameter in the evolution of infection. This parameter is pivotal in identifying whether the disease endures or dissipates within the population. Furthermore, we employed the normalized sensitivity index to conduct a sensitivity analysis of $\mathbf{R}_0$ for several model parameters. The impact of different parameters on the $\mathbf{R}_0$ has been analyzed as well. This analysis enabled us to pinpoint the control parameters significantly impacting the progression of infection.

Moreover, we utilized the two-step Lagrange interpolation method to perform numerical simulations across various fractional order values (α) in the proposed fractional model. This numerical approach not only validated our theoretical results but also provided significant insights into the dynamical behavior of the model influenced by fractional order. Our numerical results highlighted the substantial impact of increasing the lockdown scaling factor σ and decreasing the transmission rate β on reducing the number of COVID-19 infections. Furthermore, these findings offer crucial insights for intervention strategies, especially concerning lockdown measures, effectively managing COVID-19 transmission, and reducing the transmission rate. Implementing isolation and quarantining susceptible also emerged as effective strategies to curtail transmission.

While the fractional order COVID-19 model has furnished valuable insights into the epidemic transmission process and identified critical factors for its spread, a more detailed analysis requires extending the model along with some additional factors. Future research work should incorporate various fractional derivatives, such as fractal-fractional, Atangana-Beta derivative, Caputo-Fabrizio, and more. These extensions will pave the way for more comprehensive and in-depth studies in the field.

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RESEARCH ARTICLE

The effect of a psychological scare on the dynamics of the tumor-immune interaction with optimal control strategy

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ABSTRACT

Contracting cancer typically induces a state of terror among the individuals who are affected. Exploring how chemotherapy and anxiety work together to affect the speed at which cancer cells multiply and the immune system's response model is necessary to come up with ways to stop the spread of cancer. This paper proposes a mathematical model to investigate the impact of psychological scare and chemotherapy on the interaction of cancer and immunity. The proposed model is accurately described. The focus of the model's dynamic analysis is to identify the potential equilibrium locations. According to the analysis, it is possible to establish three equilibrium positions. The stability analysis reveals that all equilibrium points consistently exhibit stability under the defined conditions. The bifurcations occurring at the equilibrium sites are derived. Specifically, we obtained transcritical, pitchfork, and saddle-node bifurcation. Numerical simulations are employed to validate the theoretical study and ascertain the minimum therapy dosage necessary for eradicating cancer in the presence of psychological distress, thereby mitigating harm to patients. Fear could be a significant contributor to the spread of tumors and weakness of immune functionality.

1. Introduction

Models are instruments utilized in medicine and science to interpret results, develop hypotheses, and plan experiments to verify them [1]. For instance, mathematical models of population dynamics are frequently represented by differences or differential equations that characterize the temporal evolution of populations [2–9]. Throughout history, ecology has predominantly employed mathematical models to offer qualitative explanations for natural patterns. An exemplary illustration of this methodology was the endeavour to elucidate species diversity through competition models [10–16]. Mathematicsematical modeling is a highly versatile instrument in the field of infectious disease epidemiology, enabling the detection of epidemic patterns, extrapolation of epidemic behaviors, and evaluation of the impact of interventions, including pharmacological treatment, immunization, quarantine, social distance, and hygiene practices, among others [17–22]. An example of a disease model is cancer, which is characterized by the proliferation of malignant cells that infiltrate other anatomical structures and currently ranks as the second most prevalent cause of mortality globally, surpassed only by cardiovascular disease. Developing novel treatment options is a burgeoning study field for scientists seeking to manage cancer effectively. Nevertheless, comprehending the

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intricacies of tumor cell proliferation and their intricate interplay with the immune system is crucial in order to devise novel therapeutic approaches. To accomplish this, researchers extensively depended on mathematical models [23–27]. Several scientists have extensively researched the mathematical modeling of tumor evolution, its interaction with different cells, and the process of tumor growth. They have achieved this by creating multiple models over the past few decades [28–33]. Cancer is amenable to a variety of treatment modalities, including chemotherapy, radiotherapy, and surgery. Chemotherapy, one of the cancer treatments, is a systematic approach that targets and eliminates cancer cells at the site of the tumor while minimizing its impact on effector-normal cells. This eliminates the ability of the tumor cells to metastasize to other anatomical sites [34–36]. For instance, De Pillis and his associates examined multiple mathematical models to quantify the effects of chemotherapy [37]. In addition, Pillis et al. devised a cancer treatment model in which they discovered that combining chemotherapy and immunotherapy can completely eradicate the tumor instead of using either therapy alone [38]. On the other hand, The initial mathematical model that incorporated the influence of fear in a predator-prey system involving two species was presented by Wang et al. in 2016 [39]. Prey animals may alter their grazing location to a more secure area and relinquish their most productive feeding sites due to predator-induced anxiety. The user's text is incomplete and lacks information [40–43]. Further, There has been a recent increase in research focusing on the importance of mathematical models for studying how fear-induced behavioral changes impact the spread of diseases [44–48]. A medical study has demonstrated that psychological stress contributes to the dissemination of cancer cells throughout the patient's body. Psychological stress causes significant dilation and intensification of blood vessels, hence promoting the migration of cancer cells and facilitating the metastasis of the disease [49]. Researchers have discovered that stress-induced hormones exacerbate the proliferation of cancer cells inside the "lymphatic system," thus facilitating their dissemination to other locations, thereby promoting the metastasis of the disease throughout the human body [50].

The present study proposes a psychological scare-cancer-immune-normal-chemotherapy model (PSCINC) regulated by systems of ordinary differential equations, drawing inspiration from the model presented in [51]. We have enhanced the model of De Pillis et al. by replacing the linear functional response with the Holling type II functional response. This modification allows us to accurately depict the eradication of tumor cells by the immune system, considering the possibility of a weakened immune system due to the presence of psychological scare of cancer. Further, there is a lack of study about the influence of fear on the immune-cancer model. Hence, we deem it imperative to examine this phenomenon, as it contributes to reducing the occurrence of catastrophic circumstances.

Further, there is a lack of study about the influence of fear on the immune-cancer model. Hence, we deem it imperative to examine this phenomenon, as it contributes to reducing the occurrence of catastrophic circumstances. Therefore, this study is dedicated to discussing the impact of anxiety on immune cancer patients, which could be a significant contributor to the spread of tumors and weakness of immune functionality. The subsequent sections of this document are organized as follows: section 2 examines the assumptions of the proposed model. The presence of potential equilibrium points is determined in section 3. Next, section 4 discusses the stability conditions of the steady states. The discussion in section 5 focuses on the global stability of equilibriums. In addition, section 6 acknowledges the local bifurcation conditions in close proximity to the fixed points. In section 7, numerical examinations are conducted to validate our analytical findings.

2. Assumptions of the model

Let's examine a system of differential equations (PSCINC) that involves immune cells I (t), tumor cells $C(t)$, normal cells $N(t)$, and chemotherapy treatment $H(t)$ represented as

$$
\begin{aligned}
\frac{dI}{dt} &= \frac{\alpha}{1+eC} + \frac{p_1 IC}{\beta_1 + C} - p_2 IC - d_1 I - d_2 IH \\
&= h_1(I, C, H) \\
\frac{dC}{dt} &= m_1 C\left(1 - k_1 C\right) - \frac{p_3 IC}{\beta_2 + C} - \gamma_1 CN - d_3 HC \\
&= h_2(I, C, N, H) \\
\frac{dN}{dt} &= m_2 N\left(1 - k_2 N\right) - \gamma_2 CN = h_3(C, N) \\
\frac{dH}{dt} &= \nu - d_4 H = h_4(H)
\end{aligned}
\tag{1}
$$

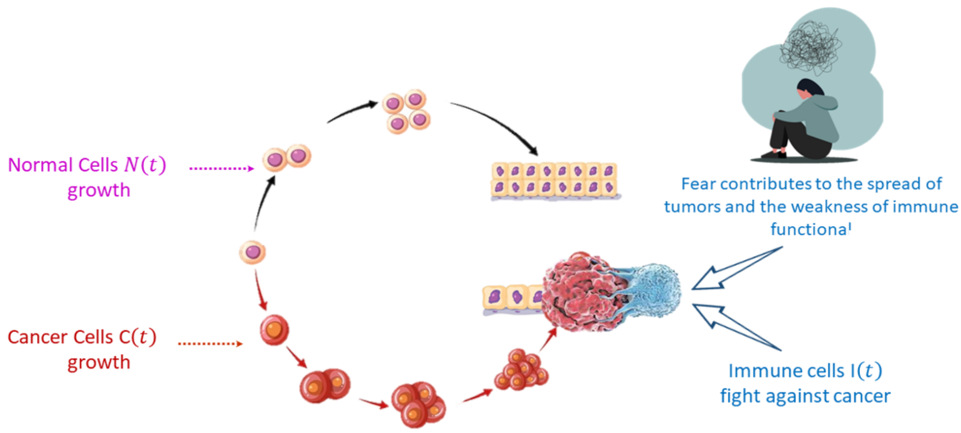
In the first equation of the PSCINC model, the term $\frac{\alpha}{1+eC}$ stands for the regular production of immune cells in the body, which is affected by the presence of cancer cells by the psychological scare factor e. Therefore,e the birth-term changes by producing fear function. The fear function is incorporated by the decreasing function $\varphi(e,C) = \frac{1}{1+eC}$, which was initially introduced by Wang et al. [46]. From the biological point of view, $\varphi(e,C)$ is appropriate since

$$\phi(0,C) = 1, \phi(e,0) = 1,$$
$$\lim_{e\to\infty}\phi(e,C) = 0,$$
$$\lim_{C\to\infty}\phi(e,C) = 0,$$
$$\frac{\partial\phi(e,C)}{\partial e} < 0, \frac{\partial\phi(e,C)}{\partial C} < 0.$$

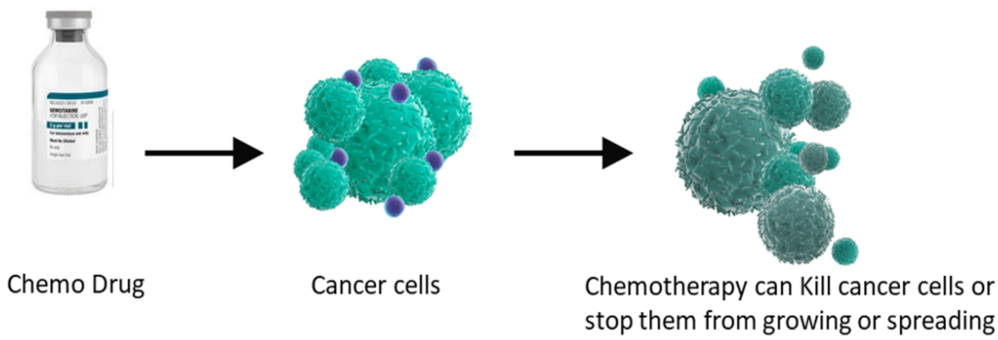
The Michaelis–Menten term $\frac{p_1\, IC}{\beta_1+C}$ signifies the existence of tumor cells that provoke the immune system's response. p_2IC indicates the immune cells' decay rate due to tumor cells. d_1I denotes the effector cells' death rate. d_2IH designates the decay rate of effector cells due to chemo-drug. In the second equation, the $(m_1C(1-k_1C))$ represents the tumor growth term. The term $\frac{p_3IC}{\beta_2+C}$ stands for the eradication of cancerous cells by the body's immune system. γ_1CN indicates the tumor cells' decay rate due to effector cells. d_3HC designates the decay rate of cancer cells due to chemo-drug. In the third equation, $m_2N(1-k_2N)$ denotes the normal cells' growth. γ_2CN represents the rate of disintegration of normal cells caused by the presence of tumor cells. In the last equation, ν is the infusion of chemotherapy drugs externally, and d_4H is the decay rate of the chemo-drug. All parameters were considered non-negative and visibly described in Table 1. Further, Figure 1 illustrates the schematic sketch of the (PSCINC) model.

The subsequent theorem establishes the positivity of all solutions of the (PSCINC) model in the positive orthant of R^4_+.

Theorem 1. *All of the solutions of the (PSCINC) model* $I(t), C(t), N(t)$ *and* $H(t)$ *with the initial conditions* $(I(0), C(0), N(0), H(0)) \in R^4_+$ *are positively invariant.*



(a) Before treatment



(b) After treatment

Figure 1. Schematic diagram of the (PSCINC) model.

Table 1. Description of (PSCINC) system's parameters.

Parameters	Denotation	Values	Source
α	A constant rate of immune cells	0.05	[47]
e	Psychological scare rate from cancer	0.1	Estimated
p_1	Maximum immune cell recruitment by tumor cells	0.1	[53]
β_1	Half-life of effector cells	0.4	[53]
p_2	Efficient elimination rate of malignant cells from effector cells	0.2	[47]
d_1	Effector cells' death rate	0.2	[53]
d_2	Decay rate of effector cells due to chemo-drug	0.09	[53]
m_1	Tumor's intrinsic growth rate	0.4	[53]
k_1	Tumor cells' carrying capacity	1.5	[53]
p_3	Maximum rate of killing the tumor cells by effector cells	0.3	[47]
β_2	Half-life of cancer cells.	0.4	[53]
γ_1	Tumor cell decay rate due to normal cells	0.2	[53]
d_3	Decay rate of cancer cells due to chemo-drug	0.05	[53]
m_2	Normal cell's intrinsic growth rate	0.35	[53]
k_2	Normal cells' carrying capacity	1	[53]
γ_2	Normal cell decay rate due to tumor cells	0.25	[53]
ν	Infusion rate of chemotherapy drugs	0.019	[53]
d_4	Decay rate of the chemo-drug	0.05	[53]

Proof. By integrating the second and third functions of the (PSCINC) model for $C(t)$ and $N(t)$ with a positive initial condition $(I(0), C(0), N(0), H(0))$, we obtain

$$C(t) = C(0)\exp\Big\{\int_0^t \Big[m_1 - m_1 k_1 C(s) - \frac{p_3 I(s)}{\beta_2 + C(s)} - \gamma_1 N(s) - d_3 H(s)\Big] ds\Big\} = Q_C > 0$$

$$N(t) = N(0)\exp\Big\{\int_0^t \Big[m_2 - m_2 k_2 N(s) - \gamma_2 C(s)\Big] ds\Big\} = Q_N > 0$$

From the first equation of the (PSCINC) model, we have

$$dI = \Big(\frac{\alpha}{1+eC} + \frac{p_1 IC}{\beta_1 + C} - p_2 IC - d_1 I - d_2 IH\Big) dt$$

$$dI \geq \Big[\frac{\alpha}{1+eQ_C} + I\Big(\frac{p_1 Q_C}{\beta_1 + Q_C} - p_2 Q_C - d_1 - \frac{d_2\nu}{d_4}\Big)\Big] dt$$

Therefore, after eliminating the non-negative terms, this produces 0000-0003-4022-8053

$$dI \geq \Big[I\Big(\frac{p_1 Q_C}{\beta_1 + Q_C} - p_2 Q_C - d_1 - \frac{d_2\nu}{d_4}\Big)\Big] dt$$

Consequently, by integrating the equation shown above for $I(t)$, these yields

$$I(t) \geq I(0)\exp\Big\{\int_0^t \Big[\Big(\frac{p_1 Q_C}{\beta_1 + Q_C} - p_2 Q_C - d_1 - \frac{d_2\nu}{d_4}\Big)\Big] ds\Big\}$$

Similarly, from the last equation of the (PSCINC) model, we get

$$dH = (\nu - d_4 H)\, dt \Longrightarrow dH \geq -d_4 H dt$$

By integrating the above equation, we get

$$H(t) \geq H(0)\, exp\Big\{\int_0^t -d_4 ds\Big\}$$

Thus, $H(t) > 0$ as $t \to \infty$.

As a result of the exponential function's definition, any solution $(I(t), C(t), N(t), H(t))$ that starts inside of R_+^4 with positive initial conditions $(I(0), C(0), N(0), H(0))$ will remain in R_+^4. □

Theorem 2. *All the solutions of the (PSCINC) model are uniformly bounded if the following condition is hold*

Proof. let $(I(0), C(0), N(0), H(0)) \in R_+^4$ be an initial condition for the (PSCINC), then, by using the Bernoulli method, we get

$$\frac{dN}{dt} = m_2 N(1 - k_2 N) - \gamma_2 CN \leq m_2 N(1 - k_2 N)$$

$$\Longrightarrow N(t) \leq \frac{1}{k_2 + N(0)e^{-m_2 t}}$$

Thus, $\lim_{t\to\infty}\sup[N(t)] \leq \frac{1}{k_2}$.

Similarly, we get

$$\lim_{t\to\infty} sup\left[C\left(t\right)\right] \leq \frac{1}{k_1},$$

Now, by using the standard comparison theory [48] and the above bound for the cancer cells, we get

$$\frac{dI}{dt} = \frac{\alpha}{1+eC} + \frac{p_1\ IC}{\beta_1 + C} - p_2 IC - d_1 I - d_2 IH$$
$$\leq \alpha - d_1 I \Longrightarrow \lim_{t\to\infty} sup\left[I\left(t\right)\right] \leq \frac{\alpha}{d_1}$$

and

$$\lim_{t\to\infty} \sup\left[H\left(t\right)\right] \leq \frac{\nu}{d_4}.$$

Therefore, the corresponding domain region for the (PSCINC) model is

$$\varphi = \Big\{ \left(I, C, N, H\right) \in R_+^4 : I\left(t\right) \leq \frac{\alpha}{d_1}, C\left(t\right) \leq \frac{1}{k_1}, N\left(t\right) \leq \frac{1}{k_2}, H(t) \leq \frac{\nu}{d_4} \Big\}.$$

□

3. Equilibria analysis

This section will delve into finding the possible equilibrium and analyzing the system's stability, specifically its stability in the vicinity of equilibrium. To accomplish this, we compute $\frac{dI}{dt} = \frac{dC}{dt} = \frac{dN}{dt} = \frac{dH}{dt} = 0$ and get the following equilibrium in two cases:

(1) No treatment case: in this case, we have two equilibrium points given by
   (a) The cancer-free or healthy point $A_0 = (I_0, 0, N_0)$, where $I_0 = \frac{\alpha}{d_1}$ and $N_0 = \frac{1}{k_2}$.
   (b) The endemic or treatment-free equilibrium point $A_1 = (I_1,\ C_1,\ N_1)$ here $N_1 = \frac{m_2 - \gamma_2 C_1}{m_2 k_2}$, $I_1 = \frac{-\alpha(\beta_1 + C_1)}{r_1 C_1 + r_2 C_1^2 - r_3 C_1^3 - r_4}$ where

$$r_1 = p_1 - p_2\beta_1 - d_1 - ed_1\beta_1,$$
$$r_2 = ep_1 - p_2 - e\beta_1 p_2 - ed_1,$$
$$r_3 = ep_2,$$
$$r_4 = d_1\beta_1,$$
$$r_5 = m_1 k_1 - \frac{\gamma_1\gamma_2}{m_2 k_2},$$
$$r_6 = m_1 - \frac{\gamma_1}{k_2},$$

and C_1 is the root of the following equation

$$f_1\left(C\right) = a_1 C^5 + a_2 C^4 + a_3 C^3 + a_4 C^2 + a_5 C + a_6, = 0,$$

where,

$$a_1 = r_3 r_5,$$
$$a_2 = \left(r_5\left(\beta_2 r_3 - r_2\right) - r_3 r_6\right)$$
$$a_3 = -\left(r_5\left(r_1 + r_2\beta_2\right) + r_6\left(\beta_2 r_3 - r_2\right)\right).$$
$$a_4 = \left(r_5\left(r_4 + r_1\beta_2\right) + r_6\left(r_1 + \beta_2 r_2\right)\right).$$
$$a_5 = \left(\alpha p_3 - r_6\left(r_4 + r_1\beta_2\right) + \beta_2 r_4 r_5\right).$$
$$a_6 = \left(\alpha\beta_1 p_3 - \beta_2 r_4 r_6\right).$$

Clearly, $f_1\left(0\right) = \left(\alpha\beta_1 p_3 - \beta_2 r_4 r_6\right)$, and

$$\begin{aligned} f_1\left(k_1\right) = {} & r_3 r_5 k_1^5 + \left(r_5\left(\beta_2 r_3 - r_2\right) - r_3 r_6\right) k_1^4 \\ & - \left(\ r_5\left(r_1 + r_2\beta_2\right) + r_6\left(\beta_2 r_3 - r_2\right)\right) k_1^3 \\ & + \Big(\ r_5\Big(r_4 - r_1\beta_2\Big) + r_6\left(r_1 + \beta_2 r_2\right)\Big) k_1^2 \\ & + \left(\alpha p_3 - r_6\left(r_4 - r_1\beta_2\right) + \beta_2 r_4 r_5\right) k_1 \\ & + \alpha\beta_1 p_3 - \beta_2 r_4 r_6. \end{aligned}$$

Therefore, by the intermediate value theorem [55], $f_1\left(C\right)$ has a positive root, say C_1 in the interval $(0,\ k_1)$ if one of the following conditions is satisfied

$$f_1\left(0\right)\ <\ 0\ and\ f_1\left(k_1\right) > 0,$$
$$f_1\left(0\right)\ >\ 0\ and\ f_1\left(k_1\right) < 0.$$

Now, for I_1 and N_1 to be positive, the following two conditions must be satisfied:

$$\begin{aligned} & m_2 > \gamma_2\ C_1 \\ & r_1 C_1 + r_2 C_1^2 < r_3 C_1^3 + r_4 \end{aligned} \qquad (2)$$

(2) After treatment case: in this case, we have one positive equilibrium point $A_2 = (I_2, C_2,\ N_2, H_2)$ here

$$H_2 = \frac{\nu}{d_4}, N_2 = \frac{m_2 - \gamma_2 C_2}{m_2 k_2}, I_2 = \frac{-\alpha(\beta_1 + C_2)}{-z_0 C_2^3 - z_1 C_2^2 + z_2 C_2 - z_3}$$

where

$$z_0 = ep_2, z_1 = p_2 - ep_1 + e\beta_1 p_2 + ed_1 + \frac{e\nu d_2}{d_4},$$
$$z_2 = p_1 - p_2\beta_1 - d_1 - ed_1\beta_1 - \frac{\nu d_2}{d_4} - \frac{e\nu d_2\beta_1}{d_4},$$
$$z_3 = d_1\beta_1 + \frac{\nu d_2\beta_1}{d_4},$$

$$z_4 = m_1 k_1 - \frac{\gamma_1 \gamma_2}{m_2 k_2},$$
$$z_5 = \frac{\gamma_1}{k_2} - m_1 + \frac{\nu d_3}{d_4},$$

and C_2 is the root of the following equation

$$f_2(C) = b_1 C^5 + b_2 C^4 + b_3 C^3 + b_4 C^2 + b_5 C + b_6 = 0,$$

where

$$b_1 = z_0 z_4,$$
$$b_2 = \left(z_4 (z_1 + z_0 \beta_2) + z_0 z_5 \right).$$
$$b_3 = \left(z_4 (z_1 \beta_2 - z_2) + z_5 (z_1 + z_0 \beta_2) \right).$$
$$b_4 = \left(z_4 (z_3 - \beta_2 z_2) + z_5 (z_1 \beta_2 - z_2) \right).$$
$$b_5 = \left(\beta_2 z_3 z_4 + z_5 (z_3 - \beta_2 z_2) + \alpha p_3 \right).$$
$$b_6 = \beta_2 z_3 z_5 + \alpha \beta_1 p_3.$$

Clearly,

$$f_2(0) = \beta_2 z_3 z_5 + \alpha p_3 \beta_1$$

and

$$\begin{aligned} f_2(k_1) = {} & z_0 z_4 k_1^5 \\ & + \left(z_4 (z_1 + z_0 \beta_2) + z_0 z_5 \right) k_1^4 \\ & + \left(z_4 (z_1 \beta_2 - z_2) + z_5 (z_1 + z_0 \beta_2) \right) k_1^3 \\ & + \left(z_4 (z_3 - \beta_2 z_2) + z_5 (z_1 \beta_2 - z_2) \right) k_1^2 \\ & + \left(\beta_2 z_3 z_4 + z_5 (z_3 - \beta_2 z_2) + \alpha p_3 \right) k_1 \\ & + \beta_2 z_3 z_5 + \alpha \beta_1 p_3. \end{aligned}$$

Therefore, by the intermediate value theorem, $f_2(C)$ has a positive root, say C_2 in the interval $(0, k_1)$ if one of the following conditions is satisfied

$$f_2(0) < 0 \text{ and } f_2(k_1) > 0,$$
$$f_2(0) > 0 \text{ and } f_2(k_1) < 0.$$

For I_2 and N_2 to be positive, the following two conditions must be satisfied:

$$\begin{aligned} & m_2 > \gamma_2 C_2 \\ & z_2 C_2 < z_0 C_2^3 + z_1 C_2^2 + z_3 \end{aligned} \tag{3}$$

Since $N = 0$ indicates that the patients are deceased, we exclude cases where $N = 0$ from consideration. In order to analyze the linear stability of the system at the three equilibrium points mentioned above, it is necessary to calculate the Jacobian matrix of the system, and the Jacobian is

$$J = \begin{bmatrix} j_{11} & j_{12} & 0 & j_{14} \\ j_{21} & j_{22} & j_{23} & j_{24} \\ 0 & j_{32} & j_{33} & 0 \\ 0 & 0 & 0 & j_{44} \end{bmatrix} \tag{4}$$

here.

$$j_{11} = \frac{p_1 C}{\beta_1 + C} - p_2 C - d_1 - d_2 H,$$
$$j_{12} = \frac{-e\alpha}{(1 + eC)^2} + \frac{p_1 \beta_1 I}{(\beta_1 + C)^2} - p_2 I,$$
$$j_{14} = -d_2 I,$$
$$j_{21} = \frac{-p_3 C}{\beta_2 + C},$$
$$j_{22} = m_1 (1 - 2k_1 C) - \frac{p_3 \beta_2 I}{(\beta_2 + C)^2} - \gamma_1 N - d_3 H,$$
$$j_{23} = -\gamma_1 C, j_{24} = d_3 C,$$
$$j_{32} = -\gamma_2 N, j_{33} = m_2 - 2m_2 k_2 N - \gamma_2 C,$$
$$j_{44} = -d_4.$$

- The Jacobian matrix at $A_0 = (I_0, 0, N_0)$ is given as:

$$J(A_0) = \begin{bmatrix} -d_1 & -e\alpha - \frac{p_1 \alpha}{\beta_1 d_1} - \frac{p_2 \alpha}{d_1} & 0 \\ 0 & m_1 - \frac{p_3 \alpha}{\beta_2 d_1} - \frac{\gamma_1}{k_2} & 0 \\ 0 & -\frac{\gamma_2}{k_2} & -m_2 \end{bmatrix} \tag{5}$$

Then, the eigenvalues of $J(A_0)$ are $\lambda_1^0 = -d_1 < 0$, $\lambda_2^0 = m_1 - \frac{p_3 \alpha}{\beta_2 d_1} - \frac{\gamma_1}{k_2}$ and $\lambda_3^0 < 0$. Therefore, A_0 is asymptotic stable whenever if

$$m_1 < \frac{p_3 \alpha}{\beta_2 d_1} + \frac{\gamma_1}{k_2}$$

- The Jacobian matrix at $A_1 = (I_1, C_1, N_1)$ is given as:

$$J(A_1) = \begin{pmatrix} a_{11}^{[1]} & a_{12}^{[1]} & 0 \\ a_{21}^{[1]} & a_{22}^{[1]} & a_{23}^{[1]} \\ 0 & a_{32}^{[1]} & a_{33}^{[1]} \end{pmatrix} \tag{6}$$

where

$$a_{11}^{[1]} = \frac{p_1 C_1}{\beta_1 + C_1} - p_2 C_1 - d_1,$$
$$a_{12}^{[1]} = \frac{-e\alpha}{(1 + eC_1)^2} + \frac{p_1 \beta_1 I_1}{(\beta_1 + C_1)^2} - p_2 I_1,$$
$$a_{21}^{[1]} = \frac{-p_3 C_1}{\beta_2 + C_1},$$
$$a_{22}^{[1]} = m_1 - 2m_1 k_1 C_1 - \frac{p_3 \beta_2 I_1}{(\beta_2 + C_1)^2} - \gamma_1 N_1,$$

$$a_{23}^{[1]} = -\gamma_1 C_1,$$
$$a_{32}^{[1]} = -\gamma_2 N_1,$$
$$a_{33}^{[1]} = m_2 - 2m_2 k_2 N_1 - \gamma_2 C_1.$$

So, the eigenvalues of $J(A_2)$ are the roots of the following equation

$$\left(\lambda^3 + U_1 \lambda^2 + U_2 \lambda + U_3 \right) = 0 \tag{7}$$

where:

$$U_1 = -\left(a_{11}^{[1]} + a_{22}^{[1]} + a_{33}^{[1]}\right)$$

$$U_2 = -\Big(-a_{11}^{[1]}\left(a_{22}^{[1]} + a_{33}^{[1]}\right) - a_{22}^{[1]}a_{33}^{[1]} + a_{23}^{[1]}a_{32}^{[1]} + a_{12}^{[1]}a_{21}^{[1]}\Big)$$

$$U_3 = \left(a_{11}^{[1]}\left(a_{23}^{[1]}a_{32}^{[1]} - a_{22}^{[1]}a_{33}^{[1]}\right) + a_{12}^{[1]}a_{21}^{[1]}a_{33}^{[1]}\right)$$

$$U_1U_2 - U_3 = \Big(\left(a_{11}^{[1]} + a_{22}^{[1]} + a_{33}^{[1]}\right)\Big(-a_{11}^{[1]}\left(a_{22}^{[1]} + a_{33}^{[1]}\right) - a_{22}^{[1]}a_{33}^{[1]} + a_{23}^{[1]}a_{32}^{[1]} + a_{12}^{[1]}a_{21}^{[1]}\Big)\Big) - \left(a_{11}^{[1]}\left(a_{23}^{[1]}a_{32}^{[1]} - a_{22}^{[1]}a_{33}^{[1]}\right) + a_{12}^{[1]}a_{21}^{[1]}a_{33}^{[1]}\right)$$

Thus, according to the Routh-Hurwitz rule [56], A_1 will be asymptotically stable if $U_1 > 0, U_3 > 0$ and $U_1U_2 > U_3$.

- The Jacobian matrix at $A_2 = (I_2, C_2, N_2, H_2)$ is given as:

$$J(A_2) = \begin{bmatrix} a_{11}^{[2]} & a_{12}^{[2]} & 0 & a_{14}^{[2]} \\ a_{21}^{[2]} & a_{22}^{[2]} & a_{23}^{[2]} & a_{24}^{[2]} \\ 0 & a_{32}^{[2]} & a_{33}^{[2]} & 0 \\ 0 & 0 & 0 & a_{44}^{[2]} \end{bmatrix} \quad (8)$$

where,

$$a_{11}^{[2]} = \frac{p_1C_2}{\beta_1 + C_2} - p_2C_2 - d_1 - d_2H_2,$$

$$a_{12}^{[2]} = \frac{-e\alpha}{(1+eC_2)^2} + \frac{p_1\beta_1I_2}{(\beta_1 + C_2)^2} - p_2I_2, a_{14}^{[2]} = -d_2I_2,$$

$$a_{21}^{[2]} = \frac{-p_3C_2}{\beta_2 + C_2},$$

$$a_{22}^{[2]} = m_1 - 2m_1k_1C_2 - \frac{p_3\beta_2I_2}{(\beta_2 + C_2)^2} - \gamma_1N_2 - d_3H_2,$$

$$a_{23}^{[2]} = -\gamma_1C_2, a_{24}^{[2]} = -d_3C_2,$$

$$a_{32}^{[2]} = -\gamma_2N_2,$$

$$a_{33}^{[2]} = m_2 - 2m_2k_2N_2 - \gamma_2C_2,$$

$$a_{44}^{[2]} = -d_4.$$

So, the eigenvalues of $J(A_2)$ are the roots of the following equation

$$(-d_4 - \lambda)\left(\lambda^3 + D_1\lambda^2 + D_2\lambda + D_3\right) = 0 \quad (9)$$

where,

$$D_1 = -\left(a_{11}^{[2]} + a_{22}^{[2]} + a_{33}^{[2]}\right)$$

$$D_2 = -\Big(-a_{11}^{[2]}\left(a_{22}^{[2]} + a_{33}^{[2]}\right) - a_{22}^{[2]}a_{33}^{[2]} + a_{23}^{[2]}a_{32}^{[2]} + a_{12}^{[2]}a_{21}^{[2]}\Big)$$

$$D_3 = \left(a_{11}^{[2]}\left(a_{23}^{[2]}a_{32}^{[2]} - a_{22}^{[2]}a_{33}^{[2]}\right) + a_{12}^{[2]}a_{21}^{[2]}a_{33}^{[2]}\right)$$

$$D_1D_2 - D_3 = \Big(\left(a_{11}^{[2]} + a_{22}^{[2]} + a_{33}^{[2]}\right)\Big(-a_{11}^{[2]}\left(a_{22}^{[2]} + a_{33}^{[2]}\right) - a_{22}^{[2]}a_{33}^{[2]} + a_{23}^{[2]}a_{32}^{[2]} + a_{12}^{[2]}a_{21}^{[2]}\Big)\Big) - \left(a_{11}^{[2]}\left(a_{23}^{[2]}a_{32}^{[2]} - a_{22}^{[2]}a_{33}^{[2]}\right) + a_{12}^{[2]}a_{21}^{[2]}a_{33}^{[2]}\right).$$

Thus, according to the Routh-Hurwitz rule, A_2 will be asymptotically stable on the condition that $D_1 > 0, D_3 > 0$ and $D_1D_2 > D_3$.

4. Global stability at the cancer-free steady state

To reach a healthy state, in this section, we will examine the global stability surrounding A_0 to explore the dynamics of the (PSCINC) system at regions far from the equilibrium point A_0.

Theorem 3. *A_0 is a GAS provided the following conditions hold:*

$$\left.\begin{array}{c} m_1k_1 \geq max\left\{\frac{2}{d_1}\left(\frac{-\alpha e}{1+eC} + \frac{p_1\ I}{\beta_1+C} - p_2I\right)^2, \frac{2\gamma_2{}^2}{m_2k_2}\right\} \\ m_1 < \frac{p_3I}{\beta_2+C} + \gamma_1N \end{array}\right\} \quad (10)$$

Proof. Let's define a Lyapunov function [57] for the (PSCINC) model at A_0 as follows: $L(t) = \frac{(I-I_0)^2}{2} + C + \left(N - N_0 - N_0ln\frac{N}{N_0}\right)$, where $L(t)$ is a positive definite about A_0. Thus,

$$\frac{dL}{dt} = (I - I_0)\frac{dI}{dt} + \frac{dC}{dt} + \left(\frac{N - N_0}{N}\right)\frac{dN}{dt}$$

$$= (I - I_0)\Big(\frac{\alpha}{1+eC} + \frac{p_1\ IC}{\beta_1 + C} - p_2IC - d_1I - \alpha + d_1I_0\Big) + \left(m_1C - m_1k_1C^2 - \frac{p_3IC}{\beta_2 + C} - \gamma_1CN\right) + (N - N_0)\left(m_2\left(1 - k_2N\right) - \gamma_2C\right).$$

Therefore,

$$\frac{dL}{dt} = (I - I_0) \left(\frac{-\alpha eC}{1+eC} + \frac{p_1 \ IC}{\beta_1 + C} - p_2 IC - d_1 (I - I_0)\right) + \left(m_1 C - m_1 k_1 C^2 - \frac{p_3 IC}{\beta_2 + C} - \gamma_1 CN\right) + (N - N_0)\left(-m_2 k_2 (N - N_0) - \gamma_2 C\right).$$

i.e.,

$$\frac{dL}{dt} = C (I - I_0) \left(\frac{-\alpha e}{1+eC} + \frac{p_1 \ I}{\beta_1 + C} - p_2 I\right) - d_1 (I - I_0)^2 + \left(m_1 C - m_1 k_1 C^2 - \frac{p_3 IC}{\beta_2 + C} - \gamma_1 CN\right) - m_2 k_2 (N - N_0)^2 - \gamma_2 C (N - N_0).$$

$$\Longrightarrow \frac{dL}{dt} = -\frac{m_1 k_1}{2} C^2 + C (I - I_0) \left(\frac{-\alpha e}{1+eC} + \frac{p_1 \ I}{\beta_1 + C} - p_2 I\right) - d_1 (I - I_0)^2 - \frac{m_1 k_1}{2} C^2 - \gamma_2 C (N - N_0) - m_2 k_2 (N - N_0)^2 + C\left(m_1 - \frac{p_3 I}{\beta_2 + C} - \gamma_1 N\right)$$

$$\Longrightarrow \frac{dL}{dt} \leq -\left(\sqrt{\frac{m_1 k_1}{2}} C + \sqrt{d_1} (I - I_0)\right)^2 - \left(\sqrt{\frac{m_1 k_1}{2}} C + \sqrt{m_2 k_2} (N - N_0)\right)^2 + C\left(m_1 - \frac{p_3 I}{\beta_2 + C} - \gamma_1 N\right)$$

Therefore, ${}^{dL}/_{dt} < 0$, and hence $L(t)$ is a Lyapunov function under condition 10. □

Thus, the cancer-free steady state A_0 fulfills the requirements for local stability, rendering the point globally stable. From a biological perspective, chemotherapy refers to the process of selectively eliminating tumor cells if conditions (10) are met.

5. Local bifurcation

This section examines the local bifurcation conditions close to steady states by applying Sotomayor's rule for local bifurcation [58, 59].

Theorem 4. *For* $m_1{}^* = \frac{p_3 \alpha}{\beta_2 d_1} + \frac{\gamma_1}{k_2}$, *the (PSCINC) model, at* A_0 *has*

(1) *No saddle-node bifurcation (SNB).*
(2) *A transcritical bifurcation (TB) if*

$$\left(T^{[0]}\right)^T \left[D^2 h_{m_1} (A_0, m_1{}^*) \left(S^{[0]}, S^{[0]}\right)\right] \neq 0. \quad (11)$$

(3) *A pitchfork bifurcation (PB) if condition (11) is violated where the notation in (11) will be introduced during the proof.*

Proof. At $m_1{}^* = \frac{p_3 \alpha}{\beta_2 d_1} + \frac{\gamma_1}{k_2}$, $J(A_0)$ has a zero eigenvalue $\lambda_2^0 = 0$. Therefore, $J(A_0)$ at m_1^* becomes

$$J^* (A_0) = \begin{bmatrix} -d_1 & -e\alpha - \frac{p_1 \alpha}{\beta_1 d_1} - \frac{p_2 \alpha}{d_1} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\gamma_2}{k_2} & -m_2 \end{bmatrix}$$

Now, let $S^{[0]} = \left(s_1^{[0]}, s_2^{[0]}, s_3^{[0]}\right)^T$ and $T^{[0]} = \left(t_1^{[0]}, t_2^{[0]}, t_3^{[0]}\right)^T$ represent the eigenvectors corresponding to the zero eigenvalue of $J^*(A_0)$ and $J^{*T}(A_0)$ respectively. Direct computation gives

$$S^{[0]} = \left(\frac{-\left(\beta_1 (ed_1 + p_2) + p_1\right) \ \alpha}{d_1^2 \beta_1}, 1, \frac{-\gamma_2}{m_2 k_2}\right)^T$$

and

$$T^{[0]} = (\ 0, 1, 0)^T .$$

Now, let $h = (h_1(I,C), h_2(I,C,N), h_3(C,N))^T$, then differentiating h with respect to m_1 gives:

$$\frac{\partial h}{\partial m_1} = \left(\frac{\partial h_1}{\partial m_1}, \frac{\partial h_2}{\partial m_1}, \frac{\partial h_3}{\partial m_1}\right) = (0, C(1 - k_1 C, 0),$$

$$h_{m_1} (A_0, \ m_1{}^*) = (0,0,0).$$

Hence,

$$T^{[0]^T} h_{m_1} (A_0, m_1{}^*) = (0,1,0)\,(0,0,0)^T = 0$$

That means the (SNB) cannot happen at $m_1{}^*$. Subsequently, since

$$T^{[0]^T} h_{m_1} (A_0, m_1{}^*) = 0$$

$$T^{[0]^T} \left[D h_{m_1} (A_0, m_1{}^*)\, S^{[0]}\right] = (0,1,0) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{-(\beta_1 (ed_1 + p_2) + p_1) \ \alpha}{d^2 \beta_1} \\ 1 \\ \frac{-\gamma_2}{m_2 k_2} \end{pmatrix} = 1 \neq 0$$

$$T^{[0]^T}\left[D^2h_{m_1}\left(A_0,{m_1}^*\right)\left(S^{[0]},S^{[0]}\right)\right]$$
$$=(0,1,0)\left(2s_1^{[0]}\left(\frac{p_1\left(1-I_0s_1^{[0]}\right)}{\beta_1}-\left(p_2+2e^2\alpha s_1^{[0]}\right)\right),\frac{p_3\left(1+\left(2I_0-\beta_2\right)\right)}{\beta_2^2}\right.$$
$$\left.-2\left(m_1^*k_1-\gamma_1s_3^{[0]}\right),-s_3^{[0]}\left(\gamma_2+2m_2k_2s_3^{[0]}\right)\right)^T$$
$$=\left(\frac{p_3\left(1+s_2^{[0]}\left(2\mathrm{I}_0-\beta_2\right)\right)}{\beta_2^2}-2\left({m_1}^*k_1-\gamma_1s_3^{[0]}\right)\right).$$

This means the required conditions for (TB) are satisfied under condition (11). Finally, if condition (11) is not satisfied, then.

$$\left(T^{[0]}\right)^TD^3h_{m_1}\left(A_0,m_1^*\right)\left(S^{[0]},S^{[0]},S^{[0]}\right)=\frac{2p_3\left(2\beta_2s_1^{[0]}-1-3I_0\right)}{\beta_2^3}.$$

□

Theorem 5. *For*

$$\gamma_1{}^*=\frac{-a_{11}^{[1]^2}\left(a_{22}^{[1]}+a_{33}^{[1]}\right)-2a_{22}^{[1]}a_{33}^{[1]}a_{11}^{[1]}}{C_1\left(a_{22}^{[1]}a_{32}^{[1]}+a_{32}^{[1]}a_{33}^{[1]}\right)}-\frac{-a_{22}^{[1]^2}\left(a_{11}^{[1]}+a_{33}^{[1]}\right)+\left(a_{11}^{[1]}+a_{22}^{[1]}\right)}{C_1\left(a_{22}^{[1]}a_{32}^{[1]}+a_{32}^{[1]}a_{33}^{[1]}\right)}-\frac{\left(-a_{33}^{[1]^2}+a_{12}^{[1]}a_{21}^{[1]}\right)}{C_1\left(a_{22}^{[1]}a_{32}^{[1]}+a_{32}^{[1]}a_{33}^{[1]}\right)}$$

where ${\gamma_2}^*>0$*, and the formulas of* $a_{ij}^{[2]}$ *are given in (8), the (PSCINC) model at* A_1 *has a (SNB) if*

$$\left(T^{[1]}\right)^T\left[D^2h_{\gamma_1}\left(A_1,\gamma_1^*\right)\left(S^{[1]},S^{[1]}\right)\right]\neq0 \quad (12)$$

Proof. According to $J(A_1)$, given by (6), the (PSCINC) model at A_1 has a zero eigenvalue, say $\lambda_2^2=0$, at ${\gamma_1}^*$ and the Jacobian matrix $J^*(A_1)=J(A_1,{\gamma_1}^*)$,becomes:

$$J^*\left(A_1\right)=\begin{bmatrix}\eta_{11}&\eta_{12}&0\\\eta_{21}&\eta_{22}&\eta_{23}\\0&\eta_{32}&\eta_{33}\end{bmatrix},$$

here,

$$\eta_{11}=\frac{p_1C_1}{\beta_1+C_1}-p_2C_1-d_1,$$
$$\eta_{12}=\frac{-e\alpha}{\left(1+eC_1\right)^2}+\frac{p_1\beta_1I_1}{\left(\beta_1+C_1\right)^2}-p_2I_1,$$
$$\eta_{21}=\frac{-p_3}{\beta_2+C_1},$$
$$\eta_{22}=m_1-2m_1k_1C_1-\frac{p_3\beta_2I_1}{\left(\beta_2+C_1\right)^2}-{\gamma_1}^*N_1,$$
$$\eta_{23}=-{\gamma_1}^*C_1,$$
$$\eta_{32}=-\gamma_2N_3,\ \eta_{33}=m_2-2m_2k_2N_1-\gamma_2C_1.$$

Now, let

$$S^{[1]}=\left(s_1^{[1]},s_2^{[1]},s_3^{[1]}\right)^T$$

and

$$T^{[1]}=\left(t_1^{[1]},t_2^{[1]},t_3^{[1]}\right)^T$$

represent the eigenvectors corresponding to the zero eigenvalue of $J^*(A_1)$ and $J^{*T}(A_1)$ respectively. Direct computation gives

$$S^{[1]}=\left(\frac{-\eta_{12}}{\eta_{11}},1,\frac{-\eta_{32}}{\eta_{33}}\right)^T$$

and

$$T^{[1]}=\left(\frac{-\eta_{21}}{\eta_{11}},1,\frac{-\eta_{23}}{\eta_{33}}\right)^T$$

where $\eta_{11}\neq0$ and $\eta_{33}\neq0$.
Subsequently, since

$$T^{[1]^T}h_{\gamma_1}\left(A_1,{\gamma_1}^*\right)=\left(\frac{-\eta_{21}}{\eta_{11}},1,\frac{-\eta_{23}}{\eta_{33}}\right)\left(0,-C_1N_1,0\right)^T=-C_1N_1\neq0$$

$$\left(T^{[1]}\right)^T\left[D^2h_{\gamma_1}\left(A_1,\gamma_1^*\right)\left(S^{[1]},S^{[1]}\right)\right]$$
$$=\left(\frac{-\eta_{21}}{\eta_{11}},1,\frac{-\eta_{23}}{\eta_{33}}\right)$$
$$\left(\frac{2p_1\beta_1({s_1}^{[1]}-I_1{s_2}^{[1]}){s_2}^{[1]}}{\left(\beta_1+C_1\right)^2}-2p_2{s_1}^{[1]}{s_2}^{[1]}\right.$$
$$+\frac{2e^2\alpha\left({s_2}^{[1]}\right)^2}{\left(1+eC_1\right)^3},\frac{p_3{s_2}^{[1]}({s_2}^{[1]}-{s_1}^{[1]}\beta_2)}{\left(\beta_2+C_1\right)^2}$$
$$+\frac{2p_3\beta_2I_1\left({s_2}^{[1]}\right)^2}{\left(\beta_2+C_1\right)^3}-2{s_2}^{[1]}({\gamma_1}^*+m_1k_1{s_2}^{[1]},$$
$$\left.-\left({s_2}^{[1]}\gamma_2+2m_2k_2\right)\right)^T$$
$$=\left(\left(\frac{2p_1\beta_1({s_1}^{[1]}-I_1{s_2}^{[1]}){s_2}^{[1]}}{\left(\beta_1+C_1\right)^2}-2p_2{s_1}^{[1]}{s_2}^{[1]}+\frac{2e^2\alpha\left({s_2}^{[1]}\right)^2}{\left(1+eC_1\right)^3}\right)\frac{-\eta_{21}}{\eta_{11}}\right.$$
$$+\left(\frac{p_3{s_2}^{[1]}({s_2}^{[1]}-{s_1}^{[1]}\beta_2)}{\left(\beta_2+C_1\right)^2}+\frac{2p_3\beta_2I_1\left({s_2}^{[1]}\right)^2}{\left(\beta_2+C_1\right)^3}-2{s_2}^{[1]}({\gamma_1}^*+m_1k_1{s_2}^{[1]}\right)$$
$$\left.-\left({s_2}^{[1]}\gamma_2+2m_2k_2\right)\left(\frac{-\eta_{23}}{\eta_{33}}\right)\right)$$

Hence, condition (12) guarantees that the second condition of saddle-node bifurcation is satisfied. Therefore, the (PSCINC) model has SNB at A_1 with the parameter γ_1^*. □

Theorem 6. *For*

$$\gamma_2{}^* = \frac{-a_{11}^{[2]^2}\left(a_{22}^{[2]}+a_{33}^{[2]}\right)-a_{22}^{[2]^2}\left(a_{11}^{[2]}+a_{33}^{[2]}\right)}{\left(a_{23}^{[2]}a_{33}^{[2]}+a_{22}^{[2]}a_{23}^{[2]}\right)N_2} + \frac{\left(a_{11}^{[2]}+a_{22}^{[2]}\right)\left(-a_{33}^{[2]^2}+a_{12}^{[2]}a_{21}^{[2]}\right)}{\left(a_{23}^{[2]}a_{33}^{[2]}+a_{22}^{[2]}a_{23}^{[2]}\right)N_2} - \frac{2a_{11}^{[2]}a_{22}^{[2]}a_{33}^{[2]}}{\left(a_{23}^{[2]}a_{33}^{[2]}+a_{22}^{[2]}a_{23}^{[2]}\right)N_2}$$

where $\gamma_2{}^* > 0$, and the formulas of $a_{ij}^{[2]}$ are given in (8), the (PSCINC) model at A_2 has a (SNB) if

$$\left(T^{[2]}\right)^T\left[D^2h_{\gamma_2}\left(A_2,\gamma_2^*\right)\left(S^{[2]},S^{[2]}\right)\right] \neq 0 \quad (13)$$

Proof. According to $J(A_2)$, given by (8), the (PSCINC) model at A_2 has a zero eigenvalue, say $\lambda_2^3 = 0$, at $\gamma_2{}^*$ and the Jacobian matrix $J^*(A_2) = J(A_2, \gamma_2{}^*)$, becomes:

$$J^*(A_2) = \begin{bmatrix} \varsigma_{11} & \varsigma_{12} & 0 & \varsigma_{14} \\ \varsigma_{21} & \varsigma_{22} & \varsigma_{23} & \varsigma_{24} \\ 0 & \varsigma_{32} & \varsigma_{33} & 0 \\ 0 & 0 & 0 & \varsigma_{44} \end{bmatrix}$$

$$\begin{aligned}
\varsigma_{11} &= \frac{p_1C_2}{\beta_1+C_2} - p_2C_2 - d_1 - d_2H_2,\\
\varsigma_{12} &= \frac{-e\alpha}{(1+eC_2)^2} + \frac{p_1\beta_1I_2}{(\beta_1+C_2)^2} - p_2I_2,\\
\varsigma_{13} &= 0,\\
\varsigma_{14} &= -d_2I_4,\\
\varsigma_{21} &= \frac{-p_3}{\beta_2+C_2},\\
\varsigma_{22} &= m_1 - 2m_1k_1C_2 - \frac{p_3\beta_2I_2}{(\beta_2+C_2)^2}\\
&\quad - \gamma_1N_2 - d_3H_2,\\
\varsigma_{23} &= -\gamma_1C_2,\\
\varsigma_{24} &= d_3C_2,\\
\varsigma_{32} &= -\gamma_2{}^*N_2,\\
\varsigma_{33} &= m_2 - 2m_2k_2N_2 - \gamma_2{}^*C_2,\\
\varsigma_{44} &= -d_4.
\end{aligned}$$

Now, let

$$S^{[2]} = \left(s_1^{[2]}, s_2^{[2]}, s_3^{[2]}, s_4^{[2]}\right)^T$$

and

$$T^{[2]} = \left(t_1^{[2]}, t_2^{[2]}, t_3^{[2]}, t_4^{[2]}\right)^T$$

represent the eigenvectors corresponding to the zero eigenvalue of $J^*(A_2)$ and $J^{*T}(A_2)$ respectively. Direct computation gives

$$S^{[2]} = \left(\frac{\varsigma_{22}\varsigma_{33}-\varsigma_{23}\varsigma_{32}}{\varsigma_{21}\varsigma_{32}}, \frac{-\varsigma_{33}}{\varsigma_{32}}, 1, 0\right)^T$$

and

$$T^{[2]} = \left(\frac{\varsigma_{22}\varsigma_{33}-\varsigma_{23}\varsigma_{32}}{\varsigma_{12}\varsigma_{23}}, \frac{-\varsigma_{33}}{\varsigma_{23}}, 1, \left[\frac{\varsigma_14(\varsigma_{23}\varsigma_{32}-\varsigma_{22}\varsigma_{33})+\varsigma_{12}\varsigma_{33}\varsigma_{24}}{\varsigma_{12}\varsigma_{23}\varsigma_{44}}\right]\right)^T$$

where $\varsigma_{12} \neq 0$.

$$T^{[2]^T}h_{\gamma_2}(A_2,\gamma_2{}^*) = \left(\frac{\varsigma_{22}\varsigma_{33}-\varsigma_{23}\varsigma_{32}}{\varsigma_{12}\varsigma_{23}}, \frac{-\varsigma_{33}}{\varsigma_{23}}, 1, \left[\frac{\varsigma_14(\varsigma_{23}\varsigma_{32}-\varsigma_{22}\varsigma_{33})+\varsigma_{12}\varsigma_{33}\varsigma_{24}}{\varsigma_{12}\varsigma_{23}\varsigma_{44}}\right]\right)^T \left((0,0,-C_2N_2,0)^T\right)^T = -C_2N_2 \neq 0.$$

$$\begin{aligned}
&\left(T^{[2]}\right)^T\left[D^2h_{\gamma_2}\left(A_2,\gamma_2^*\right)\left(S^{[2]},S^{[2]}\right)\right]\\
&= \left(\frac{\varsigma_{22}\varsigma_{33}-\varsigma_{23}\varsigma_{32}}{\varsigma_{12}\varsigma_{23}}, \frac{-\varsigma_{33}}{\varsigma_{23}}, 1, \left[\frac{\varsigma_{14}(\varsigma_{23}\varsigma_{32}-\varsigma_{22}\varsigma_{33})+\varsigma_{12}\varsigma_{33}\varsigma_{24}}{\varsigma_{12}\varsigma_{23}\varsigma_{44}}\right]\right)\\
&\left(\frac{2p_1\beta_1(s_1{}^{[2]}-I_2s_2{}^{[2]})s_2{}^{[2]}}{(\beta_1+C_2)^2} - 2p_2s_1{}^{[2]}s_2{}^{[2]} + \frac{2e^2\alpha\left(s_2{}^{[2]}\right)^2}{(1+eC_2)^3},\right.\\
&\frac{p_3s_2{}^{[2]}(s_2{}^{[2]}-s_1{}^{[2]}\beta_2)}{(\beta_2+C_2)^2} + \frac{2p_3\beta_2I_2\left(s_2{}^{[2]}\right)^2}{(\beta_2+C_2)^3} - 2s_2{}^{[2]}(\gamma_1+m_1k_1s_2{}^{[2]},\\
&\left. -\left(s_2{}^{[2]}\gamma_2{}^*+2m_2k_2\right), 0\right)^T
\end{aligned}$$

□

$$= \Bigg(\Bigg(\frac{2p_1\beta_1({s_1}^{[2]} - I_2{s_2}^{[2]}){s_2}^{[2]}}{(\beta_1 + C_2)^2} - 2p_2{s_1}^{[2]}{s_2}^{[2]} + \frac{2e^2\alpha\left({s_2}^{[2]}\right)^2}{(1+eC_2)^3}\Bigg)$$

$$\Bigg(\Bigg(\frac{\varsigma_{22}\varsigma_{33} - \varsigma_{23}\varsigma_{32}}{\varsigma_{12}\varsigma_{23}}\Bigg) + \frac{p_3{s_2}^{[2]}({s_2}^{[2]} - {s_1}^{[2]}\beta_2)}{(\beta_2 + C_2)^2} + \frac{2p_3\beta_2 I_2\left({s_2}^{[2]}\right)^2}{(\beta_2 + C_2)^3} - 2{s_2}^{[2]}(\gamma_1 + m_1k_1{s_2}^{[2]}\Bigg)$$

$$\left(\frac{-\varsigma_{33}}{\varsigma_{23}}\right) - \left({s_2}^{[2]}{\gamma_2}^* + 2m_2k_2\right)\Bigg).$$

Hence, condition (13) guarantees that the second condition of saddle-node bifurcation is satisfied. Therefore, the (PSCINC) model has SNB at A_2 with the parameter γ_2^*.

6. Optimal control

This section focuses on analyzing the model following the administration of chemotherapy treatment at a certain time. From a biomedical standpoint, we have included the notion of optimum control in the model. For this purpose, we should look into the problem with a control strategy that can lessen the health hazard for the patient. Therefore, we propose and analyze the optimal control problem applicable to model (PSCINC) to determine the optimal dose of chemotherapy to control the tumor. We decide on control inputs v of cellular chemotherapy, included in the fourth equation of the (PSCINC) model, to be supplied from an external source at different times.

So, let us assume that the time-dependent form of our considered model is given in (1) with the following initial conditions for the model set:

So, let us assume that the time-dependent form of our considered model is given in (1) with the following initial conditions for the (PSCINC) system set:

$$I(0) = I_0,\ C(0) = C_0, N(0) = N_0, H(0) = H_0, \quad (14)$$

The objective function, which is to be minimized, is defined as follows:

$$\Omega(\tau) = \int_0^{t_f}[I(t) + C(t) + \varepsilon_1\nu^2(t)]dt, \quad (15)$$

The constants ε_1 represent the weight factors of the respective terms. These are utilized to equalize the magnitude of the phrases. The ideal selection of control variable ν will effectively reduce tumor density and maximize immune density simultaneously, while also minimizing any unfavorable side effects within a set time frame. The initial component of the integrand function represents the overall quantity of tumor cells, the subsequent component of the integrand function represents the overall quantity of immune cells, and the last component of the integrand function indicates the efficacy of the administered medications on the organism. Here, we employ an optimum control problem to the model to minimize the administration of chemotherapeutic drugs, aiming to mitigate side effects and shorten the patient's recovery period. Here, we set up an optimal control ν^* such that

$$\Omega(\nu^*) = \min\ \{\Omega(\nu):\ \nu \in \Delta\}, \quad (16)$$

where $\Delta = \{\nu : \text{measurable},\ 0 \le \nu \le 1, t \in [0, t_f]\}$ is the admissible control set.

6.1. The existence of optimal control

In this sub-section, we analyze the existence of an optimal control of the (PSCINC) model (1). The property of super solutions $\bar{I}, \bar{C}, \bar{N}$, and $\bar{H}$ of the model (1) is that trajectories given by

$$\begin{aligned} \frac{d\bar{I}}{dt} &= \alpha - d_1\bar{I}, \\ \frac{d\bar{C}}{dt} &= m_1\bar{C} - p_3 I, \\ \frac{d\bar{N}}{dt} &= m_2\bar{N}, \\ \frac{d\bar{H}}{dt} &= \nu - d_4\bar{H}, \end{aligned} \quad (17)$$

are bounded. In vector form, we can express the above system (17) as:

$$\begin{pmatrix}\bar{I}\\ \bar{C}\\ \bar{N}\\ \bar{H}\end{pmatrix}' \le \begin{pmatrix}-d_1 & 0 & 0 & 0\\ -p_3 & m_1 & 0 & 0\\ 0 & 0 & m_2 & 0\\ 0 & 0 & 0 & -p_4\end{pmatrix}\begin{pmatrix}\bar{I}\\ \bar{C}\\ \bar{N}\\ \bar{H}\end{pmatrix} + \begin{pmatrix}\alpha\\ 0\\ 0\\ \nu\end{pmatrix}$$

Since this is a linear system with bounded coefficients and the time frame is limited, so, we can conclude that the solutions $\bar{I}$, $\bar{C}$,$\bar{N}$, and $\bar{H}$, of the above system are bounded. Using the theorem proposed by Lukes [60], we found that the admissible control class and the corresponding state equations with assumed initial conditions are non-empty. Also, by the definition of the set Δ, it is clear that the control set Δ is convex and closed. Since the state solutions are bounded, hence, the right-hand sides of the state system (1) are continuous and bounded by a sum of the bounded controls and the states.

Now, we examine the convexity of the integrand of $\Omega(\nu)$ on Δ and that it is bounded below by $\tau_1\nu^2(t) - \tau_2$ with $\tau_1, \tau_2 > 0$. Let p, q be distinct elements of Ω and $0 \leq Y \leq 1$. We have to show that $\Omega\left(p_1 Y+(1-Y) p_2, q_1 Y+(1-Y) q_2\right) \leq (1-Y) \Omega\left(p_1, q_1\right)+Y \Omega\left(p_2, q_2\right)$ where, $\Omega(\nu)=I(t)+C(t)+\varepsilon_1 \nu^2(t)$,

To establish it, we proceed as follows:

$$
\begin{aligned}
&\Omega\left(p_1 Y+(1-Y) q_1, p_2 Y+(1-Y) q_2\right) \\
&-(1-Y) \Omega\left(p_1, p_2\right)+Y \Omega\left(q_1, q_2\right) \\
&=C(t)+I(t)+\varepsilon_1\left(p_1 Y+(1-Y) q_1\right)^2 \\
&-Y\left\{C(t)+I(t)+\varepsilon_1 p_1^2\right\}-(1-Y)\{C(t)+I(t) \\
&+\varepsilon_1 q_1^2\}
\end{aligned}
$$

$$
\begin{aligned}
&=I(t)+C(t)+\varepsilon_1\left(p_1^2 Y^2+2 p_1 q_1 Y(1-Y)\right. \\
&\left.+(1-Y)^2 q_1^2\right)-Y\left\{I(t)+C(t)+\varepsilon_1 p_1^2\right\} \\
&-\left\{I(t)+C(t)+\varepsilon_1 q_1^2\right\}+Y\left\{I(t)+C(t)+\varepsilon_1 q_1^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&=\varepsilon_1 p_1^2 Y^2+2 \varepsilon_1 p_1 q_1 Y(1-Y)+\varepsilon_1(1-Y)^2 q_1^2 \\
&-\varepsilon_1 p_1^2 Y-\varepsilon_1 q_1^2+\varepsilon_1 q_1^2 Y \\
&=\varepsilon_1 p_1^2 Y^2+2 \varepsilon_1 p_1 q_1 Y-2 \varepsilon_1 p_1 q_1 Y^2 \\
&+\varepsilon_1\left(1-2 Y+Y^2\right) q_1^2-\varepsilon_1 p_1^2 Y-\varepsilon_1 q_1^2+\varepsilon_1 q_1^2 Y \\
&=\varepsilon_1 p_1^2 Y^2-\ 2 \varepsilon_1 p_1 q_1 Y^2+\ \varepsilon_1 q_1^2 Y^2 \\
&-\varepsilon_1 p_1^2 Y+2 \varepsilon_1 p_1 q_1 Y-\ \varepsilon_1 q_1^2 Y
\end{aligned}
$$

$=-\ \varepsilon_1\left(p_2-q_2\right)^2 Y(1-Y)$ [Since, $(Y-1) \leq 0$, and if $\varepsilon_1 \geq 0$],

and

$$
\begin{aligned}
I(t)+C(t)+\varepsilon_1 \nu^2(t) \geq \varepsilon_1 \nu^2(t) &\geq \tau_1 \nu^2(t) \\
&\geq \tau_1 \nu^2(t)-\tau_2 .
\end{aligned}
$$

This shows that $\tau_1 \nu^2(t)\ -\tau_2$ is a lower bound of $\Omega(\tau, \mu)$. This verifies that there exists an optimal control ν^* for which $\Omega\left(\nu^*\right)$=min $\Omega\left(\nu^*\right)=$ min $\{\Omega(\nu): \nu \in \Delta\}$ From the above analysis and conclusion, we state the following theorem.

Theorem 7. *Subject to the system (1), with initial conditions $I(0)=I_0, C(0)=C_0, N(0)=N_0, and H(0)=\nu_0$, the objective functional*

$$
\Omega(\nu)\ =\int_0^{t_f}\left[I(t)+C(t)+\varepsilon_1 \nu^2(t)\right] d t
$$

admits an optimal control ν^* such that $\Omega\left(\nu^*\right)=\min \{\Omega(\nu): \nu \in \Delta\}$, where $\Delta=(\nu) \nu$are piecewise continuous, $0 \leq \nu \leq 1, t \in [0, t_f]$.

6.2. Characterization of the optimal control

For applying the Pontryagin maximum principle [46], we introduced the four co-state variables ξ_i $(i=1,2,3,4)$. The Hamiltonian function is given by

$$
h=I+C+\varepsilon_1 \nu^2+\ \xi_1 \dot{I}+\ \xi_2 \dot{C}+\ \xi_3 \dot{N}+\xi_4 \dot{H} \quad (18)
$$

With substitution from (1) into (18), we get

$$
\begin{aligned}
h*=&\ I+C+\varepsilon_1 \nu^2 \\
&\xi_1\left(\frac{\alpha}{1+e C}+\frac{p_1\ I C}{\beta_1+C}-p_2 I C-d_1 I-d_2 I H\right) \\
&+\xi_2\left(m_1 C\left(1-k_1 C\right)-\frac{p_3 I C}{\beta_2+C}-\gamma_1 C N-d_3 H C\right) \\
&+\xi_3\left(m_2 N\left(1-k_2 N\right)-\gamma_2 C N\ \right)+\xi_4\left(\nu-d_4 H\right),
\end{aligned}
$$

The Hamiltonian equations are:

$$
\dot{\xi}_1=-\frac{\partial h*}{\partial I}, \dot{\xi}_2=-\frac{\partial h*}{\partial C}, \dot{\xi}_3=-\frac{\partial h*}{\partial N}, \dot{\xi}_4=-\frac{\partial h*}{\partial H}, \quad (19)
$$

where, $\xi_i(t), i=1,2,3,4$ are the adjoint functions to be determined suitably.

The form of the adjoint equations and transversality conditions are standard results from Pontryagin's Maximum Principle [61]. The adjoint system can be written in the form:

$$
\begin{aligned}
\dot{\xi}_1=&-\frac{\partial h*}{\partial I} \\
&=-1-\xi_1\left(\frac{p_1 C}{\beta_1+C}-p_2 C-d_1-d_2 H\right) \\
&+\xi_2 \frac{p_3 I C}{\beta_2+C},
\end{aligned}
$$

$$
\begin{aligned}
\dot{\xi}_2=&-\frac{\partial h*}{\partial C} \\
&=-1+\xi_1\left(\frac{-e \alpha}{(1+e C)^2}+\frac{p_1 \beta_1 I}{\left(\beta_1+C\right)^2}-p_2 I\right) \\
&-\xi_2\left(m_1-2 C m_1 k_1-\frac{p_3 \beta_2 I}{\left(\beta_2+C\right)^2}-\gamma_1 N-d_3 H\right) \\
&+\xi_3 \gamma_2 N,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\xi}_3=&-\frac{\partial h*}{\partial N} \\
&=\xi_2 \gamma_1 C-\xi_3\left(m_2-2 m_2 k_2 N-\gamma_2 C\right),
\end{aligned}
$$

$$
\dot{\xi}_4=-\frac{\partial h*}{\partial H}=\xi_1 d_2 I+\xi_2 d_3 C+d_4 \xi_4,
$$

The transversality conditions are $\xi_i\left(t_f\right)=0$, for $i=1,2,3,4$.

The condition dictate the necessary optimum control functions is

$\frac{\partial h*}{\partial \nu}=0$.

Hence, we get

$$
\nu^*(t)=-\frac{\xi_4}{2 \varepsilon_1} ; \nu=\nu^*(t) \quad (20)
$$

By using the bounds for the control $\nu^*(t)$ from (20), we get

$$\nu^* = \begin{cases} -\frac{\xi_4}{2\varepsilon_1}, & if\ 0 \leq -\frac{\xi_4}{2\varepsilon_1} \leq 1 \\ 0, & if \quad -\frac{\xi_4}{2\varepsilon_1} \leq 0 \\ 1, & if \quad \frac{\xi_4}{2\varepsilon_1} \geq 1 \end{cases}$$

In compact notation, we have

$$\nu^* = \min\left\{\max\left\{0, -\frac{\xi_4}{2\varepsilon_1}\right\}, 1\right\}, \tag{21}$$

Based on the analysis and conclusion presented above, the subsequent theorem is derived.

Theorem 8. *For optimal control ν^* and corresponding state variable solutions $I^*(t), C^*(t), N^*(t)$ and $H^*(t)$ that minimize over Δ, there exist specific adjoint variables $\xi_i(t)$, $i = 1, 2, 3, 4$ satisfying the following system:*

$$\begin{aligned}
\dot{\xi}_1 &= -1 - \xi_1\left(\frac{p_1\ C}{\beta_1 + C} - p_2C - d_1 - d_2H\right) \\
&\quad + \xi_2\frac{p_3IC}{\beta_2 + C}, \\
\dot{\xi}_2 &= -1 + \xi_1\left(\frac{-e\alpha}{(1 + eC)^2} + \frac{p_1\ \beta_1 I}{(\beta_1 + C)^2} - p_2I\right) \\
&\quad - \xi_2(m_1 - 2Cm_1k_1 - \frac{p_3\beta_2 I}{(\beta_2 + C)^2} \\
&\quad - \gamma_1 N - d_3H) \ + \xi_3\gamma_2N, \\
\dot{\xi}_3 &= \xi_2\gamma_1C - \xi_3\left(m_2 - 2m_2k_2N - \gamma_2C\right), \\
\dot{\xi}_4 &= \xi_1 d_2I + \xi_2d_3C + d_4\xi_4,
\end{aligned} \tag{22}$$

subject to the transversality conditions

$$\xi_i\left(t_f\right) = 0,\ i\ = 1, 2, 3, 4.$$

Furthermore, the subsequent properties are valid:

$$\tau^* = \min\left\{\max\left\{0, -\frac{\xi_4}{2\varepsilon_1}\right\}, 1\right\}$$

7. Numerical Analysis

Numerical verification is essential for completing analytical studies. In this section, we visually confirmed the accuracy of our analytical findings for the (PSCINC) system using the software MATLAB. This verification holds significant practical significance. The simulations were conducted using the parameter values specified below [53].

$\alpha = 0.05$, $e = 0.1, p_1 = 0.1, \beta_1 = 0.4, p_2 = 0.2,$
$d_1 = 0.2, d_2 = 0.09, m_1 = 0.4,\ k_1 = 1.5,$
$p_3 = 0.3, \beta_2 = 0.4,\ \gamma_1 = 0.2, d_3 = 0.05,$
$m_2 = 0.35, k_2 = 1; \gamma_2 = 0.25, \nu = 0.019,$
$d_4 = 0.05.$

Now, we will consider five scenarios to comprehend the dynamic behavior of the (PSCINC) model and assess the influence of chemotherapy treatment and psychological anxiety on tumor suppression. Subsequently, the outcomes of the five cases will be juxtaposed for comparison. The five cases are:

7.1. Case I: the healthy case

In this scenario, we examine the interaction dynamics between healthy cells $N(t)$ and immune cells $I(t)$ in the absence of chemotherapy treatment and psychological nervousness, i.e., where $\nu = 0$ and $e = 0$. Figure 2 depicts the (PSCINC) model with a cancer-free equilibrium point and a single positive equilibrium at $A_0 = (2, 0, 2.38, 0)$ Furthermore, regardless of the initial values, the solution initially experiences growth or decline before converging asymptotically to A_0 after approximately thirty days.

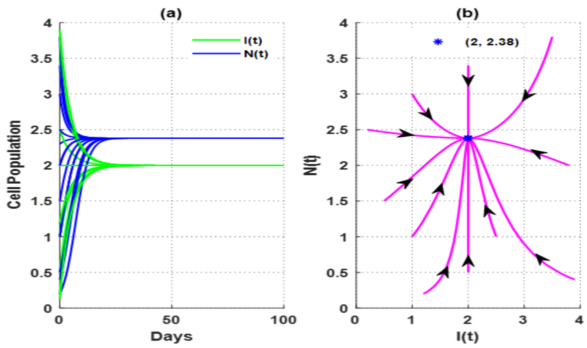


Figure 2. The dynamics of (PSCINC) model with $C = 0, \nu = 0$ and $e = 0$.

7.2. Case II: no treatment case

Here, we examine the behavior of the (PSCINC) model in the absence of treatment and the psychological scare. Figure 3 illustrates the performance of the (PSCINC) model where $\nu = 0$ and $e = 0$. All initial conditions lead to the convergence of the system to a treatment-free equilibrium point $A_1 = (I_1, C_1, N_1, 0) = (0.25, 0.13, 0.9, 0)$. In addition, the population of immune cells steadily diminishes as the number of tumor cells gradually increases. Furthermore, this case clearly demonstrates that eradicating tumor cells is unattainable without a well-defined therapeutic strategy.

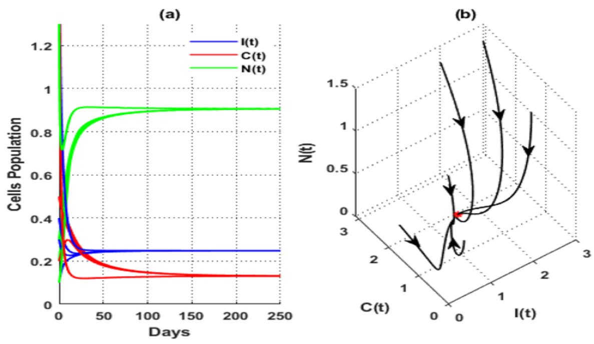


Figure 3. The dynamics of the (PSCINC) model with $\nu = 0$ and $e = 0$.

7.3. Case III: psychological scare case

The objective of this case is to demonstrate the impact of anxiety on the interaction between cancer cells and immune cells in the absence of chemotherapy drugs. Figure 4 explains the performance of the (PSCINC) model where $\nu = 0$ with various values of e. The relationship between rising anxiety and declining immune function is evident. As a result, the tumor cells significantly grow; therefore, external treatment is needed.

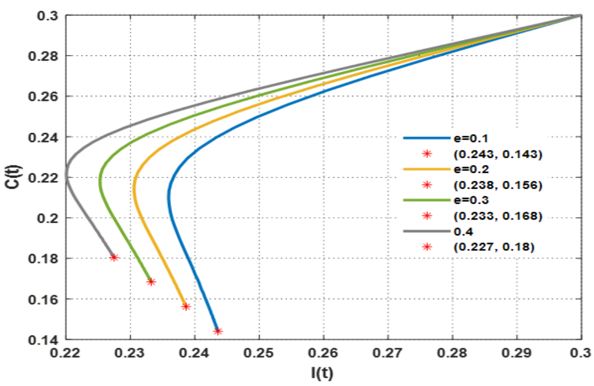


Figure 4. The dynamics of the ($PSCINC$) model with $\nu = 0$ and various value of e.

7.4. Case IV: a treatment case

In this instance, we will examine the intricacies of the (PSCINC) system when subjected to chemo-drug. Figure.5 clearly depicts the global stability characteristics of the positive steady state $A_2 = (I_2, C_2, N_2, H_2) = (0.2, 0.14, 0.89, 0.38)$. The administration of chemotherapy leads to a substantial decrease in tumor cells within the body compared to past instances. In addition, chemotherapy also adversely affects the immune cells, decreasing the quantity of immune cells compared to the previous cases. Considering those mentioned above, additional doses are necessary to achieve a state devoid of tumors.

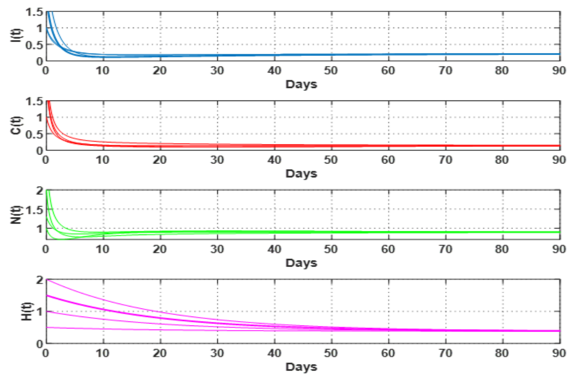


Figure 5. The dynamics of the (PSCINC) model with treatment case.

7.5. Case V: a minimum dosage of chemo-drug

This case aims to examine the effects of modifying the number of chemotherapy doses required to achieve a healthy state. Figure 5 clarifies the performance of the (PSCINC) model with various values of ν. The solution of the (PSCINC) system asymptotically converges to A_2 when v is less than 0.14. Conversely, the system tends towards a cancer-free state A_0 when $\nu = 0.14$. Thus, a value of $\nu = 0.14$ is the minimum dosage of chemotherapy necessary to achieve a condition devoid of cancer.

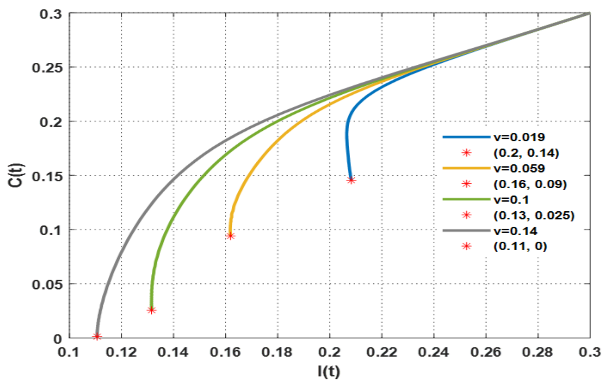


Figure 6. The dynamics of the (PSCINC) model with various values of ν

8. Conclusion

It has been looked at how an ODE mathematical model for tumor growth works, which includes how immune cells interact with tumor cells and how psychological scares and chemotherapy drugs work. The fundamental attributes of the model's solutions, including positivity and boundedness, were established. A stability analysis was conducted on the system under consideration to investigate the model's dynamic behavior. Our research indicates that the constant state devoid of tumors is stable globally under particular conditions. This suggests that the prescribed treatment can eliminate tumor cells from the body for a specific tumor growth rate.

The numerical simulations validate the analytical findings. Precisely, the threshold values for the transcritical bifurcation are calculated, indicating the point at which cancer transitions from persisting to eradicating. Additionally, numerical analysis reveals that when the tumor size is modest, the prescribed chemotherapy drug can effectively eliminate tumor cells from the body with a minimal minimum dose. Nonetheless, a constraint of our model is that prolonged treatment and a substantial dosage of medications are necessary to eradicate large tumors, both of which can be detrimental to the patient's health.

Our upcoming research will focus on augmenting the immune system by regular vitamin intake or the utilization of stem cells.

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An abstract is a concise summary of the whole paper, not just the conclusions. The abstract should be no more than 250 words and convey the following:

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Journal article

Author, A.A., & Author, B. (Year). Title of article. Title of Journal, Vol(Issue), pages.

Castles, F.G., Curtin, J.C., & Vowles, J. (2006). Public policy in Australia and New Zealand: The new global context. Australian Journal of Political Science, 41(2), 131–143.

Book

Author, A. (Year). Title of book. Publisher, Place of Publication.

Mercer, P.A., & Smith, G. (1993). Private Viewdata in the UK. 2nd ed. Longman, London.

Chapter

Author, A. (Year). Title of chapter. In: A. Editor and B. Editor, eds. Title of book. Publisher, Place of publication, pages.

Bantz, C.R. (1995). Social dimensions of software development. In: J.A. Anderson, ed. Annual review of software management and development. CA: Sage, Newbury Park, 502–510.

Internet document

Author, A. (Year). Title of document [online]. Source. Available from: URL [Accessed (date)].

Holland, M. (2004). Guide to citing Internet sources [online]. Poole, Bournemouth University. Available from: http://www.bournemouth.ac.uk/library/using/guide_to_citing_internet_sourc.html [Accessed 4 November 2004].

Newspaper article

Author, A. (or Title of Newspaper) (Year). Title of article. Title of Newspaper, day Month, page, column.

Independent (1992). Picking up the bills. Independent, 4 June, p. 28a.

Thesis

Author, A. (Year). Title of thesis. Type of thesis (degree). Name of University.

Agutter, A.J. (1995). The linguistic significance of current British slang. PhD Thesis. Edinburgh University.

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Tables and figures (illustrations) should be embedded in the text at the appropriate points, rather than at the end. A short descriptive title should appear above each table with a clear legend and any footnotes suitably identified below.

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8. Maximum five AMS Classification number (http://www.ams.org/mathscinet/msc/msc2010.html) of the study have been provided after keywords.
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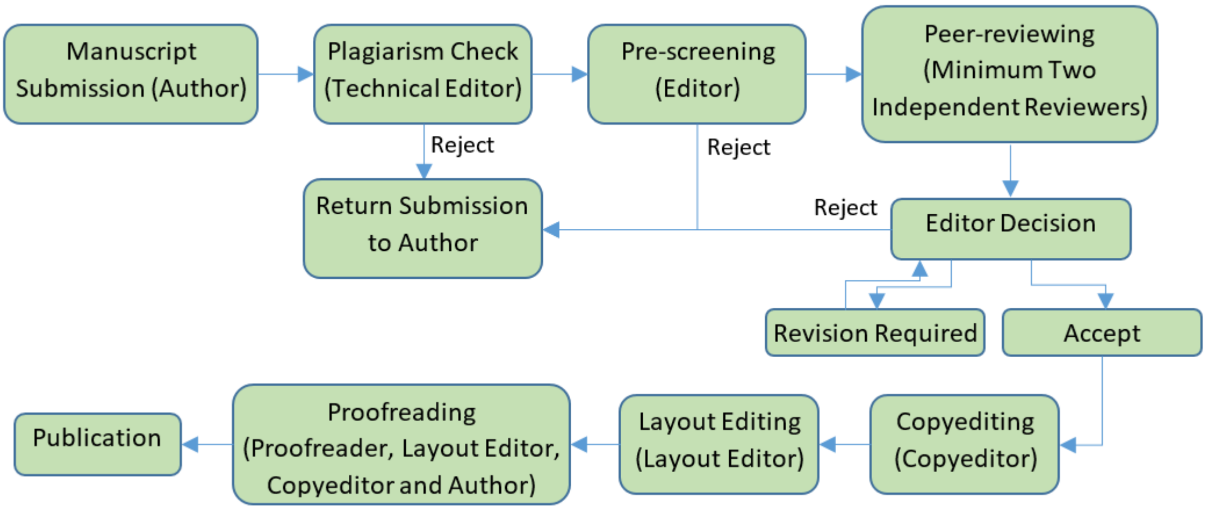

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Reference:

Kleinert S & Wager E (2011). Responsible research publication: international standards for editors. A position statement developed at the 2nd World Conference on Research Integrity, Singapore, July 22-24, 2010. Chapter 51 in: Mayer T & Steneck N (eds) Promoting Research Integrity in a Global Environment. Imperial College Press / World Scientific Publishing, Singapore (pp 317-28). (ISBN 978-981-4340-97-7) [Link].

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- Researchers should adhere to publication requirements that submitted work is original, is not plagiarised, and has not been published elsewhere.
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Reference:

Wager E & Kleinert S (2011) Responsible research publication: international standards for authors. A position statement developed at the 2nd World Conference on Research Integrity, Singapore, July 22-24, 2010. Chapter 50 in: Mayer T & Steneck N (eds) Promoting Research Integrity in a Global Environment.

Imperial College Press / World Scientific Publishing, Singapore (pp 309-16). (ISBN 978-981-4340-97-7) [Link].

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Peer review in all its forms plays an important role in ensuring the integrity of the scholarly record. The process depends to a large extent on trust and requires that everyone involved behaves responsibly and ethically. Peer reviewers play a central and critical part in the peer-review process as the peer review assists the Editors in making editorial decisions and, through the editorial communication with the author, may also assist the author in improving the manuscript.

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Reference:

Homes I (2013). COPE Ethical Guidelines for Peer Reviewers, March 2013, v1 [Link].

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