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RESEARCH ARTICLE

Certain Saigo type fractional integral inequalities and their q-analogues

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ABSTRACT

The main purpose of the present article is to introduce certain new Saigo fractional integral inequalities and their q-extensions. We also studied some special cases of these inequalities involving Riemann- Liouville and Erdélyi-Kober fractional integral operators.

1. Introduction

Fractional calculus has great importance in mathematical analysis, also in consideration of its numerous applications in modeling and in applications. Recently there have been several significant contributions to the theory of fractional operators [1].

From last decades various types of integral inequalities have attracted the attention of many mathematicians [2–6] and also fractional integral inequalities have been found many interesting applications in the fields of engineering and physics.

Very recently in 2019, Ekinci and Ozdemir [2] have studied Hermite-Hadamard type inequalities involving intermediate values of $| f' |$ by using Riemann-Liouville fractional operator and Butt et al. [3] established some new integral inequalities involving Caputo fractional derivatives for exponential s-convex functions. In this sequence Kizil and Ardiç [4] have introduced inequalities for strongly convex functions via Atangana-Baleanu integral operators. Later in 2022, Kalsoom *et.al.* [5] proposed few new inequalities of Ostrowski type by means of newly derived identity and considered some special cases. Our present work is fully motivated by the mentioned work.

Saigo fractional integral operator is one of the important operators of fractional calculus theory due to involving the Gauss hypergeometric function ${}_2F_1(.)$. This operator has already found various applications in solving problems in the theory of special functions, integral transforms and theory of inequalities.

Before setting out the main findings of our article, it is useful to briefly review the contributions on which it is based.

We say that, fixed $v \in \mathbb{R}$ a real valued function $g(x)$ defined for $x > 0$ belongs to the function space C_v, if there exists a real number $q > v$ such

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that $g(x) = x^q\Phi(x)$, where $\Phi(x) \in C(0,1)$. Moreover, for $m \in \mathbb{R}a$ we say that $g(x)$ belongs to the function space C_v^m , if $g^m \in C_v$.

For $a > 0$, Riemann-Liouville fractional integral operator of a function g such that $g \in C_v(, v \geq -1)$ defined as follows [7]:

$$R_{0,y}^a\{g(y)\} = \frac{1}{\Gamma(a)}\int_0^y (y-t)^{a-1}g(t)dt, \qquad (1)$$

here, $y > 0$.

For $a > 0$, Erdélyi-Kober fractional integral operator of a function g such that $g \in C_v(v \geq -1)$ defined as follows [8]:

$$K_{0,y}^{a,b}\{g(y)\} = \frac{y^{-a-b}}{\Gamma(a)}\int_0^y t^b(y-t)^{a-1}g(t)dt, \quad (2)$$

here, $b \in \mathbb{R}$.

For $a > 0$, Saigo fractional integral operator of a function g, such that $g \in C_v(v \geq -1)$ defined as follows [9]:

$$I_{0,y}^{a,a',b}\{g(y)\} = \frac{y^{-a-a'}}{\Gamma(a)}\times \int_0^y (y-t)^{a-1}{}_2F_1\left(a+a', -b, a, 1-\frac{t}{y}\right)g(t)dt, \qquad (3)$$

here, $a', b \in \mathbb{R}$, and ${}_2F_1(r_1, r_2, r_3; z)$ is classical Gauss hypergeometric function defined in [10].

$${}_2F_1(r_1; r_2; r_3; z) = \sum_{n=0}^{\infty}\frac{(r_1)_n(r_2)_n}{(r_3)_n}\frac{z^n}{n!}, \qquad (4)$$

where $(s)_k$ denotes the Pochhammer symbol (shifted factorial) defined as follows [10,11]:

$$(s)_k := \frac{\Gamma(s+k)}{\Gamma(s)} = \begin{cases} 1 & (k=0; s \in \mathbb{C}\setminus\{0\}) \\ s(s+1)\cdots(s+k-1) & (k \in \mathbb{N}; s \in \mathbb{C}) \end{cases} \qquad (5)$$

By observing, we note that Saigo fractional integral operators contains both Reimann-Liouville fractional integral operators as well as Erdélyi-Kober fractional integral operators.

Remark 1. *(i)Taking $a' = -a$ in the equations* (3) *then, we get Riemann-Liouville fractional integral operator* (1).

$$R_{0,y}^a\{g(y)\} = I_{0,y}^{a,-a,b}\{g(y)\} \qquad (6)$$

Remark 2. *(ii)Taking $a' = 0$ in the equations* (3) *then, we get Erdélyi-Kober fractional integral operator* (2).

$$K_{0,y}^{a,b}\{g(y)\} = I_{0,x}^{a,0,b}\{g(y\} \qquad (7)$$

Fractional integral inequalities are an important tool to prove the key result, the uniqueness of solutions of fractional partial differential equations and fractional boundary value problems. Also, they give information about the boundness of the solutions of partial differential equations and fractional boundary value problems. These features have led many researchers in the area of integral inequalities to analyze some more extensions and generalizations by involving fractional calculus integral operators.

Introduce the following functional:

$$\begin{aligned} T(k,l,m,n) = &\int_c^d n(t)dt\int_c^d m(t)k(t)l(t)dt \\ &+ \int_c^d m(t)dt\int_c^d n(t)k(t)l(t)dt \\ &-\left(\int_c^d n(t)k(t)dt\right)\left(\int_c^d m(t)l(t)dt\right) \\ &-\left(\int_c^d m(t)k(t)dt\right)\left(\int_c^d n(t)l(t)dt\right) \end{aligned} \qquad (8)$$

here $k, l : [c,d] \to \mathbb{R}$ are two integrable functions defined on the interval $[c,d]$ and $m(t)$ and $n(t)$ are positive integrable functions defined on $[c,d]$.

Consider two functions Φ and Ψ defined on $[c,d]$, then they are synchronous on $[c,d]$, if they satisfies the following inequality:

$$(\Phi(t)-\Phi(s))(\Psi(t)-\Psi(s)) \geq 0, \qquad (9)$$

for arbitrary $t, s \in$[c,d], then from [12,13], we observe that

$$T(\Phi, \Psi, m, n) \geq 0, \qquad (10)$$

If the inequality defined in (9) is reversed, then functions Φ and Ψ are called asynchronous on [c,d] and satisfies the following inequality:

$$. \quad (\Phi(t)-\Phi(s))(\Psi(t)-\Psi(s)) \leq 0, \qquad (11)$$

for any $t, s \in$[c,d].

From [14], we have the Chebyshev inequality for the special case when $m(t) = \Psi(t)$, for any t,s $\in [c,d]$, .

The functional $T(k,l,m,n)$ defined in (8) has drawn many researchers' attentions due to the wide range of applications in engineering mathematics, statistical probability, transform theory, and probability and numerical quadrature. From all these applications, the functional $T(k,l,m,n)$ (8) has also been used to produce many integral inequalities (see, e.g., [15–22]; for more details regarding very recent work, we can refer [23])

In the last few years, many researchers have more attention to the q-calculus and fractional q-differential equations due to many applications of the q-calculus in physics, statistics and mathematics. The q-calculus is also called the quantum calculus can be dated back to 1908, Jackson's work [24] and fractional q-calculus is the q-analogous of the ordinary fractional calculus. Recently, q-calculus operators have been applied in various fields like optimal control problems, ordinary fractional calculus, solutions of the q-difference (differential),q-transform analysis and q-integral equations, and many more such areas.

In 1966, Al-Salam gives the idea of fractional q-calculus by introducing the q-analogue of Cauchy's formula ([25–27]). Then, in 1969 Agrawal [28] studied some fractional q-integral operators and q-derivatives and their basic properties. Then later in 2007, Rajkovic et al. [29] extended the notion of the left fractional q-integral operators and fractional q-derivatives by introducing variable lower limit and proved the semi-group properties. In the sequence, Isogawa et al. [30] studied various basic properties of fractional q-derivatives.

For $0 <| q |< 1$ the q-shifted factorial is defined as [31]:

$$(b;q)_k = \begin{cases} 1 & (k=0) \\ \prod_{s=0}^{k-1}(1-bq^s) & (k\in\mathbb{N}), \end{cases} \tag{12}$$

here, $b, q \in \mathbb{C}$ and $b \neq q^{-l} (l \in N_0)$.

For $k \in N_0$, q-shifted factorial with negative subscript is defined as follows:

$$(b;q)_k = \frac{1}{(1-bq^{-1})(1-bq^{-2})(1-bq^{-3})...(1-bq^{-k})}. \tag{13}$$

From (12) and (13), we can conclude that:

$$(b;q)_\infty = \prod_{s=0}^{\infty}(1-bq^s), \tag{14}$$

here, $b, q \in \mathbb{C}$

By using the equations (12), (13) and (14), we observe that:

$$(b;q)_\infty = \frac{(b;q)_\infty}{(bq^k;q)_\infty}, \tag{15}$$

here, $k \in \mathbb{N}$.

Then from above equations, for any complex number β,

$$(b;q)_\beta = \frac{(b;q)_\infty}{(bq^\beta;q)_\infty}, \tag{16}$$

here, only the principal value of q^β is valid for the above equation.

For the power function $(c-d)^m$, we can define its q-analogy as follows:

$$\begin{aligned} (c-d)_q^m &= c^m\left(\frac{d}{c};q\right)_m \quad (m\in\mathbb{N}) \\ &= c^m\frac{\left(\frac{d}{c};q\right)_\infty}{\left(\frac{d}{c}q^m;q\right)_\infty} \quad (c\neq 0) \\ &= c^m\prod_{l=0}^{\infty}\left[\frac{1-\left(\frac{d}{d}\right)q^l}{1-\left(\frac{d}{c}\right)q^{l+m}}\right]. \end{aligned} \tag{17}$$

From above (17), we conclude that:

$$(c-d)_q^m = \begin{cases} 1 & (m=0) \\ (c-d)(c-dq)\cdots(c-dq^{m-1}) & (m\in\mathbb{N}). \end{cases} \tag{18}$$

In 1910, Jackson was the first researcher who introduced q-derivative and q-integral in systematic way.

The q-derivative of a function $g(x)$ is defined as [31]:

$$D_q\{g(x)\} = \frac{d_q}{d_qx}\{g(x)\} = \frac{g(qx)-g(x)}{qx-x}. \tag{19}$$

From above, we observe and notice that

$$\lim_{q\to 1} D_q\{g(x)\} = \frac{d}{dx}\{g(x)\}, \tag{20}$$

if, given function $g(x)$ is differentiable.

The q-integral of a function $g(x)$ is defined as [31]:

$$\int_0^t g(x)d_qx = t(1-q)\sum_{l=0}^{\infty}q^lg(tq^l), \tag{21}$$

$$\int_t^\infty g(x)d_qx = t(1-q)\sum_{l=0}^{\infty}q^{-l}g(tq^{-l}), \tag{22}$$

$$\int_0^\infty g(x)d_qx = t(1-q)\sum_{l=-\infty}^{\infty}q^lg(q^l). \tag{23}$$

For $0 < q < 1$, q-gamma function is given by [31]:

$$\Gamma_q(b) = \frac{(q;q)_\infty}{(q^b;q)_\infty}(1-q)^{(1-b)}. \tag{24}$$

For $b > 0$, q-analogue of Riemann-Liouville fractional integral operator of a function $g(x)$ defined as [28]:

$$R_q^b\{g(x)\} = \frac{x^{b-1}}{\Gamma_q(b)}\int_0^x \left(\frac{qt}{x};q\right)_{b-1}g(t)d_qt, \tag{25}$$

where,$(b,q)_\beta$ is given in the equation (16).

For, $a > 0$ and $b \in R$ and $0 < q < 1$, q-analogue of the Erdélyi-Kober fractional operator is defined as [28]:

$$K_q^{a,b}\{f(x)\} = \frac{x^{b-1}}{\Gamma_q(a)} \int_0^x \big(\frac{qt}{x};q\big)_{a-1} t^b f(t) d_q t. \tag{26}$$

For $a > 0$, a' and $b \in \mathbb{R}$, q-analogue of Saigo's fractional integral is defined as [32]:

$$I_q^{a,a',b}\{f(x)\} = \frac{x^{-a'-1}}{\Gamma_q(a)} \int_0^x \big(\frac{qt}{x};q\big)_{a-1} \sum_{k=0}^{\infty} \frac{(q^{a+a'};q)_k (q^{-b};q)_k}{(q^{-a};q)_k (q,q)_k} q^{(b-a')k}(-1)^k q^{-\binom{k}{2}} \big(\frac{t}{x}-1\big)_q^k f(t) d_q t \tag{27}$$

2. Certain inequalities involving Saigo type fractional integral operator

In this section, we introduce some inequalities involving the Saigo type fractional integral operator and their special cases.

Theorem 1. *Assume u and v are two positive integrable and synchronous mapping on $[0,\infty]$. Suppose $\exists$ four positive integrable mappings m_1, m_2, n_1 and n_2 such that:*

$$\begin{aligned} &0 < m_1(t) \le u(t) \le m_2(t),\\ &0 < n_1(t) \le v(t) \le n_2(t),\\ &(t \in [0,x], x > 0). \end{aligned} \tag{28}$$

Then the following inequality holds true:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1n_2u^2\}(x) \times I_{0,x}^{a,a',b}\{m_1m_2v^2\}(x)\\ &\le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{(m_1n_1+m_2n_2)uv\}(x)\Big)^2 \end{aligned} \tag{29}$$

Proof. By using the relations that are given in (28), for $t \in [0,x], \forall x > 0$, we can easily have:

$$\left(\frac{m_2(t)}{n_1(t)} - \frac{u(t)}{v(t)}\right) \ge 0 \tag{30}$$

$$\left(\frac{u(t)}{v(t)} - \frac{m_1(t)}{n_2(t)}\right) \ge 0 \tag{31}$$

If we product (30) and (31) side by side , we can write

$$\left(\frac{m_2(t)}{n_1(t)} - \frac{u(t)}{v(t)}\right)\left(\frac{u(t)}{v(t)} - \frac{m_1(t)}{n_2(t)}\right) \ge 0$$

Then we have:

$$\begin{aligned} &\big(m_1(t)n_1(t) + m_2(t)n_2(t)\big)u(t)v(t)\\ &\ge n_1(t)n_2(t)u^2(t) + m_1(t)m_2(t)v^2(t). \end{aligned} \tag{32}$$

Consider the following function $F(x,t)$ defined by:

$$\begin{aligned} F(x,t) =& \frac{x^{-a-a'}(x-t)^{a-1}}{\Gamma(a)}\\ &\times {}_2F_1\left(a+a', -b, a, 1-\frac{t}{x}\right),\\ &(t \in (0,x); x > 0). \end{aligned} \tag{33}$$

Then multiplying both sides of (32), by $F(x,t)$ defined by (33) and integrating the resulting inequality with respect to t from 0 to x and using the definition (3), we have:

.

$$\begin{aligned} &I_{0,x}^{a,a',b}\{(m_1n_1+m_2n_2)uv\}(x)\\ &\ge I_{0,x}^{a,a',b}\{(n_1n_2)u^2\}(x) + I_{0,x}^{a,a',b}\{(m_1m_2)v^2\}(x) \end{aligned} \tag{34}$$

Let us recall the A.M -G.M inequality, i.e $(a+b) \ge 2\sqrt{ab}$, $a,b \in \mathbb{R}^+$. By applying this classical inequality to (34), we obtain:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{(m_1n_1+m_2n_2)uv\}(x)\\ &\ge 2\sqrt{I_{0,x}^{a,a',b}\{(n_1n_2)u^2\}(x) \times I_{0,x}^{a,a',b}\{(m_1m_2)v^2\}(x)} \end{aligned} \tag{35}$$

By making use of some necessary operations, we deduce that:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1n_2u^2\}(x) \times I_{0,x}^{a,a',b}\{m_1m_2v^2\}(x)\\ &\le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{(m_1n_1+m_2n_2)uv\}(x)\Big)^2 \end{aligned} \tag{36}$$

This complete the proof of Theorem 1. □

If we substitute $a' = -a$ and $a' = 0$ in above results we get following special cases of the inequalities respectively.

Corollary 1. *For Riemann-Liouville fractional integral operator the following inequality holds true:*

$$\begin{aligned} &R_{0,x}^{a}\{n_1n_2u^2\}(x) \times R_{0,x}^{a}\{m_1m_2v^2\}(x)\\ &\le \frac{1}{4}\Big(R_{0,x}^{a}\{(m_1n_1+m_2n_2)uv\}(x)\Big)^2 \end{aligned} \tag{37}$$

Corollary 2. *For Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$\begin{aligned} &K_{0,x}^{a,b}\{n_1n_2u^2\}(x) \times K_{0,x}^{a,b}\{m_1m_2v^2\}(x)\\ &\le \frac{1}{4}\Big(K_{0,x}^{a,b}\{(m_1n_1+m_2n_2)uv\}(x)\Big)^2. \end{aligned} \tag{38}$$

Theorem 2. *Consider u and v are two positive integrable and synchronous mapping on $[0,\infty]$. Assume $\exists$ four positive integrable mapping m_1, m_2, n_1 and n_2 such that:*

$$\begin{aligned} &0 < m_1(t) \le u(t) \le m_2(t),\\ &0 < n_1(t) \le v(t) \le n_2(t),\\ &(t \in [0,x], x > 0). \end{aligned} \tag{39}$$

Then the following inequality holds true:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1 n_2\}(x) I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\quad \times I_{0,x}^{a,a',b}\{m_1 m_2\}(x) I_{0,x}^{a,a',b}\{v^2\}(x)\\ &\quad \le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{n_1 v\}(x) I_{0,x}^{a,a',b}\{m_1 u\}(x)\\ &\quad + I_{0,x}^{a,a',b}\{n_2 v\}(x) I_{0,x}^{a,a',b}\{m_2 u\}(x)\Big)^2. \end{aligned} \tag{40}$$

Proof. With similar steps to the proof of the previous Theorem, if we consider the inequalities are given in (39), we have

$$\left(\frac{m_2(t)}{n_1(s)} - \frac{u(t)}{v(s)}\right) \ge 0, \tag{41}$$

$$\left(\frac{u(t)}{v(s)} - \frac{m_1(t)}{n_2(s)}\right) \ge 0. \tag{42}$$

Then we can write above inequality as the following form:

$$\left(\frac{m_1(t)}{n_2(s)} + \frac{m_2(t)}{n_1(s)}\right)\frac{u(t)}{v(s)} \ge \frac{u^2(t)}{v^2(s)} + \frac{m_1(t)m_2(t)}{n_1(s)n_2(s)}. \tag{43}$$

If we multiply both sides of (43), by $n_1(s)n_2(s)v^2(s)$, we get

$$\begin{aligned} &m_1(t)u(t)n_1(s)v(s) + m_2(t)u(t)n_2(s)v(s)\\ &\quad \ge n_1(s)n_2(s)u^2(t) + m_1(t)m_2(t)v^2(s). \end{aligned} \tag{44}$$

Then on multiplying both sides of the equation (44), by $F(x,t)$ defined in (33) and integrating with respect to t from 0 to x, and using the definition (3), we have

$$\begin{aligned} &n_1(s)v(s) I_{0,x}^{a,a',b}\{m_1 u\}(x)\\ &\quad + n_2(s)v(s) I_{0,x}^{a,a',b}\{m_2 u\}(x)\\ &\quad \ge n_1(s)n_2(s) I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\quad\quad + v^2(s) I_{0,x}^{a,a',b}\{m_1 m_2\}(x). \end{aligned} \tag{45}$$

Again multiplying both sides of the equation (45), by $F(x,s)$ defined in (33), a nd integrating with respect to s from 0 to x and using the definition (3), we have

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1 v\}(x) I_{0,x}^{a,a',b}\{m_1 u\}(x)\\ &+ I_{0,x}^{a,a',b}\{n_2 v\}(x) I_{0,x}^{a,a',b}\{m_2 u\}(x)\\ &\ge I_{0,x}^{a,a',b}\{n_1 n_2\}(x) I_{0,x}^{a,a',b}\{u^2\}(x)\\ &+ I_{0,x}^{a,a',b}\{v^2\}(x) I_{0,x}^{a,a',b}\{m_1 m_2\}(x). \end{aligned} \tag{46}$$

Now, using the AM-GM inequality, we have:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1 v\}(x) I_{0,x}^{a,a',b}\{m_1 u\}(x)\\ &+ I_{0,x}^{a,a',b}\{n_2 v\}(x) I_{0,x}^{a,a',b}\{m_2 u\}(x)\\ &\quad \ge 2\Big\{I_{0,x}^{a,a',b}\{n_1 n_2\}(x) I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\quad\quad \times I_{0,x}^{a,a',b}\{v^2\}(x) I_{0,x}^{a,a',b}\{m_1 m_2\}(x)\Big\}^{\frac{1}{2}}. \end{aligned} \tag{47}$$

By making use of some necessary operations, we deduce that:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1 n_2\}(x) I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\times I_{0,x}^{a,a',b}\{m_1 m_2\}(x) I_{0,x}^{a,a',b}\{v^2\}(x)\\ &\le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{n_1 v\}(x) I_{0,x}^{a,a',b}\{m_1 u\}(x)\\ &+ I_{0,x}^{a,a',b}\{n_2 v\}(x) I_{0,x}^{a,a',b}\{m_2 u\}(x)\Big)^2. \end{aligned} \tag{48}$$

This proofs the Theorem (2). ☐

On putting $a' = -a$ and $a' = 0$ in above results we get following special cases of the inequalities respectively.

Corollary 3. *For Riemann-Liouville fractional integral operator the following inequality holds true:*

$$\begin{aligned} &R_{0,x}^{a}\{n_1 n_2\}(x) R_{0,x}^{a}\{u^2\}(x)\\ &\times R_{0,x}^{a}\{m_1 m_2\}(x) R_{0,x}^{a}\{v^2\}(x)\\ &\le \frac{1}{4}\Big(R_{0,x}^{a}\{n_1 v\}(x) R_{0,x}^{a}\{m_1 u\}(x)\\ &+ R_{0,x}^{a}\{n_2 v\}(x) R_{0,x}^{a}\{m_2 u\}(x)\Big)^2 \end{aligned} \tag{49}$$

Corollary 4. *For Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$
\begin{aligned}
&K^{a,b}_{0,x}\{n_1n_2\}(x)K^{a,b}_{0,x}\{u^2\}(x)\\
&\times K^{a,b}_{0,x}\{m_1m_2\}(x)K^{a,b}_{0,x}\{v^2\}(x)\\
&\leq \frac{1}{4}\Bigg(K^{a,b}_{0,x}\{n_1v\}(x)K^{a,b}_{0,x}\{m_1u\}(x)\\
&+K^{a,b}_{0,x}\{n_2v\}(x)K^{a,b}_{0,x}\{m_2u\}(x)\Bigg)^2
\end{aligned}
\tag{50}
$$

3. Saigo type fractional q-integral inequalities

Here, we established some q-integral inequalities involving q-Saigo type fractional integral operator which are the q-analogues of the Theorems proved in the previous section.

Theorem 3. *Consider $0 < q < 1$, let u and v are two positive integrable and synchronous mapping on $[0, \infty]$. Assume $\exists$ four positive integrable mapping m_1, m_2, n_1 and n_2 such that:*

$$
\begin{aligned}
&0 < m_1(t) \leq u(t) \leq m_2(t),\\
&0 < n_1(t) \leq v(t) \leq n_2(t),\\
&(t \in [0, x], x > 0).
\end{aligned}
\tag{51}
$$

Then the following inequality holds true:

$$
\begin{aligned}
&I^{a,a',b}_q\{(m_1n_1 + m_2n_2)uv\}(x)\\
&\geq I^{a,a',b}_q\{(n_1n_2)u^2\}(x)\\
&+I^{a,a',b}_q\{(m_1m_2)v^2\}(x).
\end{aligned}
\tag{52}
$$

Proof. To prove our result we need to recall function with their conditions defined by Choi [33],

$$
\begin{aligned}
H(t, x, u(x); a, a', b; q) = &\frac{x^{-a'-1}}{\Gamma_q(a)}\big(\frac{qt}{x}, q\big)_{a-1}\\
&\sum_{k=0}^{\infty}\frac{(q^{a+a'}, q)_k(q^{-b}, q)_k}{(q^{-a}, q)_k(q, q)_k}\\
&\times q^{(b-a')k}(-1)^k q^{-\binom{k}{2}}\big(\frac{t}{x}-1\big)^k_q u(t)
\end{aligned}
\tag{53}
$$

where $x > 0, 0 \leq t \leq x$; $a > 0$, $a', b \in \mathbb{R}$ with $a + a' > 0$ and $b < 0$, $0 < q < 1$, $u : [0, \infty) \to [0, \infty)$ it is seen that

$$
H(t, x, u(x); a, a', b; q) \geq 0. \tag{54}
$$

Then from (44), we have

$$
\begin{aligned}
&\big(m_1(t)n_1(t) + m_2(t)n_2(t)\big)u(t)v(t)\\
&\geq n_1(t)n_2(t)u^2(t) + m_1(t)m_2(t)v^2(t).
\end{aligned}
\tag{55}
$$

Now multiplying both sides of (55) by $H(t, x, 1; a, a', b; q)$ given in (53) together with (54) and taking q-integration with respect to t from 0 to x with aid of (27), we get our desired result.

$$
\begin{aligned}
&I^{a,a',b}_q\{(m_1n_1 + m_2n_2)uv\}(x)\\
&\geq I^{a,a',b}_q\{(n_1n_2)u^2\}(x) + I^{a,a',b}_q\{(m_1m_2)v^2\}(x)
\end{aligned}
\tag{56}
$$

□

If we substitute $a' = -a$ and $a' = 0$ in above results we get following special cases of the inequalities respectively.

Corollary 5. *For q-analogue of Riemann-Liouville fractional integral operator the following inequality holds true:*

$$
\begin{aligned}
&R^a_q\{(m_1n_1 + m_2n_2)uv\}(x)\\
&\geq R^a_q\{(n_1n_2)u^2\}(x) + R^a_q\{(m_1m_2)v^2\}(x)
\end{aligned}
\tag{57}
$$

Corollary 6. *For q-analogue of Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$
\begin{aligned}
&K^{a,b}_q\{(m_1n_1 + m_2n_2)uv\}(x)\\
&\geq K^{a,b}_q\{(n_1n_2)u^2\}(x) + K^{a,b}_q\{(m_1m_2)v^2\}(x)
\end{aligned}
\tag{58}
$$

Theorem 4. *Let $0 < q < 1$, consider u and v are two positive integrable and synchronous mapping on $[0, \infty]$. Assume $\exists$ four positive integrable mapping m_1, m_2, n_1 and n_2 such that:*

$$
\begin{aligned}
&0 < m_1(t) \leq u(t) \leq m_2(t),\\
&0 < n_1(t) \leq v(t) \leq n_2(t),\\
&(t \in [0, x], x > 0).
\end{aligned}
\tag{59}
$$

Then the following inequality holds true:

$$
\begin{aligned}
&I^{a,a',b}_q\{n_1v\}(x)I^{a,a',b}_q\{m_1u\}(x)\\
&+I^{a,a',b}_q\{n_2v\}(x)I^{a,a',b}_q\{m_2u\}(x)\\
&\geq I^{a,a',b}_q\{n_1n_2\}(x)I^{a,a',b}_q\{u^2\}(x)\\
&+I^{a,a',b}_q\{v^2\}(x)I^{a,a',b}_q\{m_1m_2\}(x).
\end{aligned}
\tag{60}
$$

Proof. From (44), we have

$$
\begin{aligned}
&m_1(t)u(t)n_1(s)v(s) + m_2(t)u(t)n_2(s)v(s)\\
&\geq n_1(s)n_2(s)u^2(t) + m_1(t)m_2(t)v^2(s).
\end{aligned}
\tag{61}
$$

Then on multiplying both sides of the equation (61), by $H(t,x,1;a,a',b;q)$ defined in (53) together with (54) and taking q-integration with respect to t from 0 to x with aid of (27)

$$\begin{aligned} &n_1(s)v(s)I_q^{a,a',b}\{m_1u\}(x)\\ &\quad+n_2(s)v(s)I_q^{a,a',b}\{m_2u\}(x)\\ &\quad\geq n_1(s)n_2(s)I_q^{a,a',b}\{u^2\}(x)\\ &\quad\quad+v^2(s)I_q^{a,a',b}\{m_1m_2\}(x) \end{aligned} \tag{62}$$

Again multiplying both sides of the equation (62), by $H(t,x,1;a,a',b;q)$ defined in (53) together with (54) and taking q-integration with respect to s from 0 to x with aid of (27), we get our desired result.

$$\begin{aligned} &I_q^{a,a',b}\{n_1v\}(x)I_q^{a,a',b}\{m_1u\}(x)\\ &+I_q^{a,a',b}\{n_2v\}(x)I_q^{a,a',b}\{m_2u\}(x)\\ &\geq I_q^{a,a',b}\{n_1n_2\}(x)I_q^{a,a',b}\{u^2\}(x)\\ &\quad+I_q^{a,a',b}\{v^2\}(x)I_q^{a,a',b}\{m_1m_2\}(x) \end{aligned} \tag{63}$$

□

By setting $a'=-a$ and $a'=0$ in above results we get following special cases of the inequalities respectively.

Corollary 7. *For q-analogue of Riemann-Liouville fractional integral operator the following inequality holds true:*

$$\begin{aligned} &R_q^a\{n_1v\}(x)R_q^a\{m_1u\}(x)\\ &+R_q^a\{n_2v\}(x)R_q^a\{m_2u\}(x)\\ &\geq R_q^a\{n_1n_2\}(x)R_q^a\{u^2\}(x)\\ &\quad+R_q^a\{v^2\}(x)R_q^a\{m_1m_2\}(x) \end{aligned} \tag{64}$$

Corollary 8. *For q-analogue of Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$\begin{aligned} &K_q^{a,b}\{n_1v\}(x)K_q^{a,b}\{m_1u\}(x)\\ &+K_q^{a,b}\{n_2v\}(x)K_q^{a,b}\{m_2u\}(x)\\ &\geq K_q^{a,b}\{n_1n_2\}(x)K_q^{a,b}\{u^2\}(x)\\ &\quad+K_q^{a,b}\{v^2\}(x)K_q^{a,b}\{m_1m_2\}(x) \end{aligned} \tag{65}$$

4. Concluding remark

We summarize our research work by mentioning that all the results derived in this paper are novel and important. Firstly, we have established certain inequalities involving Saigo type fractional integral operator and derived some special cases of it. Then we have derived q-analogues of the inequalities involving Saigo type fractional integral operator that means certain q-integral inequalities. Some special cases of q-integral inequalities are also derived. We also notice that when q approaches to 1 then the resulting inequalities presented in Section 3, are become those demonstrated in Section 2.

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RESEARCH ARTICLE

A simple method for studying asymptotic stability of discrete dynamical systems and its applications

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ABSTRACT

In this work, we introduce a simple method to investigate the asymptotic stability of discrete dynamical systems, which can be considered as an extension of the classical Lyapunov's indirect method. This method is constructed based on the classical Lyapunov's indirect method and the idea proposed by Ghaffari and Lasemi in a recent work. The new method can be applicable even when equilibia of dynamical systems are non-hyperbolic. Hence, in many cases, the classical Lyapunov's indirect method fails but the new one can be used simply. In addition, by combining the new stability method with the Mickens' methodology, we formulate some nonstandard finite difference (NSFD) methods which are able to preserve the asymptotic stability of some classes of differential equation models even when they have non-hyperbolic equilibrium points. As an important consequence, some well-known results on stability-preserving NSFD schemes for autonomous dynamical systems are improved and extended. Finally, a set of numerical examples are performed to illustrate and support the theoretical findings.

1. Introduction

Many important processes and phenomena in real-world situations can be mathematically modeled by autonomous dynamical systems described by differential equations associated with the classical and fractional derivative operators [1–8]. While differential equation models with the classical derivatives have been formed and studied for a long time [1, 3, 5, 6, 8], mathematical models based on fractional differential equations have been strongly developed in recent years (see, for example, [9–28]). The stability analysis of differential equation models has been a central and prominent problem with many useful applications.

In this work, we consider general time-continuous dynamical systems of the form

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0 \in \mathbb{R}^n,$$

where $y : [0, T] \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ are real-valued functions satisfying appropriate conditions to guarantee the existence and uniqueness of solutions of the system (see, for instance, [1,3,5,6,8]).

The stability analysis of the system (1) has played a prominent role in both theory and practice, especially in control theory and mathematical epidemiology [5, 29]. The continuous version of the classical Lyapunov's indirect method can be considered as the most successful approach to this problem (see [29] or also [1,3,5,6,8]). This method studies the asymptotic stability of an equilibrium

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point by analyzing the associated Jacobian matrix with respect to the left half-plane. More precisely, an equilibrium point y^* is locally asymptotically stable if all the eigenvalues of the Jacobian matrix $J(y^*)$ lie strictly in the left half-plane; and y^* is unstable if any of the eigenvalues lie in the strict right half-plane. Clearly, the Lyapunov's indirect theorem is only applicable for hyperbolic equilibrium points. Here, an equilibrium point y^* is said to be hyperbolic if none of the eigenvalues of $J(y^*)$ lies on the imaginary axis; otherwise, y^* is said to be non-hyperbolic. Hence, the method fails to determine the asymptotic stability of non-hyperbolic equilibrium points. This leads to a big restriction of the application of the method. For this reason, in a recent work [30], Ghaffari and Lasemi constructed a new method to examine the stability of continuous dynamical systems, which is based on the classical Lyapanov's indirect method. However, it studies the stability of an equilibrium point by analyzing the associated Jacobian matrix at a deleted neighborhood of the equilibrium point instead of at the equilibrium point. As a consequence, the new method can be applicable for non-hyperbolic equilibrium points in many cases. Therefore, the weakness of the classical theorem can be improved.

Similarly to the continuous version, the discrete version of the Lyapunov's indirect method can be considered as a powerful and effective approach to the stability problem of discrete dynamical systems (see, for instance, [5, 31]). This method investigates the stability of an equilibrium point by considering the position of eigenvalues of the associated Jacobian matrix with respect to the unit circle. More specifically, an equilibrium point is asymptotically stable if all the eigenvalues of the Jacobian matrix lie strictly inside the unit circle and is unstable if any of the eigenvalues lie outside the unit circle. Consequently, the method is only applicable when none of the eigenvalues of the Jacobian matrix lies on the unit circle. In this case, equilibrium points are said to be hyperbolic.

Motivated and inspired by the above reason, in this work we introduce a new and simple method to analyze the asymptotic stability of discrete dynamical systems, which can be considered as an extension of the classical Lyapunov's indirect method. This method is constructed based on the classical Lyapunov's indirect method and the idea proposed by Ghaffari and Lasemi in [30]. It is worth noting that the new method can be applicable even when equilibia of dynamical systems are non-hyperbolic. Consequently, in many cases, the classical Lyapunov's indirect method fails but the new theorem can be used simply. In addition, a relation between the new method and the classical Lyapunov's indirect one is also provided.

To illustrate the applicability of the new theorem, we combine it with the Mickens' methodology [32–36] to construct nonstandard finite difference (NSFD) methods, which have ability to preserve the asymptotic stability of some differential equation models even when they possess non-hyperbolic equilibrium points. We recall that the concept of NSFD schemes was first introduced by Mickens in 1980 to overcome drawbacks of standard finite difference ones [32–36]. Nowadays, NSFD schemes have been widely used as a powerful and efficient class of numerical methods for solving differential equations arising in real-world situations. We refer the readers to [32–39] and [40–54] for good reviews and some recent notable works related to NSFD schemes, respectively. Recently, we have successfully developed the Mickens' methodology to construct NSFD schemes for differential equation models arising in real-world applications [55–60]. In the construction of NSFD schemes, one of the most important problem is to formulate NSFD schemes preserving the asymptotic stability of equilibrium points of differential equation models (see, for instance, [43,51,55,61–64]). A common approach to this problem is the use of the continuous and discrete versions of the classical Lyapunov's indirect method. Following this approach, the continuous version is first used to determine the stability of equilibria, and then, the discrete version is applied to analyze the stability of NSFD schemes. However, as mentioned before, the classical Lyapunov' indirect method fails to conclude the asymptotic stability of non-hyperbolic equilibrium points. So, the construction of NSFD schemes for differential equation models having non-hyperbolic equilibrium points is still a challenge. This challenge was mentioned in some well-known works [43,61,62]. An indispensable condition in the previous results on stability-preserving NSFD methods [43,51,55,61–64] is that all equilibrium points of differential equation models must be hyperbolic. This problem leads to a big restriction in the application of these NSFD methods.

For the above reason, by combining the new stability theorem with the Mickens' methodology, we formulate some NSFD methods which can preserve the asymptotic stability of some classes of differential equation models even when they have non-hyperbolic equilibrium points. Consequently, the applicability of the new method is shown

and the stability-preserving NSFD schemes formulated in [43, 55, 61, 62] are improved and extended. Therefore, the new method is reliable and it has advantages over the classical one. In numerical examples, we will see that in many cases the classical method is not working but the new method proves helpful.
The plan of this work is as follows:
In Section 2, some concepts and preliminaries are provided. The new stability method is introduced in Section 3. In Section 4, we construct stability-preserving NSFD schemes for some classes of differential equation models having non-hyperbolic equilibrium points. Numerical examples are performed in Section 5. Some conclusions and remarks are presented in the last section.

2. Preliminaries

In this section, we provide some concepts and preliminaries related to stability theory of dynamical systems and NSFD methods, which will be used in the next sections.

2.1. Stability of dynamical systems

The following theorem is known as the Lyapunov's indirect method for continuous dynamical systems.

Theorem 1. *([3, Theorem 4.7]) Let $y^* = 0$ be an equilibrium point for the nonlinear system*

$$\frac{dy}{dt} = f(y), \tag{1}$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial y}(y)\bigg|_{y=0}.$$

Then,

(1) *The origin is asymptotically stable if $Re\lambda_i < 0$ for all eigenvalues of A.*
(2) *The origin is unstable if $Re\lambda_i > 0$ for one or more of the eigenvalues of A.*

Definition 1. *([8, Definition 2.3.6]) An equilibrium point y^* of the system (1) is said to be hyperbolic if none of the eigenvalues of $df(y^*)$ lies on the imaginary axis.*

The following extension of Theorem 1 was proposed by Ghaffari and Lasemi in [30].

Theorem 2. *Let N be a deleted neighborhood of origin that contains no equilibrium points of the system (1). Let y_0 be the initial condition inside N:$\{i.e, y_0 \in N\}$, and $A = \frac{\partial f}{\partial y}(y)\Big|_{y=y_0}$, then;*

(1) *The origin is asymptotically stable if for any y_0 in N all eigenvalues of A are in the open left-half complex plane.*
(2) *The origin is unstable if for any y_0 in N one or more of the eigenvalues of A are in the open right-half complex plane.*

We now consider a general dynamical system governed by difference equations of the form

$$y_{n+1} = g(y_n), \quad y_0 = c \in \mathbb{R}^n, \tag{2}$$

where $G : D \to \mathbb{R}^n$ and $D \in \mathbb{R}^n$ is the domain of definition of g.

Definition 2. *([8, Definition 1.3.6]) An equilibrium point y^* of the system (2) is said to be hyperbolic if none of the eigenvalues of $dg(y^*)$ lie on the unit circle.*

Theorem 3. *([8, Theorem 1.3.7]) Let $g \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$. Then an equilibrium point y^* of the system (2) is asymptotically stable of the eigenvalues of $dg(y^*)$ lie strictly inside the unit circle. If any of the eigenvalues lie outside the unit circle the equilibrium point is unstable.*

2.2. Nonstandard finite difference methods

Consider a one-step numerical scheme with a step size h, that approximates the solution $y(t_n)$ of the system (1) in the form:

$$D_h(y_n) = F_h(f; y_n), \tag{3}$$

where $D_h(y_n) \approx dy/dt, F(f; y_n) \approx f(y)$, and $t_n = t_0 + nh$. The following definition is derived from the Mickens' methodology.

Definition 3. *(See [37, Definition 1], [45, Definition 3.3], [64, Definition 3]) The one-step finite-difference scheme (3) for solving System (1) is a NSFD method if at least one of the following conditions is satisfied:*

- *$D_h(y_n) = \frac{y_{n+1} - y_n}{\phi(h)}$, where $\phi(h) = h + \mathcal{O}(h^2)$ is a non-negative function*
- *$F(f, y_n) = g(y_n, y_{n+1}, h)$, where $g(y_n, y_{n+1}, h)$ is a non-local approximation of the right-hand side of System (1).*

Definition 4. *([61, Definition 4]) The finite-difference method is called "weakly" nonstandard if the traditional denominator h in the first-order discrete derivative $D_h(y_n)$ is replaced by a non-negative function $\phi(h)$ such that $\phi(h) = h + \mathcal{O}(h^2)$.*

The advantage and power of NSFD schemes over the standard ones are expressed in the following definitions.

Definition 5. *(See [37, Definition 2]) Assume that the solutions of Eq.* (1) *satisfy some property* $\mathcal{P}$. *The numerical scheme* (3) *is called (qualitatively) stable with respect to property* $\mathcal{P}$ *(or* $\mathcal{P}$*-stable), if for every value of* $h > 0$ *the set of solutions of* (3) *satisfies property* $\mathcal{P}$.

Definition 6. *(See [34]) Consider the differential equation* $y' = f(y)$. *Let a finite difference scheme for the equation be* $y_{n+1} = F(y_n, h)$. *Let the differential equation and/or its solutions have property* $\mathcal{P}$. *The discrete model equation is dynamically consistent with the differential equation if it and/or its solutions also have property* $\mathcal{P}$.

3. New stability method for discrete dynamical systems

In this section, we introduce a new method to study the asymptotic stability of discrete dynamical systems and give a relation between it and the Lyapunov's indirect method.

Theorem 4. *Assume that* $y^* \in \mathbb{R}^n$ *is an equilibrium point of the dynamical system* (2), *that is,* $g(y^*) = y^*$. *Let* N^* *be a deleted neighborhood of the equilibrium* y^* *that contains no equilibrium points of the system. Let* y_0 *be any point belonging to* N *and denote* $A^* = \frac{\partial g}{\partial y}(y)\Big|_{y=y_0}$. *Then,*

(1) *The equilibrium point* y^* *is asymptotically stable if for any* y_0 *in* N^* *all eigenvalues of* A^* *lie strictly inside the unit circle.*
(2) *The equilibrium* y^* *is unstable if for any* y_0 *in* N^* *one or more of the eigenvalues of* A^* *lie outside the unit circle.*

Remark 1. *The proof of this Theorem is based on the proof of the classical Lyapunov's indirect method (see, for instance, [3, 8, 31]).*

Proof. **Proof of Part (i).** First, it follows from the mean value theorem that

$$
\begin{aligned}
g_i(g(y)) &= g_i(y) + \frac{\partial g_i}{\partial y}(\xi_i)(g(y) - y) \\
&= g_i(y) + \frac{\partial g_i}{\partial y}(y_0)(g(y) - y) \\
&\quad + \left(\frac{\partial g_i}{\partial y}(\xi_i) - \frac{\partial g_i}{\partial y}(y_0)\right)(g(y) - y),
\end{aligned}
$$

where ξ_i is a point in the line segment connecting $g(y)$ to the y. Hence, we can write

$$
g(g(y)) = g(y) + A^*(g(y) - y) + h(y), \tag{4}
$$

where

$$
\begin{aligned}
A^* &= \frac{\partial g}{\partial y}(y)\Big|_{y=y_0}, \\
h_i(y) &= \left(\frac{\partial g_i}{\partial y}(\xi_i) - \frac{\partial g_i}{\partial y}(y_0)\right)(g(y) - y),
\end{aligned}
$$

for $i = 1, 2, \ldots, n$ and $h_i(y)$ is the ith row of $h(y)$. The function $h_i(y)$ satisfies

$$
|h_i(y)| \le \left\| \frac{\partial g_i}{\partial y}(\xi_i) - \frac{\partial g_i}{\partial y}(y_0) \right\| \big\|(g(y) - y)\big\|.
$$

By continuity of $(\partial g/\partial y)$, we obtain that

$$
\frac{\|h(y)\|}{\|g(y) - y\|} \to 0 \quad \text{as} \quad \|y - y_0\| \to 0.
$$

Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$
\|h(y)\| \le \epsilon \|g(y) - y\| \quad \text{if} \quad \|y - y_0\| < \delta. \tag{5}
$$

We now use the function

$$
V(y) = \big(g(y) - y\big)^T R\big(g(y) - y\big),
$$

as a Lyapunov function candidate for the nonlinear system (2), where R is a symmetric positive definite matrix. The variation of V relative to (2) is given by

$$
\begin{aligned}
\Delta V(y) &:= V(g(y)) - V(y) \\
&= \big[g(g(y)) - g(y)\big]^T R\big[g(g(y)) - g(y)\big] \\
&\quad - \big[g(y) - y\big]^T R\big[g(y) - y\big].
\end{aligned}
$$

From (4), we have that

$$
\begin{aligned}
&\Delta V(y) \\
&= \Big[A^*\big(g(y) - y\big) + h(y)\Big]^T R\Big[A^*\big(g(y) - y\big) + h(y)\Big] \\
&- \big(g(y) - y\big)^T R\big(g(y) - y\big) \\
&= \big(g(y) - y\big)^T \big(A^{*T} R A^* - R\big)\big(g(y) - y\big) \\
&+ 2(h(y))^T R A^*\big(g(y) - y\big) + (h(y))^T R h(y).
\end{aligned}
$$

Since all eigenvalues of the matrix A^* lie strictly inside the unit circle, for every positive definite symmetric matrix T, there is a unique symmetric and positive definite matrix R such that (see Theorem 4.30 in [31] or Lemma B.12 in [8])

$$A^{*T}RA^* - R = -T, \tag{6}$$

which implies that

$$\begin{aligned}&\big(g(y)-y\big)^T\big(A^{*T}RA^*-R\big)\big(g(y)-y\big)\\&=\big(g(y)-y\big)^T(-T)(g(y)-y)\\&\le -\lambda_{\min}(T)\|g(y)-y\|^2,\end{aligned}$$

where $\lambda_{\min}(T)$ denotes the minimum eigenvalue of matrix T. Note that $\lambda_{\min}(T)$ is real and positive because T is symmetric and positive definite. Therefore,

$$\begin{aligned}\Delta V(y) &\le -\lambda_{\min}(T)\|g(y)-y\|^2\\&\quad+2h^T(y)RA^*\big(g(y)-y\big)+h^T(y)Rh(y).\end{aligned}$$

It follows from the estimate (5) that

$$\begin{aligned}&2(h(y))^TRA^*\big[g(y)-y\big]\\&\le 2\|h(y)\|\|R\|\|A^*\|\|g(y)-y\|\\&\le 2\epsilon\|A^*\|\|R\|\|g(y)-y\|^2,\\&(h(y))^TRh(y)\le\|R\|\|h(y)\|^2\le\|R\|\epsilon^2\|g(y)-y\|^2.\end{aligned}$$

for all $\|y-y_0\|<\delta$. Thus,

$$\begin{aligned}&\Delta V(y)\\&<\Big(-\lambda_{\min}(T)+2\epsilon\|A^*\|\|R\|+\epsilon^2\|R\|\Big)\|g(y)-y\|^2\end{aligned}$$

for all $\|y-y_0\|<\delta$. We now choose ϵ small enough such that $\lambda_{\min}(T) > 2\epsilon\|A^*\|\|R\|+\epsilon^2\|R\|$. Then, $\Delta V(y)<0$. Therefore, for any $y_0\in N$ there always exists $\epsilon>0$ such that $\Delta V(y)<0$. Thus, by the classical Lyapunov's direct method, we conclude that the equilibrium y^* is asymptotically stable. The proof of this part is complete.

Proof of part (ii). Assume that at y_0 the matrix A^* has an eigenvalue which lies outside the unit circle. By [31, Corollary 4.31], then there exists a real symmetric matrix R that is not positive semidefinite for which $A^{*T}RA^*-R=-T$ is negative define. Thus, the Lyapunov function $V(y)=\big(g(y)-y\big)^TR\big(g(y)-y\big)$ is negative at points arbitrarily close to the origin. Furthermore, we also obtain

$$\begin{aligned}\Delta V(y)&=-\big(g(y)-y\big)^TT\big(g(y)-y\big)\\&\quad+2\big(g(y)-y\big)^T(A^*)^TRh(y)+V(h(y)).\end{aligned}$$

Similarly to the proof of Part (i), if we choose ϵ small enough then $\Delta V(y)\le-\gamma\|g(y)-y\|^2$ for some $\gamma>0$. Therefore, by [31, Theorem 4.27], the equilibrium y^* is unstable. The proof of this part is complete. □

Remark 2. *From the continuity of polynomial roots (see [65, Theorem 3.9.1]), it is easy to verify that if the classical Lyapunov's indirect method is applicable, so is Theorem 4. In other words, the classical Lyapunov's indirect theorem is a consequence of Theorem 4.*

Example 1. *Consider the difference equation*

$$y_{n+1}=y_n+ay_n^3,\quad a\in\mathbb{R}. \tag{7}$$

The equation (7) has a unique equilibrium point $y^*=0$. The Jacobian matrix at y^* is given by $J(y^*)=1$. So, y^* is non-hyperbolic and the classical Lyapunov's indirect method fails to conclude the stability of y^*. However, Theorem 4 is applicable. Indeed, let $y_0\neq0$. The Jacobian matrix at y_0 is given by

$$J(y_0)=1+3ay_0^2.$$

Hence, by Theorem 4, we conclude that:

(1) If $a>0$, then y^* is unstable.
(2) If $a<0$, then y^* is asymptotic stable.

4. Stability-preserving NSFD methods

In this section, we construct NSFD methods which can preserve the stability of not only hyperbolic equilibrium points but also non-hyperbolic equilibrium ones of the system (1). For this purpose, we introduce the following hypotheses for the system (1):

(H1) The set of equilibrium points of the system (1) is finite.

(H2) For each equilibrium point, there is a deleted neighborhood in which none of the eigenvalues of the Jacobian matrix lies on the imaginary axis.

The hypothesis (H2) means that Theorem 4 is applicable for the system (1). Obviously, this condition is satisfied automatically for hyperbolic equilibrium points.

Theorem 5. *Assume that the hypotheses (H1) and (H2) are satisfied for the system* (1). *Then, the following NSFD scheme*

$$\frac{y_{n+1}-y_n}{\phi(h)}=\Big[I-\frac{\phi(h)}{2}\frac{\partial f}{\partial y}(y_n)\Big]^{-1}f(y_n) \tag{8}$$

is dynamically consistent with respect to the asymptotic stability of the system (1).

Proof. Suppose that y^* is an equilibrium point of the system (1) and N is a deleted neighborhood of y^*. For each $y_0\in N$, let us denote by $\lambda_i(y_0)$ and $\mu_i(y_0)$ $(1\le i\le n)$ are eigenvalues of $\dfrac{\partial f}{\partial y}(y_0)$ and $\dfrac{\partial g}{\partial y}(y_0)$, respectively, where g is given by

$$g(y_n)=y_n+\phi\Big[I-\frac{\phi(h)}{2}\frac{\partial f}{\partial y}(y_n)\Big]^{-1}f(y_n).$$

Then, we have

$$\mu_i(y_0) = \Big(1 + \frac{\phi}{2}\lambda_i(y_0)\Big)\Big(1 - \frac{\phi}{2}\lambda_i(y_0)\Big)^{-1}.$$

Hence, $|\mu_i(y_0)| < 1$ if and only if

$$\Big|1 + \frac{\phi}{2}\lambda_i(y_0)\Big| < \Big|1 - \frac{\phi}{2}\lambda_i(y_0)\Big|,$$

or equivalently,

$$2\phi Re(\lambda(y_0)) < 0. \tag{9}$$

We consider two cases of the stability of y^*.

Case 1. y^* is an asymptotically stable equilibrium point of (1). Then, by Theorem 2, there is a deleted neighborhood N of y^* in which $Re(\lambda_i(y_0)) < 0$ for all $i = 1, 2, \dots, n$. Therefore, the inequality (9) is satisfied for all $y_0 \in N$. By Theorem 4, we conclude that y^* is an asymptotically stable equilibrium point of (8).

Case 2. y^* is an unstable equilibrium point of (1). Then, there is a deleted neighborhood N of y^* such that for all $y_0 \in N$, there exists some j $(1 \le j \le n)$ for which $Re(\lambda_j(y_0)) > 0$. Consequently, the inequality (9) does not hold. Therefore, by Theorem 4, y^* is an unstable stable equilibrium point of (8).

Combining Case 1 and Case 2, we conclude that the scheme (8) preserves the stability of the system (1) for all finite step sizes. The proof is complete. □

Remark 3.

- *If $\phi(h)$ is small enough, then $I - \frac{\phi}{2}\frac{\partial f}{\partial y}(y_n) \approx I$. Hence, the existence of the solution of the scheme* (8) *is ensured. To make sure the scheme* (8) *is defined for all finite step sizes, we can use the following family of nonstandard denominator functions*

$$\phi(h) = \frac{1 - e^{-\tau h}}{\tau}, \quad \tau > 0$$

  since they are bounded from above by τ^{-1}. Note that the standard denominator function $\phi(h) = h$ is not bounded from above for $h > 0$.
- *In the case it is hard to determine $\Big[I - \frac{\phi}{2}\frac{\partial f}{\partial y}(y_n)\Big]^{-1}$, we can compute the numerical solutions of the scheme* (8) *as follows.*
  (1) *Set $\delta_n = y_{n+1} - y_n$.*
  (2) *Solve the following linear system*

$$\Big[I - \frac{\phi}{2}\frac{\partial f}{\partial y}(y_k)\Big]\delta_n = \phi f(y_n).$$

  (3) *Compute $y_{n+1} = y_n + \delta_n$.*

The following theorem is proved similarly to Theorem 5.

Theorem 6. *Assume that the hypotheses (H1) and (H2) are satisfied for the system* (1). *Then, the nonstandard implicit trapezoidal scheme*

$$\frac{y_{n+1} - y_n}{\phi(h)} = \frac{1}{2}f(y_n) + \frac{1}{2}f(y_{n+1}) \tag{10}$$

and the nonstandard implicit midpoint scheme

$$\frac{y_{n+1} - y_n}{\phi(h)} = f\left(\frac{y_n + y_{n+1}}{2}\right) \tag{11}$$

are dynamically consistent with respect to the asymptotic stability of the system (1).

Remark 4. *The numerical schemes* (8), (10) *and* (11) *can preserve the asymptotic stability of the system* (1) *for all denominator functions $\phi(h) = h + \mathcal{O}(h^2)$. When $\phi(h) = h$, these schemes becomes standard ones. However, in real-world applications, differential equation models possess not only the stability but also other essential mathematical features, for examples, the positivity. Therefore, nonstandard denominator functions are needed for dynamics consistency. Moreover, they can ensure the existence of the solutions of the schemes* (10) *and* (11).

5. Some applications and numerical experiments

In this section, we conduct numerical simulations to illustrate and support the theoretical findings.

Example 2. *Consider the following scalar differential equation*

$$\dot{y} = ay^3, \quad a \in \mathbb{R}. \tag{12}$$

In this case, the equation has a unique equilibrium point $y^* = 0$, which is non-hyperbolic. It was shown in [30] that

(1) if $a > 0$, y^* is unstable;
(2) if $a < 0$, y^* is asymptotically stable.

Note that the set $\mathbb{R}_+ := \{y \in \mathbb{R} | y \ge 0\}$ is a positively invariant set of the equation (12). Therefore, our objective is to construct an NSFD scheme, which is dynamically consistent with respect to the positivity and stability of (12). For convenience, we only consider the case $a < 0$. The case $a \ge 0$ can be considered in a same way.

Applying the Mickens' methodology, we obtain the following NSFD scheme for (12)

$$\frac{y_{n+1} - y_n}{\phi(h)} = ay_{n+1}y_n^2,$$

or equivalently

$$y_{n+1} = \frac{y_n}{1 - \phi a y_n^2}. \tag{13}$$

The equation (13) implies that $y_n \ge 0$ for all $n \ge 1$ whenever $y_0 \ge 0$. So, the positivity of

(12) is preserved. We now analyze the stability of (13). The Jacobian matrix associated with (13) is given by

$$J(y) = \frac{1+\phi a y^2}{1-\phi a y^2}.$$

Hence, $J(0) = 1$. In this case, $y^* = 0$ is a non-hyperbolic equilibrium point. So, the classical Lyapunov's indirect method fails to conclude the stability of y^*. However, by Theorem 4 we have that y^* is asymptotically stable since

$$J(y) = 1 + \frac{2\phi a y^2}{1-\phi a y^2} \in (-1, 1) \quad \text{for all} \quad y \neq 0.$$

Consequently, we obtain a positvity and stability preserving NSFD scheme for the equation (12).

Example 3. *Consider the following nonlinear system*

$$\begin{aligned} \dot{x} &= -x^3 - x + y, \\ \dot{y} &= x - 2y^3 - y. \end{aligned} \tag{14}$$

The system (14) has a unique equilibrium point, that is, $E^* = (0, 0)$. Moreover,

$$J(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Hence, E^* is a non-hyperbolic equilibrium point. So, the classical Lyapunov's indirect method cannot conclude the stability of E^*. However, by using a Lyapunov function given by

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

we have

$$\dot{V} = x\dot{x} + y\dot{y} = -x^4 - 2y^4 - (x-y)^2.$$

Hence, E^* is asymptotically stable. Also, since

$$\begin{aligned} \dot{x}\big|_{x=0} &= y \geq 0, \\ \dot{y}\big|_{y=0} &= x \geq 0, \end{aligned}$$

we conclude that the set $\mathbb{R}_2^+$ is a positively invariant set of (14) (see Theorem B.7 in [66]).

Our object is to construct an NSFD scheme preserving the positivity and stability of the system (14). For this purpose, applying the Mickens' methodology, we propose the following NSFD scheme for (14)

$$\begin{aligned} \frac{x_{n+1} - x_n}{\phi(h)} &= -x_{n+1}x_n^2 - x_{n+1} + y_n, \\ \frac{y_{n+1} - y_n}{\phi(h)} &= x_n - 2y_{n+1}^2 y_n - y_{n+1}. \end{aligned} \tag{15}$$

The system of difference equations (15) can be rewritten in the explicit form

$$\begin{aligned} x_{n+1} &= \frac{x_n + \phi y_n}{1 + \phi + \phi x_n^2}, \\ y_{n+1} &= \frac{y_n + \phi x_n}{1 + \phi + 2\phi y_n^2}, \end{aligned}$$

which implies that the set $\mathbb{R}_+^2$ is a positively invariant set of (15).

We now investigate the stability of (15). The system (15) has a unique equilibrium point, that is $E^* = (0, 0)$. The Jacobian matrix associated with (15) is

$$J(x,y) = \begin{pmatrix} \dfrac{1+\phi-\phi x^2 - 2\phi^2 xy}{(1+\phi+\phi x^2)^2} & \dfrac{\phi}{1+\phi+\phi x^2}, \\[2ex] \dfrac{\phi}{1+\phi+2\phi y^2} & \dfrac{1+\phi-2\phi y^2 - 4\phi^2 xy}{(1+\phi+2\phi y^2)^2} \end{pmatrix}. \tag{16}$$

Hence,

$$J(0,0) = \begin{pmatrix} \dfrac{1}{1+\phi} & \dfrac{\phi}{1+\phi} \\ \dfrac{\phi}{1+\phi} & \dfrac{1}{1+\phi}. \end{pmatrix}.$$

This implies that $E^* = (0, 0)$ is a non-hyperbolic equilibrium point. So, it is not suitable to use the classical Lyapunov's indirect method for investigating the stability of E^*. For this reason, we will apply Theorem 4. By some simple algebraic manipulations, we have

$$\begin{aligned} &Trace(J(x,y)) < 1, \\ &1 + Trace(J(x,y)) + \det(J(x,y)) > 0, \\ &1 - Trace(J(x,y)) + \det(J(x,y)) > 0, \end{aligned}$$

for all (x, y) in some appropriate deleted neighborhood of the origin. By the Jury condition [1], all eigenvalues of $J(x, y)$ lie strictly inside the unit circle. Consequently, the stability of E^* is proved.

We now compare the NSFD scheme (15) with the standard Euler and second-order Runge-Kutta (RK2) schemes. Figures 1 and 2 depict numerical solutions generated by the Euler and RK2 schemes. It is clear that the obtained numerical solutions are negative. So, the positivity of the system is violated.

Conversely, from Figures 3-5, we observe that the numerical solutions obtained by the NSFD scheme (15) preserves the positivity and stability of the system for all the chosen step sizes. Also, the dynamics of the numerical solutions does not dependent on the chosen step sizes.

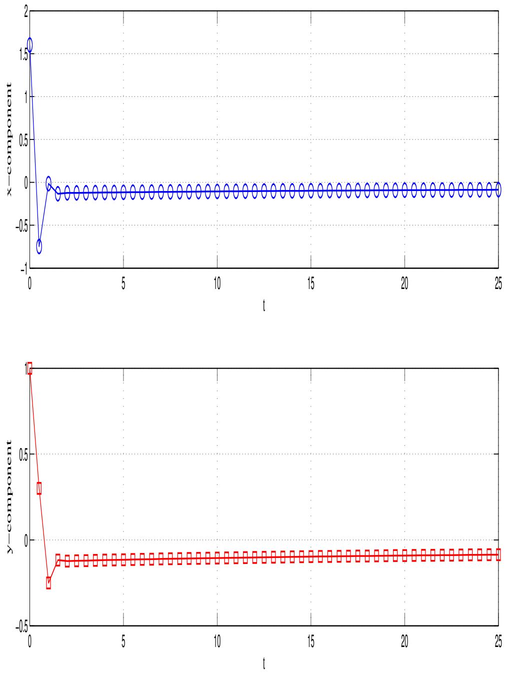

Figure 1. The numerical solutions obtained by the Euler scheme with $h = 0.5$ after 50 iterations in Example 3.

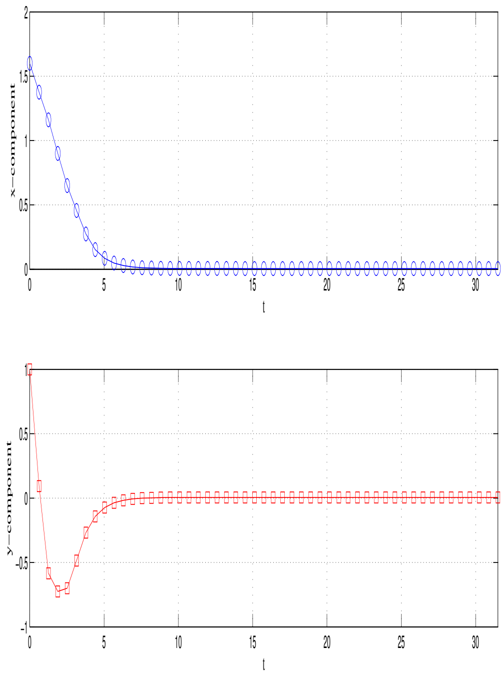

Figure 2. The numerical solutions obtained by the RK2 scheme with $h = 0.63$ after 50 iterations in Example 3.

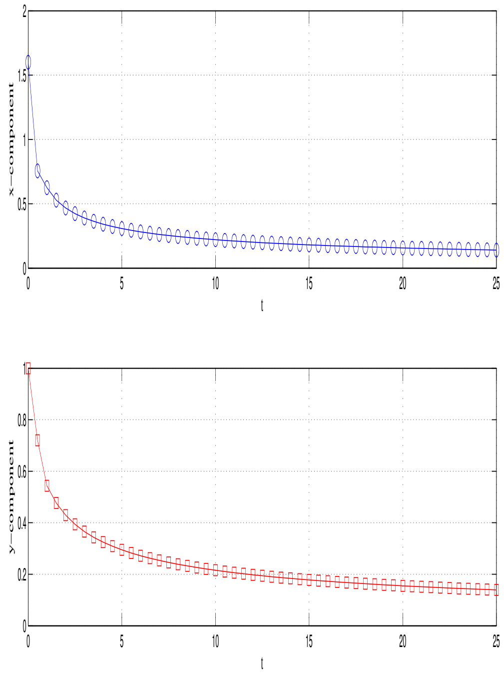

Figure 3. The numerical solutions obtained by the NSFD scheme with $h = 0.5$ after 50 iterations in Example 3.

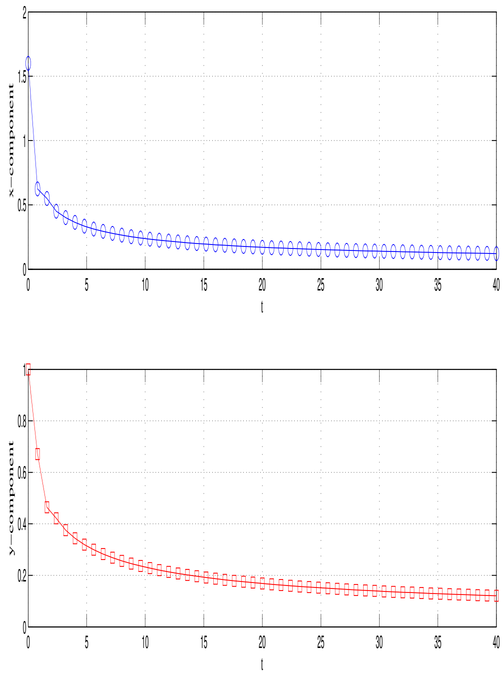

Figure 4. The numerical solutions obtained by the NSFD scheme with $h = 0.8$ after 50 iterations in Example 3.

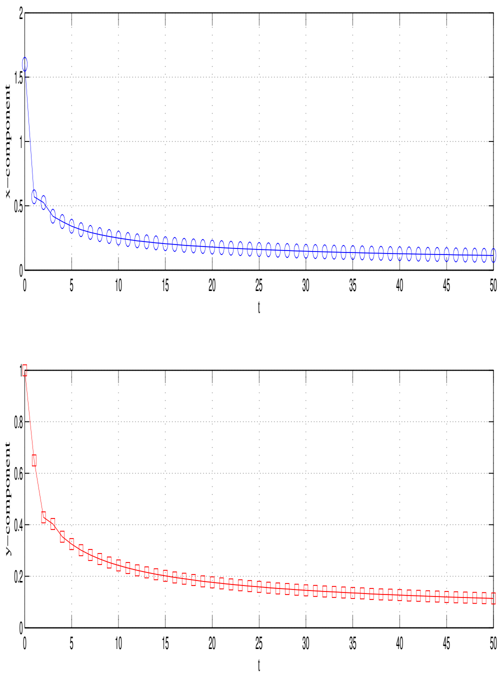

Figure 5. The numerical solutions obtained by the NSFD scheme with $h = 1$ after 50 iterations in Example 3.

If applying the scheme (8) for the system (14) we obtain

$$\begin{pmatrix} \dfrac{x_{n+1}-x_n}{\phi} \\ \\ \dfrac{y_{n+1}-y_n}{\phi} \end{pmatrix} = \begin{pmatrix} 1+\dfrac{\phi}{2}(3x_n^2+1) & -\dfrac{\phi}{2} \\ \\ -\dfrac{\phi}{2} & 1+\dfrac{\phi}{2}(6y_n^2+1) \end{pmatrix} \times \begin{pmatrix} -x_n^3-x_n+y_n \\ \\ x_n-2y_n^3-y_n \end{pmatrix}. \tag{17}$$

The scheme (17) is defined for all denominator functions ϕ since

$$\det \begin{pmatrix} 1+\dfrac{\phi}{2}(3x_n^2+1) & -\dfrac{\phi}{2} \\ \\ -\dfrac{\phi}{2} & 1+\dfrac{\phi}{2}(6y_n^2+1) \end{pmatrix} > 0.$$

Example 4. *Consider the nonlinear system*

$$\begin{aligned} \dot{x} &= -x^5 - x + y, \\ \dot{y} &= -x - y^3 - y. \end{aligned} \tag{18}$$

It is easy to verify that the system (18) has a unique equilibrium point $E^* = (0,0)$, which is non-hyperbolic. However, by a Lyapunov function given by $V(x,y) = x^2 + y^2$, we have that E^* is asymptotically stable. Our objective is to construct an NSFD scheme which is dynamically consistent with respect to the stability of the system (18). For this purpose, we propose the following NSFD scheme

$$\begin{aligned} \frac{x_{n+1}-x_n}{\phi(h)} &= -x_{n+1}x_n^4 - x_{n+1} + y_n, \\ \frac{y_{n+1}-y_n}{\phi(h)} &= -x_n - y_{n+1}y_n^2 - y_{n+1}. \end{aligned} \tag{19}$$

The explicit form of the scheme (19) is given by

$$\begin{aligned} x_{n+1} &= \frac{x_n + \phi y_n}{1+\phi+\phi x_n^4}, \\ y_{n+1} &= \frac{y_n - \phi x_n}{1+\phi+\phi y_n^2}. \end{aligned}$$

The trivial equilibrium point $E^* = (0,0)$ is also a non-hyperbolic equilibrium point of the scheme (19). So, the classical Lyapunov's indirect method fails to conclude the stability of E^*. However, by the new theorem 4, we can show that E^* is a asymptotically stable equilibrium point of the NSFD scheme (19). Figures 6-8 sketch numerical solutions generated by the NSFD scheme (19) with three different step sizes. In these figures, each blue curve represents a phase plane corresponding to a specific initial data, the red circle represents the position of the stable equilibrium point and the yellow arrows show the evolution of the model. Clearly, the stability of the system (18) is confirmed.

We can also obtain a stability-preserving numerical scheme for the system (18) by using the scheme (8). In this case, the scheme (8) is given by

$$\begin{pmatrix} \dfrac{x_{n+1}-x_n}{\phi(h)} \\ \\ \dfrac{y_{n+1}-y_n}{\phi(h)} \end{pmatrix} = \begin{pmatrix} 1+\dfrac{\phi}{2}(5x_n^4+1) & -\dfrac{\phi}{2} \\ \\ \dfrac{\phi}{2} & 1+\dfrac{\phi}{2}(3y_n^2+1) \end{pmatrix} \times \begin{pmatrix} -x_n^5-x_n+y_n \\ \\ -x_n-y_n^3-y_n \end{pmatrix}. \tag{20}$$

Note that

$$\det\begin{pmatrix} 1+\dfrac{\phi}{2}(5x_n^4+1) & -\dfrac{\phi}{2} \\ \dfrac{\phi}{2} & 1+\dfrac{\phi}{2}(3y_n^2+1)\end{pmatrix} > 0,$$

which implies that the scheme (20) is defined for all denominator function $\phi(h)$.

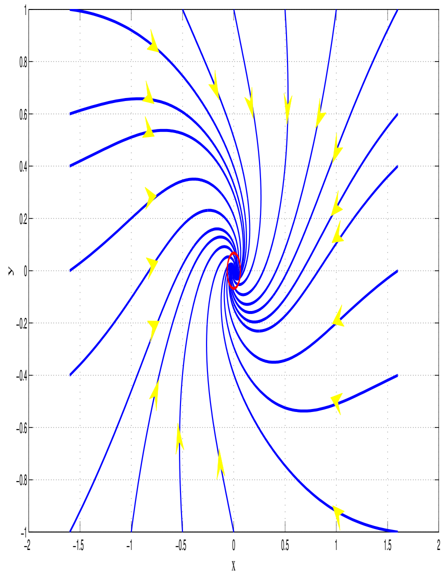

Figure 6. The numerical solutions generated by the NSFD scheme with $h = 0.01$ and $t \in [0, 100]$ in Example 4.

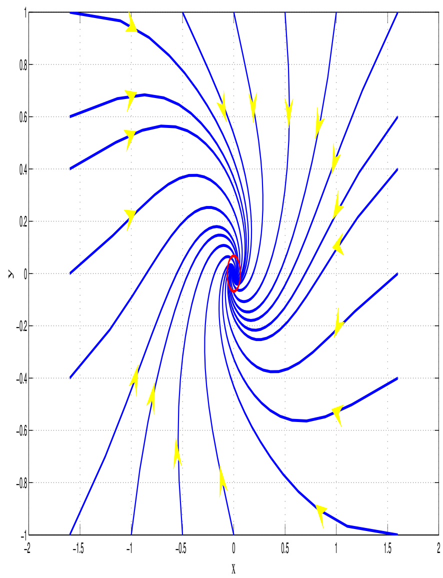

Figure 7. The numerical solutions generated by the NSFD scheme with $h = 0.1$ and $t \in [0, 100]$ in Example 4.

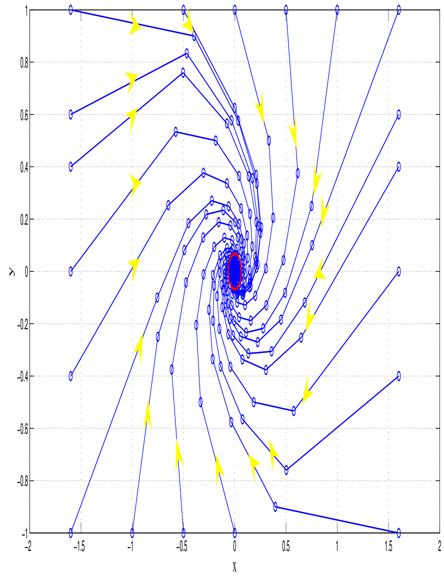

Figure 8. The numerical solutions generated by the NSFD scheme with $h = 0.5$ and $t \in [0, 100]$ in Example 4.

Example 5. *Consider the following system ([30])*

$$\begin{aligned}\dot{x} &= -x^3 + y,\\ \dot{y} &= -4x - y^3.\end{aligned} \tag{21}$$

It was proved in [30] that this system has a unique equilibrium point $E^* = (0, 0)$, which is non-hyperbolic and also asymptotically stable. Numerical solutions generated by the standard Euler and RK2 schemes are sketched in Figures 9-11. Clearly, these schemes cannot preserve the dynamics of the system (21). We now utilize the NSFD scheme (8) to solve the system (21). In this case, we have

$$I - \frac{\phi}{2}\frac{\partial f}{\partial y} = \begin{pmatrix} 1+\dfrac{3\phi}{2}x^2 & -\dfrac{\phi}{2} \\ 2\phi & 1+\dfrac{3\phi}{2}y^2\end{pmatrix},$$

which implies that

$$\det\left(I-\frac{\phi}{2}\frac{\partial f}{\partial y}\right) = 1+\frac{3\phi}{2}x^2+\frac{3\phi}{2}y^2+\frac{9\phi^2}{4}x^2y^2+\phi^2 > 0.$$

Hence, the scheme (8) is defined for all denominator functions and step sizes. Numerical solutions obtained by the NSFD scheme (8) with $\phi(h) = 1 - e^{-h}$ are depicted in Figures 12-14. It is clear that the dynamics of the system (21) is preserved.

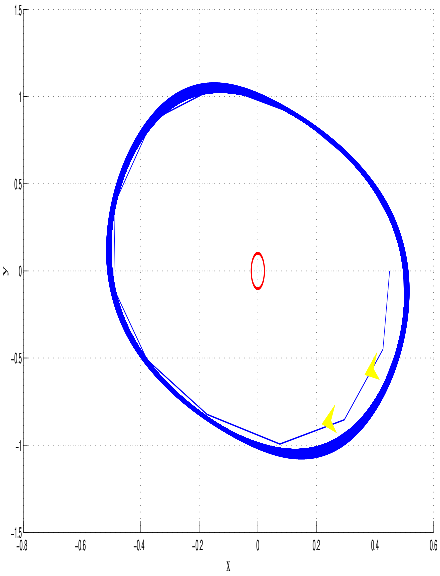

Figure 9. The numerical solution generated by the Euler scheme with $h = 0.2$ and $t \in [0, 1000]$ in Example 5.

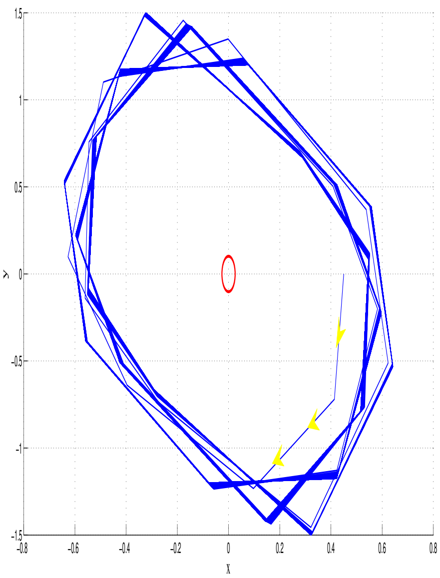

Figure 10. The numerical solution generated by the Euler scheme with $h = 0.4$ and $t \in [0, 1000]$ in Example 5.

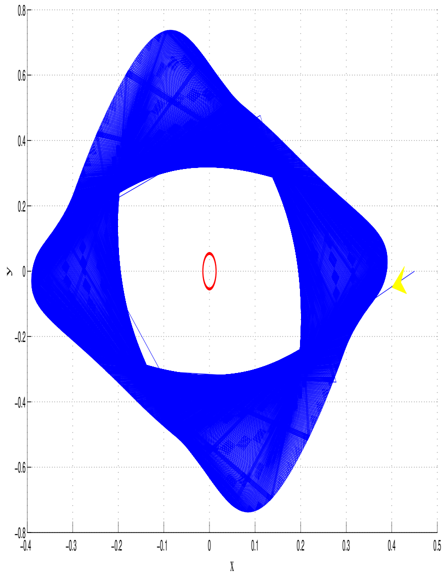

Figure 11. The numerical solutions generated by the RK2 scheme with $h = 0.7$ and $t \in [0, 980]$ in Example 5.

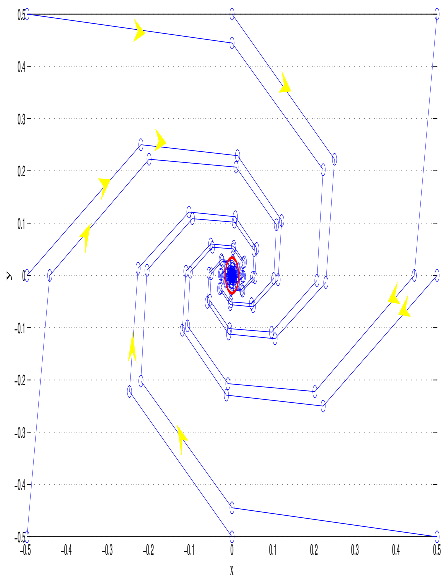

Figure 12. The numerical solutions generated by the NSFD scheme with $h = 1.0$ and $t \in [0, 1000]$ in Example 5.

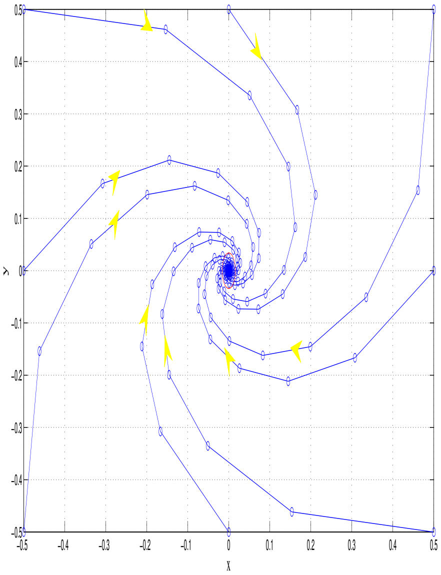

Figure 13. The numerical solutions generated by the NSFD scheme with $h = 0.5$ and $t \in [0, 1000]$ in Example 5.

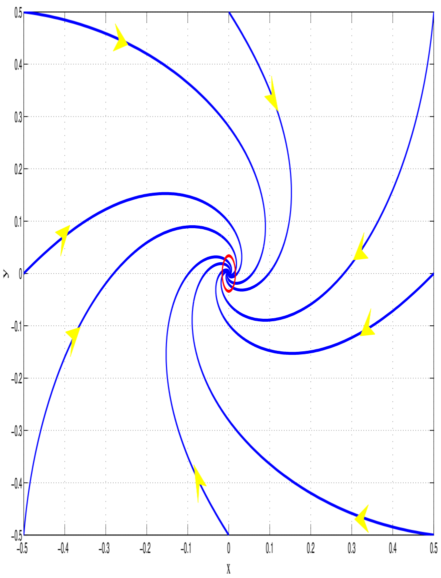

Figure 14. The numerical solutions generated by the NSFD scheme with $h = 0.01$ and $t \in [0, 1000]$ in Example 5.

Example 6 (Stabilization of nonlinear systems by feedback)**.** *Consider the following discrete dynamical systems described by the nonlinear difference equation*

$$y_{n+1} = y_n + a y_n^3, \quad a > 0. \tag{22}$$

It was proved in Example 1 that the equilibrium point $y^* = 0$ is unstable. Our objective is to find a control that stabilizes this system. More clearly, we need to determine a feedback control $u_n = h(y_n)$ in such a way that $y^* = 0$ of the corresponding closed-loop system is asymptotically stable. For this purpose, we consider

$$u_n = C y_n^3, \quad C \in \mathbb{R}.$$

Then, the corresponding closed-loop system is given by

$$y_{n+1} = y_n + (a + C) y_n^3. \tag{23}$$

The Jacobian matrix of (23) *evaluating at $y^* = 0$ is*

$$J(0) = 1.$$

Consequently, the classical Lyapunov stability theorem fails to conclude the stability of (23)*. However, the new method (Theorem 4) can be used easily. Indeed, the Jacobian matrix* (23) *is given by*

$$J(y) = 1 + 3(a + C)y^2,$$

which implies that $J(y) < 1$ if $C > -a$. On the other hand, $J(y) > -1$ whenever $y^2 < \dfrac{-2}{3(a + C)}$. Therefore, by using Theorem 4 we deduce that (23) *is locally asymptotically stable if $C > -a$. This means that the desired feedback control u_n is determined.*

Let us consider a more complicated system. Consider the following nonlinear system

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{3}x_n^3, \\ y_{n+1} &= y_n + \frac{1}{2}x_n^2 + \frac{1}{5}y_n^5. \end{aligned} \tag{24}$$

This system has a unique equilibrium point $E^* = (x^*, y^*) = (0, 0)$. The Jacobian of the system is given by

$$J(x, y) = \begin{pmatrix} 1 + x^2 & 0 \\ x & 1 + y^4. \end{pmatrix}$$

Therefore, the classical Lyapunov stability theorem cannot conclude the stability of E^*. However, E^* is unstable by applying Theorem 4.

To stabilize the system (24)*, we use a feedback control $u_n = (\alpha x_n^3, \beta y_n^5)$, where $\alpha, \beta \in \mathbb{R}$. Then, the closed-loop system is given by*

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{3}x_n^3 + \alpha x_n^3, \\ y_{n+1} &= y_n + \frac{1}{2}x_n^2 + \frac{1}{5}y_n^5 + \beta y_n^5. \end{aligned} \tag{25}$$

The Jacobian matrix of (25) *is*

$$J(x, y) = \begin{pmatrix} 1 + (3\alpha + 1)x^2 & 0 \\ x & 1 + (5\beta + 1)y^4. \end{pmatrix}$$

Hence, the classical Lyapunov stability theorem is not applicable to determine the stability of (25)*,* *but it follows from Theorem 4 that the closed-loop system is locally asymptotically stable if*

$$\alpha < -\frac{1}{3}, \qquad \beta < -\frac{1}{5}.$$

Hence, the system (24) *is stabilized.*

6. Conclusions and remarks

In this work, based on the classical Lyapunov's indirect method and the idea proposed by Ghaffari and Lasemi in [30], we have introduce a new and simple method for investigating the asymptotic stability of discrete dynamical systems (Theorem 4), which can be considered as an extension of the classical Lyapunov's indirect method. It is worth noting that the new method can be applicable even when equilibia of dynamical systems are non-hyperbolic. Hence, in many cases, the classical Lyapunov's indirect method fails but the new one can be used simply. Next, using the new theorem, we have constructed NSFD methods which are able to preserve the asymptotic stability of differential equation models having non-hyperbolic equilibrium points (Theorems 5 and 6). As an important consequence, some well-known results on positivity-preserving NSFD schemes for autonomous dynamical systems formulated in [43, 55, 61, 62] have been improved and extended. Finally, a set of numerical examples are performed to illustrate and support the theoretical findings.

In the near future, we will study practice applications of the new method to problems arising in control theory, economic and applied sciences. In addition, extensions of the new stability method for nonlinear systems associated with fractional-order operators will be also considered.

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RESEARCH ARTICLE

Observer design for a class of irreversible port Hamiltonian systems

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ABSTRACT

In this paper we address the state estimation problem of a particular class of irreversible port Hamiltonian systems (IPHS), which are assumed to be partially observed. Our main contribution consists to design an observer such that the augmented system (plant + observer) is strictly passive. Under some additional assumptions, a Lyapunov function is constructed to ensure the stability of the coupled system. Finally, the proposed methodology is applied to the gas piston system model. Some simulation results are also presented.

1. Introduction

Port Hamiltonian systems (PHS) encompass a very large class of systems including electrical, mechanical, and in general multi-energy systems [1–4]. This formalism has been suggested as a way for modeling and analysis of free and controlled physical systems, due mainly to its essential feature of underlying the crucial role played by the energy function, the interconnection structure, and the dissipation in the control of the system.

Although the PHS frame expresses the first principle of thermodynamics (the conservation of the energy), it is not suited for systems describing irreversible phenomena, as it is necessary to express the irreversible entropy creation, i.e. the second principle of thermodynamics. To solve this problem, the PHS frame has been revised and many quasi-PHS formulations have been presented in [5–7]. In [7], the PHS frame has been extended to a class of systems called IPHS. These systems are defined with respect to a skew symmetric structure matrix, and have the advantage of representing the first and the second principles of thermodynamics as theoretical properties of the system. (The reader is referred to [7] for more details on the IPHS construction and properties).

In most realistic problems, we do not have full information about the system state. Hence, the need to estimate the unknown part of the vector state is of great interest. For PHS many research papers have been developed to investigate the observer design problem [8–12]. In [10, 11], an observer design method based on passivation of the error dynamics is presented. By combining the interconnection and damping assignment method and the dissipativity theory, two observer design strategies are proposed in [8]. In [12], a full order observer design method based on contraction analysis is suggested for a particular class of PHS. For the class of systems considered in this paper (IPHS) regarding the control, a globally stabilizing controller preserving the IPHS structure in closed loop is proposed in [13] and [14]. In [15], an energy shaping and damping injection IPHS

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controller is constructed for an IPHS. Concerning the state estimation problem, to our modest knowledge, there is no observer design method developed for IPHS.

In this paper, our contribution is to present an observer design method for a class of IPHS by extending the approach suggested in [10,11] for PHS to the IPHS setting. Although our methodology is following that of [11], it is not obvious or simple to establish the same results for our class of systems. Some specific hypotheses are introduced in order to take into account the conservation of energy and the positivity of entropy production. It is assumed that the system is partially observed and that the observations are depending on the measured state only. That case is the most popular in practice and does not constitute any restriction as the availability of all state variables measurements is infrequent. Our observer is globally exponentially stable, and it is a copy of the original system in which the vector state components are directly the estimates of the plant ones.

The main advantage of the present study is that it is the first approach devoted to the observer design problem of IPHS. Unlike to [10,11] where the irreversibility is not considered, in this paper some specific hypotheses are introduced in order to take into account the conservation of energy and the positivity of entropy production. In addition, the use of the passivity technique renders the observer more stronger and robust against perturbations. Although the efficiency of our design method has been proven, the proposed strategy is restricted to minimum phase systems.

The rest of the paper is organized as follows. In section 2, a brief overview of the considered IPHS, the used observer, and some motivation will be given. Section 3 will be devoted to the description of our main result. In section 4, an application of the proposed approach on the gas piston system model will be presented. The paper is wrapped up in section 5 with a summary and an outlook.

2. Irreversible port Hamiltonian systems

Irreversible Port Hamiltonian Systems (IPHS) have been introduced in [7] as an extension of port Hamiltonian systems. In particular, the IPHS formulation is used to express simultaneously the energy conservation and the irreversible entropy creation. This article will be limited to the class of IPHS given by the following definition.

Definition 1. *The input affine representation of IPHS is defined by the dynamic equation and the output relation:*

$$\begin{aligned} \dot{x} &= R(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x}) J \frac{\partial U}{\partial x} + g(x, \frac{\partial U}{\partial x}) u(t), \quad (1) \\ y &= g^T(x, \frac{\partial U}{\partial x}) \frac{\partial U}{\partial x}(x) \end{aligned}$$

where:

(1) $x(t) \in \mathbb{R}^n$ *is the state vector.*
(2) $u(t) \in \mathbb{R}^m$ *is an input time dependent function,* $g(x, \frac{\partial U}{\partial x}) \in \mathbb{R}^{n\times m}$.
(3) $U(x) \in \mathbb{R}$, $S(x) \in \mathbb{R}$ *represent respectively the internal energy (the Hamiltonian) and the entropy functions.*
(4) $J \in \mathbb{R}^n \times \mathbb{R}^n$ *is a constant skew symmetric matrix, the structure matrix of the Poisson bracket* $\{.,.\}_J$*, where* $\{S, U\}_J = \frac{\partial S^T}{\partial x}(x) J \frac{\partial U}{\partial x}(x)$.
(5) $R(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x})$ *is the product of a positive definite function* γ *and the Poisson bracket of* S *and* U.

$$R(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x}) = \gamma(x, \frac{\partial U}{\partial x})\{S, U\}_J \quad (2)$$

with $\gamma(x, \frac{\partial U}{\partial x}) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $\gamma \geq 0$, *a non linear positive function.*

By construction, it is clear that IPHS satisfy the first principle of thermodynamics (conservation of energy):

$$\frac{dU}{dt} = y^T u, \quad (3)$$

which expresses that system (1) is lossless dissipative with energy supply rate $y^T u$ (See e.g. [13]). Moreover, IPHS obey the second principle of thermodynamics (positivity of the internal entropy production):

$$\begin{aligned} \frac{dS}{dt} &= R(x, \frac{\partial U}{\partial x}, \frac{\partial S}{\partial x}) \frac{\partial S^T}{\partial x} J \frac{\partial U}{\partial x} \quad (4) \\ &+ \frac{\partial S^T}{\partial x} g(x, \frac{\partial U}{\partial x}) u(t), \\ &= \gamma(x, \frac{\partial U}{\partial x})\{S, U\}_J^2 + (g^T(x, \frac{\partial U}{\partial x}) \frac{\partial S}{\partial x})^T u, \end{aligned}$$

where $\gamma(x, \frac{\partial U}{\partial x})\{S, U\}_J^2 = \sigma(x, \frac{\partial U}{\partial x}) \geq 0$, and $\{S, U\}_J^2 = \{S, U\}_J^T \{S, U\}_J$, (see [13], [16] for more details).

The energy and entropy functions are usually extensive variables. They satisfy the additivity [17]

$$\begin{aligned} S(X, Y) &= S(X) + S(Y), \\ U(X, Y) &= U(X) + U(Y), \end{aligned}$$

where X and Y are two states. In addition, they satisfy the scaling relation [17]

$$\begin{aligned} S(\lambda X) &= \lambda S(X), \\ U(\lambda X) &= \lambda U(X), \end{aligned}$$

where λ is an arbitrary scaling function.

In most realistic problems, we do not have full information about the system state. Hence, the need of estimating the unknown part of the vector state is of great interest. This motivates our observer design method in which the state of the original system will be decomposed into two parts. One is measured and hence it is selected to be the output. The other is non measured and it will be estimated by the observer.

3. Problem formulation

In this note we address the partial state observer problem of IPHS of the form:

$$\left\{ \begin{array}{ll} \dot{x} = & R(x_1, \frac{\partial U}{\partial x_1}, \frac{\partial S}{\partial x_1})J(x_1)\frac{\partial U}{\partial x} + g(x_1, \frac{\partial U}{\partial x_1})u(t), \\ y = & x_1. \end{array} \right. \tag{5}$$

Where

$J = \begin{bmatrix} J_1(x_1) & N(x_1) \\ -N^T(x_1) & J_2(x_1) \end{bmatrix}$, $g(x_1, \frac{\partial U}{\partial x_1}) = \begin{bmatrix} g_1(x_1, \frac{\partial U}{\partial x_1}) \\ \\ g_2(x_1, \frac{\partial U}{\partial x_1}) \end{bmatrix}$,

$x = (x_1, x_2) \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^p$ is the measured state, $x_2 \in \mathbb{R}^{n-p}$ is the unmeasured state, $u \in \mathbb{R}^m$ is the input (where m, n and p are integers such that $1 \leq p < n$ and $m \leq n$). It is assumed that the system (5) is forward complete, that is trajectories are defined for all $t \geq 0$. The matrices $J_1 \in \mathbb{R}^{p\times p}$, $J_2 \in \mathbb{R}^{(n-p)\times(n-p)}$ are skew symmetric, $N \in \mathbb{R}^{p\times(n-p)}$, $g_1 \in \mathbb{R}^{p\times m}$ and $g_2 \in \mathbb{R}^{(n-p)\times m}$. $U : \mathbb{R}^p \times \mathbb{R}^{n-p} \longrightarrow \mathbb{R}$ is the internal energy of the system. $S : \mathbb{R}^p \times \mathbb{R}^{n-p} \longrightarrow \mathbb{R}$ is the entropy function. The energy and entropy functions are assumed to satisfy

$$U(x) = U_1(x_1) + U_2(x_2), \tag{6}$$
$$S(x) = S_1(x_1) + S_2(x_2), \tag{7}$$

where $U_1 : \mathbb{R}^p \longrightarrow \mathbb{R}$, and $U_2 : \mathbb{R}^{n-p} \longrightarrow \mathbb{R}$ are two energy functions. $S_1 : \mathbb{R}^p \longrightarrow \mathbb{R}$, and $S_2 : \mathbb{R}^{n-p} \longrightarrow \mathbb{R}$ are two entropy functions.

Our aim is to design a full order observer for system (5) in the following form:

$$\begin{aligned} \dot{\hat{x}} &= R(\hat{x}_1, \frac{\partial U}{\partial \hat{x}_1}, \frac{\partial S}{\partial \hat{x}_1})J(\hat{x}_1)\frac{\partial U}{\partial \hat{x}}(\hat{x}) \\ &+ g(\hat{x}_1, \frac{\partial U}{\partial \hat{x}_1})u(t) + L(\hat{x}_1)v, \end{aligned} \tag{8}$$

where $L(\hat{x}_1) = \begin{bmatrix} L_1(\hat{x}_1) \\ L_2(\hat{x}_1) \end{bmatrix}$, $\hat{x} = (\hat{x}_1, \hat{x}_2)$, $\hat{x}_1 \in \mathbb{R}^p$, $\hat{x}_2 \in \mathbb{R}^{n-p}$, $v \in \mathbb{R}^p$.

Where $v = -k(y, \hat{x})y_d + v_d$, y_d and v_d are desired output and input respectively, and $k : \mathbb{R}^p \times \mathbb{R}^n \longrightarrow \mathbb{R}^{+*}$ is a continuous scalar function.

Following ([15], page 20), the total energy of the augmented system composed of (5) and (8) is $U(x, \hat{x}) = U(x) + U(\hat{x})$. This result represents an extension of the composition theory of dirac structures from the port Hamiltonian case to the irreversible one. This result states that the energy of any two port controlled Hamiltonian systems or more is the sum of the energy function of each system. See ([4], page 241) for more details.

The time derivative of the energy of the augmented system may be defined as follows

$$\begin{aligned} \dot{U}(x, \hat{x}) = & \frac{\partial U^T}{\partial x} g(x_1, \frac{\partial U}{\partial x_1})u + \frac{\partial U^T}{\partial \hat{x}} g(\hat{x}_1, \frac{\partial U}{\partial \hat{x}_1})u \\ - & \frac{\partial U^T}{\partial \hat{x}} Lky_d + \frac{\partial U^T}{\partial \hat{x}} Lv_d. \end{aligned}$$

Then under the conditions

$$\begin{aligned} [\frac{\partial U^T}{\partial x} g(x_1, \frac{\partial U}{\partial x_1}) + \frac{\partial U^T}{\partial \hat{x}} g(\hat{x}_1, \frac{\partial U}{\partial \hat{x}_1})]u &= 0, \\ \frac{\partial U^T}{\partial \hat{x}} L &= y_d^T, \end{aligned}$$

we get

$$\dot{U}(x, \hat{x}) = y_d^T v_d - k y_d^T y_d \leq y_d^T v_d, \tag{9}$$

and hence the feedback law $v = -k(y, \hat{x})y_d + v_d$ makes the augmented system composed of (5) and (8) strictly passive, with respect to the manifold $M = \{(x, \hat{x}) : x = \hat{x}\}$, from the new input v_d to the new output y_d. In that case, system (8) is called a passivity based observer for system (5).

Recall a fundamental characterization of passive systems. A system of the form $\dot{x} = f(x) + g(x)u$, $y = h(x)$, $x \in \mathbb{R}^n$ satisfies the KYP property if there exists a nonnegative function $U : \mathbb{R}^n \to \mathbb{R}$, with $U(0) = 0$, such that

$$\begin{aligned} (\nabla U(x))^T f(x) &\leq 0, \\ (\nabla U(x))^T g(x) &= h^T(x), \end{aligned}$$

see [18] for more details.

In order to solve the observer design problem, we shall find gains L_1, L_2, and some function k such that the augmented system is strictly passive (for more details on the passivity definition and its applications see [10,14,18,19]) with respect to a certain manifold that will be specified later. In this manifold, the unmeasurable state can be reconstructed, and hence global exponential stability of the system can be obtained by letting $v_d \equiv 0$. Note that a nonlinear observer is sensitive to measurement disturbances. In [10], it is shown that the passivity property can be used to modify the nonlinear injection gain in order to make the observer robust with respect to measurement disturbances.

4. Observer design

In the beginning of this section , we state the conditions which will make the augmented system strictly passive from the input v_d to the output y_d. To this end, we follow the same idea as in [11] by using the equivalence between the next two statements established in [18]:

(1) Any affine control system can be rendered strictly passive by a smooth static state feedback.
(2) The system has a vector relative degree $\{1,\dots,1\}$ and is globally minimum phase.

We recall the relative degree is equal to the number of times that one has to differentiate the system in order to have the input explicitly appearing. Moreover, a system is said to be globally minimum phase if its zero dynamics are globally asymptotically stable. See [18] for more details.

Note that in the study of passive systems, the concepts of relative degree and zero dynamics arise naturally. In particular, in our setting, we assume that the system has a vector relative degree $\{1,\dots,1\}$ in order to ensure the existence of the system zero dynamics.

We make in the sequel the two following assumptions.

Assumption 1. *For any $Z=\hat{x}_2-x_2$, there exist $Q=Q^T>0,\ Q\in\mathbb{R}^{(n-p)\times(n-p)}$ such that:*

$$\frac{\partial U}{\partial \hat{x}_2}=\frac{\partial U}{\partial x_2}+QZ. \tag{10}$$

Assumption 2. *There exists a smooth globally invertible matrix $L_1(x_1)\in\mathbb{R}^{p\times p}$ and a smooth matrix $L_2(x_1)\in\mathbb{R}^{(n-p)\times p}$ such that:*

$$B^T(x_1)+B(x_1)>\delta I_{(n-p)\times(n-p)},\qquad \delta>0, \tag{11}$$

holds for all x_1, where:

$$B(x_1)=L_2(x_1)L_1^{-1}(x_1)R(x_1,\frac{\partial U}{\partial x_1},\frac{\partial S}{\partial x_1})N(x_1).$$

We are now ready to state the passivation result:

Lemma 1. *Assume that assumptions* (1) *and* (2) *are satisfied. Then:*

(1) *The augmented system $(x,\hat{x})$ has a vector relative degree $\{1,\dots,1\}$ with respect to the input v and the output $y_d=\hat{x}_1-x_1$.*
(2) *The zero dynamics of the augmented system $(x,\hat{x})$ with respect to the output y_d renders the manifold $\mathcal{P}=\{(x_1,x_2,\hat{x}_2):\hat{x}_2=x_2\}$ positively invariant and globally exponentially attractive.*

Proof. (1) Now, we compute the derivative of y_d as:

$$\dot{y}_d=RN[\frac{\partial U}{\partial \hat{x}_2}-\frac{\partial U}{\partial x_2}]+L_1v.$$

As v is the considered input and L_1 is invertible by assumption for all x_1, the result is achieved.

(2) The zero dynamics of the augmented system with respect to the output y_d consist of (5) and the equations:

$$0=RN[\frac{\partial U}{\partial \hat{x}_2}-\frac{\partial U}{\partial x_2}]+L_1v, \tag{12}$$

$$\begin{aligned}\dot{\hat{x}}_2 &= -RN^T\frac{\partial U}{\partial x_1}+RJ_2\frac{\partial U}{\partial \hat{x}_2}+g_2u \\ &+ L_2(x_1)v,\end{aligned} \tag{13}$$

we note that these zero dynamics are defined uniformly for all $u\in\mathbb{R}^m$.

Now, consider the manifold $\mathcal{P}$ and denote $Z=\hat{x}_2-x_2$. By using (12), we compute the derivative of Z along (5) and (13). We get

$$\dot{Z}=[RJ_2-L_2L_1^{-1}RN]QZ. \tag{14}$$

Then by skew symmetry of RJ_2 and the use of assumption (2), we have the positive invariance of the manifold $\mathcal{P}$.

Now consider the Lyapunov function

$$V=\frac{1}{2}(\frac{\partial U}{\partial \hat{x}_2}-\frac{\partial U}{\partial x_2})^TQ^{-1}(\frac{\partial U}{\partial \hat{x}_2}-\frac{\partial U}{\partial x_2}), \tag{15}$$

then by assumptions (1) and (2) we obtain

$$\begin{aligned}\dot{V} &= \frac{1}{2}\dot{Z}^TQZ+\frac{1}{2}Z^TQ\dot{Z}\\ &= \frac{1}{2}[Z^TQ(-RJ_2-B^T+RJ_2-B)QZ]\\ &= -\frac{1}{2}Z^TQ[B^T+B]QZ\\ &\le -\delta\frac{\lambda_m^2(Q)}{\lambda_M(Q)}V.\end{aligned}$$

where $\lambda_m(Q)$ and $\lambda_M(Q)$ denotes respectively the smallest and the largest eigenvalue of Q. Thus the system exponentially decays to zero with convergence rate $\delta \frac{\lambda_m^2(Q)}{\lambda_M(Q)}$.

□

Remark 1. *In the last lemma we mean by zero dynamics, the dynamics of the augmented system composed by the observer and the plant restricted to the set of initial conditions such that the corresponding output $y_d = \hat{x}_1 - x_1$ is zero (which implies that $\hat{x}_1 = x_1$).*
Hence the observer (8), will be defined by

$$\begin{aligned} \dot{\hat{x}} &= R(x_1, \frac{\partial U}{\partial x_1}, \frac{\partial S}{\partial x_1}) J(x_1) \frac{\partial U}{\partial \hat{x}}(\hat{x}) \quad (16) \\ &+ g(x_1, \frac{\partial U}{\partial x_1}) u(t) + L(x_1) v, \end{aligned}$$

where $L(x_1) = \begin{bmatrix} L_1(x_1) \\ L_2(x_1) \end{bmatrix}$.
We note that this definition differs from the usual understanding of zero dynamics, as the input $u(t)$ still appearing in the equations.

The following assumption will play a crucial role in our analysis.

Assumption 3. *There exists a smooth function $\beta : \mathbb{R}^p \longrightarrow \mathbb{R}^{n-p}$ such that*

$$L_2(x_1) L_1^{-1}(x_1) = \frac{\partial \beta}{\partial x_1}(x_1) \quad (17)$$

holds for all $x_1 \in \mathbb{R}^p$.

Remark 2. (1) *Assumption 1 is important in the development of our approach. It allows us to easily demonstrate the positive invariance and the exponential stability of the manifold $\mathcal{P}$. This assumption is very crucial and will be helpful in the choice of our example given in section 5. Moreover, it expresses a relation between states variables and co-energy variables, and means that any co-energy variable $\frac{\partial U}{\partial x_2}(\frac{\partial U}{\partial \hat{x}_2})$ may be linearized with respect to the state x_2 ($\hat{x}_2$).*
(2) *Assumption 2 is usually satisfied since L_1 and L_2 represent degrees of freedom. The choice of L_1 and L_2 is done such that the augmented system has a vector relative degree and is globally minimum phase.*
(3) *In assumption 3, a matching condition is defined and has to be solved. This condition requires that the selected gains L_1 and L_2 should be integrable. This assumption will be used to achieve the attractivity of the manifold.*

Now, we proceed to the design of the feedback law and consequently to the construction of the full order observer.

Theorem 1. *Assume $g_1 \equiv 0$.*
Then, the augmented system (5), (8) expressed in the coordinates $(x_1, x_2; \xi_1, \xi_2)$ where

$$\begin{aligned} \xi_1 &= \hat{x}_1 - x_1, \quad (18) \\ \xi_2 &= \hat{x}_2 - x_2 - \{\beta(\hat{x}_1) - \beta(x_1)\} \quad (19) \end{aligned}$$

has global normal form with respect to the input v and the output y_d.
Moreover, the feedback law defined by

$$v = -(\alpha + \phi_1 + \phi_2^2)\xi_1 + v_d, \quad (20)$$

where $\alpha > 0$ and $\phi_i(\xi_1, \hat{x}_1, \hat{x}_2)$, $i = 1, 2$ are non negative scalar functions, renders the system strictly passive with respect to the manifold $\mathcal{P}$, uniformly for all $u \in \mathbb{R}^m$, from the input v_d to the output ξ_1 with the storage function being given by

$$W(\xi_1, \xi_2) = \frac{1}{2}\xi_2^T Q \xi_2 + \frac{1}{2}\xi_1^T X \xi_1,$$

where $X \in \mathbb{R}^{p\times p}$, and $Q \in \mathbb{R}^{n-p\times n-p}$.

Proof. We define the functions $F_i(\xi_1, \xi_2, x_1, x_2) = F_i$, $i = 1, 2, 3$, as:

$$\begin{aligned} F_1 &= \hat{f}_1 - f_1 \quad (21) \\ F_2 &= \hat{f}_2 - f_2 \quad (22) \end{aligned}$$

$$F_3 = \frac{\partial \beta}{\partial \hat{x}_1}(\hat{x}_1)\hat{f}_1 - \frac{\partial \beta}{\partial x_1}(x_1) f_1; \quad (23)$$

where $f_i(x_1, x_2) = f_i$, $f_i(\hat{x}_1, \hat{x}_2) = \hat{f}_i$, $i = 1, 2$, $F_1 \in \mathbb{R}^p$, $F_2 \in R^{n-p}$,

$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = R(x_1, \frac{\partial U}{\partial x_1}, \frac{\partial S}{\partial x_1}) J \frac{\partial U}{\partial x},$$

and hence using the assumption $g_1 \equiv 0$, the system dynamic may be expressed in the coordinate transformation (ξ_1, ξ_2) as

$$\begin{aligned} \dot{\xi}_1 &= F_1(\xi_1, \xi_2, x_1, x_2) + L_1(\hat{x}_1), \\ \dot{\xi}_2 &= (F_2 - F_3)(\xi_1, \xi_2, x_1, x_2). \end{aligned}$$

In addition, we have

$$F_i = F_i|_{x_2 = x_2 + \xi_2} + F_i|_{\xi_1 = 0},$$

where $F_i|_{x_2=x_2+\xi_2} = F_i(\xi_1, \xi_2, x_1, x_2 + \xi_2)$, $F_i|_{\xi_1=0} = F_i(0, \xi_2, x_1, x_2)$. We note that when $\xi_1 = 0$, then $F_i(\xi_1, \xi_2, x_1, x_2 + \xi_2) = 0$. Hence, the augmented system assumes its global normal form as we have the existence of continuous matrix functions $A_1(\xi_1, x_1, x_2) \in R^{p\times p}$, and $A_i(\xi_1, x_1, x_2) \in R^{n-p\times p}$, for $i = 2, 3$ achieving

$$F_i(\xi_1, \xi_2, x_1, x_2 + \xi_2) = A_i(\xi_1, x_1, x_2)\xi_1. \quad (24)$$

Now, There exist non-negative continuous scalar functions $\psi_i(\xi_1, x_1, x_2 + \xi_2)$, $i = 1, 2, 3$ such that,

$$\|A_i(\xi_1, x_1, x_2 + \xi_2)\| \leq \psi_i(\xi_1, x_1, x_2 + \xi_2), \quad (25)$$

holds for all $\xi_1, x_1, x_2+\xi_2$, where $\|.\|$ is the induced norm of any general matrix.

The next inequalities will be used to demonstrate that the system is strictly passive with respect to the input v_d and the output y_d:

$$\begin{aligned} &\|\xi_2^T Q(F_2 - F_3)(\xi_1, \xi_2, x_1, x_2 + \xi_2)\| \leq \\ &\zeta\{\psi_2 + \psi_3\}(\xi_1, x_1, x_2 + \xi_2)\sqrt{\delta}\|\xi_1\|\|\xi_2\| \end{aligned} \quad (26)$$

where $\zeta = \frac{\lambda_M(Q)}{\sqrt{\delta}}$, and $\lambda_M(Q)$ is the largest eigenvalue of Q.

$$\begin{aligned} &\|\xi_1^T X F_1(\xi_1, \xi_2, x_1, x_2 + \xi_2)\| \leq \\ &\lambda_M(X)\psi_1(\xi_1, x_1, x_2 + \xi_2)\|\xi_1\|^2, \end{aligned} \quad (27)$$

$$\begin{aligned} &\|\xi_1^T X F_1(0, \xi_2, x_1, x_2)\| \leq \\ &\zeta\lambda_M(X)\|R\|\|N\|\sqrt{\delta}\|\xi_1\|\|\xi_2\|, \end{aligned} \quad (28)$$

Now consider the feedback law v (20) with $\phi_1 = \lambda_M(X)\psi_1$ and $\phi_2 = \zeta(\{\psi_2 + \psi_3\} + \lambda_M(X)\|N\|\|R\|)$, and the storage function W. Using the inequalities (26), (27) and (28) we obtain:

$$\begin{aligned} \dot{W} \leq &- \alpha\|\xi_1\|^2 + \xi_1^T v_d - \frac{3}{4}\delta\|\xi_2\|^2 - \\ &-\{\frac{1}{2}\sqrt{\delta}\|\xi_2\| - \|\xi_1\|\phi_2\}^2 \end{aligned}$$

Thus, we get the result. □

5. Application

We consider a pure ideal gas contained in a cylinder closed by a piston and submitted to gravity. The thermodynamic properties of this system may be decomposed into the properties of the piston in the gravitation field and the properties of the perfect gas. See [16] for more details.

The total energy of the system is:

$$U(x) = TS - PV + H_{mec}(z, p), \quad (29)$$

where $x = [S, V, z, p]^T$ is the vector of state variable, S denotes the entropy variable, V is the volume variable, z is the altitude of the piston and p its kinetic momentum. $H_{mec}(z, p) = \frac{1}{2}mp^2 + mgz$ represents the energy of the piston. The co-energy variables are defined by the gradient of the total energy:

$$\begin{aligned} \frac{\partial U}{\partial S} &\triangleq T \\ \frac{\partial U}{\partial V} &\triangleq -P \\ \frac{\partial U}{\partial z} &= mg = F_g \\ \frac{\partial U}{\partial p} &\triangleq v \end{aligned} \quad (30)$$

where T is the temperature, P is the pressure, F_g is the gravity force and v is the velocity of the piston. This system may be written in the state space representation form (5) as follows:

$$\frac{d}{dt}\begin{bmatrix} S \\ V \\ z \\ p \end{bmatrix} = \underbrace{R\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & A\frac{T}{\nu v} \\ 0 & 0 & 0 & \frac{T}{\nu v} \\ -1 & -A\frac{T}{\nu v} & -\frac{T}{\nu v} & 0 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} T \\ -P \\ F \\ v \end{bmatrix}}_{\nabla_x U}$$

where A denotes the area of the piston, and

$$\begin{aligned} R &= R(x, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial S}), \\ &= \gamma(x, \frac{\partial U}{\partial x})\{S, U\}_J, \\ &= \frac{\nu v}{T}, \\ &= R(x_1, \frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial S}), \end{aligned}$$

and

$$J = J(x_1),$$

such that $x_1 = [S, V, p]^T$ and $x_2 = z$.

In order to stay in the context of partial state observability, we assume x_1 to be measured and x_2 is non measured. If we let $(\hat{x}_1, \hat{x}_2)$ be the state estimates and define their dynamic as in (8), the error of the system may be expressed as $(e_1, e_2, e_3, e_4) = (\hat{S}, \hat{V}, \hat{p}, \hat{z}) - (S, V, p, z)$.

The assumption 1 is satisfied:

$$\frac{\partial U}{\partial \hat{x}_2} - \frac{\partial U}{\partial x_2} = \rho A g(\hat{z} - z),$$

where g is the gravity force, ρ is the density.

Now let $L_1 = I_3$, where I_3 is the identity matrix of order 3, and $L_2 = [0, 1, 1]$. Then, assumption 2 is clearly satisfied as we have:

$$B^T(x_1) + B(x_1) = 2A + 2 > 0.$$

The function β of assumption 3 will be defined as: $\beta(x_1) = V + p$. To express the system in its global normal form, we use the following change of coordinates:

$$\begin{aligned}
\xi_1 &= \hat{x}_1 - x_1, \\
&= [\hat{S} - S, \hat{V} - V, \hat{p} - p]^T, \\
&= [\xi_{11}, \xi_{12}, \xi_{13}]^T, \\
\xi_2 &= \hat{x}_2 - x_2 - (\beta(\hat{x}_1) - \beta(x_1)), \\
&= \hat{z} - z - (\hat{V} - V) - (\hat{p} - p),
\end{aligned}$$

where $\xi_{11} = \hat{S} - S$, $\xi_{12} = \hat{V} - V$, and $\xi_{13} = \hat{p} - p$. We choose the total energy as

$$W(\xi_1, \xi_2) = \frac{1}{2}\xi_{11}^2 + \frac{1}{2}\xi_{12}^2 + \frac{1}{2}\xi_{13}^2 + \frac{1}{2}\xi_2^T\xi_2.$$

Now, as all tools are available, we shall design the feedback law (20). Firstly, the functions f_1 and f_2 are given by

$$f_1(x_1, x_2) = \begin{bmatrix} \nu\frac{v^2}{T} \\ Av \\ -\nu v + AP - \rho Agz \end{bmatrix}, \quad f_2 = v.$$

Then

$$F_1 = \begin{bmatrix} \nu(\frac{\hat{v}^2}{\hat{T}} - \frac{v^2}{T}) \\ A(\hat{v} - v) \\ -\nu(\hat{v} - v) + A(\hat{P} - P) - \rho Ag(\hat{z} - z) \end{bmatrix},$$

$$F_2 = \hat{v} - v,$$

$$F_3 = (A - \nu)(\hat{v} - v) + A(\hat{P} - P) - \rho Ag(\hat{z} - z).$$

Using the inequalities (26), (27), and (28) we get

$$\begin{aligned}
\phi_1 =& max(Anr\frac{|T|}{|V\hat{V}|} + \rho Ag, Anr\frac{T_0}{c|\hat{V}|} + T_0\nu\frac{|\hat{v}^2|}{c|T\hat{T}|}, \\
&\rho Ag + (A + \nu + \frac{|V + \hat{V}|}{|T|})sup_p(\|\nabla v\|))
\end{aligned}$$

and

$$\begin{aligned}
\phi_2 =& max(Anr\frac{T_0}{|V\hat{V}|} + \rho Ag, Anr\frac{T_0}{c|\hat{V}|}, sup_p(\|\nabla v\|)+ \\
&+\rho Ag) + 1 + A + \nu\frac{|v|}{|T|}.
\end{aligned}$$

Therefore, we get the expression of the feedback law (20) v as

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = -(\alpha + \phi_1 + \phi_2^2)\xi_1 + \begin{pmatrix} v_{d_1} \\ v_{d_2} \\ v_{d_3} \end{pmatrix},$$

where $\xi_1 = (\xi_{11}, \xi_{12}, \xi_{13})^T = (\hat{S} - S, \hat{V} - V, \hat{p} - p)^T$.

The simulations below address respectively the entropy, volume, the altitude of piston and the kinetic momentum. The plant curves are in red and the observer ones are in black.

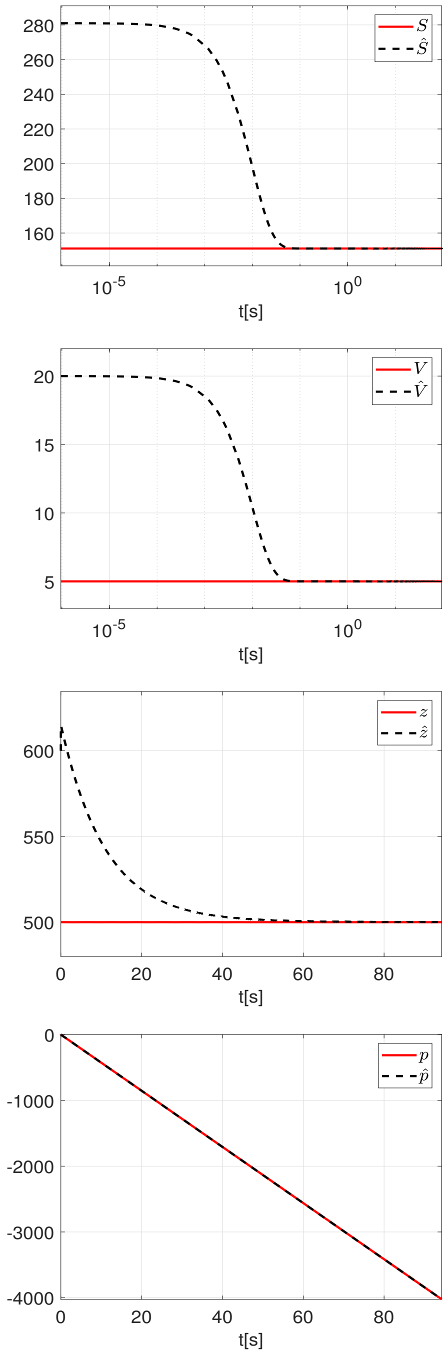

Figure 1. Open-Loop trajectories for the gas piston model and the observer.

The simulation results for the gas piston system model and the observer are given under the initial conditions:

	S_0	$V_0(l)$	p_0	z_0	α
Plant	151.077	5	0	500	0
Observer	281	20	0	600	10

The other parameters are chosen as: $g = 10m/S^2$, $n = 0.1002$ mol, $\nu = 0.05$, $A = 0.01m^2$, $T_0 = 600$ K, $c = 180$ j/Kg/K. $r = 8.31$ $jmol^{-1}K^{-1}$ is the universal gas constant.

6. Conclusion and Outlook

In this note, we have proposed a passivity based observer for a special class of irreversible port Hamiltonian systems. The observer design is done in two steps: The first one is the passivation of the system. In this step we check if assumption 1 is satisfied. Then, the matrices L_1 and L_2 are chosen in such a way to fulfil assumption 2. Finally, we compute the function β by using assumption 3.

The basic idea of the second step is to express the system in its global normal form and compute the feedback law v by using the procedure described in Theorem 1. Finally, the result has been applied to the gas piston system model considered in [7], and some simulation results of the studied example are presented. Since our study involves time derivatives, future works will tackle the investigation of the proposed observer design to the study of fractional differential operators (see [20–28]).

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RESEARCH ARTICLE

The effect of marketing and R&D expenditures on firm profitability and stock return: Evidence from BIST

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ABSTRACT

This study aims to determine the effects of R&D and marketing expenditures of companies that force marketing and finance to act together on stock return, return on assets, and return on equity. To this end, the quarterly frequency data of nine companies that were continuously traded in the BIST Technology Index between March 2009 and December 2020 were examined with panel-data analysis. In line with the purpose of the research, analyzes were carried out in three different models. First of all, we determined which tests should be performed on the models based on the cross-sectional dependence, homogeneity/heterogeneity, and panel unit root test results obtained for the established models. The results of panel least squares test carried out to determine the effect of R&D and marketing expenditures on stock return showed that the effect of R&D expenditures on stock return was not statistically significant while marketing expenditures had a positive and significant effect on stock return. Analyzes should be continued with cointegration tests according to the characteristics of the two models established to determine the effect of R&D and marketing expenditures on return on assets and return on equity. The results implied a positive and significant relationship between R&D expenditures and return on both assets and equity. While no statistically significant relationship was found between marketing expenditures and return on assets, there was a positive and significant relationship between marketing expenditures and return on equity.

1. Introduction

The interface (approach) of marketing and finance emerges as an important and functional research area that helps to demonstrate the accountability of the marketing department and its activities in the management processes of companies and to create an interdisciplinary bridge for finance and accounting [1]. Srinivasan and Hanssens published the primary study in marketing and finance in 2009. Since then, this research area has become an area of great interest [1]. The discipline of Marketing-Finance has a high-level relationship with marketing with regard to areas such as both asset pricing and corporate finance. This research area focuses on the relationships between marketing-related issues and metrics, including the behavior of financial market participants such as economic and financial analysts, investors, and creditors. The main purpose of this research discipline is to emphasize the significance of marketing considering the investors as stakeholders in order to highlight that marketing and finance should also be taken into account in managerial decisions about firm processes [2].

Marketing departments and their activities are generally carried out in a structure where expenditures are made in companies and return on these expenditures are obtained in the long term. This phenomenon makes it compulsory to evaluate marketing-related activities in managerial processes and to measure them with rational metrics. The high-budget structure of marketing investments and the inability to quantify their return are considered an important leadership problem for the senior managers of the companies. Therefore, the accountability of marketing comes to the fore. The accountability of marketing is defined as the measurement and optimization of the contribution of marketing investments to the performance and value of a firm

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[1]. Marketing includes investment, expenditure, and managerial decision-making processes for a firm within the framework of customer value elements called product, price, promotion, and place (i.e., the 4Ps of marketing). These processes can be defined as marketing inputs. As for marketing outputs, the effect of marketing performance indicators on firm profitability and stock value may be cited [3-4]. It is of utmost importance to reveal the relationship between these input and output elements of marketing in a measurable way in order to provide a quantitative perspective to the decision-making mechanisms of the managers.

The current study aims to investigate the effect of the expenditures made by companies for marketing and therefore R&D/innovation investments on stock return and profitability. To this end, our study primarily touched on the relationship between marketing and R&D expenditures and profitability and stocks theoretically. We presented a summary table by examining the studies conducted on the relevant subject in the literature. Then, the models created to determine the effects of marketing and R&D expenditures on stock return, return on assets, and return on equity were investigated with panel-data analysis method. The findings obtained as a result of the analysis were reported and interpreted. In addition, the results were discussed and evaluated.

2. Conceptual framework

Recent advances in digital channels, alongside data explosion and the emergence of marketing automation, the globalization of markets, and the rise of customer experience as a key priority for companies have increased the significance of understanding how potential marketing outcomes have impacted and may impact firm profitability and firm value [5]. In marketing, innovation is considered to be an important factor that generates firm value, primarily in the market and also in the stock market [6]. Recently, a significant number of studies has focused on the effects of companies' marketing practices and marketing-oriented innovative assets and actions (search engine marketing practices, R&D investments, patents, new product launches, etc.) on the financial performance and value of the firm [7-12].

Operational processes (production, marketing, general management, etc.) in businesses are highly dynamic with the presence of constant innovations. The development and change of the abilities, capabilities, and activities in these processes require firm management to keep up with these changes. It is an important question to be answered by managers whether these practices and investments generate any return for the firm. When the academic literature is examined, various studies are focusing on measuring the effect of these operational activities and practices on the performance of companies in financial proportions [13]. Profitability arises as the most important indicator of firm performance in research. Therefore, the term "performance" is generally used when referring to profitability for companies [14].

The main purpose of examining the relationship between marketing and finance is to investigate the degree to which markets function smoothly [15]. However, there are two different difficulties in determining how successfully this goal can be achieved. *The first difficulty* is related to the capital. The difficulty means that investment decisions must be motivated by *"long-term factors"* (rather than short-term cash flows, for example, without long-term contributions). Therefore, a firm needs investment performance measures that have been proven to create long-term value with regard to management performance. *The second difficulty* is the evaluations to be performed in marketing practices to distinguish between *"effective marketing" and "ineffective marketing"*. To ensure the effectiveness of marketing practices, *inputs* include decisions about marketing actions called "product, price, promotion, and place (4P)" while *outputs* include several potential key performance indicators or metrics for marketing. Expenditure on these marketing practices may affect profitability [3] and stock return [4], thus firm performance. As Abramson et al. [3] and Shulze et al. [4] pointed out, operational activities and expenditures (marketing and innovation expenditures) are of great significance for stock return and profitability, both conceptually and with regard to firm management processes. Analyzing the relationship between these variables is crucial both in order to provide an important input for the decision-making processes of financial investors and to demonstrate its effect on the smooth functioning of the stock markets. Klingenberg et al. [13] suggest that the data is obtained either from the financial reports of publicly traded companies or in the form of perceptual data through surveys in order to analyze the relationship between a firm's operational practices (marketing and innovation) and its performance. The researchers claim that there are inconsistencies in the results obtained with these data (data obtained by making use of financial reports and survey data concurrently) [13]. Therefore, marketing and R&D expenditures, stock return, and profitability are examined through secondary data in the present study.

3. Literature

The studies conducted in the last 20 years in the national and international literature on this research area and their findings are summarized in Table 1.

Table 1. Summary of the literature on the research area

-Author(s) **-Sample** **-Time Period**	**Variables**	**Method**	**Findings**
Wakelin [16] UK stock exchange 1945-1983	Innovation and R&D investments	Least squares method	Separating the firms according to their innovation histories, the rate of return to R&D is much higher for innovative than non-innovative firms.
Hanel and St-pierre [17] Firms in the S&P compustant database 1972-1991	R&D expenditures and operating profit	Regression analysis	It has been determined that R&D has a direct, positive effect on profitability.
Öztürk [18] BIST firms 2002-2006	Market and book value of the firm's equity, monopoly power and R&D investments	Multiple regression analysis	It has been determined that R&D investments have statistically significant and positive effects on firm value.
Çifci et al. [19] BIST firms 2000-2008	Marketing expenses, general management expenses, total asset size and net profit/loss for the period	Panel data analysis	According to the findings of the study; marketing expenditures, general administration expenditures and total asset size have positive impacts on the performance of the business and it has been identified that among them the most important variable is the marketing expenditure.
Ehie and Olibe [20] US manufacturing and service firms 1990-2007	R&D expenditures and market value	Regression analysis	It has been determined that R&D investments contribute positively to firm performance.
Parcharidis and Varsakelis [21] Athens stock exchange manufacturing and computer firms 1996-2004	R&D expenditures and Tobin's q	Panel data analysis	It has been determined that R&D investments have an effect on the market value of the firms.
Topuz and Akşit [22] BIST Food industry 2000-2013	Marketing sales and distribution expenses, return on stock	Panel regression analysis	It has been determined that marketing expenditures have a positive effect on stock returns.
Doğan and Mecek [23] BIST Manufacturing Industry 200-2012	Marketing expenditures, return on assets, return on equity and Tobin's Q	Multiple regression and correlation analysis	A positive and statistically significant relationship was found between marketing expenditures and firm value.
Yücel and Ahmetoğulları [24] BIST technology, software and informatics sector 2000-2014	R&D expenses, change in net income and earnings per share	Regression analysis	There is a positive relationship between the change in R&D expenditures and the change in net profit for the same period. In addition, it has been determined that the effect of R&D expenses on earnings per share has a lag of three periods.
Alper and Aydoğan [25] BIST Chemical industry 2001-2014	R&D expenditures, return on assets, return on equity, firm size and financial leverage ratio	Dinamic panel data analysis	Study findings demonstrated that R&D expenses affected corporate financial performance positively and significantly with one year lag.
Işık et al. [26] BIST firms 2008-2014	R&D spending, sales and profitability	Panel data analysis	The analysis results show that; R&D spending have a positive and significant effect on profitability and sales.
Öztürk and Dülgeroğlu [27] BIST Manufacturing Industry 2007-2015	Marketing expenditure, general administrative expense, and sales	Panel regression analysis	It has been determined that the sales performance is stronger in companies whose marketing expenses are higher than their administrative expenses.
Polat and Elmas [28] BIST Metal Goods, Machinery and Equipment Production industry 2007-2015	R&D investments, profitability in sales and assets, growth and logarithm, liabilities/assets	Panel data analysis	The effect of R&D investments on firm performance has been determined as negative.
Lee et al. [29] Arts and culture firms in the USA 2003-2013	Marketing expenditure and total revenue	Regression analysis	It has been determined that marketing expenditures have a positive effect on total revenue.
Özer and Gülençer [30] Borsa Istanbul cement sector 2009-2013	R&D expenditure and intensity, marketing expenditure and intensity, stock value	Panel regression analysis	It was found that marketing expenditures had a positive effect on the stock value, and although R&D expenditures did not have a significant effect on the stock value directly, it was concluded that the intensity of R&D expenditures positively affected the stock value.

Table 1. *continued*

Serçek et al. [31] BIST Tourism Sector 2012-2015	Marketing expenses/net sales, marketing expenses/cost of operations and sales, debt ratio, firm size, return on asset, return on equity and operating cash flow	Panel data analysis	A statistically insignificant relationship was found between marketing expenses and profitability.
Yıldırım and Sakarya [32] BIST technology and informatics sector 2009-2016	R&D expenditures, return on assets and return on equity	Panel data analysis	It has been concluded that R&D expenditures have a significant and positive effect on the return on assets and equity.
Ayaydın et al. [33] Borsa Istanbul Technology 2008-2018	R&D investments, MV/BV, earnings per share and P/E	Dynamic panel data analysis	The results of the analysis indicated that there is a positive relationship between R & D investments and MV/BV, earnings per share and P/E.
Aydın and Kaya Aydın [34] Airline companies of various countries selected by convenience sampling method 2016	Revenue passenger kilometer, liquidity, Skytrax ranking and fleet numbers	Stochastic frontier analysis	According to the analysis, as the liquidity of the companies increase, the revenue passenger kilometer decreases. As the number of Skytrax ranking increases, revenue passenger kilometer decreases.
Liu et al. [35] Chinese manufacturing firms listed on the Shenzhen and Shanghai Stock Exchange 2012-2016	R&D investment and intensity, Tobin's q	Tobit regression analysis	It has been observed that R&D expenditures have an inverted U-shaped relationship with firm value, and increases in R&D investments exceeding a certain point are likely to result in lower firm value.

There is a significant number of studies in the literature analyzing the effect of marketing expenditures and R&D expenditures directly related to marketing on firm profitability and stock value. Some of these studies are presented in Table 1 with a systematic point of view. The studies in Table 1 were analyzed based on the author and year of the study, information about the samples used in the study, the years covered with the analyzed data, variables examined in the study, analysis methods, and information about the results of the study. Table 1 shows that:

- The relationship between marketing expenditures and firm profitability has been investigated by many authors from 2001 to 2020.
- When the scope of samples is analyzed, companies within the scope of *"BIST, British Stock Exchange, S&P Database, and Shenzhen and Shanghai Stock Exchange"* were investigated. In these stock exchanges and databases, many different fields and industries have been studied such as manufacturing industry, computer companies, food industry, technology, software and information industry, chemical industry, metal goods and machinery industry, art and culture companies, cement industry, tourism industry, technology industry, and airline companies.
- When the variables included in the analyzes are examined, the independent variables such as *"productivity, R&D expenditures, firm monopoly power, marketing expenditures, sales and distribution expenditures, general administrative expenditures, R&D expenditure intensity, and paid passenger mileage"* are associated with dependent variables including *"operating profitability, book value, net profit/loss for a given period, market value, Tobin's Q value, stock return, return on equity, change in net profit, profit per stock, firm size, financial leverage ratio, total revenue, and operating cash flow"*.
- The methods used during the analyzes consist of many causal and relational analyzes such as *"Least Squares Estimation, Regression Analysis, Multiple Regression Analysis, Panel-Data Analysis, Cross-Sectional/Stepwise Regression Analysis, Correlation Analysis, Dynamic Panel-Data Analysis, Stochastic Analysis of Boundary, and Tobit Regression Analysis"*.
- The results of the research have revealed *some positive and significant relationships between marketing and R&D expenditures and firm profitability and stock return*.

4. Dataset and method

Within the scope of the present study examining the effects of companies' marketing and R&D expenditures on stock return and profitability, we conducted research on the companies included in the BIST Technology Index. While analyzing the time period between 2009 and 2020, we determined that 9 companies were traded in the index continuously during the aforementioned period and included these companies in the sample. The dataset of the research consists of quarterly R&D and marketing expenditures, stock returns, return on assets, and return on equity of 9 companies in the BIST Technology Index during the period between March 2009 and December 2020. When the studies in the literature on the relationship or effect between firm performance and marketing and R&D expenditures are examined, the studies focusing on the relationship between innovative corporate/operational practices and firm performance are in the majority. On the other hand, some other studies analyze the R&D and marketing expenditures of the companies for the activities in the product development and marketing processes. The current study examines the impact of "marketing and R&D expenditures" on both the market and the return on assets (ROA) and return on equity (ROE) of companies. Therefore, the present study differs from other studies and contributes to the literature by comparing the non-operational performance indicators (stock value and stock return) and operational performance indicators (ROA and ROE) of R&D and marketing expenditures. The variables included in the study were obtained from the financial statements published on the official website of Borsa İstanbul's Public Disclosure Platform (www.kap.org.tr). They are separated into dependent and independent variables and presented in Table 2 together with their abbreviations in the analysis.

Table 2. Variables in the model

	Variable name	**Abbrevi-ation**	**Data Period**
Dependent Variables	Stock Return	SR	2009:03 – 2020:12
	Return on Asset	ROA	2009:03 – 2020:12
	Return on Equity	ROE	2009:03 – 2020:12
Independent Variables	R&D Expenditures	R&D	2009:03 – 2020:12
	Marketing Expenditures	ME	2009:03 – 2020:12

We preferred to use panel-data analysis since both the variables belonging to the companies and the time-series data of these variables were present in the study. This is because the panel-data analysis method allows the time-series data of the cross-sectional observations of each firm in the sample to be combined and analyzed.

The research models of the current study, conducted to determine the effect of R&D and marketing expenditures on stock return, return on assets, and return on equity, were formed as follows:

$$SR = \beta_0 + \beta_1(\text{ME}) + \beta_2(\text{R\&D}) + \varepsilon \quad \textit{(model 1)}$$

$$ROA = \beta_0 + \beta_1(\text{ME}) + \beta_2(\text{R\&D}) + \varepsilon \quad \textit{(model 2)}$$

$$ROE = \beta_0 + \beta_1(\text{ME}) + \beta_2(\text{R\&D}) + \varepsilon \quad \textit{(model 3)}$$

5. Results

EViews 12, Stata 15, and Gauss programs were used in the present study, and three models established for the purpose of the research were analyzed sequentially. In the first stage of panel-data analysis, cross-sectional dependence tests should be performed. This is because some authors state that the results obtained in the analyzes carried out without considering the cross-sectional dependence will be biased and inconsistent [36]. In addition, it is possible to determine which unit root tests are suitable to apply to the variables based on the results of the cross-sectional dependence test.

5.1. Analysis results of Model 1

$$\boldsymbol{SR} = \beta_0 + \beta_1(\mathbf{ME}) + \beta_2(\mathbf{R\&D}) + \boldsymbol{\varepsilon}$$

The dependent variable of Model 1 is stock return, and its independent variables consist of marketing expenditures and R&D expenditures. The results of cross-sectional dependence test of these variables are presented in Table 3.

Table 3. Cross-sectional dependence test results of Model 1

	SR		ME		R&D	
Test	Stat.	p	Stat	p	Stat	P
B-P LM	62.47	0.00	739.85	0.00	78.76	0.00
P LM	3.11	0.00	82.94	0.00	5.04	0.00
Bias-cs LM	3.02	0.00	82.85	0.00	4.94	0.00
P CD	3.47	0.00	21.05	0.0	0.48	0.62

Abbreviations: B-P LM: Breusch-Pagan LM,
P LM: Pesaran scaled LM, B-cs LM: Bias-corrected scaled LM,
P CD: Pesaran CD
H_0:No Cross Section Dependency, p. %5

The cross-sectional dependence tests given in Table 3 have various characteristics depending on the use scenario. For instance, it is assumed that the test developed by Breusch and Pagan [38] (Breusch-Pagan test) will be used when the time dimension (T) is larger than the cross-sectional dimension (N) [39]. Since the time dimension (T=12 years*4 periods) of the present study was larger than the cross-sectional dimension (N=9 companies), the Breusch-Pagan LM cross-sectional dependence test results were evaluated. As a result, H_0 is not supported since the result of test statistics for all variables is $p<0.05$. Therefore, there is a cross-sectional dependence in the series. For this reason, it is appropriate to conduct second generation unit root tests in the further phases of the analysis. The results of the second generation unit root tests Bai and Ng's PANIC and Pesaran's CIPS are presented in Table 4.

Table 4. Unit root test results of Model 1

	SR		ME		R&D	
Test	Stat	P	Stat	P	Stat	P
B-NG	1.47	0.00	1.15	0.00	1.99	0.00
P CIPS	-5.08	< 0.01	-3.78	< 0.01	-2.79	< 0.01

Abbreviations: B-NG: Bai and NG – PANIC, P CIPS: Pesaran CIPS H_0:No Unit Root, p. %5

Table 4 demonstrates that Bai and Ng's unit root test results were $p<0.05$ for all variables. Therefore, the variables did not contain a unit root. In other words, the series was stationary at level I(0). Since $p<0.01$ was obtained for all variables in the Pesaran's CIPS unit root test results, H_0 was not supported, and it was confirmed that the series was stationary at level. For this reason, the Panel Least Squares Method (LSM) should be used in the further phases of the analysis. In order to utilize the Panel LSM, it is necessary to determine the fixed, random or pooled effects the model includes. The analysis should be performed once the suitable effect is selected. The results of the tests performed to examine the influences of these effects both on time and horizontal dimension are given in Table 5.

Table 5. Panel OLS effect test results

		Statistics	P
Cross Section	Random Effect (Hausman)	0.1503	0.9276
	Fixed Effect (Chow F)	1.4873	0.1597
	Pooled Effect (LM Breusch Pagan)	0.4458	0.5043
Period	Random Effect (Hausman)	1.8882	0.3890
	Fixed Effect (Chow F)	0.8665	0.7182
	Pooled Effect (LM Breusch Pagan)	0.5271	0.4678

Based on the results in Table 5, all significance values were determined to be $p>0.05$. Therefore, all H_0 hypotheses are supported. The hypotheses of the Hausman test are "H_0: Random effect, H_1: Fixed effect" [40], the hypotheses of the Chow F-test are "H_0: Pooled effect, H_1: Fixed effect", and the hypotheses of the LM test are "H_0: Pooled effect, H_1: Random effect". While the Hausman test carried out for both cross-section and period indicates that the model includes random effects, Chow F-test and LM Breusch-Pagan test results demonstrate that the model contains pooled effects. Since the majority of the tests showed that the pooled effect was suitable for the model, the least squares method was used under the pooled effects for both cross-section and period. The panel LSM results are given in Table 6.

Table 6. Panel OLS results

Variables	Coefficients	Std. Error	t-statistics	P
R&D	-3.61E-07	1.61E-06	-0.2250	0.8220
ME	15.3489	1.69E-06	-0.3011	0.0035
C	30.8184	11.8743	2.5953	0.0098

The results in Table 6 enabled us to determine that marketing expenditures had a positive (15.3489) and significant ($p<0.05$) effect on stock return. On the other hand, R&D expenditures had no statistically significant effect on stock return. Based on these findings, the model coefficients extracted in line with the purpose of the study are as follows:

$$SR = 30.8184 + 15.3489(\text{ME}) + \varepsilon$$

5.2. Analysis results of Model 2

$$\boldsymbol{ROA} = \beta_{\mathbf{0}} + \beta_{\mathbf{1}}(\mathbf{ME}) + \beta_{\mathbf{2}}(\mathbf{R\&D}\ ) + \boldsymbol{\varepsilon}$$

Since the cross-sectional dependence tests of the independent variables of Model 2 were performed during the analyzes of Model 1 and the cross-sectional dependence was established, only the cross-sectional dependence tests of the dependent variable, return on assets (ROA), were performed for Model 2. The relevant test results are presented in Table 7.

Table 7. Cross-sectional dependence test results of Model 2

	ROA	
Test	Statistics	P
Breusch-Pagan LM	213.5667	0.0000
Pesaran scaled LM	20.9264	0.0000
Bias-corrected scaled LM	20.8306	0.0000
Pesaran CD	3.4724	0.5581

Since the Breusch-Pagan LM test statistic was $p<0.05$ for the variable ROA, there was a cross-sectional dependence in the series. In addition, the results of the cross-sectional dependence test for the remains of Model 2 are given in Table 8.

Table 8. Cross-sectional dependence test results of Model 2

Test	**Statistics**	**P**
Breusch-Pagan LM	162.6277	0.0000
Pesaran scaled LM	14.9232	0.0000
Pesaran CD	0.0629	0.9498

The results in Table 8 prove that the presence of cross-sectional dependence in Model 2 was established. Therefore, analyzes should be continued with second generation unit root tests. As the independent variables of all models are the same and the unit root test was carried out for the independent variables in Model 1, unit root test was performed only for the dependent variable of Model 2 at this stage. The results of unit root test performed for ROA are shown in Table 9.

Table 9. Unit root test results of Model 2

	ROA		First Difference ROA	
Test	Statistics	p	Statistics	p
Bai and NG – PANIC	1.3947	0.1631	0.1526	0.0000
Pesaran CIPS	-2.6594	>0.10	-4.8151	<0.01

The unit root test results indicated that the variable ROA had unit root at level and the tests were repeated with the first difference of the series. Accordingly, the variable ROA was I(1). The independent variables of the model are I(0) while the dependent variable is I(1). Therefore, the variables have stationarity at different levels. At this stage, it is suitable to perform homogeneity/heterogeneity tests. The results of the

Hsiao panel homogeneity test are presented in Table 10.

Table 10. Panel homogeneity test results of Model 2

Hypotheses	F Statistics	p
H_1	5.4483	1.32E-10
H_2	1.3298	0.0266
H_3	9.5272	3.81E-12

Specification Tests of Hsiao (1986)
H1 = Null Hypothesis : panel is homogeneous vs Alternative Hypothesis : H2
H2 = Null Hypothesis : H3 vs Alternative Hypothesis : panel is heterogeneous
H3 = Null Hypothesis : panel is homogeneous vs Alternative Hypothesis : panel is partially homogeneous

Table 10 demonstrates that all hypotheses have a value of $p<0.05$ at a significance level of 5%. Therefore, the H_0 hypotheses are not supported. As a result, we determined that not all slope coefficients in Model 2 have equal cross-sectional coefficients. Therefore, the coefficients in the model have a heterogeneous structure. The findings obtained up to this stage of the analysis for Model 2 indicate the presence of cross-sectional dependence, heterogeneity, and stationarity of the variables at different levels. Based on all these results, second generation cointegration tests should be carried out in the further phases of the analysis. Table 11 presents the Westerlund ECM cointegration test results for Model 2.

Table 11. Cointegration test results of Model 2

	Statistics	noCD p value	Bootstrap P value
g-tau	0.330	0.001	0.031
g-alpha	0.788	0.007	0.038
p-tau	-1.222	0.003	0.042
p-alpha	-1.403	0.002	0.044

The bootstrap results of Westerlund ECM g-Tau and g-Alpha tests should be evaluated with regard to heterogeneity and cross-sectional dependence [37]. Since the results had a value of $p<0.05$ at a significance level of 5%, the series were cointegrated. Cointegration coefficients should be determined at the last stage of the analysis for the cointegrated variables. Panel AR Distributed Lag Models (Mean Group) Common Correlated Effects (Panel ARDL MG-CCE) is the panel cointegration estimator that should be applied based on the previously specified characteristics of the model such as cross-sectional dependence, heterogeneity, I(1) for the dependent variable, and I(0) for the independent variables. Panel ARDL MG-CCE test results are presented in Table 12.

The results in Table 12 show that no statistically significant relationship was found between marketing expenditures and return on assets. On the other hand, R&D expenditures have a positive (1.6907) and significant ($p<0.05$) relationship with return on assets. Based on the findings, the model 2 coefficients equation extracted in line with the purpose of the study is as follows:

$$ROA = 4.4047 + 1.6907(\text{R\&D}) + \varepsilon$$

Table 12. Panel ARDL MG-CCE test results of Model 2

ROA	Coefficients	Std. Error	Z	P
R&D	1.6907	5.6987	2.9633	0.0032
ME	5.8813	5.9908	0.9814	0.3269
C	4.4047	0.4159	10.5899	0.0000
R^2	0.979			
Adj. R^2	0.967			
F-Stat	336.45 (0.000)			

5.3. Analysis results of Model 3

$$\boldsymbol{ROE} = \beta_0 + \beta_1(\mathbf{ME}) + \beta_2(\mathbf{R\&D}) + \boldsymbol{\varepsilon}$$

The cross-sectional dependence test results of the dependent variable, return on equity (ROE), of Model 3 established for the purpose of the study are shown in Table 13.

Table 13. Cross-sectional dependence test results of Model 3

	ROE	
Test	Statistics	P
Breusch-Pagan LM	169.7672	0.0000
Pesaran scaled LM	15.7646	0.0000
Bias-corrected scaled LM	15.6688	0.0000
Pesaran CD	-0.7353	0.4621

Since the Breusch-Pagan LM test statistic was $p<0.05$ for the variable ROE, there was a cross-sectional dependence in the series. However, cross-sectional dependence for the remains of Model 3 was established based on the Breusch-Pagan LM test results (statistics: 183.8536 and p: 0.000). Therefore, the analyzes should be continued with second generation unit root tests that must be carried out for cross-sectional dependence. The unit root test results for ROE are presented in Table 14.

Table 14. Unit root test results of Model 3

	ROE		First Difference ROE	
Test	Statistics	p	Statistics	P
Bai and NG – PANIC	1.4867	0.6643	5.4442	0.0000
Pesaran CIPS	-6.6594	>0.10	-2.4196	<0.05

The unit root test results in Table 14 demonstrated that the variable ROE was not stationary at level and the tests were repeated with the first difference of the variable. As a result, we determined that the variable ROE was I(1). Following this stage, the analyzes were continued with homogeneity/heterogeneity tests. The Hsiao test results are given in Table 15.

Table 15. Panel homogeneity test results of Model 3

Hypotheses	F Statistics	P
H_1	7.4469	2.12E-15
H_2	4.6172	2.05E-05
H_3	9.6172	2.87E-12

Specification Tests of Hsiao (1986)
H1 = Null Hypothesis : panel is homogeneous vs Alternative Hypothesis : H2
H2 = Null Hypothesis : H3 vs Alternative Hypothesis : panel is heterogeneous
H3 = Null Hypothesis : panel is homogeneous vs Alternative Hypothesis : panel is partially homogeneous

Since all hypotheses had a value of $p<0.05$ at a significance level of 5% in Table 15, the coefficients in the model had a heterogeneous structure. As all the characteristics in Model 2 are also valid for Model 3, all tests in Model 2 were repeated for Model 3 after this stage. Therefore, the first test performed in the continuation of the analysis is the second generation cointegration test. Table 16 demonstrates the cointegration test results for Model 3.

Table 16. Cointegration test results of Model 3

	Statistics	Bootstrap p value
g-tau	0.622	0.003
g-alpha	0.510	0.014

The results of the Westerlund ECM test, one of the second generation cointegration tests carried out with regard to heterogeneity and cross-sectional dependence, had a value of $p<0.05$ at a significance level of 5%. Therefore, the series were cointegrated. Cointegration coefficients should be determined at the last stage of the analysis for the cointegrated variables. Panel AR Distributed Lag Models (Mean Group) Common Correlated Effects (Panel ARDL MG-CCE) test is the panel cointegration estimator that should be applied due to the characteristics of the model such as cross-sectional dependence, heterogeneity, and stationarity of the variables at different levels. Panel ARDL MG-CCE test results are presented in Table 17.

Table 17. Panel ARDL MG-CCE test results of Model 3

ROE	Coefficients	Std. Error	Z	P
R&D	8.7508	1.2007	0.7314	0.0049
ME	2.8407	1.2607	2.2540	0.0247
C	13.1640	0.8750	15.0443	0.0000
R^2	0.8987			
Adj. R^2	0.8769			
F-Stat	411.51 (0.000)			

The results in Table 17 point out a statistically significant and positive relationship between R&D expenditures and return on equity. In addition, the relationship between marketing expenditures and return on equity was positive and significant. Based on the findings, the model 3 coefficients equation extracted in line with the purpose of the study is as follows:

$$ROE = 13.1640 + 2.8407(\text{ME}) + 8.7508(\text{R\&D}) + \varepsilon$$

6. Conclusion

Expenditures and investments made for R&D/innovation are crucial indicators for companies in particular and countries in general. This is because the growth, development, and sustainability of countries depend on the R&D investments made by the companies and the emergence of products with high added value as a result [41]. For this reason, companies that have more added value, especially in the technology industry, are supported with R&D investments. Most studies emphasize that R&D activities and expenditures, which are of strategic importance for companies to gain competitive advantage, grow, and be efficient, are also crucial for the markets [42-47]. The reason is that the positive or negative perception of the expenditures by the markets and investors may affect the stock prices in publicly traded companies. In addition, these expenditures may contribute to firm profitability, as well as have a destructive effect on profitability if they become excessive. On the other hand, the effect of marketing expenditures on firm performance, profitability, and stock value is one of the most frequently studied research areas in the literature. The common idea is that the effectiveness of marketing activities is more important than their amount [48]. However, most of the effective activities are proportional to the amount of expenditure.

The present study examines the effects of R&D and marketing expenditures on stock return and profitability and includes research with panel data analysis for companies in the BIST Technology Industry. The quarterly frequency data of R&D expenditures, marketing expenditures, stock return, return on assets, and return on equity were included in the analyzes in three different models in line with the purpose of the study. *Model 1* focuses on "the effect of R&D and marketing expenditures on stock return". Cross-sectional dependency test, the first test to be carried out in panel data analysis for the model, was performed and second generation unit root tests were completed to determine the cross-sectional dependence in the series. Since all the variables were stationary at level, the analysis was continued with panel least squares test. After ensuring that the suitable effect for the model is pooled effect, the coefficients obtained as a result of the findings from the panel least squares method were included in the model:

$$SR = 30.8184 + 15.3489(\text{ME}) + \varepsilon$$

As a result, the effect of R&D expenditures on stock return ($p>0.05$) was not statistically significant; however, we determined that marketing expenditures had a positive and significant effect on stock return.

Cross-sectional dependency tests were performed for *Model 2* that was created to determine the "effect of R&D and marketing expenditures on ROA". After the cross-sectional dependence was established, the analyzes were continued with second-generation unit root tests. Since ROA, the dependent variable of Model 2, is I(1) and the independent variables are I(0), we decided to continue the analyzes with cointegration tests. In order to determine which cointegration tests are suitable, it is necessary to confirm whether the slope coefficients in the model are homogeneous or heterogeneous in the first place. To this end, the Hsiao test was performed and the coefficients had a heterogeneous structure. Due to the aforementioned characteristics of Model 2, it has been appropriate to continue the analyzes with the Panel AR Distributed Lag Models (Mean Group) Common Correlated Effects (Panel ARDL MG-CCE) test. As a result, no statistically significant relationship was found between marketing expenditures and ROA. On the other hand, R&D expenditures have a positive (1.6907) and significant ($p<0.05$) relationship with ROA. Based on the obtained findings, the Model 2 coefficients equation was formed as follows:

$$ROA = 4.4047 + 1.6907(\text{R\&D E}) + \varepsilon$$

As for *Model 3*, the presence of cross-sectional dependence was established, the dependent variable, ROE, was I(1), the independent variables were I(0), and the homogeneity/heterogeneity test demonstrated that the model coefficients had a heterogeneous structure. Due to the aforementioned characteristics of Model 3, this model was further analyzed with the Panel ARDL MG-CCE test. As a result, a statistically significant and positive relationship was found between both R&D and marketing expenditures and ROE. Therefore, the Model 3 coefficients equation was formed as follows:

$$ROE = 13.1640 + 2.8407(\text{ME}) + 8.7508(\text{R\&D}) + \varepsilon$$

When our findings and the literature are compared, R&D expenditures had a positive effect on profitability, as suggested by [17], [24-26], [32] as well. These studies support the result of the current study. On the other hand, the finding obtained by [28] contradicts the result of the current study. The finding indicating that marketing expenditures have a positive effect on stock return is in line with the result obtained by [22]. The positive effect of marketing expenditures on firm performance was also found by [19], [23], which supports our results. On the other hand, [31] found a statistically insignificant relationship between marketing expenditures and profitability, which contradicts our results.

Companies' R&D and marketing expenditures have an impact on both the markets and the level of profitability. R&D and marketing investments are of utmost importance for maintaining sustainability for companies that can keep up with the requirements of the period and cope with intense competition. For this reason, companies should plan their expenditures and allocate appropriate amounts of R&D and marketing budgets, and governments should support these activities.

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RESEARCH ARTICLE

Novel approach for nonlinear time-fractional Sharma-Tasso-Olever equation using Elzaki transform

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ABSTRACT

In this article, we demonstrated the study of the time-fractional nonlinear Sharma-Tasso-Olever (STO) equation with different initial conditions. The novel technique, which is the mixture of the *q*-homotopy analysis method and the new integral transform known as Elzaki transform called, *q*-homotopy analysis Elzaki transform method (*q*-HAETM) implemented to find the adequate approximated solution of the considered problems. The wave solutions of the STO equation play a vital role in the nonlinear wave model for coastal and harbor designs. The demonstration of the considered scheme is done by carrying out some examples of time-fractional STO equations with different initial approximations. *q*-HAETM offers us to modulate the range of convergence of the series solution using $\hbar$, called the auxiliary parameter or convergence control parameter. By performing appropriate numerical simulations, the effectiveness and reliability of the considered technique are validated. The implementation of the new integral transform called the Elzaki transform along with the reliable analytical technique called the *q*-homotopy analysis method to examine the time-fractional nonlinear STO equation displays the novelty of the presented work. The obtained findings show that the proposed method is very gratifying and examines the complex nonlinear challenges that arise in science and innovation.

1. Introduction

Fractional calculus (FC) is an incipient tool in the field of mathematics with strong execution in the diverse areas of science and engineering. FC is defined as the generalization of the classical calculus where we study the integral and differential operators of fractional order, even can be lengthened to a complex set. In the past few decades, many mathematical minds have strengthened this concept and designed various fractional differential and integral operators [1, 2]. The progressive functioning of the demonstration of the classical derivatives is done using the nonlocality of the fractional operators. Fractional operators are undeniably used to define sophisticated memory and a range of objects that may be studied using normal mathematical methods such as classical differential calculus. Latterly, fractional operators with nonlocality have been demonstrated and foreseen in the absence of a singular kernel. However, we are still at the initial stage of implementing the concept of FC in various areas of research. Nowadays, FC is a very promising tool due to its larger applications in the dynamics of complex nonlinear phenomena.

The idea of fractional calculus has its origin in the correspondence between *L'hospital* and *Leibniz*. Additionally, it was shown that FC is much more suitable to handle most complex real-world issues than classical calculus. Fractional calculus's richness in applied research has grown over time. Several studies have now proved its potential to deal with a variety of issues., particularly in the fields of science domains like robotics [3], viscoelasticity [4], image processing [5],

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biological population models [6], and several more [7-27]. Compare to the integer-order differential equations, fractional counterparts are much more reserved to get adequate exact solutions for highly nonlinear problems. For this purpose, many numerical and analytical techniques are developed to solve this category of problems.

Along with the development of the classical theory in physics, the concept of fractional calculus and its operators has dragged much attention due to its importance in applied physics such as plasma physics, chemical kinematics, fluid mechanics, optical fibres, probability, statistics, etc. Although it has a long history, in recent decades, scientists have been attracted to fractional differential equations (FDE) due to its extensive applications in wide areas of science and engineering upon which few systems which are inherently nonlinear in nature are much studied by physicists, mathematicians, engineers, meteorologists, etc.

Nonlinear fractional differential equations (NLFDEs) which describe the change in the variables over time was difficult to solve and unpredictable and are most commonly approximated by linear equations. The basic common approach to solve NLFDEs is either to change the variables so that, the solution for the equation will become simpler like the linear equation or transform the problem that can result in a linear equation. Sometimes, the problem will be converted into one or more ordinary differential equation(s) which may or may not be solvable further. For example, weather forecasting is one of the non-linear behaviour systems in which, some parameters are complete of random behaviour, where simple changes in one part of the system produce complex results throughout the system. Resulting in difficulty with accurate long-term weather forecasts even with current advanced technology. Therefore, the investigation of the exact solutions for NLFDEs plays an important role in the study of a nonlinear system of equations such as Navier–Stokes equations of fluid dynamics, Nonlinear optics, Nonlinear Schrödinger equation, Boltzmann equation, General relativity, Van der Pol oscillator, etc.

The inquisition of soliton results of complex nonlinear evolution equations has great significance in the examination of the nonlinear field. These solutions are very informative towards the essential nonlinear science aspects. In this article, we are investigating the nonlinear time-fractional STO equation [28] given as follows:

$$D_t^{\alpha} u(x,t) + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0,\ t > 0,\ 0 < \alpha \leq 1, \tag{1}$$

where a is the random real constant, u is the dependent variable, t and x are the temporal and spatial variables respectively. The STO equation is similar to the KdV equation which can describe evolutionary physics phenomena and interaction with nonlinear waves, like continuum mechanics, fluid dynamics, solitons and turbulence, aerodynamics, etc. The STO equation incorporates the double nonlinear term and linear dispersive term. The solution of the STO equation has been acquired by numerous methods. The Backlund transformation and Hirota's direct method have been implemented to get the fusion and fission of the solitary wave solutions. It's been revealed that the fission of solutions is obtained for $a < 0$ and when $a > 0$ waves depict only the fusion of solutions [29]. The potential symmetries and the generalized symmetries of the STO equation are studied in [30, 31]. Furthermore, to examine the soliton solutions of nonlinear PDEs are analyzed by numerous effective methods so far, like Hirota's method [32], Scattering transformation [33], the First integral method [34, 35], Kudryashov method [36], Extended homoclinic test function method [37, 38], Functional variable method [39], Ansatz method and simplest equation approach [40-42], and others. Various researchers across the globe have given many methods and approaches to solve the nonlinear differential equations among which, Sharma–Tasso–Olver equation which is popularly known as the STO equation has not been much investigated. With this motivation, this work highlights the new generalized novel approach for the nonlinear time-fractional Sharma-Tasso-Olever equation using the Elzaki transform.

To solve linear and nonlinear problems, a semi-analytical tool, known as the homotopy analysis method (HAM) is a very efficient scheme recommended and demonstrated by *Liao* [43-45]. Further, for solving nonlinear problems, the q-homotopy analysis method (q-HAM) as a furnished concept of HAM was introduced by El-Tavil and Hussain [46, 47]. Latterly, the combination of the semi-analytical schemes with the Laplace transform is hired to scrutinize nonlinear equations such as Abel integral equation [48], nonlinear fractional shock wave equation [49], nonlinear boundary value problem on the semi-infinite domain [50], two-dimensional Burger's equation [51], class of nonlinear differential equations [52], nonlinear fractional Zakharov-Kuznetsov equation [53], fractional Klein-Gordon-Schrödinger equations [54], fractional coupled Burger's equations [55], and so on.

The study of the nonlinear STO equation using various numerical and analytical techniques is covered in a large body of literature. The innovative aspect of the current study is the investigation of the nonlinear time-fractional Sharma-Tasso-Olever equation utilizing a powerful analytical tool known as the q-homotopy analysis Elzaki transform method. The primary goal of this work is to use the new integral transform known as the Elzaki transform to investigate the fractional behaviour of the problem under consideration. The presented work has not been performed before using the considered algorithm.

In the present work, we investigate the reliability and effectiveness of the q-homotopy analysis Elzaki transform method (q-HAETM) [56] for solving the time-fractional nonlinear STO equation. The considered technique is the amalgamation of the Elzaki transform (ET) scheme and the q-homotopy analysis method (q-HAM). The Elzaki transform is the new integral transform obtained by the classical Fourier integral, which was presented by Tarig Elzaki [57] to alleviate the procedure of addressing the solutions for ordinary and partial differential equations. The combination of an Elzaki transform with the decomposition algorithm is applied to solve the numerous nonlinear partial differential equations [58], ADM Elzaki and VIM Elzaki [59], homotopy perturbation Elzaki transform method [60], the nonlinear regularized long-wave models are studied with the help of Elzaki transform in [61], and so on. The benefits of the q-HAETM include not requiring discretization, linearization, perturbations, or any rigid assumptions, significantly reducing the complexity of complex computations, promising a wide convergence region, offering a non-local effect, and not requiring complex polynomials, integrations, or physical parameter calculations. To limit the convergence zone and frequent convergence of the obtained solution to a minimum tolerable region, the studied approach is also natured by auxiliary and homotopy parameters. It produces more digestible outcomes for the identical grid point and series solution sequence. Additionally, the technology under consideration preserves greater accuracy despite requiring less time, making it incredibly efficient and trustworthy. The feasibility and optimism of the considered strategy are demonstrated by its capacity to provide highly precise precision, a large convergence range, and a straightforward solution technique.

The rest of the work is organized as follows: Section 2 covers prefaces of the fractional integral in Reimann-Liouville sense, ET, and Caputo fractional derivative. The fundamental notion of the investigated methodology is explained in Section 3, and the results for the time-fractional STO equation are discussed in Section 4. Plots are used to explain the responsiveness and pattern of the acquired fractional-order findings. The numerical simulations of the results obtained using q-HAETM are cited in comparison with ADM, HPM, and OHAM. The final section contains comments on the findings obtained.

2. Preliminaries

Here we present some basic notions of Fractional operators and the Elzaki transform:

Definition 1. The fractional Riemann-Liouville integral of a function $f(t) \in C_\mu (\mu \geq -1)$, is presented [1] by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\vartheta)^{\alpha-1} f(\vartheta) d\vartheta, \tag{2}$$

$$J^0 f(t) = f(t). \tag{3}$$

Definition 2. The derivative with fractional order α of $f \in C_{-1}^n$ in the Caputo sense [1] is:

$$D_t^\alpha f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\vartheta)^{n-\alpha-1} f^{(n)}(\vartheta) d\vartheta, & \alpha \in (n-1, n), n \in \mathbb{N}. \end{cases} \tag{4}$$

Definition 3. The Elzaki transform (ET) of a function $f(t)$ is demarcated as follows [57]:

$$E\{f(t)\} = \tilde{f}(s) = s \int_0^\infty e^{-\frac{t}{s}} f(t) dt.$$

The ET of some basic functions are given below [57]

$E\{t^n\} = n!\, s^{n+2}$, where $n = 0,1,2,3, \ldots$

$E\{e^{at}\} = \frac{s^2}{1-as}$,

$E\{\sin(at)\} = \frac{as^3}{1+a^2s^2}$,

$E\{\cos(at)\} = \frac{as^2}{1+a^2s^2}$,

$E\{\sinh(at)\} = \frac{as^3}{1-a^2s^2}$,

$E\{\cosh(at)\} = \frac{as^2}{1-a^2s^2}$.

Definition 4. The ET of a derivative in Eq. (4) is presented as [60]

$$E[D_t^\alpha f(t)] = \frac{\tilde{f}(s)}{s^\alpha} - \sum_{r=0}^{n-1} s^{2-\alpha+r} f^{(r)}(0), \quad (n-1 < \alpha \leq n), \tag{5}$$

where $\tilde{f}(s)$ denote the ET of the function $f(t)$.

3. The basic concept of the q-homotopy analysis Elzaki transform method (q-HAETM)

Consider the following nonlinear fractional PDE involving linear (N) and nonlinear (R) operators to illustrate the basic principle of the considered method:

$$D_t^\alpha\, \mathcal{U}(x,t) + R\, \mathcal{U}(x,t) + N\, \mathcal{U}(x,t) = f(x,t), \quad 0 < \alpha \leq 1, \tag{6}$$

where $D_t^\alpha \mathcal{U}(x,t)$is the Liouville-Caputo fractional derivative of $\mathcal{U}(x,t), f(x,t)$ is the source term. Currently, hiring the ET on Eq. (6) leads to

$$\frac{1}{s^\alpha} E[\mathcal{U}(x,t)] - \sum_{k=0}^{n-1} s^{2-\alpha+k} \mathcal{U}^{(k)}(x,0) + E[R\mathcal{U}(x,t)] + E[N\mathcal{U}(x,t)] = E[f(x,t)], \tag{7}$$

By reducing Eq. (7), we get

$$E[\mathcal{U}(x,t)] - s^\alpha \sum_{k=0}^{n-1} s^{2-\alpha+k} \mathcal{U}^k(x,0) + s^\alpha \{E[R\mathcal{U}(x,t)] + E[N\mathcal{U}(x,t)] - E[f(x,t)]\} = 0. \tag{8}$$

The nonlinear operator N is defined under the homotopy analysis approach as follows

$$N[\varphi(x,t;q)] = E[\varphi(x,t;q)] - s^{\alpha}\sum_{k=0}^{n-1} s^{\alpha-k-1}\varphi^{(k)}(x,t;q)(0^{+}) + s^{\alpha}\{E[R\varphi(x,t;q)] + E[N\varphi(x,t;q)] - E[f(x,t)]\}, \quad (9)$$

where E is the Elzaki transform and $\varphi(x,t;q)$ is a real function of $x,t,$ and q(embedding parameter) $\in \left[0,\frac{1}{n}\right]$ $(n \geq 1)$.

The homotopy is defined as:

$$(1-nq)E[\varphi(x,t;q) - \mathcal{U}_0(x,t)] = \hbar q H(x,t) N[\varphi(x,t;q)], \quad (10)$$

where $\mathcal{U}_0(x,t)$ is an initial guess of $\mathcal{U}(x,t)$,$\hbar \neq 0$ is an auxiliary parameter. For $q=0$ and $q=1/n$, respectively we have:

$$\varphi(x,t;0) = \mathcal{U}_0(x,t), \quad \varphi\left(x,t;\frac{1}{n}\right) = \mathcal{U}(x,t). \quad (11)$$

As a result, by changing q from 0 to $\frac{1}{n}$, the solution $\varphi(x,t;q)$ converges from $\mathcal{U}_0(x,t)$ to $\mathcal{U}(x,t)$. The function $\varphi(x,t;q)$ can then be enlarged with the utilization of the Taylor theorem across q.

$$\varphi(x,t;q) = \mathcal{U}_0(x,t) + \sum_{m=1}^{\infty}\mathcal{U}_m(x,t)q^m, \quad (12)$$

with

$$\mathcal{U}_m(x,t) = \frac{1}{m!}\frac{\partial^m\varphi(x,t;q)}{\partial q^m}|_{q=0}. \quad (13)$$

The series (12) joins at $q=\frac{1}{n}$, resulting in the fundamental nonlinear equation, and it's one of the solutions of the type, by selecting the$n$ and $\hbar$ (auxiliary parameter) the initial guess $\mathcal{U}_0(x,t)$ and $H(x,t)$ properly.

$$\mathcal{U}(x,t) = \mathcal{U}_0(x,t) + \sum_{m=1}^{\infty}\mathcal{U}_m(x,t)\left(\frac{1}{n}\right)^m. \quad (14)$$

Then divide by $m!$ by differentiating Eq. (10) m times with respect to q. Finally, we derive the deformation equation of order m as follows for q=0.

$$E[\mathcal{U}_m(x,t) - K_m\mathcal{U}_{m-1}(x,t)] = \hbar H(x,t)\Re_m(\vec{\mathcal{U}}_{m-1}). \quad (15)$$

and the vectors considered in the form as

$$\vec{\mathcal{U}}_m = \{\mathcal{U}_0(x,t), \mathcal{U}_1(x,t), \dots, \mathcal{U}_m(x,t)\}. \quad (16)$$

Eq. (15) is the recursive equation that may be represented by the effect of the inverse Elzaki transform

$$\mathcal{U}_m(x,t) = K_m\mathcal{U}_{m-1}(x,t) + \hbar E^{-1}\left[H(x,t)\Re_m(\vec{\mathcal{U}}_{m-1})\right], \quad (17)$$

where

$$\Re_m(\vec{\mathcal{U}}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(x,t;q)]}{\partial q^{m-1}}|_{q=0}. \quad (18)$$

and

$$K_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \quad (19)$$

Finally, we find the component-wise q-HAETM series solution using Eq. (17).

4. Solution for nonlinear Sharma-Tasso-Olever equation of fractional order

The investigation of the following examples witnesses the efficacy and resolution of the contemplated scheme.

4.1. Example 1

The Sharma-Tasso-Olever equation

$$D_t^{\alpha}u(x,t) + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0. \quad (20)$$

with the starting solution

$$u(x,0) = \frac{2k(\tanh(kx)+w)}{w\tanh(kx)+1}. \quad (21)$$

Introduce ET on Eq. (20) along with the starting solution in (21), which leads to

$$E[u(x,t)] - s^2\left\{\frac{2k(\tanh(kx)+w)}{w\tanh(kx)+1}\right\} + s^{\alpha}E\{3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx}\} = 0. \quad (22)$$

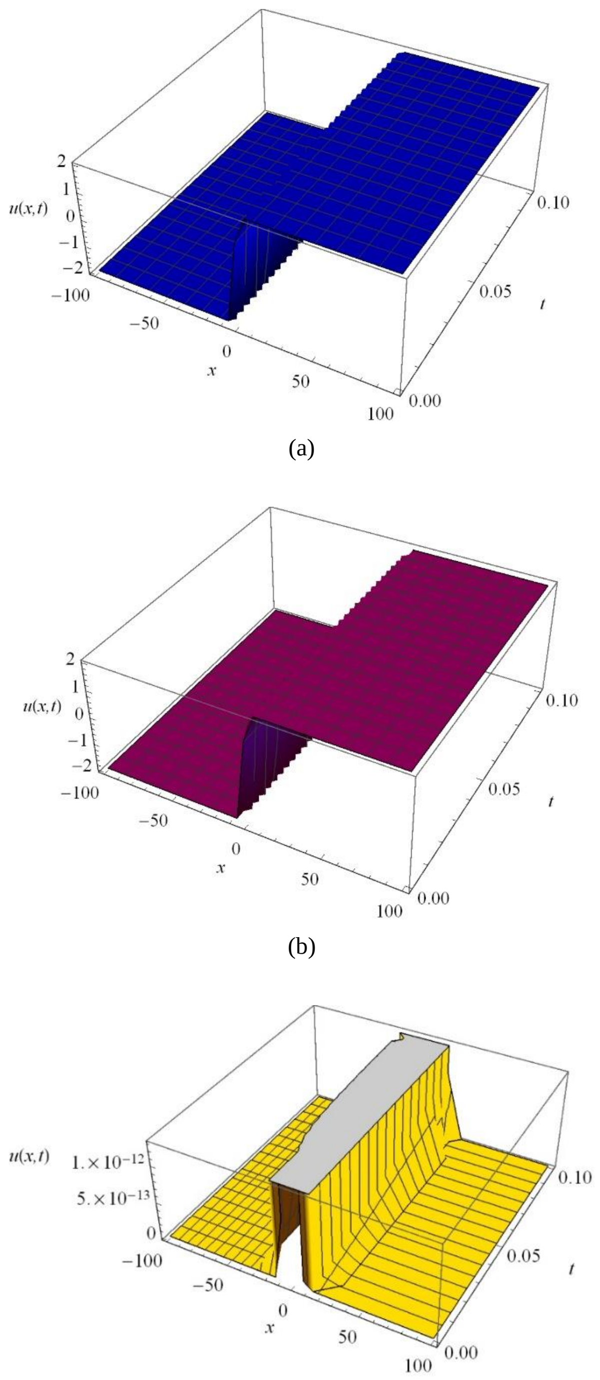
The nonlinear operator N is defined as

$$N[\varphi(x,t;q)] = E[\varphi(x,t;q)] - s^2\left\{\frac{2k(\tanh(kx)+w)}{w\tanh(kx)+1}\right\} + s^{\alpha}E\left\{3a\frac{\partial\varphi^2(x,t;q)}{\partial x} + 3a\varphi^2(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x} + 3a\varphi(x,t;q)\frac{\partial^2\varphi(x,t;q)}{\partial x^2} + a\frac{\partial^3\varphi(x,t;q)}{\partial x^3}\right\}. \quad (23)$$

The m[th] order deformation equation is

$$E[u_m(x,t) - K_m u_{m-1}(x,t)] = \hbar\Re_m[\vec{u}_{m-1}], \quad (24)$$

where



(a)

(b)

(c)

Figure 1. 3D plots of (a) q-HAETM solution (b) Exact solution (c) Absolute error= $|u_{Exact} - u_{App.}|$ at $\alpha = 1, n = 1, a = 1, k = 1, w = 0.5$, and $\hbar = -1$.

$$\Re_m[\vec{u}_{m-1}] = E[u(x,t)] - \left(1 - \frac{K_m}{n}\right) s^2 \left\{\frac{2k(\tanh(kx)+w)}{w\tanh(kx)+1}\right\} + s^\alpha E\left\{3a\sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x}\frac{\partial u_{m-i-1}}{\partial x} + 3a\sum_{i=0}^{m-1}\sum_{j=0}^{i} u_i u_{i-j}\frac{\partial u_{m-i-1}}{\partial x} + 3a\sum_{i=0}^{m-1} u_i \frac{\partial^2 u_{m-i-1}}{\partial x^2} + a\frac{\partial^3 u_{m-1}}{\partial x^3}\right\}. \quad (25)$$

Apply inverse ET on Eq. (24), we obtain

$$u_m(x,t) = K_m u_{m-1}(x,t) + \hbar E^{-1}\{\Re_m[\vec{u}_{m-1}]\}, \quad (26)$$

From Eq. (26), we arrive at:

$$u_0(x,t) = \frac{2k(\tanh(kx)+w)}{w\tanh(kx)+1},$$

$$u_1(x,t) = \frac{8ak^4(w^2-1)\hbar t^\alpha}{\Gamma(\alpha+1)(w\sinh(kx)+\cosh(kx))^2},$$

$$u_2(x,t) = \frac{8ak^4(w^2-1)\hbar(n+\hbar)t^\alpha}{\Gamma(\alpha+1)(w\sinh(kx)+\cosh(kx))^2} + \frac{8a^2k^7(w^2-1)\hbar^2t^{2\alpha}\text{sech}^6(kx)(4(w^3+w)\cosh(4kx))}{\Gamma(\alpha+1)\Gamma(2\alpha+1)(w\tanh(kx)+1)^6} - \frac{8a^2k^7(w^2-1)\hbar^2t^{2\alpha}\text{sech}^6(kx)16w(w^2-1)\cosh(2kx)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)(w\tanh(kx)+1)^6} - \frac{2\sinh(2kx)((w^4+6w^2+1)\cosh(2kx)-4w^4+4)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)(w\tanh(kx)+1)^6},$$

$\vdots$

Finally, after getting further iterative terms, the essential series solution of Eq. (20) is presented by

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \quad (27)$$

By taking $n = 1$, $\alpha = 1$, and $\hbar = -1$ then the attained solution $\sum_{m=1}^{N} u_m(x,t)\left(\frac{1}{n}\right)^m$, will end up with the exact solution $u(x,t) = \frac{2k(\tanh(k(x-4ak^2t))+w)}{w\tanh(k(x-4ak^2t))+1}$ which is of the Sharma-Tasso-Olever equation as $N \to \infty$.

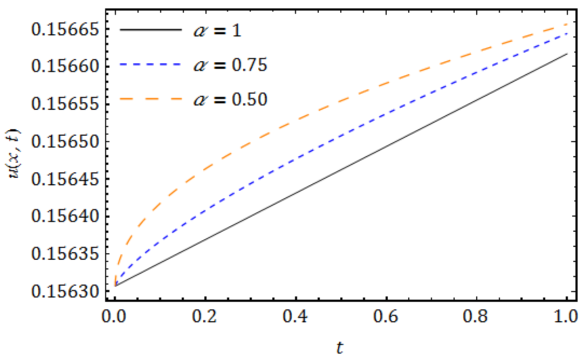
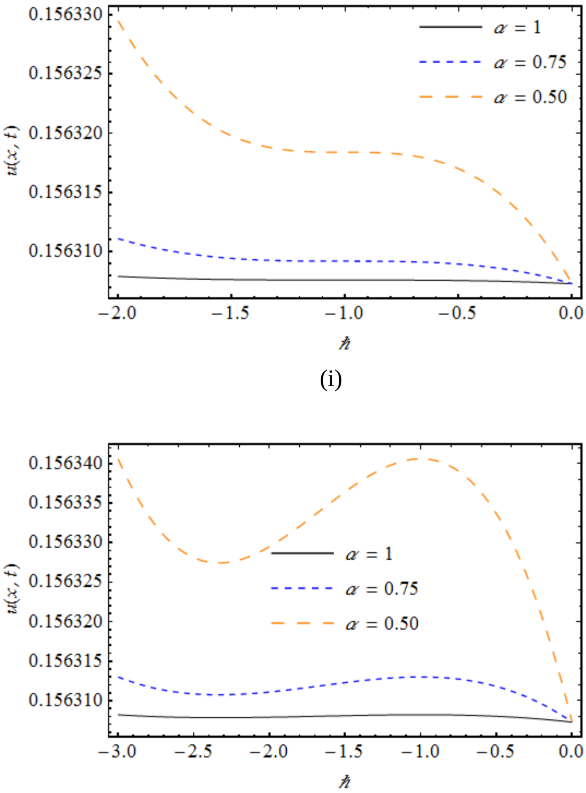


Figure 2. $u(x,t)$ versus t for contemplated, Ex. 1. when $\hbar = -1$, $x = 5$, $a = 1$, $k = 0.1$, $w = 0.5$, and $n = 1$ for distinct α.

Table 1. Absolute errors of ADM, HPM, OHAM [63], and the q-HAETM for Ex. 1 at $\boldsymbol{\alpha = 1, n = 1, k = 1, a = 1, \hbar = -1, w = 0.5}$ and $\boldsymbol{t = 0.01}$.

x	ADM	HPM	OHAM	q-HAETM
2	5.3799×10^{-3}	5.3799×10^{-3}	4.6088×10^{-3}	3.8991×10^{-3}
3	2.4002×10^{-3}	2.4002×10^{-3}	6.4466×10^{-4}	5.3356×10^{-4}
4	9.4208×10^{-4}	9.4208×10^{-4}	8.7636×10^{-5}	7.2319×10^{-5}
5	3.5464×10^{-4}	3.5464×10^{-4}	1.1867×10^{-5}	9.7893×10^{-6}
6	1.3156×10^{-4}	1.3156×10^{-4}	1.6062×10^{-6}	1.3248×10^{-6}
7	4.8547×10^{-5}	4.8547×10^{-5}	2.1737×10^{-7}	1.7930×10^{-7}
8	1.7879×10^{-5}	1.7879×10^{-5}	2.9419×10^{-8}	2.4266×10^{-8}
9	6.5802×10^{-6}	6.5802×10^{-6}	3.9814×10^{-9}	3.2840×10^{-9}
10	2.4211×10^{-6}	2.4211×10^{-6}	5.3883×10^{-10}	4.4445×10^{-10}



(i)

(ii)

Figure 3. $\hbar$-curve for the acquired solution $y(x,t)$ versus $\hbar$ for considered Ex. 1 when (i) $n=1$ and (ii) $n=2$ when $a=1, k=0.1, w=0.5, t=0.001, x=5$ for distinct α.

Table 2. Absolute errors of ADM, HPM, OHAM [63], and the *q*-HAETM for Ex. 1 at $\boldsymbol{\alpha=1, n=1, k=1, a=1, \hbar=-1, w=0.5}$ and $\boldsymbol{t=0.001}$.

x	ADM	HPM	OHAM	*q*-HAETM
2	7.2096×10^{-4}	7.2096×10^{-4}	4.6795×10^{-4}	3.8602×10^{-4}
3	5.3361×10^{-4}	5.3361×10^{-4}	6.5443×10^{-5}	5.2797×10^{-5}
4	2.3942×10^{-4}	2.3942×10^{-4}	8.8962×10^{-6}	7.1556×10^{-6}
5	9.4126×10^{-5}	9.4126×10^{-5}	1.2046×10^{-6}	9.6860×10^{-7}
6	3.5453×10^{-5}	3.5453×10^{-5}	1.6305×10^{-7}	1.3109×10^{-7}
7	1.3154×10^{-5}	1.3154×10^{-5}	2.2066×10^{-8}	1.7741×10^{-8}
8	4.8545×10^{-6}	4.8545×10^{-6}	2.9864×10^{-9}	2.4010×10^{-9}
9	1.7879×10^{-6}	1.7879×10^{-6}	4.0416×10^{-10}	3.2494×10^{-10}
10	6.5802×10^{-7}	6.5802×10^{-7}	5.4698×10^{-11}	4.3975×10^{-11}

4.2. Example 2

The Sharma-Tasso-Olever equation

$$D_t^{\alpha}u(x,t)+3au_x^2+3au^2u_x+3auu_{xx}+au_{xxx}=0, \tag{28}$$

with initial conditions

$$u(x,0)=-\sqrt{2}\sqrt{\mathrm{B}_0}\tan\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right). \tag{29}$$

Introduce ET on Eq. (28) along with the starting solution in (29), which leads to

$$E[u(x,t)]+s^2\left\{\sqrt{2}\sqrt{B_0}\tan\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right)\right\}+s^{\alpha}E\{3au_x^2+3au^2u_x+3auu_{xx}+au_{xxx}\}=0\}. \tag{30}$$

The nonlinear operator N is defined as

$$N[\varphi(x,t;q)]=E[\varphi(x,t;q)]+s^2\left\{\sqrt{2}\sqrt{\mathrm{B}_0}\tan\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right)\right\}+s^{\alpha}E\left\{3a\frac{\partial\varphi^2(x,t;q)}{\partial x}+3a\varphi^2(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x}+3a\varphi(x,t;q)\frac{\partial^2\varphi(x,t;q)}{\partial x^2}+a\frac{\partial^3\varphi(x,t;q)}{\partial x^3}\right\}. \tag{31}$$

The mth order deformation equation is

$$E[u_m(x,t)-K_mu_{m-1}(x,t)]=\hbar\Re_m[\vec{u}_{m-1}], \tag{32}$$

where

$$\Re_m[\vec{u}_{m-1}]=E[u(x,t)]+\left(1-\frac{K_m}{n}\right)s^2\left\{\sqrt{2}\sqrt{\mathrm{B}_0}\tan\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right)\right\}+s^{\alpha}E\left\{3a\sum_{i=0}^{m-1}\frac{\partial u_i}{\partial x}\frac{\partial u_{m-i-1}}{\partial x}+3a\sum_{i=0}^{m-1}\sum_{j=0}^{i}u_iu_{i-j}\frac{\partial u_{m-i-1}}{\partial x}+3a\sum_{i=0}^{m-1}u_i\frac{\partial^2u_{m-i-1}}{\partial x^2}+a\frac{\partial^3u_{m-1}}{\partial x^3}\right\}. \tag{33}$$

Apply inverse ET on Eq. (32), we obtain

$$u_m(x,t)=K_mu(x,t)+\hbar E^{-1}\{\Re_m[\vec{u}_{m-1}]\}. \tag{34}$$

From Eq. (34), we arrive at:

$$u_0(x,t)=-\sqrt{2}\sqrt{\mathrm{B}_0}\tan\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right),$$

$$u_1(x,t)=-\frac{2a\mathrm{B}_0{}^2\hbar t^{\alpha}\sec^2\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right)}{\Gamma(\alpha+1)},$$

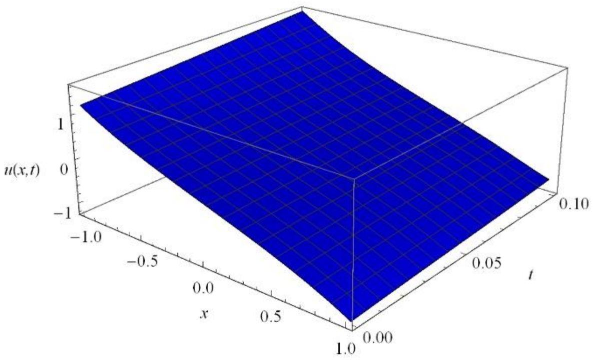
$$u_2(x,t)=-\frac{2a\mathrm{B}_0{}^2\hbar(n+\hbar)t^{\alpha}\sec^2\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right)}{\Gamma(\alpha+1)}-\frac{a^2\mathrm{B}_0{}^{7/2}\hbar^2t^{2\alpha}\sec^6\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right)\left(\sqrt{2}\,\Gamma(\alpha+1)(8\sin(\sqrt{2}\sqrt{\mathrm{B}_0}x)+\sin(2\sqrt{2}x))\right)}{2\,\Gamma(2\alpha+1)}+\frac{a^2\mathrm{B}_0{}^{7/2}\hbar^2t^{2\alpha}\sec^6\left(\frac{\sqrt{\mathrm{B}_0}x}{\sqrt{2}}\right)\sqrt{2}\,\Gamma(\alpha+1)24a\mathrm{B}_0{}^{3/2}\hbar t^{\alpha}}{2\,\Gamma(2\alpha+1)},$$

$$\vdots$$

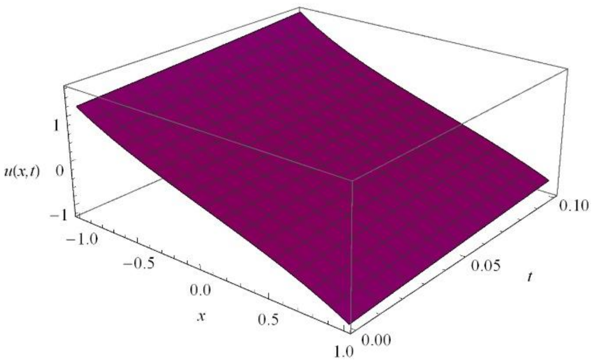
Finally, after getting further iterative terms, the essential series solution of Eq. (28) is presented by

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{35}$$

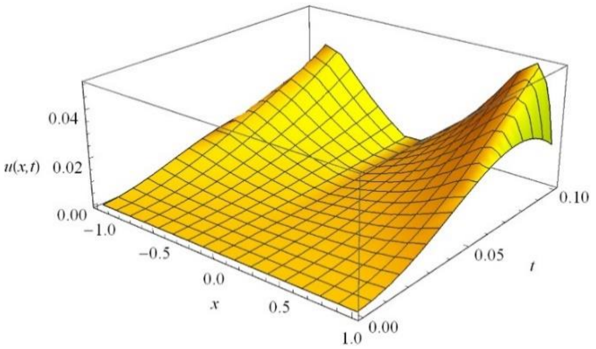
If we set $n=1$, $\alpha=1$, and $\hbar=-1$ then the secure solution $\sum_{m=1}^{N}u_m(x,t)\left(\frac{1}{n}\right)^m$, converges to exact solution $u(x,t)=-\sqrt{2\mathrm{B}_0}\tan\left(\frac{1}{2}\sqrt{2B_0}\left(x-\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)}\right)\right)$ of the integer-order Sharma-Tasso-Olever equation as $N\to\infty$.



(a)



(b)



(c)

Figure 4. Surfaces of **(**a) q-HAETM solution (b) Exact solution (c) Absolute error= $|u_{Exact} - u_{App.}|$ at $\hbar = -1, B_0 = 1, \lambda = 2, a = 1, n = 1$, and $\alpha = 1$.

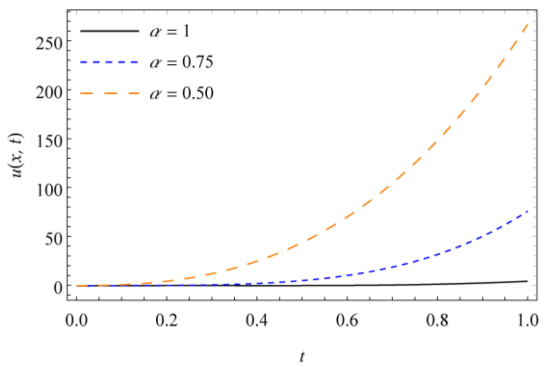
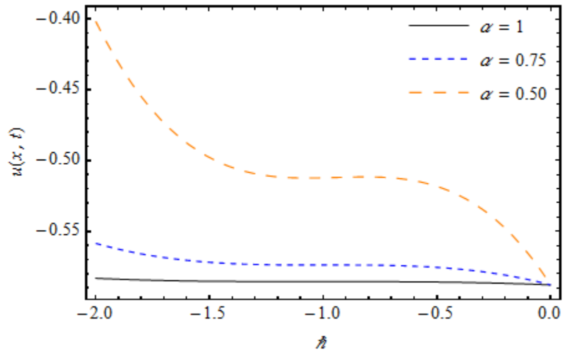
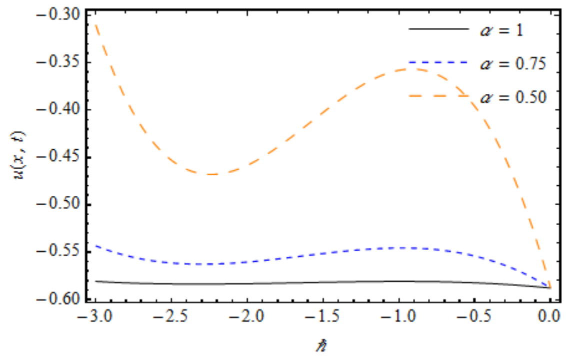


Figure 5. $u(x,t)$ versus t for Ex. 2 at $\hbar = -1, x = 5, \mathrm{B}_0 = 1, \lambda = 2, a = 1$, and $n = 1$ for distinct α.



(i)



(ii)

Figure 6. A plot of approximate solution $u(x,t)$ with respect to $\hbar$ for Ex. 2 when (i) $n = 1$ and (ii) $n = 2$ when $x = 5, \mathrm{B}_0 = 1, \lambda = 2, a = 1$, and $t = 0.001$ for distinct α.

Table 3. Numerical simulations for Ex. 2 at $n = 1, \alpha = 1, \hbar = -1, \mathrm{B}_0 = 1, a = 1, \lambda = 2$ for various values of x and at $t = 0.001, t = 0.01$.

t	x	$\alpha = 1$	$\alpha = 0.75$	$\alpha = 0.5$
0.001	0.1	3.0758×10^{-7}	1.5406×10^{-5}	7.9499×10^{-4}
	0.2	6.3401×10^{-7}	3.1368×10^{-5}	1.5953×10^{-3}
	0.3	1.0070×10^{-6}	4.9479×10^{-5}	2.4768×10^{-3}
	0.4	1.4581×10^{-6}	7.1155×10^{-5}	3.4833×10^{-3}
	0.5	2.0307×10^{-6}	9.8300×10^{-5}	4.6578×10^{-3}
0.01	0.1	3.4407×10^{-5}	6.1324×10^{-4}	1.0980×10^{-2}
	0.2	6.8106×10^{-5}	1.1773×10^{-3}	2.1147×10^{-2}
	0.3	1.0657×10^{-4}	1.8089×10^{-3}	3.1485×10^{-2}
	0.4	1.5292×10^{-4}	2.5468×10^{-3}	4.1785×10^{-2}
	0.5	2.1137×10^{-4}	3.4366×10^{-3}	5.1021×10^{-2}

4.3. Example 3

The Sharma-Tasso-Olever equation

$$D_t^{\alpha} u(x,t) + a{u^3}_x + \frac{3}{2} a{u^2}_{xx} + a u_{xxx} = 0, \qquad (36)$$

with initial conditions

$$u(x,0) = \sqrt{\frac{1}{a}} \tanh\left(\sqrt{\frac{1}{a}} x\right). \qquad (37)$$

Introduce ET on Eq. (36) along with the starting solution in (37), which leads to

$$E[u(x,t)] - s^2\left\{\sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}x\right)\right\} + s^\alpha E\left\{a{u^3}_x + \frac{3}{2}a{u^2}_{xx} + au_{xxx}\right\} = 0\}. \tag{38}$$

The nonlinear operator N is defined as

$$N[\varphi(x,t;q)] = E[\varphi(x,t;q)] - s^2\left\{\sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}x\right)\right\} + s^\alpha E\left\{a\frac{\partial\varphi^3(x,t;q)}{\partial x} + \frac{3}{2}a\frac{\partial^2\varphi^2(x,t;q)}{\partial x^2} + a\frac{\partial^3\varphi(x,t;q)}{\partial x^3}\right\}. \tag{39}$$

The m[th] order deformation equation is

$$E[u_m(x,t) - K_m u_{m-1}(x,t)] = \hbar\Re_m[\vec{u}_{m-1}]. \tag{40}$$

where

$$\Re_m[\vec{u}_{m-1}] = E[u(x,t)] + \left(1 - \frac{K_m}{n}\right)s^2\left\{\sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}x\right)\right\} + s^\alpha E\left\{a\sum_{i=0}^{m-1}\sum_{j=0}^{i}\frac{\partial u_i}{\partial x}\frac{\partial u_{i-j}}{\partial x}\frac{\partial u_{m-i-1}}{\partial x} + \frac{3}{2}a\sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x^2}\frac{\partial^2 u_{m-i-1}}{\partial x^2} + a\frac{\partial^3 u_{m-1}}{\partial x^3}\right\}. \tag{41}$$

Apply inverse ET on Eq. (40), we obtain

$$u_m(x,t) = K_m u(x,t) + \hbar E^{-1}\{\Re_m[\vec{u}_{m-1}]\}. \tag{42}$$

From Eq. (42), we arrive at:

$$u_0(x,t) = \sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}x\right),$$

$$u_2(x,t) = \frac{\hbar t^\alpha\left(a\cosh\left(4\sqrt{\frac{1}{a}}x\right) - 2(a-3)\cosh\left(2\sqrt{\frac{1}{a}}x\right) - 3a - 4\right)\operatorname{sech}^6\left(\sqrt{\frac{1}{a}}x\right)}{2a^2\Gamma(\alpha+1)},$$

$$u_1(x,t) = \frac{\hbar(n+\hbar)t^\alpha(a\cosh(4\sqrt{\frac{1}{a}}x) - 2(a-3)\cosh(2\sqrt{\frac{1}{a}}x) - 3a - 4)\operatorname{sech}^6(\sqrt{\frac{1}{a}}x)}{2a^2\Gamma(\alpha+1)} - \frac{1}{4\Gamma(\alpha+1)\Gamma(2\alpha+1)}\left(\Gamma(\alpha+1)\cosh^5\left(\sqrt{\frac{1}{a}}x\right)\left(-52a^2\cosh\left(6\sqrt{\frac{1}{a}}x\right) + a^2\cosh\left(8\sqrt{\frac{1}{a}}x\right) + 4(149a^2 + 149a - 588)\cosh\left(2\sqrt{\frac{1}{a}}x\right) + 4(7a^2 - 223a + 72)\cosh\left(4\sqrt{\frac{1}{a}}x\right) + 515a^2 + 60a\cosh\left(6\sqrt{\frac{1}{a}}x\right) + 1548a + 2160\right)\right)\left(\left(\frac{1}{a}\right)^{7/2}\hbar^2 t^{2\alpha}\operatorname{sech}^{15}\left(\sqrt{\frac{1}{a}}x\right)\right) - \frac{4\left(\frac{1}{a}\right)^5\hbar^3 t^{3\alpha}\sinh\left(\sqrt{\frac{1}{a}}x\right)\tanh\left(\sqrt{\frac{1}{a}}x\right)\operatorname{sech}^{15}\left(\sqrt{\frac{1}{a}}x\right)}{4\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}$$

$$\frac{\left((12-8a)\cosh\left(2\sqrt{\frac{1}{a}}x\right) + a\cosh\left(4\sqrt{\frac{1}{a}}x\right) - 9(a+2)\right)^2}{4\Gamma(\alpha+1)^2\Gamma(3\alpha+1)},$$

$$\vdots$$

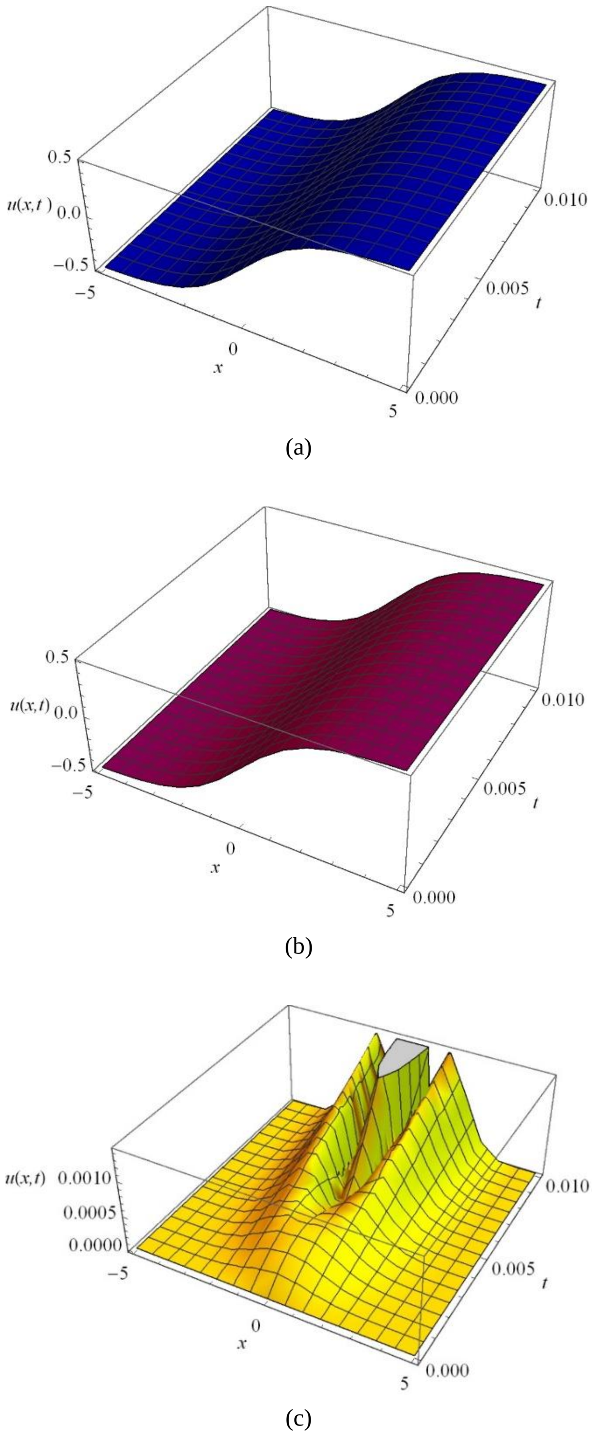


Figure 7. (a) 3D plot for q-HAETM solution (b) surface of exact solution (c) approximated solution surface at $\hbar = -1.858$, $a = 4, n = 1$ and $\alpha = 1$.

Finally, after getting further iterative terms, the essential series solution of Eq. (36) is presented by

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{43}$$

If we set $n = 1$, $\alpha = 1$, and $\hbar = -1$ then the secure solution $\sum_{m=1}^{N} u_m(x,t)\left(\frac{1}{n}\right)^m$, converges to exact solution $u(x,t) = \sqrt{\frac{1}{a}}\tanh\left(\sqrt{\frac{1}{a}}(x-t)\right)$ of the integer-order Sharma-Tasso-Olever equation as $N \to \infty$.

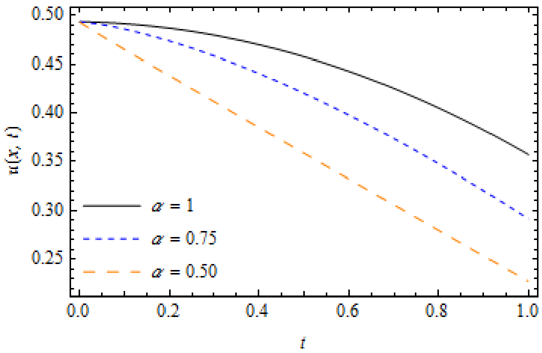
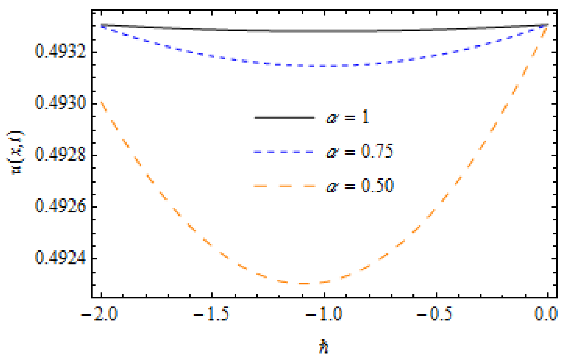
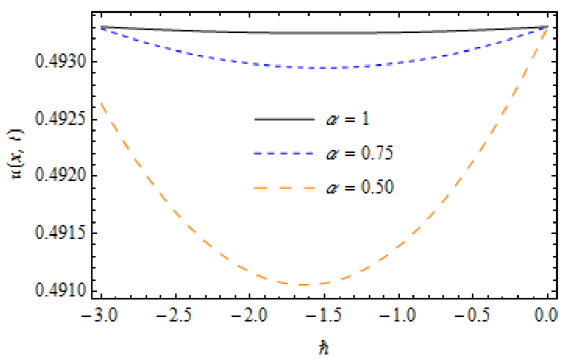


Figure 8. $u(x,t)$ versus t for the contemplated Ex. 3 at $\hbar = -1.858, x = 5, a = 4$, and $n = 1$ for distinct of α.



(i)



(ii)

Figure. 9. $\hbar$-curve for acquired solution $u(x,t)$ for Ex. 3 when (i) $n = 1$ and (ii) $n = 2$ when $x = 5, a = 4$, and $t = 0.001$ for distinct α.

Table 4. Numerical simulations for Ex. 3 at $n = 1, \alpha = 1, \hbar = -1, B_0 = 1, a = 1, \lambda = 2$ for various values of x and at $t = 0.001, t = 0.002$.

t	x	$\alpha = 1$	$\alpha = 0.75$	$\alpha = 0.5$
	5	2.8246×10^{-7}	4.1132×10^{-5}	4.9465×10^{-4}
	4	4.2905×10^{-7}	9.2494×10^{-5}	8.1866×10^{-4}
0.001	3	8.7092×10^{-6}	1.3618×10^{-4}	7.2756×10^{-4}
	2	5.7987×10^{-5}	3.6881×10^{-5}	8.8362×10^{-3}
	1	1.9745×10^{-4}	4.3659×10^{-4}	2.9978×10^{-2}
	5	8.1686×10^{-7}	7.4104×10^{-5}	8.4707×10^{-4}
	4	6.5000×10^{-7}	1.5605×10^{-4}	1.2792×10^{-3}
0.002	3	1.9479×10^{-5}	1.5950×10^{-4}	2.2392×10^{-3}
	2	1.2665×10^{-4}	3.8463×10^{-4}	1.8766×10^{-2}
	1	3.6421×10^{-4}	1.5111×10^{-3}	6.0264×10^{-2}

5. Numerical results and discussion

The numerical research for non-integer order STO equations using the q-HAETM is implemented in the current part. In terms of absolute error, Figure 1 depicts the resemblance of the solution obtained using the discussed method to the precise solution for Eq. (20). We can see that the obtained solution and the exact solution are the best matches with each other. The recommended technique's conclusion for Eq. (20) is plotted against time in figure 2. The solution increases with an increase in time for considering various fractional orders. The performance of n with $\hbar$ in an accomplished outcome of the provided method is shown in figure 3. The optimal region for the convergence of the obtained series solution in terms of the auxiliary parameter $\hbar$ can be depicted in figure 3. Table 1 and Table 2 cite the accurateness of the considered method in comparison with various methods namely, ADM, HPM, and OHAM through absolute errors at $t = 0.001$ and $t = 0.01$ respectively. The link between the results obtained by the proposed method in terms of absolute error and the precise answer for Ex. 2 is depicted in figure 4. We can compare both obtained solution and the exact solution to check the accuracy of the projected algorithm. The deed of the safe results of Ex. 2 with the change in time t is depicted in figure 5. As we can see, the solution increases with an increase in time t. The effectiveness of n in the produced solution by the proposed algorithm is shown in figure 6. Also, the solution is affected by various fractional orders with the time t. However, we attained a better accuracy rate with the consideration of fractional order differential operators too. This shows that the presented scheme is highly suitable to deal with nonlinear fractional differential equations. The approximated error results acquired for various values of α with the help of the considered scheme are cited in Table 3. Figure 7 depicts the relationship between the results acquired by the q-HAETM concerning absolute error and the exact solution for Ex. 3. Figure 8 is decorated with the variation of attained solution with time t. According to the considered initial approximation, the solution gradually decreases with an increase in time t. The performance of the embedding parameter ($\hbar$) for distinct values of n in the secured solution by the proposed strategy is shown in figure 9. Table 4 cites the accuracy of the obtained solutions in terms of absolute error.

6. Conclusion

In this paper, we have demonstrated how to solve the nonlinear time-fractional STO equation using the effective q-HAM with the Elzaki transform. We have examined three examples with distinct starting solutions to prove the significance as well as the effectiveness of the considered scheme. Moreover, we can compare the obtained results with the exact solutions to witness the same. The rate of convergence of the obtained series solution to the exact solution is accelerated with the help of optimal values of convergence control parameter $\hbar$. Presented numerical simulations guarantee results with higher accuracy. The numerical simulations are executed by using the considered technique in comparison with the other schemes like ADM, HPM, and OHAM in terms of approximated errors. The secure outputs indicate that a considered methodology was used to generate a standardized analytical solution. In this study, the detailed analysis of the fractional behaviour of the nonlinear STO equation and its solution is achieved by considering different initial approximations. The process of finding the solution for the considered problem using the Elzaki transform was effortless. The proposed approach is effective in delivering a simple solution, a critical convergence zone, and a non-local influence. Finally, we claim that our proposed technique is incredibly dependable and can be applied to large study classifications relating to fractional-order nonlinear scientific methods, which aid us in better understanding the nonlinear compound phenomena in linked domains of innovation and science.

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RESEARCH ARTICLE

Approximate controllability for Riemann-Liouville fractional differential equations

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ABSTRACT

The article objectifies the approximate controllability of fractional nonlinear differential equations having Riemann-Liouville derivatives. The nonlinear term involved in the equation, also depending on the control parameter $u(\cdot)$, is considered to be locally Lipschitz. First, the existence of solutions is deduced using Lipschitz condition, semigroup theory and fixed point approach. Then sufficient condition for approximate controllability of the system is established using Cauchy convergence through iterative and approximate techniques. The theory of semigroup together with probability density function has been utilized to reach the desired conclusions. Lastly, an application is provided to support the proposed methodology.

1. Introduction

The scope of this article revolves around the underneath system:

$$D^{\eta}_{\tau}z(\tau) = Az(\tau) + Bu(\tau) + g(\tau, z(\tau), u(\tau)), \quad (1)$$
$$\tau \in (0, a], \quad 0 < \eta \leq 1$$
$$I^{1-\eta}_{\tau} z(\tau)|_{\tau=0} = z_0 \in Z,$$

where D^{η}_{τ} indicates the Riemann-Liouville η^{th} order derivative. $A : D(A) \subseteq Z \to Z$ generates a C_0 - semigroup $T(\tau)(\tau \geq 0)$ on Z. $z(\tau)$ and $u(\tau)$ takes value in Banach spaces Z and U respectively. The linear map B is defined from $L^q([0, a]; U)$ to $L^q([0, a]; Z)$, $q > \frac{1}{\eta}$. g is a function from $[0, a] \times Z \times U \to Z$.

The study of fractional calculus has long been admired from past three decades. The first work, exclusively committed to the study of fractional calculus, is the book by Oldham and Spanier [1], 1974. Fractional derivatives serves as an exemplary mechanism for the interpretation of hereditable properties and memory of profuse scientific, physical and engineering phenomena. On account of finer accuracy and precision over integer-order models, fractional derivatives accelerates its applications in diffusion process, biological mathematical models, aerodynamics, viscoelasticity, electrical engineering, signal and image processing, control theory, heat equation, electricity mechanics, electrodynamics of complex medium, etc. (see [2–10]).

In domain of fractional calculus, Riemann-Liouville and Caputo type derivatives have maintained to be the centre of attention for numerous analysts. Riemann-Liouville derivative shows supremacy over Caputo in the sense that it allows the function involved to bear discontinuity at origin. Also, in turn, doesn't allow the use of traditional initial conditions, the initial conditions involved in Riemann-Liouville case are integral initial conditions. Heymans and Podlubny [11] were the ones accredited for the manifestation of physical significance to the initial conditions used in regard of Riemann-Liouville fractional order viscoelastic systems.

Controllability is the qualitative property of steering any dynamical system from initial arbitrary position to any desired final position utilizing

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appropriate control functions within stipulated time. Control theory, being a multidisciplinary branch stemmed from mathematics to engineering, has wide-ranging implementation in robotics, aeronautical and automobile engineering, image processing, biomathematical modelling and appreciably more. Control theory, in spaces of infinite and finite dimensions, have thoroughly been discussed in [12] and [13] respectively. The conception of controllability was first initiated and established by Kalman [14] in 1963, and since then it is the matter of prime importance for the researchers worldwide. In due course, profuse types of controllability were examined by the researchers in the past. Approximate controllability of semilinear fractional systems involving Caputo derivative was established by Sakthivel in [9] by supposing C_0- semigroup $T(t)$ to be compact and the nonlinear function involved to be uniformly bounded and continuous, Devies in [15] established exact and null controllability for linear systems, Mahmudov in [16] designed partial approximate controllability for Caputo type frcational order systems, Klamka in [17] mannered constrained controllability. Wen & Zhou [18] discussed complete and approximate controllability of semilinear system for Caputo derivative with control in the nonlinear part. The results of existence and controllability for various differential systems of integral and fractional order involving Riemann-Liouville and Caputo derivatives have closely been demonstrated in many artefacts (refer [3, 6, 9, 17, 19–31] and references therein). The article [32] discusses about the numerical treatment of fractional heat equation. S. N. Bora [33] recently established the approximate controllability for semilinear Hilfer fractional evolution equations by relaxing the compactness of the semigroup generated. Vijayakumar, Nisar & Shukla [34–40] established important results of controllability and approximate controllability of fractional evolution systems involving other new fractional derivatives like Atangana-Baleanu derivative and Hilfer derivative.

This artefact explores the study for Riemann-Liouville differential systems involving control function in the nonlinear part and is drafted as: Section 2 gives the briefing for basic results and definitions. Results for the existence of solutions are apparent in Section 3. Section 4 accords with the sufficient assumptions and controllability conditions. Section 5 presents an application validating the proposed methodology. Section 6 concludes the article by summarizing the present findings along with discussing the futuristic scope.

2. Preliminaries

This segment revisits several fundamental concepts and definitions which are beneficial for the smooth study of the paper. The considered Banach space is

$$C_{1-\eta}([0,a];Z) = \{z : \tau^{1-\eta}z(\tau) \in C([0,a];Z)\}$$

equipped with the norm

$$\|z\|_{C_{1-\eta}} = \sup_{\tau\in[0,a]} \{\tau^{1-\eta}\|z(\tau)\|_Z\},$$

where $C([0,a];Z)$ indicates the set of all continuous functions defined from $[0,a]$ to Z. For C_0 - semigroup $T(\tau)$, let $M = \sup_{\tau\in[0,a)} \|T(\tau)\| < \infty$.

Definition 1. *[4] The Riemann-Liouville $\eta^{th}-$order fractional integral is written in terms of the following integral*

$$I_\tau^\eta z(\tau) = \frac{1}{\Gamma(\eta)}\int_0^\tau (\tau-r)^{\eta-1}z(r)dr, \qquad \eta > 0,$$

where Γ denotes the gamma function.

Definition 2. *[4] The fractional $\eta^{th}-$order Riemann-Liouville derivative is defined by the following expression*

$$D_\tau^\eta z(\tau) = \frac{1}{\Gamma(n-\eta)}\left(\frac{d}{d\tau}\right)^n \int_0^\tau (\tau-r)^{n-\eta-1}z(r)dr,$$

where $0 \le n-1 < \eta < n$.

Definition 3. *[4] A function of the complex variable w defined by*

$$E_\eta(w) = \sum_{i=0}^\infty \frac{w^i}{\Gamma(\eta i+1)}$$

is known as the Mittag-Leffler function in one parameter.

Definition 4. *[41] A mild solution of the system (1) is a function $z \in C_{1-\eta}([0,a];Z)$ satisfying the underneath integral equation:*

$$\begin{aligned} z(\tau) &= \tau^{\eta-1}T_\eta(\tau)z_0 \\ &+ \int_0^\tau (\tau-r)^{\eta-1}T_\eta(\tau-r)Bu(r)dr \\ &+ \int_0^\tau (\tau-r)^{\eta-1}T_\eta(\tau-r)g(r,z(r),u(r))dr. \end{aligned} \tag{2}$$

where

$$T_\eta(\tau) = \eta\int_0^\infty \Theta\xi_\eta(\Theta)T(\tau^\eta\Theta)d\Theta,$$

$$\xi_\eta(\Theta) = \frac{1}{\eta}\Theta^{-1-\frac{1}{\eta}}\varpi_\eta(\Theta^{-\frac{1}{\eta}}),$$

$$\varpi_\eta(\Theta) = \frac{1}{\pi}\sum_{n=1}^\infty \Theta^{-n\eta-1}(-1)^{n-1}\frac{\Gamma(1+n\eta)}{n!}sin(n\pi\eta)$$

with $\Theta \in (0,\infty)$ *and domain of the probability density function* $\xi_\eta(\Theta)$ *is* $(0,\infty)$, *i.e.,* $\xi_\eta(\Theta) \geq 0$ *and* $\int_0^\infty \xi_\eta(\Theta)d\Theta = 1$.

Definition 5. *Let* $z(\tau,u)$ *be a mild solution of the system(1) at time* τ *corresponding to a control* $u(.) \in U$. *The set* $K_a(g) = \{z(a,u) \in Z; u(.) \in U\}$ *is known as the reachable set for final time a. If* $K_a(g)$ *becomes dense in Z, the system (1) is approximately controllable on [0,a].*

Lemma 1. *[31] The operator* $T_\eta(\tau)$ *possesses the underneath properties:*

(i) *For every fixed* $\tau \geq 0$, *operator* $T_\eta(\tau)$ *is linear and bounded, which means, for any* $z \in Z$,
$$\|T_\eta(\tau)z\| \leq \frac{M}{\Gamma(\eta)}\|z\|.$$

(ii) *Operator* $T_\eta(\tau)(\tau \geq 0)$ *is strongly continuous.*

3. Existence of mild solution

This segment establishes the existence and uniqueness of mild solution for the system (1) utilizing the Banach fixed point approach along with the generalised Gronwall's inequality. The results are based on the below mentioned hypotheses:

(H1) A function $\psi(\cdot)$ exists in $L^q([0,a];\mathbb{R}^+)$, $q > \frac{1}{\eta}$, and a constant $b > 0$, such that $\|g(\tau,z,u)\| \leq \psi(\tau) + b\tau^{1-\eta}\|z\|_Z + \|u\|_U$ for a.e. $\tau \in [0,a]$ and all $z \in Z$.

(H2) A constant $\Bbbk > 0$ exists in a way satisfying $\|g(\tau,z,u) - g(\tau,y,v)\| \leq \Bbbk\big[\|z-y\|_Z + \|u-v\|_U\big]$ $\forall\ z,y \in Z$ and $\forall\ u,v \in U$.

Theorem 1. *The nonlinear system (1) admits a unique mild solution in* $C_{1-\eta}([0,a];Z)$ *for each control* $u(.) \in L^q([0,a];U)$, *provided the hypotheses H(1)-H(2) hold true.*

Proof. Consider the operator G as
$$(Gz)(\tau) = \tau^{\eta-1}T_\eta(\tau)z_0 + \int_0^\tau (\tau-r)^{\eta-1}T_\eta(t-r)\big[Bu(r) + f(r,z(r),u(r))\big]dr. \quad (3)$$

It is unchallenging to confirm that G maps $C_{1-\eta}([0,a];Z)$ into itself under the hypotheses $H(1) - H(2)$.

It is now required to prove G^m is a contraction operator on $C_{1-\eta}([0,a];Z)$ for some $m \in \mathbb{N}$.

For any $z,y \in C_{1-\eta}([0,a];Z)$ and $\tau \in [0,a]$, it is

$$\begin{aligned}
&\tau^{1-\eta}\|(Gz)(\tau) - (Gy)(\tau)\|_{C_{1-\eta}} \\
&\leq \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)\big[g(r,z(r),u(r)) - g(r,y(r),u(r))\big]\|dr \\
&\leq \frac{\tau^{1-\eta}M}{\Gamma(\eta)}\int_0^\tau (\tau-r)^{\eta-1}\|g(r,z(r),u(r)) - g(r,y(r),u(r))\|_Z dr \\
&\leq \frac{\tau^{1-\eta}M\Bbbk}{\Gamma(\eta)}\int_0^\tau (\tau-r)^{\eta-1}r^{\eta-1}r^{1-\eta}\|z(r)-y(r)\|_Z dr \\
&\leq \frac{\tau^{1-\eta}M\Bbbk}{\Gamma(\eta)}\|z-y\|_{C_{1-\eta}}\int_0^\tau r^{\eta-1}(\tau-r)^{\eta-1}dr \\
&\leq \frac{\Gamma(\eta)M\Bbbk\tau^\eta}{\Gamma(2\eta)}\|z-y\|_{C_{1-\eta}}.
\end{aligned} \quad (4)$$

Further, by applying induction on m and using (3), (4), it leads to

$$\tau^{1-\eta}\|(G^m z)(\tau) - (G^m y)(\tau)\| \leq \frac{\Gamma(\eta)(\Bbbk M a^\eta)^m}{\Gamma[(m+1)\eta]}\|z-y\|_{C_{1-\eta}}$$

Therefore, G^m is shown as a contraction operator on $C_{1-\eta}([0,a];Z)$ with the inequality obtained as

$$\|G^m z - G^m y\|_{C_{1-\eta}} \leq \frac{\Gamma(\eta)(\Bbbk M a^\eta)^m}{\Gamma(m+1)\eta}\|z-y\|_{C_{1-\eta}} \quad (5)$$

where $\frac{(\Bbbk M a^\eta)^m}{\Gamma(m+1)\eta}$ becomes the m^{th} term of the two parameter Mittag-Leffler series $E_{\eta,\eta}(M\Bbbk a^\eta) = \sum_{i=0}^\infty \frac{(M\Bbbk a^\eta)^i}{\Gamma(i\eta+\eta)}$. The series converges uniformly on $[0,a]$, thus for sufficiently large m,

$$\frac{\Gamma(\eta)(\Bbbk M a^\eta)^m}{\Gamma(m+1)\eta} < 1.$$

It is evident through generalisation of Banach fixed point theorem and (5) that G possess a unique fixed point $z(\cdot)$ on $C_{1-\eta}([0,a];Z)$ which serves as the requisite solution of system (1). □

4. Controllability results

Defining the underneath operators:

The Nemytskil operator
$$\Omega_g : C_{1-\eta}([0,a];Z) \to L^q([0,a];Z)$$
is defined as

$$\Omega_g(z)(\tau) = g(\tau, z(\tau), u(\tau)),$$
$$z(\cdot) \in C_{1-\eta}([0,a]; Z).$$

and the bounded linear operator

$$\mathbb{F} : L^q([0,a]; Z) \to Z$$

as

$$\mathbb{F}f = \int_0^a (a-r)^{\eta-1} T_\eta(a-r) f(r) dr,$$
$$f(\cdot) \in L^q([0,a]; Z).$$

The hypotheses mentioned below are made to prove the approximate controllability for the considered system (1):

(H3) A constant $\Bbbk' > 0$ exists in a way satisfying $\|g(\tau, z, u) - g(\tau, y, v)\| \le \Bbbk' \left[\tau^{1-\eta}\|z-y\|_Z + \|u-v\|_U\right] \quad \forall\, z, y \in Z,\ u, v \in U$ and $\tau \in [0,a]$.

(H4) The operator B is bounded below, i.e., a constant $\ell > 0$ exists satisfying $\|u\| \le \ell \|Bu\| \quad \forall u \in U$.

(H5) For any $\epsilon > 0$ and $\vartheta(\cdot) \in L^q([0,a], Z)$, $\exists$ a $u(\cdot) \in L^q([0,a]; U)$ satisfying $\|\mathbb{F}\vartheta - \mathbb{F}Bu\|_Z < \epsilon$, $\|Bu(\cdot)\|_{L^q([0,a];Z])} < \aleph \|\vartheta(\cdot)\|_{L^q([0,a];Z)}$, where $\aleph$ is a constant independent of $\vartheta(\cdot) \in L^q([0,a]; Z)$,

$$\frac{M\aleph\Bbbk'}{\Gamma(\eta)}\Big(\frac{aq-a}{q\eta-1}\Big)^{\frac{q-1}{q}} \times (1+\Bbbk'\ell)E_\eta(M\Bbbk' a) + \aleph\Bbbk'\ell < 1. \tag{6}$$

Lemma 2. *Assuming the hypotheses (H1), (H3) and (H4) hold true for the considered function g, then every mild solution of the control system (1) meets the inequalities stated below for any* $u, v \in L^q([0,a]; U)$*:*

$$\|z(\cdot; 0, z_0, u)\|_{C_{1-\eta}} \le kE_\eta(Mab),$$
$$\|z(\cdot) - y(\cdot)\|_{C_{1-\eta}} \le \varrho E_\eta(M\Bbbk' a)\|Bu - Bv\|_{L^q},$$

where

$$k = \frac{M}{\Gamma(\eta)}\Bigg[\|z_0\| + \Big(\frac{q-1}{q\eta-1}\Big)^{\frac{q-1}{q}}(\|Bu\|_{L^q} + \|\psi\|_{L^q} + \|u\|_{L^q})a^{1-\frac{1}{q}}\Bigg],$$

$$\varrho = \frac{M}{\Gamma(\eta)}\Big(\frac{q-1}{q\eta-1}\Big)^{\frac{q-1}{q}}(1+\Bbbk'\ell)a^{1-\frac{1}{q}}.$$

Proof. Let z be a mild solution of system (1) in accord with control $u(\cdot) \in L^q([0,a]; U)$ on $C_{1-\eta}([0,a]; Z)$, then

$$z(\tau) = \tau^{\eta-1}T_\eta(\tau)z_0 + \int_0^\tau (\tau-r)^{\eta-1}T_\eta(\tau-r)Bu(r)dr + \int_0^\tau (\tau-r)^{\eta-1}T_\eta(\tau-r)g(r, z(r), u(r))dr$$

For $\tau \in [0,a]$,

$$\begin{aligned}
\tau^{1-\eta}\|z(\tau)\| &\le \|T_\eta(\tau)z_0\| \\
&+ \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)Bu(r)\|dr \\
&+ \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)g(r,z(r),u(r))\|dr \\
&\le \frac{M}{\Gamma(\eta)}\Bigg[\|z_0\| + \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|Bu(r)\|_Z dr \\
&+ \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}[\psi(r) + br^{1-\eta}\|z(r)\|_Z + \|u(r)\|_U]dr\Bigg] \\
&\le \frac{M}{\Gamma(\eta)}\Bigg[\|z_0\| + \Big(\frac{aq-a}{q\eta-1}\Big)^{\frac{q-1}{q}}(\|Bu\|_{L^q} + \|\psi\|_{L^q} \\
&+ \|u\|_{L^q}) + ba^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}r^{1-\eta}\|z(r)\|_Z dr\Bigg].
\end{aligned} \tag{7}$$

Thus,

$$\tau^{1-\eta}\|z(\tau)\| \le k + \frac{Mba^{1-\eta}}{\Gamma(\eta)} \times \int_0^\tau (\tau-r)^{\eta-1}[r^{1-\eta}\|z(r)\|]dr.$$

Using generalised Gronwall's inequality ([42]), it concludes to

$$\tau^{1-\eta}\|z(\tau)\| \le kE_\eta(Mab).$$

Therefore,

$$\|z\|_{C_{1-\eta}} = \sup_{\tau\in[0,a]} \tau^{1-\eta}\|z(\tau)\|_Z \le kE_\eta(Mab)$$

Now,

$$
\begin{aligned}
&\tau^{1-\eta}\|z(\tau)-y(\tau)\| \\
&\leq \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)[Bu(r)-Bv(r)]\|dr \\
&\quad+\tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)[g(r,z(r),u(r)) \\
&\quad-g(r,y(r),v(r))]\|dr \\
&\leq \frac{M}{\Gamma(\eta)}\Bigg[\tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|Bu(r)-Bv(r)\|dr \\
&\quad+\tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\big(\Bbbk'[r^{1-\eta}\|z(r)-y(r)\|_Z \\
&\quad+\|u(r)-v(r)\|_U]\big)dr\Bigg] \\
&\leq \frac{M}{\Gamma(\eta)}\Bigg[\left(\frac{q-1}{q\eta-1}\right)^{\frac{q-1}{q}}(1+\Bbbk'\ell)\|Bu(r)-Bv(r)\|a^{1-\frac{1}{q}} \\
&\quad+\Bbbk' a^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}r^{1-\eta}\|z(r)-y(r)\|dr\Bigg].
\end{aligned} \tag{8}
$$

Thus,

$$
\begin{aligned}
&t^{1-\eta}\|z(\tau)-y(\tau)\|_Z \\
&\leq \varrho\|Bu(r)-Bv(r)\|_Z \\
&\quad+\frac{M\Bbbk' a^{1-\eta}}{\Gamma(\eta)}\int_0^\tau (\tau-r)^{\eta-1}r^{1-\eta}\|z(r)-y(r)\|dr.
\end{aligned}
$$

Again, using generalised Gronwall's identity [42], we have

$$\tau^{1-\eta}\|z(\tau)-y(\tau)\|_Z \leq \varrho E_\eta(M\Bbbk' a)\|Bu-Bv\|_{L^q}$$

Hence,

$$\|z-y\|_{C_{1-\eta}} \leq \varrho E_\eta(M\Bbbk' a)\|Bu-Bv\|_{L^q}$$

This accomplishes the proof. □

Theorem 2. *The nonlinear control system (1) becomes approximately controllable, provided the hypotheses* $(H1)$ *and* $(H3)-(H5)$ *hold true and* A *generates the differentiable semigroup* $T(t)$.

Proof. It is well known that domain of A, $D(A)$ is dense in Z. Thus, to manifest approximate controllability of nonlinear control system (1), it is adequate to claim that $D(A)\subset \overline{K_a(g)}$, i.e., for any given $\epsilon>0$ and $\lambda\in D(A)$, a control $u_\epsilon\in L^q([0,a];U)$ can be found satisfying

$$\|\lambda^*-\mathbb{F}(Bu_\epsilon)-\mathbb{F}(\Omega_g(z_\epsilon))\|_Z \leq \epsilon,$$

where $z_\epsilon(t)$ is a mild solution of system(1) in accord with the control $u_\epsilon(t)$ and

$$\lambda-a^{\eta-1}T_\eta(a)z_0=\lambda^*\in D(A)$$

Let $\epsilon>0$ be given and $u_1\in L^q([0,a];U)$. Then by hypothesis (H5), there exists $u_2\in L^q([0,a];U)$ satisfying

$$\|\lambda^*-\mathbb{F}(\Omega_g(z_1))-\mathbb{F}(Bu_2)\|_Z \leq \frac{\epsilon}{2^2}$$

where $z_1(\tau)=z(\tau,u_1)$. Denote $z_2(\tau)=z(\tau,u_2)$, again by hypothesis (H5), $\exists\,\omega_2\in L^q([0,a];U)$ satisfying

$$\|\mathbb{F}[\Omega_g(z_2)-\Omega_g(z_1)]-\mathbb{F}(B\omega_2)\|_Z \leq \frac{\epsilon}{2^3}$$

and

$$
\begin{aligned}
&\|B\omega_2\|_{L_p} \\
&\leq \aleph\|\Omega_g(z_2)-\Omega_g(z_1)\|_{L_p} \\
&\leq \aleph\Bbbk'\left[\tau^{1-\eta}\|z_2-z_1\|+\|u_2-u_1\|\right] \\
&\leq \aleph\Bbbk'\Big[\varrho E_\eta(M\Bbbk' a)\|Bu_2-Bu_1\|+\ell\|Bu_2-Bu_1\|\Big] \\
&\leq \Bigg[\frac{M\aleph\Bbbk'}{\Gamma(\eta)}\left(\frac{q-1}{q\eta-1}\right)^{1-\frac{1}{q}}(1+\Bbbk'\ell)a^{1-\frac{1}{q}}E_\eta(M\Bbbk' a) \\
&\quad+\aleph\Bbbk'\ell\Bigg]\|Bu_2-Bu_1\|_{L^q}.
\end{aligned}
$$

Now, define

$$u_3(\tau)=u_2(\tau)-\omega_2(\tau),\quad u_3(\tau)\in U,$$

then

$$
\begin{aligned}
&\|\lambda^*-\mathbb{F}\Omega_g(z_2)-\mathbb{F}Bu_3\|_Z \\
&\qquad\leq \|\lambda^*-\mathbb{F}\Omega_g(z_1)-\mathbb{F}Bu_2\|_Z \\
&\qquad\quad+\|\mathbb{F}B\omega_2-[\mathbb{F}\Omega_g(z_2)-\mathbb{F}\Omega_g(z_1)]\|_Z \\
&\qquad\leq \left(\frac{1}{2^2}+\frac{1}{2^3}\right)\epsilon
\end{aligned}
$$

By applying inductions, a sequence $\{u_n\}$ in $L^q([0,a];U)$ is obtained such that

$$
\begin{aligned}
&\|\lambda^*-\mathbb{F}\Omega_g(z_n)-\mathbb{F}Bu_{n+1}\|_Z \\
&\qquad< \left(\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\ldots+\frac{1}{2^{n+1}}\right)\epsilon,
\end{aligned}
$$

where $z_n(\tau)=z(\tau,u_n(\tau))$ and

$$
\begin{aligned}
&\|Bu_{n+1}-Bu_n\|_{L^q} \\
&\quad< \Bigg[\frac{M\aleph\Bbbk'}{\Gamma(\eta)}\left(\frac{aq-a}{q\eta-1}\right)^{1-\frac{1}{q}}(1+\Bbbk'\ell)E_\eta(M\Bbbk' a) \\
&\qquad+\aleph\Bbbk'\ell\Bigg]\|Bu_n-Bu_{n-1}\|_{L^q}
\end{aligned}
$$

By (6), it is evident that the sequence $\{Bu_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $L^q([0,a];Z)$. Thus, for any $\epsilon > 0$, a positive integer n_0 can be found satisfying

$$\|\mathbb{F}Bu_{n_0+1} - \mathbb{F}Bu_{n_0}\|_Z < \frac{\epsilon}{2}.$$

Now,

$$\begin{aligned}
&\|\lambda^* - \mathbb{F}\Omega_g(z_{n_0}) - \mathbb{F}Bu_{n_0}\|_Z \\
&\quad \le \|\lambda^* - \mathbb{F}\Omega_g(z_{n_0}) - \mathbb{F}Bu_{n_0+1}\|_Z \\
&\qquad + \|\mathbb{F}Bu_{n_0+1} - \mathbb{F}Bu_{n_0}\|_Z \\
&\quad \le \left(\frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{n_0+1}}\right)\epsilon + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Hence, the approximate controllability of (1) is proved. □

5. Example

Examine the below mentioned initial value problem for $\tau \in (0, 1]$ and $x \in [0, \pi]$:

$$\begin{aligned}
D_\tau^{\frac{2}{3}} z(\tau, x) &= \frac{\partial^2}{\partial x^2} z(\tau, x) + u(\tau) \\
&\quad + g(\tau, z(\tau, x), u(\tau)), \qquad (9) \\
z(\tau, 0) &= z(\tau, \pi) = 0, \\
I_{0^+}^{\frac{1}{3}} z(\tau, x)|_{\tau=0} &= z_0(x),
\end{aligned}$$

Take $Z = U = L^2([0, \pi])$ and $A : D(A) \subset Z \to Z$ as

$$Az = z''$$

where

$$D(A) = \left\{ z \in Z \ | \ z, \frac{\partial z}{\partial x} \text{ are absolutely continuous}, \frac{\partial^2 z}{\partial x^2} \in Z \text{ and } z(0) = 0 = z(\pi) \right\}$$

Then, A can be expressed as

$$Az = \sum_{m=1}^{\infty} (-m^2)\langle z, \alpha_m\rangle \alpha_m, \quad z \in D(A)$$

where $\alpha_m(x) = \sqrt{\frac{2}{\pi}} sin\ mx$ ($m \in \mathbb{N}$) are the eigen functions corresponding to the eigen values $-m^2$ respectively and $\{\alpha_1, \alpha_2, ...\}$ is a basis of Z.
A differentiable semigroup $T(\tau)(\tau > 0)$ in Z having A as its infinitesimal genaerator is expressed as

$$T(\tau)z = \sum_{m=1}^{\infty} exp^{-m^2\tau}\langle z, \alpha_m\rangle \alpha_m, \quad z \in Z$$
$$\text{and} \quad \|T(\tau)\| \le e^{-1} < 1 = M.$$

Let us choose the nonlinear function g as

$$\begin{aligned}
g(\tau, z(\tau, x), u(\tau)) &= 1 + \tau^2 + \beta\tau^\gamma \\
&\quad \times [z(\tau, x) + \sin z(\tau, x) + u(\tau)],
\end{aligned}$$

where β and γ are constants with $-1 \le \beta \le 1$ and $\gamma \ge 1 - \eta$. Now,

$$\begin{aligned}
&\|g(\tau, z(\tau, x), u(\tau))\| \\
&\quad \le 1 + \tau^2 + |\beta|\tau^\gamma \left[\left\|[z(\tau, x) + \sin z(\tau, x)]\right\| + \|u(\tau)\|\right] \\
&\quad \le 1 + \tau^2 + |\beta|\tau^{\gamma+\eta-1}\tau^{1-\eta}\left[2\|z(\tau, x)\| + \|u(\tau)\|\right] \\
&\quad \le (1 + \tau^2) + 2|\beta|\tau^{1-\eta}\|z(\tau, x)\| + \|u(\tau)\|
\end{aligned}$$

and

$$\begin{aligned}
&\|g(\tau, z(\tau, x), u(\tau)) - g(\tau, y(\tau, x), v(\tau))\| \\
&\quad \le |\beta|\tau^\gamma \left[\|z(\tau, x) - y(\tau, x) + \sin z(\tau, x) - \sin y(\tau, x)\|\right. \\
&\qquad \left. + \|u(\tau) - v(\tau)\|\right] \\
&\quad \le |\beta|\tau^{\gamma+\eta-1}\tau^{1-\eta}\Bigg[\|z(\tau, x) - y(\tau, x)\| \\
&\qquad + \left\|2\cos\left(\frac{z(\tau, x) + y(\tau, x)}{2}\right)\sin\left(\frac{z(\tau, x) - y(\tau, x)}{2}\right)\right\| \\
&\qquad + \|u(\tau) - v(\tau)\|\Bigg] \\
&\quad \le |\beta|\tau^{1-\eta}\left[2\|z(\tau, x) - y(\tau, x)\| + \|u(\tau) - v(\tau)\|\right] \\
&\quad \le 2|\beta|\left[\|z(\tau, x) - y(\tau, x)\| + \|u(\tau) - v(\tau)\|\right]
\end{aligned}$$

Here, the assumptions $(H1)$ and $(H2)$ are evidently satisfied with $\psi(\tau) = 1 + \tau^2$ and $b = \Bbbk = 2|\beta|$. Moreover, assumption $(H5)$ is satisfied by choosing β sufficiently close to zero.
The abstract form of the system (1) is expressed as:

$$D_\tau^{\frac{2}{3}} \widetilde{z}(\tau) = A\widetilde{z}(\tau) + B\widetilde{u}(\tau) + g(\tau, \widetilde{z}(\tau), \widetilde{u}(\tau)), \tau \in (0, 1],$$
$$I_\tau^{\frac{1}{3}} \widetilde{z}(\tau)|_{\tau=0} = \widetilde{z}_0,$$

where $\widetilde{z}(\tau) = z(\tau, \cdot)$, $\widetilde{u}(\tau) = u(\tau, \cdot)$ and $\widetilde{z}_0 = z_0(\cdot)$.
Approximate controllability of (1) accomplishes from Theorem 2 as it is seen assumptions $H(1)$-$H(5)$ are satisfied.

6. Conclusion

In this paper, thorough analysis for existence and uniqueness, and approximate controllability of the fractional nonlinear differential system has been performed in Banach spaces. The existence and uniqueness results were established using concepts of fractional calculus, definition [41], generalised Gronwall's inequality, semigroup theory and Banach's fixed point theorem. The sufficient condition for approximate controllability was derived with the aid of Lemma 2 and iterative technique. The present findings of the paper can be extended to stochastic fractional differential equations with or without delay in state or in the control term present in the nonlinear function of the system. For some idea, see [34, 38, 39].

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RESEARCH ARTICLE

A predator-prey model for the optimal control of fish harvesting through the imposition of a tax

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ABSTRACT

This paper is devoted to the study of ecosystem based fisheries management. The model represents the interaction between prey and predator population with Holling II functional response consisting of different carrying capacities and constant intrinsic growth rates. We have considered the continuous harvesting of predator only. It is observed that if the intrinsic growth rate of predator population crosses a certain critical value, the system enters into Hopf bifurcation. Our observations indicate that tax, the management object in fisheries system play huge impacts on this system. The optimal harvesting policy is disposed by imposing a tax per unit of predator biomass. The optimal harvest strategy is determined using Pontryagin's maximum principle, which is subject to state equations and control limitations. The implications of tax are also examined. We have derived different bifurcations and global stability of the system. Finally, numerical simulations are used to back up the analytical results.

1. Introduction

Ecosystems are made up of live creatures, plants, and non-living things that coexist and 'interact' with one another. Fish are part of the marine ecosystem since they do not live in isolation. They have a strong connection to their physical, chemical, and biological environments. They rely on the environment to supply the necessary conditions for their growth, reproduction, and survival. They also serve as a food supply for other species such as seabirds and marine mammals, making them an essential component of the marine food chain.

Fishing activity has an impact not just on the fish populations, but also on the habitat in which the fish dwell. Fishing has both direct and indirect effects on the ecosystem. As a result of fishing, other species are caught and/or discarded, and fishing gear affects the seabed. Fishing can have an indirect influence on the ecosystem by harvesting fish from the marine food chain, for example. The ecosystem approach to fisheries recognises that fisheries must be managed as part of its ecosystem and that their environmental impact should be kept to a minimum. It is well known that interspecies competition between multi-fish species is vary much complicated. Competition between two fish species with the combined harvesting have been discussed in [1]. The authors in [2] have studied the combined harvesting of two-species predator-prey model with discrete time delay in fishery system. However, restricting to harvest fishes above a certain age or size only can help the fishery and prevent its extinction. Harvesting of two competing fish species in the presence of toxicity has been discussed in [3]. The authors studied bionomic equilibrium and optimal harvesting policy with help of control theory. The author in [4] have analyzed a coral

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reef ecosystem to explore the effects of changes in economic, biological, and social parameters in a multiple-species coral-reef ecosystem with adaptive harvesters. Recently, the authors in [5] and [6] have developed a bio-economic model that combines a model of competition and a model of prey-predator of multi-fish populations. They have calculated the fishing effort which maximizes the income of the fishing fleet. In ecology, a stochastic differential fishery game for a two species fish population has been studied in [7]. Recently, several aspects of the optimal harvest of a stage structured model of a fishery have been discussed in [8]. The authors also looked at how changes in costs and harvesting technologies will affect whether the optimal harvesting strategy is to target one age group, the other age group, or both age groups. The authors in [9] have emphasized the importance of age-structured modelling in practical fishery economics. In [10], a prey-predator type fishery model with solely prey harvesting was investigated. Also many papers on prey-predator model with harvesting have been studied in [11–22]. In [23] and [24], have explored deterministic chaos vs stochastic oscillation in a prey-predator model and global stability of a three-species food chain model with diffusion, respectively. Recently, in [25], local and global stability have been examined in a fractional prey-predator model in presence of harvesting rate.

Motivated by the above theoretical and experimental literatures, the dynamics of such system in which implications of tax on harvesting of predator is studied. It should be noted from the aforementioned literature review that no attempt has been made to research prey-predator fishery harvesting with taxation as a control device. The present paper investigates a dynamic reaction model in the context of a prey-predator type fishery system in which only the predator species are harvested. The imposition of a tax serves as a deterrent to fishermen while also protecting the predator from over-exploitation. The main goal of this paper is to determine the proper taxation strategy that will benefit the community as much as possible through harvesting while preventing the extinction of the predator.

The main target in present manuscript is to investigate the subsequent biological topics:

- How does carrying capacity of both population influence the prey-predator dynamics.
- Can imposing tax influence to stabilize the fishery system.
- How does constant price per unit biomass of predator influence the prey-predator dynamics.

In this paper, we present a new deterministic prey-predator model. It incorporates a feature that appears for the first time in this situation, the emergence of a Holling type II response function that has only been suggested by [26] in presence of harvesting of predator with taxation as a control instrument. The use of a Holling type II functional response is however in contrast to other models, such as [27], where the predation term of the model exhibits ratio-dependent type. In contrast to other current models [26], we account for an alternative food source for the predator, which helps in stabilizing the system. In the present article, predator do not only depend on prey but also grow logistically. In this plankton-fish interaction model, two logistic growth rates of the phytoplankton and zooplankton populations are incorporated in [28].

In this paper, a prey-predator interaction model in presence of harvesting is described in fishery management. The stability of equilibrium point is analyzed. Conditions for globally asymptotically stable of coexistence equilibrium have been studied. We also found the requirements for system instability near the coexiestence equilibrium and Hopf bifurcation. We analyze the optimal tax policy by using Pontryagin's maximum principle. To back up our analytical result, we ran numerical simulations with a set of parametric variables. The paper comes to a close with a brief conclusion.

2. The mathematical model

Let $x(t)$ be the concentration of the prey population at time t with carrying capacity K_1 and constant intrinsic growth rate r_1. Let $y(t)$ denotes the predator population at time t with carrying capacity K_2 and constant intrinsic growth rate r_2. Let α_1 be the maximal prey's ingestion rate and β_1 be the maximal conversion rate for the growth of predator population respectively ($\beta_1 \leq \alpha_1$). Let d_1, d_2 be the mortality rates of the prey and predator population respectively. To characterise the grazing phenomenon, we use the Holling type II functional form with a_1 as half saturation constant.

The predator's catchability coefficient is constant c_1 in this case and E is harvesting effort. In this paper, we consider E to be a dynamic (i.e. time-dependent) variable regulated by the equation

$$E(t) = \mu_1 Q(t), \ 0 \leq \mu_1 \leq 1, \qquad (1)$$

$$\frac{dQ}{dt} = I(t) - \gamma_1 Q(t),\ Q(0) = Q_0, \quad (2)$$

where $I(t)$ is the gross investment rate at time t. The amount of capital invested fishery at time t is $Q(t)$ and constant rate of depreciation of capital is γ_1. A regulatory agency controls exploitation of the fishery by imposing a tax $\tau_1(>0)$ per unit biomass of the predator. Here $\tau_1(<0)$ be the subsidy given to the fisherman. The net economic revenue to the fisherman is $[c_1(p_1 - \tau_1)y - C]\,E$, where p_1 is the constant price per unit biomass of predator species and C is the constant cost per unit of harvesting effort. Moreover, we assume that the gross rate of investment of capital is proportional to the net economic revenue to the fisherman. Therefore we can write

$$I(t) = \mu_2\,[c_1(p_1 - \tau_1)y - C]\,E(t),\ 0 \le \mu_2 \le 1. \quad (3)$$

Equation (3) shows that the maximum investment rate at any time equals the net economic revenue (for $\mu_2 = 1$) at that time. By virtue of (2) and (3) yield the result

$$\frac{dE}{dt} = \{\mu_1\mu_2\,[c_1(p_1 - \tau_1)y - C] - \gamma_1\}E. \quad (4)$$

In this paper, we consider tax as the management objective when discussing the impact of harvesting in the fishery system and assume $p_1 - \tau_1 > 0$. Let $m = \mu_1\mu_2$; as a result, the following system of equation is given by :

$$\begin{aligned}
\frac{dx}{dt} &= r_1x(1 - \frac{x}{K_1}) - \frac{\alpha_1 xy}{a_1 + x} - d_1x \equiv G_1,\\
\frac{dy}{dt} &= r_2y(1 - \frac{y}{K_2}) + \frac{\beta_1 xy}{a_1 + x} - d_2y - c_1yE \equiv G_2,\\
\frac{dE}{dt} &= \{m\,[c_1(p_1 - \tau_1)y - C] - \gamma_1\}E \equiv G_3,
\end{aligned} \quad (5)$$

where $G_i = G_i(x, y, E)$, i=1,2,3. The system (5) will be analyzed with the following initial conditions,

$$x(0) = X_1 \ge 0,\ y(0) = X_2 \ge 0,\ E(0) = X_3 \ge 0. \quad (6)$$

3. Some preliminary results

3.1. Positive invariance

By setting $X = (x,\ y,\ E)^T \in \mathbf{R}^3$ and $G(X) = [G_1(X),\ G_2(X),\ G_3(X)]^T$, with $G : {\mathbf{R}_+}^3 \to \mathbf{R}^3$ and $G \in C^\infty(\mathbf{R}^3)$, equation (5) becomes

$$\dot{X} = G(X), \quad (7)$$

together with $X(0) \in {\mathbf{R}_+}^3$. It is easy to check that whenever $X(0) \in {\mathbf{R}_+}^3$ with $X_i \ge 0$, for i=1, 2, 3, then $G_i(X)\ |_{X_i=0} \ge 0$. Then any solution of equation (7) with $X_0 \in {\mathbf{R}_+}^3$, say $X(t) = X(t; X_0)$, is such that $X(t) \in {\mathbf{R}_+}^3$ for all $t > 0$.

Lemma 1. *All the non negative solutions of the system (5) are ultimately bounded.*

Proof. From the first equation of the system (5) we have $\frac{dx}{dt} \le r_1x\left(1 - \frac{x}{K_1}\right)$, which gives $x(t) \to K_1$ as $t \to \infty$.
Therefore, corresponding to $\epsilon_1 > 0$, there exists $t_{\epsilon_1} > 0$ such that $x(t) \le K + \epsilon_1$ for all $t \ge t_{\epsilon_1}$.
For all $t \ge t_{\epsilon_1}$, from the second equation of (5), we have $\frac{dy}{dt} \le y\left[r_2(1 - \frac{y}{K_2}) + \frac{\beta_1(K_1+\epsilon_1)}{a_1+K_1+\epsilon_1}\right]$ and so, corresponding to $\epsilon_2 > 0$ there exists $t_{\epsilon_2} > 0$ such that $y(t) \le K_2 + \frac{\beta_1(K_1+\epsilon_1)K_2}{r_2(a_1+K_1+\epsilon_1)}$ for all $t \ge \max\{t_{\epsilon_1}, t_{\epsilon_2}\}$.
This gives, $\lim\limits_{t\to\infty}\{x(t) + y(t)\} \le K_1 + K_2 + \frac{\beta_1K_1K_2}{r_2(a_1+K_1)}$. Using the previous conditions in the third equation of system (5) we can easily to verify that E is bounded and less than some positive constant when $t \to \infty$.
Now we consider, $w(t) = x(t) + y(t) + \frac{1}{m(\rho_1-\tau_1)}E$.
The time derivative of w along the solutions of is
$\frac{dw}{dt} \le r_1x(1 - \frac{x}{K_1}) + r_2y(1 - \frac{y}{K_2}) - d_1x - d_2y - \frac{(mc+\gamma_1)}{(\rho_1-\tau_1)m}E$,
$\frac{dw}{dt} \le -D_0w + r_1x(1 - \frac{x}{K_1}) + r_2y(1 - \frac{y}{K_2})$ where $D_o = Min\{d_1, d_2, (mc + \gamma_1)\}$,
$\frac{dw}{dt} + D_0w \le \frac{r_1K_1}{4} + \frac{r_2K_2}{4}$.
Integrating the above inequality and using initial condition we get, $0 < w(t) \le w(0)e^{-D_0t} + (\frac{r_1K_1+r_2K_2}{4})(1 - e^{-D_0t})$.
As $t \to \infty$, the above inequality simplifies to
$0 < w(t) \le (\frac{r_1K_1+r_2K_2}{4})$. Hence, all the solutions of the system is uniformly bounded.

□

3.2. Equilibria

The system (5) possesses the following equilibria:
(i) The prey-predator equilibrium $S_0 = (0, 0, 0)$.
(ii) The predator free equilibrium $S_1 = (x_1, 0, 0) = (\frac{K_1(r_1-d_1)}{r_1}, 0, 0)$, which exists if $r_1 > d_1$.
(iii) The prey free equilibrium in absence of harvesting effort $S_2 = (0, y_2, 0) = (0, \frac{K_2(r_2-d_2)}{r_2}, 0)$, which exists if $r_2 > d_2$.
(iv) The prey free equilibrium in presence of harvesting effort $S_3(0, y_3, E_3)$ with $y_3 = \frac{\gamma_1+Cm}{mc_1(p_1-\tau_1)}$, $E_3 = \frac{K_2mc_1(p_1-\tau_1)(r_2-d_2)-r_2(\gamma_1+Cm)}{K_2mc_1(p_1-\tau_1)}$, which exists if $p_1 > Max\{\tau_1, \tau_1 + \frac{r_2(\gamma_1+Cm)}{K_2mc_1(r_2-d_2)}\}$.
(v) The harvesting effort free equilibrium $S_4(x_4, y_4, 0)$ with $x_4 = \frac{a_1(d_2-r_2+\frac{r_2y_4}{K_2})}{\beta_1-(d_2-r_2+\frac{r_2y_4}{K_2})}$

and $y_4 = \frac{[r_1(1-\frac{x_4}{K_1})-d_1][a_1+x_4]}{\alpha_1}$, which exists if $x_4 < \frac{r_1-d_1}{K_1}$ and $\frac{(r_2-d_2)K_2}{r_2} < y_4 < \frac{K_2(\beta_1-d_2+r_2)}{r_2}$.
(vi) The coexistence equilibrium $S^* = (x^*, y^*, E^*)$ with $y^* = \frac{\gamma_1+Cm}{mc_1(p_1-\tau_1)}$,
$E^* = \frac{r_2(1-\frac{y^*}{K_2})-\frac{\beta_1 x^*}{a_1+x^*}}{c_1} - \frac{d_2}{c_1}$ and x^* satisfies $x^{*2} + \{\frac{K_1 d_1}{r_1} + (a_1 - K_1)\}x^* + \frac{K_1\alpha(\gamma_1+Cm)}{\gamma_1 mc_1(p_1-\tau_1)} + \frac{a_1 K_1 d_1}{r_1} - a_1 K_1 = 0$.
Let x_1 and x_2 be the roots of above equation. We only consider that x_1, x_2 have only one positive root then $x_1 x_2 = \frac{K_1\alpha_1(\gamma_1+Cm)}{\gamma_1 mc_1(p_1-\tau_1)} + \frac{a_1 K_1 d_1}{r_1} - a_1 K_1 < 0 \implies \tau_1 < \frac{a_1 K_1(r_1-d_1)mc_1 p_1 - K_1\alpha_1(\gamma_1+Cm)}{aK_1(r_1-d_1)mc_1}$, and $\Delta = \{\frac{K_1 d_1}{r_1} + (a_1 - K_1)\}^2 + 4\{a_1 K_1 - \frac{a_1 K_1 d_1}{r_1} - \frac{K_1\alpha_1(\gamma_1+Cm)}{\gamma_1 mc_1(p_1-\tau_1)}\} > 0$.
Hence, x^* exists as a positive root: $x^* = \frac{1}{2}[\frac{K_1 d_1}{r_1} + (a_1 - K_1) + \sqrt{\Delta}]$. Thus, the coexistence equilibrium exists if $x^* > 0,\ y^* > 0$ *and* $E^* > 0$ i.e. $\tau_1 < Min\{\frac{a_1 K_1(r_1-d_1)mc_1 p_1 - K_1\alpha_1(\gamma_1+Cm)}{aK_1(r_1-d_1)mc_1}, p_1\}$.

3.3. Stability analysis of the system (5)

In this section, local stability analysis of the system around the biologically feasible equilibria is performed. Let $\overline{S} = (\overline{x}, \overline{y}, \overline{E})$ be any arbitrary equilibrium. Then the Jacobian matrix about $\overline{S}$ is given by

$$\overline{V} = \begin{bmatrix} \overline{v}_{11} & -\frac{\alpha_1\overline{x}}{a_1+\overline{x}} & 0 \\ \frac{a_1\beta_1\overline{y}}{(a_1+\overline{x})^2} & \overline{v}_{22} & -c_1\overline{y} \\ 0 & mc_1(p_1-\tau_1)\overline{E} & \overline{v}_{33} \end{bmatrix},$$

where $\overline{v}_{11} = r_1(1-\frac{2\overline{x}}{K_1}) - \frac{a_1\alpha_1\overline{y}}{(a_1+\overline{x})^2} - d_1$, $\overline{v}_{22} = r_2(1-\frac{2\overline{y}}{K_2}) + \frac{\beta_1\overline{x}}{a_1+\overline{x}} - d_2 - c_1\overline{E}$ and $\overline{v}_{33} = m[c_1(p_1-\tau_1)\overline{y} - C] - \gamma_1$.

By calculating the Jacobian matrix for the equilibrium S_0 of the system (5). We see that the eigenvalues of the variational matrix V_0 are $\lambda_1 = r_1 - d_1 > 0$, $\lambda_2 = r_2 - d_2 > 0$, $\lambda_3 = -mC - \gamma_1 < 0$. Clearly S_0 is always unstable. It is clear that $S_0(0,0,0)$ is always unstable.

Lemma 2. *If* $R_0 = \frac{\beta_1 K_1(r_1-d_1)}{(d_2-r_2)[a_1 r_1 + K_1(r_1-d_1)]} > 1$ *then the predator free steady state* S_1 *of the system (5) is unstable.*

Proof. Now again computing the Jacobian matrix for the equilibrium S_1 of the system (5) we find that the eigenvalues of the Jacobian matrix V_1 are $\lambda_{11} = -(mC+\gamma_1) < 0$, $\lambda_{12} = -r_1 + d_1 < 0$ and $\lambda_{13} = r_2 + \frac{\beta_1 K_1(r_1-d_1)}{a_1 r_1 + K_1(r_1-d_1)} - d_2$. It is clear that $\lambda_{13} < 0$ if $\frac{\beta_1 K_1(r_1-d_1)}{(d_2-r_2)[a_1 r_1 + K_1(r_1-d_1)]} < 1$ i.e. $R_0 < 1$ where $R_0 = \frac{\beta_1 K_1(r_1-d_1)}{(d_2-r_2)[a_1 r_1 + K_1(r_1-d_1)]}$.
So, S_1 is asymptotically stable if and only if $R_0 < 1$. Clearly if $R_0 > 1$, then predator free steady state S_1 is unstable which indicates the proof of lemma 1. ☐

Lemma 3. *There exists a feasible prey free steady state* S_2 *in absence of harvesting effort of predator of the system (5) which is unstable if*

$$R_1 = \frac{a_1 r_2(r_1-d_1)}{\alpha_1 K_2(r_2-d_2)} > 1. \tag{8}$$

Proof. Now again computing the Jacobian matrix for the equilibrium S_2 of the system (5) we find that the eigenvalues of the Jacobian matrix V_2 are $\lambda_{11}^1 = -(mC+\gamma_1) < 0$, $\lambda_{12}^{'} = -r_2 + d_2 < 0$ and $\lambda_{13}^{'} = r_1 - d_1 - \frac{\alpha_1 K_2(r_2-d_2)}{a_1 r_2}$. It is clear that $\lambda_{13}^{'} < 0$ if $\frac{a_1 r_2(r_1-d_1)}{\alpha_1 K_2(r_2-d_2)} < 1$ i.e. $R_1 < 1$ where $R_1 = \frac{a_1 r_2(r_1-d_1)}{\alpha_1 K_2(r_2-d_2)}$.
So, S_1 is asymptotically stable if and only if $R_1 < 1$. Clearly, if $R_1 > 1$, then prey free without harvesting effort steady state S_2 is unstable which indicates the proof of lemma 2. ☐

Lemma 4. *There exists a prey free steady state* S_3 *of the system (5) which is unstable if*

$$R_2 = \frac{a_1 mc_1(r_1-d_1)(p_1-\tau_1)}{\alpha_1(\gamma_1+Cm)} > 1. \tag{9}$$

Proof. Further the eigenvalues of the Jacobian matrix V_3 around the equilibrium S_3 of the system (5) are θ_1, θ_2 which are the roots of the equation $\theta^2 + \frac{r_2 y_3}{K_2}\theta + mc_1^2(p_1-\tau_1)y_3 E_3 = 0$ and $\theta_3 = r_1 - d_1 - \frac{\alpha_1 y_3}{a_1}$. Clearly, θ_1 and θ have negative real parts for equilibrium point $S_3(0, y_3, E_3)$. So, prey free equilibrium S_3 is asymptotically stable if $\theta_3 < 0$ i.e. $\frac{a_1 mc_1(r_1-d_1)(p_1-\tau_1)}{\alpha_1(\gamma_1+Cm)} < 1$, i.e. $R_2 < 1$ where $R_2 = \frac{a_1 mc_1(r_1-d_1)(p_1-\tau_1)}{\alpha_1(\gamma_1+Cm)}$. Therefore, S_3 is unstable if condition (9) i.e $R_2 > 1$ is satisfied. ☐

Lemma 5. *The harvesting effort free equilibrium of the (5) is locally asymptotically stable if* $B_i > 0$ *where i=1,2,3 and* $B_1 B_2 - B_3 > 0$.

Proof. The Jacobian matrix of system (5) around the harvesting effort free equilibrium $S_4 = (x_4, y_4, 0)$ is

$$V^* = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & n_{33} \end{bmatrix},$$

where $m_{11} = r_1 - \frac{2r_1 x_4}{K_1} - \frac{a_1\alpha_1 y_4}{(a_1+x_4)^2} - d_1 = -\frac{r_1 x_4}{K_1} + \frac{\alpha_1 x_4 y_4}{(a_1+x_4)^2}$, $m_{12} = -\frac{\alpha_1 x_4}{a_1+x_4} < 0, m_{21} = \frac{a_1\beta_1 y_4}{(a_1+x_4)^2} > 0$, $m_{22} = -\frac{r_2 y_4}{K_2} < 0$, $m_{23} = -c_1 y_4 < 0$, $m_{33} = m[c_1(p_1-\tau_1)y_4 - C] - \gamma_1 > 0$.
The characteristic equation is given by

$$Q_1^3 + B_1 Q_1^2 + B_2 Q_1 + B_3 = 0,$$

where $B_1 = -(m_{11}+m_{22}+m_{33})$, $B_2 = m_{11}m_{22}+m_{11}m_{33}+m_{22}m_{33}-m_{12}m_{21}$, $B_3 = m_{33}m_{12}m_{21}-m_{33}m_{11}m_{22}$.
Case 1: If $m_{11} < 0$, which shows that $B_3 < 0$. Then S_4 is unstable.
Case 2: If $m_{11} > 0$, Then $B_1 = -(m_{11} + m_{22} + m_{33}) > 0$ if $m_{22} < m_{11} + m_{33})$. Also $B_2 = m_{11}m_{22}+m_{11}m_{33}+m_{22}m_{33}-m_{12}m_{21} > 0$ if $m_{11}m_{33} - m_{12}m_{21} > -(m_{11}m_{22} + m_{22}m_{33})$ since $m_{11}m_{22} < 0$, $m_{11}m_{33} > 0$, $m_{22}m_{33} < 0$ and $m_{12}m_{21} < 0$. Clearly $B_3 = m_{33}m_{12}m_{21} - m_{33}m_{11}m_{22} > 0$ if $m_{33}m_{12}m_{21} > m_{33}m_{11}m_{22}$ since $m_{33}m_{12}m_{21} < 0$ and $m_{33}m_{11}m_{22} < 0$. Now $B_1B_2 - B_3 > 0$ if $B_1B_2 > B_3$. Therefore, according the Routh-Hurwitz criteria, all roots of above equation have negative real parts. Thus S_4 is locally asymptotically stable. □

Lemma 6. *The coexistence equilibrium of the system (5) is locally asymptotically stable if $\Theta_i > 0$ where i=1,2,3 and $\Theta_1\Theta_2 - \Theta_3 > 0$.*

Proof. The Jacobian matrix of system (5) around the positive interior equilibrium $S^* = (x^*, y^*, E^*)$ is

$$V^* = \begin{bmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & n_{23} \\ 0 & n_{32} & 0 \end{bmatrix},$$

where $n_{11} = r_1 - \frac{2r_1x^*}{K_1} - \frac{a_1\alpha_1y^*}{(a_1+x^*)^2} - d_1 = -\frac{r_1x^*}{K_1} + \frac{\alpha_1x^*y^*}{(a_1+x^*)^2}$, $n_{12} = -\frac{\alpha_1x^*}{a_1+x^*} < 0$, $n_{21} = \frac{a_1\beta_1y^*}{(a_1+x^*)^2} > 0$, $n_{22} = -\frac{r_2y^*}{K_2} < 0$, $n_{23} = -c_1y^* < 0$, $n_{32} = mc_1(p_1 - \tau_1)E^* > 0$.
The characteristic equation is

$$Q^3 + \Theta_1Q^2 + \Theta_2Q + \Theta_3 = 0,$$

where $\Theta_1 = -(n_{11}+n_{22})$, $\Theta_2 = n_{11}n_{22}-n_{32}n_{23}-n_{12}n_{21}$, $\Theta_3 = n_{11}n_{32}n_{23}$.
Case 1: If $n_{11} > 0$, which shows that $\Theta_3 < 0$. Then S^* is unstable.
Case 2: If $n_{11} < 0$, Then $\Theta_1 = -(n_{11} + n_{22}) > 0$ Also, $\Theta_2 = n_{11}n_{22} - n_{32}n_{23} - n_{12}n_{21} > 0$ since $n_{11}n_{22} > 0$, $n_{32}n_{23} < 0$ and $n_{12}n_{21} < 0$. Clearly, $\Theta_3 = n_{11}n_{32}n_{23} > 0$. Now $\Theta_1\Theta_2 - \Theta_3 > 0$ if $\Theta_1\Theta_2 > \Theta_3$. Therefore according the Routh-Hurwitz criteria, all roots of above equation have negative real parts. Thus S^* is locally asymptotically stable. □

The analytical results are summarized in the Table 1.

Theorem 1. *When the intrinsic growth rate of predator r_2 crosses a critical value, say r_2^*, the system (5) enters into a Hopf-bifurcation around the coexistence equilibrium, which induces oscillations of the populations.*

Proof. If the Hopf-bifurcation exists for $r_2 = r_2^*$, the following are the necessary and sufficient conditions: (i) $\Theta_i(r_2^*) > 0$, $i = 1,2,3$ (ii)$\Theta_1(r_2^*)\Theta_2(r_2^*) - \Theta_3(r_2^*) = 0$ and (iii) the eigenvalues of above characteristic equation should be of the form $\lambda_i = u_i + \mathbf{i}v_i$, and $\frac{du_i}{dr_2} \neq 0$, $i = 1,2,3$. The Hopf-bifurcation condition (iii) will now be tested by putting $\lambda = u + \mathbf{i}v$ in the above equation, we get

$$(u+\mathbf{i}v)^3+\Theta_1(u+\mathbf{i}v)^2+\Theta_2(u+\mathbf{i}v)+\Theta_3 = 0. \quad (10)$$

On distinguishing the real and imaginary parts and removing v, we get

$$8u^3+8\Theta_1u^2+2u(\Theta_1^2+\Theta_2)+\Theta_1\Theta_2-\Theta_3 = 0. \quad (11)$$

From the foregoing, it is apparent that $u(r_2^*) = 0$ iff $\Theta_1(r_2^*)\Theta_2(r_2^*)-\Theta_3(r_2^*) = 0$. Further, at $r_2 = r_2^*$, $u(r_2^*)$ is the only root, since the discriminant $8u^2+8\Theta_1u+2(\Theta_1^2+\Theta_2) = 0$ if $64\Theta_1^2 - 64(\Theta_1^2+\Theta_2) < 0$. Further, differentiating (11) with respect to r_2, we have
$24u^2\frac{du}{dr_2}+16\Theta_1u\frac{du}{dr_2}+2(\Theta_1^2+\Theta_2)\frac{du}{dr_2}+2u[2\Theta_1\frac{d\Theta_1}{dr_2}+\frac{d\Theta_2}{dr_2}] + \frac{dS}{dr_2} = 0$ where $S = \Theta_1\Theta_2 - \Theta_3$.
Since at $r_2 = r_2^*$, $u(r_2^*) = 0$ we get $\left[\frac{du}{dr_2}\right]_{r_2=r_2^*} = \frac{-\frac{dS}{dr_2}}{2(\Theta_1^2+\Theta_2)} \neq 0$.
This ensures that the above system has a Hopf-bifurcation around the coexistence equilibrium E^*. □

Theorem 2. *If the coexistence equilibrium S^* exists, then (x^*, y^*, E^*) is globally asymptotically stable in the $x - y - E$ plane.*

Proof. Let's start by defining a Lyapunov function

$$W(x,y,E) = \int_{x^*}^{x} \frac{\xi - x^*}{\xi}d\xi + D_1\int_{y^*}^{y} \frac{\eta - y^*}{\eta}d\eta + D_2\int_{E^*}^{E} \frac{\rho - E^*}{\rho}d\rho, \quad (12)$$

where D_1 and D_2 are positive constants.
It is easy to examine that $W(x, y, E)$ is zero at the equilibrium point and the positive for all other positive values of $W(x, y, E)$.
The time derivative of W along the trajectories of the subsystem is

$$\frac{dW}{dt} = \frac{dx}{dt}\left[\frac{x - x^*}{x}\right] + D_1\left[\frac{dy}{dt}\right]\left[\frac{y - y^*}{y}\right] + D_2\left[\frac{dE}{dt}\right]\left[\frac{E - E^*}{E}\right]$$

Table 1. The table depicting thresholds and stability of steady states.

Thresholds (R_0, R_1, R_2)	$S_0(0,0,0)$	$S_1(x_1,0,0)$	$S_2(0,y_2,0)$	$S_3(0,y_3,E_3)$	$S_4(x_4,y_4,0)$	$S^*(x^*,y^*,E^*)$
$R_0 < 1$	Unstable	Asymptotically stable	Not feasible	Not feasible	Not feasible	Not feasible
$R_0 > 1$, $R_1 < 1$	Unstable	Unstable	Asymptotically stable	Not feasible	Not feasible	Not feasible
$R_1 > 1$, $R_2 < 1$	Unstable	Unstable	Unstable	Asymptotically stable	Not feasible	Not feasible
$R_2 > 1$	Unstable	Unstable	Unstable	Unstable	Asymptotically stable	Not feasible
$S^* > 0, \Theta_i > 0, i = 1,2,3.,\Theta_1\Theta_2 - \Theta_3 > 0.$	Unstable	Unstable	Unstable	Unstable	Unstable	Asymptotically stable

$$= [x - x^*]\left[r_1(1 - \frac{x}{K_1}) - \frac{\alpha_1 y}{a_1 + x} - d_1\right]$$
$$+D_1\left[r_2(1 - \frac{y}{K_2}) + \frac{\beta_1 x}{a_1 + x} - d_2 - c_1 E\right]$$
$$[y - y^*] + D_2\left[mc_1(p_1 - \tau_1)y - mC\right][E - E^*]$$
$$= [x - x^*][-\frac{r_1}{K_1}(x - x^*) -$$
$$\frac{\alpha_1}{(a_1 + x)(a_1 + x^*)}[(a_1 + x^*)(y - y^*) - y^*(x - x^*)]]$$
$$+D_1\left[r_2(1 - \frac{y}{K_2}) + \frac{\beta_1 x}{a_1 + x} - d_2 - c_1 E\right][y - y^*]$$
$$+D_2\left[mc_1(p_1 - \tau_1)y - mC\right][E - E^*]$$
$$= \left[-\frac{r_1}{K_1} + \frac{\alpha_1 y^*}{(a_1 + x)(a_1 + x^*)}\right](x - x^*)^2$$
$$-\frac{\alpha_1(x - x^*)(y - y^*)}{a_1 + x}$$
$$-\frac{r_2 D_1}{K_2}(y - y^*)^2 + D_1(\frac{\beta_1 x}{a_1 + x} - \frac{\beta_1 x^*}{a_1 + x^*})(y - y^*)$$
$$-D_1 c_1(E - E^*)(y - y^*)$$
$$+D_2 m c_1(p_1 - \tau_1)(E - E^*)(y - y^*).$$

Now here we choose arbitrary constants D_1 and D_2 such as $D_1 = \frac{\alpha_1(a_1+x^*)}{a_1\beta_1}$, $D_2 = \frac{\alpha_1(a_1+x^*)}{a_1 m\beta_1\beta(p_1-\tau_1)}$. Then

$$\frac{dW(x,y,E)}{dt} = \left[-\frac{r_1}{K_1} + \frac{\alpha_1 y^*}{(a_1 + x)(a_1 + x^*)}\right]$$
$$\times (x - x^*)^2 - \frac{r_2}{K_2}\frac{\alpha_1(a_1 + x^*)}{a_1\beta_1}(y - y^*)^2.$$

Clearly, the second term is negative. Now after some calculation in first term we see that if $x^* > (K_1 - a_1) - \frac{D_1 K_1}{r_1}$ then $\frac{dW(x,y,E)}{dt} \leq 0$. Clearly, $\frac{dW(x,y,E)}{dt} = 0$ if and only if $x = x^*$ and $y = y^*$ which yields $E = E^*$. Hence, $\frac{dW(x,y,E)}{dt} = 0$ if and only if $x = x^*$, $y = y^*$ and $E = E^*$. So from Lasalle invariant principle we say that S^* is globally asymptotically stable. □

4. Optimal Taxation policy

Biologically, we care more about the coexistence equilibrium in the presence of harvesting in order to ensure the existence of both species. Our main goal is to save each species while also maximising the monetary and social benefits. The profits (revenues) received by the fisherman and regulatory agency are saved to the society through the fishery in a large society. The entire economic revenue is

$$(c_1 p_1 y - C)E = [c_1(p_1 - \tau_1)y - C]E + \tau_1 c_1 y E. \quad (13)$$

It is equal to the sum of the entire fisherman's economic revenue and the regulating agency's economic revenue. It is obvious that

$$\pi(x,y,E) = (c_1 p_1 y - C)E. \quad (14)$$

In order to maximise the present value J of a continuous time steam of revenues, we analyse optimal harvest policy

$$J = \int_0^\infty e^{-\delta t}(c_1 p_1 y - C)E dt, \quad (15)$$

where δ be the instantaneous annual rate of discount [29–31].

Now we want to discover the path tracked by $(x(t); y(t); E(t))$ with the tax policy τ_1 so that if fish populations and harvesting effort stay on this path, the regulatory authority will be assumed to have accomplished its goal.

Our goal is to use Pontryagin's maximal principle [32] to determine a tax policy $\tau_1 = \tau_1(t)$ that maximises J under the state equation (5). The control variable $\tau_1(t)$ is subjected to the constraints $min\ \tau_1 \leq \tau_1 \leq max\ \tau_1$. When $min\ \tau_1 < 0$, we can explore subsidies, which in this case would have the effect of increasing the rate of expansion of the fishery.

Hamiltonian function is defined as follows:

$$H = e^{-\delta t}(c_1 p_1 y - C)E + \lambda_1\frac{dx}{dt} + \lambda_2\frac{dy}{dt} + \lambda_3\frac{dE}{dt}. \quad (16)$$

where λ_1, λ_2 and λ_3 are the adjoint variables. Hamiltonian (16) must be maximized when $\tau_1 \in [min\ \tau_1, max\ \tau_1]$. We assume that the optimal solution does not occur at $\tau_1 = min\ \tau_1\ or\ max\ \tau_2$ which imply that constraints are not binding. Therefore, singular control is represented by

$$\frac{\delta H}{\delta \tau_1} = -\lambda_3 m c_1 y E \Longrightarrow \lambda_3 = 0. \quad (17)$$

The adjoint equations, according to Pontryagin's maximal principle, are

$$\frac{d\lambda_1}{dt} = \frac{\delta H}{\delta x}, \ \frac{d\lambda_2}{dt} = \frac{\delta H}{\delta y}, \ \frac{d\lambda_3}{dt} = \frac{\delta H}{\delta E}. \tag{18}$$

As a result of the substitution and simplification, we arrive at

$$\frac{d\lambda_1}{dt} = -\frac{\delta H}{\delta x} = -\lambda_1 \left[r_1(1 - \frac{2x}{K_1}) - \frac{a_1\alpha_1 y}{(a_1+x)^2} - d_1 \right] - \lambda_2 \frac{a_1\beta_1 y}{(a_1+x)^2}, \tag{19}$$

$$\frac{d\lambda_2}{dt} = -\frac{\delta H}{\delta y} = -e^{-\delta t} c_1 p_1 E + \lambda_1 \frac{\alpha_1 x}{a_1+x} - \lambda_2 \left[(r_2 - \frac{2r_2 y}{k_2}) + \frac{\beta_1 x}{a_1+x} - d_2 - c_1 E \right], \tag{20}$$

$$\frac{d\lambda_3}{dt} = -\frac{\delta H}{\delta E} = -(c_1p_1y - C)e^{-\delta t} + \lambda_2 c_1 y = 0. \tag{21}$$

The solution of (21) is described in the following in order to obtain an optimal equilibrium solution by considering the coexisting equilibrium as follows:

$$\lambda_2 = e^{-\delta t}(p_1 - \frac{C}{c_1 y^*}). \tag{22}$$

Let

$$A_1 = -\left[r_1 - \frac{2r_1x^*}{K_1} - \frac{a_1\alpha_1 y^*}{(a_1+x^*)^2} - d_1 \right],$$
$$A_2 = \frac{a_1\beta_1 y^*}{(a_1+x^*)^2}(p_1 - \frac{C}{c_1y^*})e^{-\delta t},$$
$$A_3 = c_1p_1E^* - \frac{A_2}{A_1+\delta}\frac{\alpha_1 x^*}{a_1+X^*} - \frac{r_2 y^*}{\alpha_2}(p_1 - \frac{C}{c_1y^*}). \tag{23}$$

Now the equations (19) and (20) can be written as

$$\frac{d\lambda_1}{dt} = A_1\lambda_1 - A_2 e^{-\delta t}\frac{d\lambda_2}{dt} = -A_3 e^{-\delta t}. \tag{24}$$

Solving the above linear differential equation we get

$$\lambda_1 = \frac{A_2}{A_1+\delta}e^{-\delta t}, \ \lambda_2 = -\frac{A_3}{\delta}e^{-\delta t}. \tag{25}$$

Substituting the value of λ_2 from (22) into (25), we get

$$(p_1 - \frac{C}{c_1y^*}) = \frac{A_3}{\delta}. \tag{26}$$

Now putting the value of x^*, y^* and E^* into (26), we get an equation for τ_1; let τ_1^* be a solution of this equation. We get the optimal equilibrium solutions $x = x(\tau_1^*)$, $y = y(\tau_1^*)$ and $E = E(\tau_1^*)$ by using the value of $\tau_1 = \tau_1^*$. As a result, we have established the existence of an optimal equilibrium solution that satisfies the necessary conditions of the maximum principle. From the above analysis carried out in this section, we observe the following.

From (21), we get

$$\lambda_2 c_1 y = (c_1p_1y - C)e^{-\delta t} = e^{-\delta t}\frac{\delta\pi}{\delta E}. \tag{27}$$

Putting the value of $\lambda_2(t)$ into (27), we get

$$c_1p_1y - \frac{A_3}{\delta}c_1y = C. \tag{28}$$

When $\delta \longrightarrow \infty$, (28) leads to the results $c_1p_1y = C$, which implies that the economic rent is completely dissipated.

(ii) By (26) we get the optimal equilibrium populations $x = x(\tau_1^*), y = y(\tau_1^*), E = E(\tau_1^*)$, hence, we have

$$\pi = (c_1p_1y - C)E = \frac{A_3}{\delta}c_1yE. \tag{29}$$

Thus π is a decreasing function of δ we, therefore, conclude that π leads to maximization when δ leads to 0.

5. Numerical simulations

In this section, some numerical simulations are performed with the help of used RK4 schemes to discuss the dynamical behavior of system (5) and to verify analytical results. To examine the dynamic of fishery system, we start with a set of parametric values (Ref. [26])

$$r_1 = 7, \ r_2 = 1 \ K_1 = 3, \ K_2 = 20, \ \alpha_1 = 1.5,$$
$$\beta_1 = 0.8, \ d_1 = 0.01, \ d_2 = 0.01, \ m = 0.8, \ c_1 = 1.2,$$
$$p_1 = 0.7, \ \tau_1 = 0.08, \ C = 0.49, \ a_1 = 0.05, \ \gamma_1 = 0.1. \tag{30}$$

Considering the parametric values, we find the coexistence equilibrium point $S^* = (2.81, 0.83, 1.44)$ which is locally asymptotically stable (cf. Figure 1a (black line)). Taking $K_2 = 200$, the system (5) exhibits oscillation around S^* (cf. Figure 1a (red line)). In case of $K_2 = 0.4$, then Figure 1a (blue line) shows the system switches to harvesting effort free equilibrium S_4.

Further from Figure 1b (black line), it follows tax per unit biomass of the predator, $\tau_1 = 0.52$, the equilibrium S_3 is locally asymptotically stable. Increasing the value of $\tau_1 = 0.7$, the system switches to prey free equilibrium S_2 in absence of harvesting effort (cf. Figure 1b (blue line)). It is observe that the system switches to stable to oscillatory behavior around S^* due to low value of $r_2 = 0.2$ (cf. Figure 1c). But in both case of $r_1 = 1$ and $K_1 = 0.3$ the system switches to prey free equilibrium simultaneously (cf. Figure 1d) . Figures 3a,3b and 3c illustrate the different steady state behaviour of each species in the system (5) for the parameter τ_1. We note that Hopf point (red star (H)) situated

Table 2. Natures of equilibrium points.

Parameters	Values	Eigenvalues	Equilibrium points
τ_1	0.436074	$(4.23719, \pm 0.900417i)$	Hopf (H)
	0.557934	$(0, -0.0901945 \pm 0.92798i)$	Limit Point (LP)
p_1	0.343926	$(4.23719, \pm .900417i)$	Hopf (H)
	0.222066	$(0, -0.0901945 \pm 0.92798i)$	Limit Point (LP)
	2.279571	$(0, -0.005825 \pm 0.693768)$	Branch Point (BP)
	4.942273	$(0, 1.52281, -10.147,)$	Branch Point (BP)
(τ_1, r_1)	$(0.046608, 2.014876)$	$(0.8937, \pm 0.934174i)$	Generalized Hopf (GH)
(p_1, r_1)	$(0.733392, 2.014876)$	$(0.8937, \pm 0.934174i)$	Generalized Hopf (GH)
	$(1.010732, 1.443315)$	$(0.655091, \pm 0.936168i)$	Generalized Hopf (GH)
(p_1, K_2)	$(1.590494, 0.669121)$	$(1.64826, \pm 0.688981i)$	Generalized Hopf (GH)

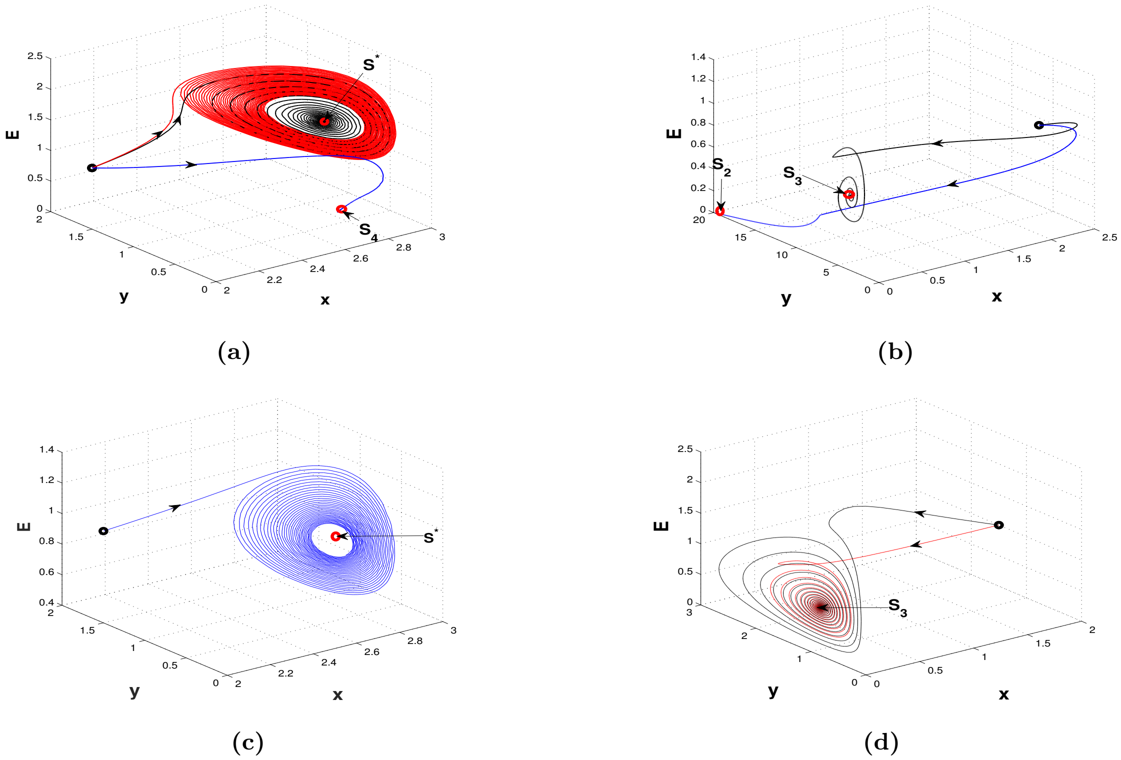
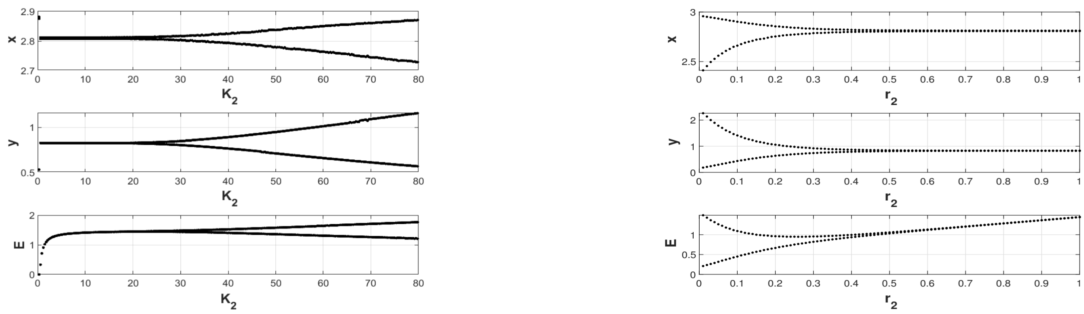


Figure 1. **(a)** Phase plane diagram showing local stability of S^* for $K_2 = 20$ (black solid line), oscillatory behavior around S^* for $K_2 = 200$ (red solid line) and local stability at S_4 for $K_2 = 0.4$ (blue solid line). **(b)** The solution of trajectory approaches to two different equilibrium points S_3 and S_2 for $\tau_1 = 0.52$ (black line) and 0.7 (blue line) respectively. **(c)** The system switches to oscillatory behavior for $r_2 = 0.2$. **(d)** Phase plane diagram indicating the local stability of S_3 for $r_1 = 1$ (red solid line) and $K_1 = 0.3$ (black line) respectively.

at 0.436074 with two complex parts of eigenvalues ≈ 0. We also observe that at that particular point, the value of first Lyapunov coefficient is positive 0.01112548 which indicates unstable limit cycle bifurcates from Hopf point. To proceed further, we have a limit point (LP) at $\tau_1 = 0.557934$ with eigenvalues $0, 0.0901945 \pm 0.92798i$. From Figures 3d, it is evident that at $\tau_1 = .4360747$ and 0.4578429 we have two Limit point cycle (LPC) and Branch Point cycle (BPC).

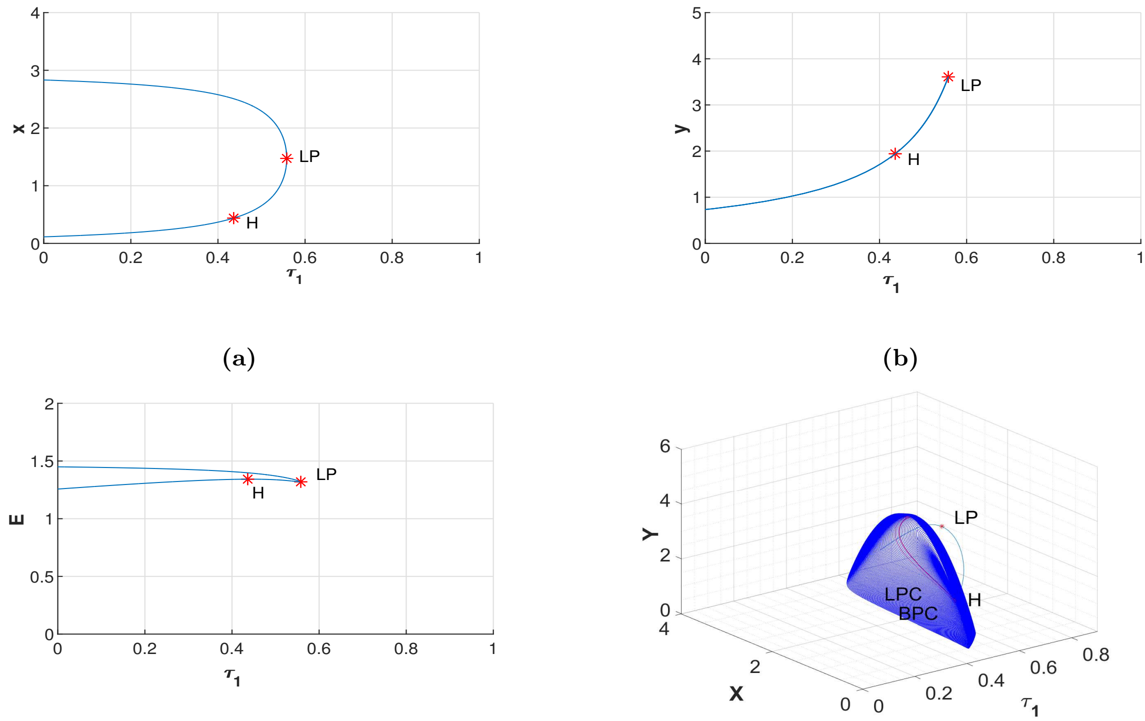
Figures 4a, 4b and 4c depict different behavior of each species when p_1 is a free parameter. Here we observe that a Hopf points (H), two Branch points (BP) and one Limit point (LP). In this case, Hopf point is situated at 0.343929 with first Lyapunov coefficients a .01112584 indicating subcritical bifurcation. Further, it is observed that one LP and two Branch points are located at 0.222066, 2.279571 and 4.942273 respectively. Starting from Hopf point and proceed further, a family of unstable limit cycle is generated (cf. Figure 4d).

To demonstrate the clear picture of changes in dynamical system when K_2 and r_2 be the free parameters, we plot two bifurcation diagrams (cf. Figure 2a, 2b) respectively. Finally, we draw two parameter bifurcation diagrams for $\tau_1 - r_1$, $p_1 - r_1$



(a) **(b)**

Figure 2. **(a)** Bifurcation diagram for K_2. **(b)** Bifurcation diagram for r_2.



(a) **(b)** **(c)** **(d)**

Figure 3. **(a)** Different steady state behavior of prey for the effect of τ_1. **(b)** Different steady state behavior of predator for the effect of τ_1. **(c)** Different steady state behavior of harvesting effort for the effect of τ_1. **(d)**The family of limit cycles bifurcating from the Hopf point H for τ_1.

and $p_1 - K_2$ (cf. Figure 5a, 5b and 5c). In each cases, we have generalized Hopf (GH) point. Actually, two branches of sub and supercritical Andronov-Hopf bifurcations split at GH point.

6. Conclusion

We investigate the interspecies competition of prey and predator in a fishery system in this work. We assumed that predator undergo exploitation due to consume of prey. This work has a dual goal, namely economic and ecological. The economic goal is to maximize monetary benefit to society while also preventing the predator from extinction. Here we implement a tax to regulate the harvesting effort in order to preserve the ecological balance.

As a result, one of the most important features of this approach is the harvesting effort and net economic revenue to the fisherman. The first step is to perform analytical conditions for the existence and stability of various steady states. We also look into the global stability of coexistence equilibrium while the tax remains certain threshold value. The outcome of global stability shows that when a tax provides a sustainable threshold value, predator are not from a body of water at a rate greater than that the species can replenish

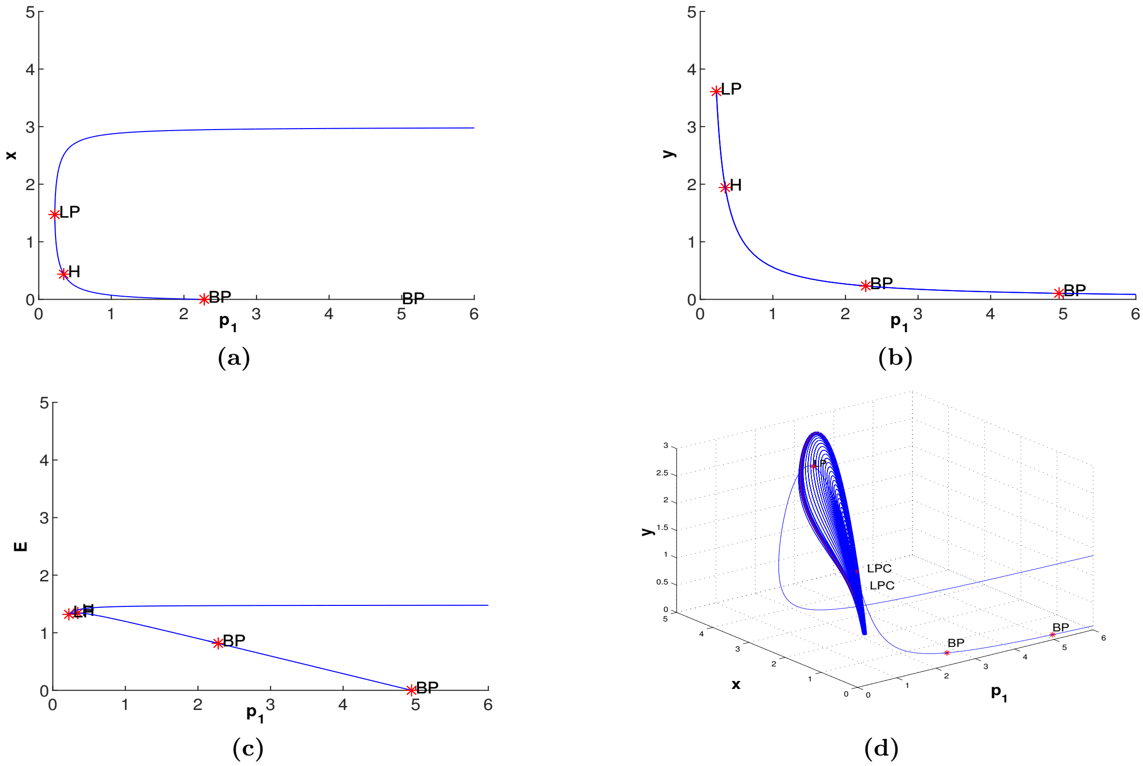


Figure 4. **(a)** Different steady state behavior of prey population for the effect of p_1. **(b)** Different steady state behavior of predator population for the effect of p_1. **(c)** Different steady state behavior of harvesting effort for the effect of p_1. **(d)**The family of limit cycles bifurcating from the Hopf point H for p_1.

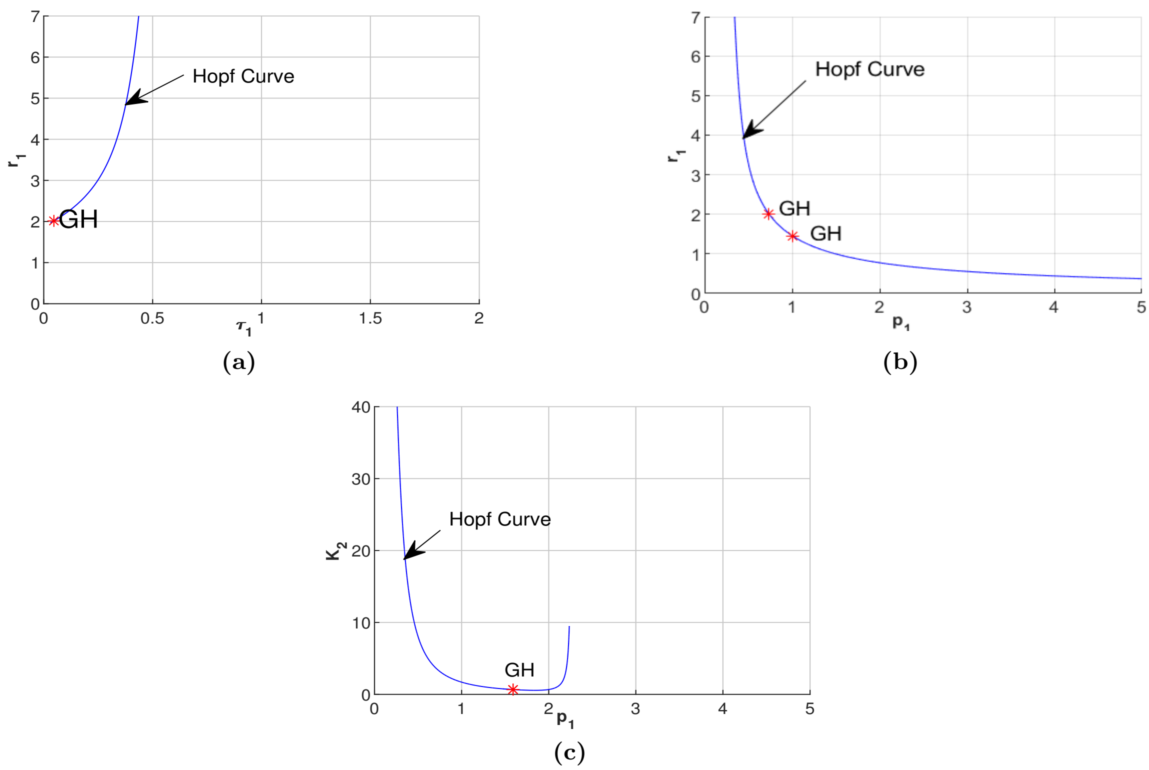


Figure 5. **(a)** The two parameters bifurcation diagram for $\tau_1 - r_1$. **(b)** The two parameters bifurcation diagram for $p_1 - r_1$. **(c)** The two parameters bifurcation diagram for $p_1 - K_2$.

its population naturally. We note the following observations by numerical simulation:

Two different scenarios are shown when changing the value of carrying capacity of predator. Here we observe that each fish species are present in the system in absence of harvesting efforts for low values of carrying capacity of predator. On the other hand, because to the high value of predators' carrying capacity, all species become unstable. Due to high tax levels, coexistence equilibrium switches to different boundary equilibrium, which is related to transcritical bifurcations. The system becomes oscillate due to low values of intrinsic growth rate of predator. Our research also shows that maintaining carrying capacity and imposing a tax on harvesting of predator are critical factors in keeping predator exploitation under control. In addition, we impose a tax to study the the optimal harvesting policy for harvesting predator. When the monetary social benefit is subject to maximisation, it is demonstrated by utilising Pontryagin's maximal principle. We established the optimal equilibrium solution by using optimal tax $\tau_1 = \tau_1^*$. It has been demonstrated that zero discounting maximises economic revenue and that an infinite discount rate causes economic rent to dissipate completely. It should be noted that in this paper, several crucial factors are disregarded, including ecological fluctuations, refuge, allee effect etc. Therefore, further study is required to meet the demands in this area.

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RESEARCH ARTICLE

The processes with fractional order delay and PI controller design using particle swarm optimization

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ABSTRACT

In this study, the stability analysis of systems with fractional order delay is presented. Besides, PI controller design using particle swarm optimization (PSO) technique for such systems is also presented. The PSO algorithm is used to obtain the controller parameters within the stability region. As it is known that it is not possible to investigate the stability of systems with fractional order delay using analytical methods such as the Routh-Hurwitz criterion. Furthermore, stability analysis of such systems is quite difficult. In this study, for stability testing of such systems, an approximation method previously introduced in the literature by the corresponding author is used. In addition, the unit step responses have been examined to evaluate the systems' performances. It should be noted that examining unit step responses of systems having fractional-order delay is not possible due to the absence of analytical methods. One of the aims of this study is to overcome this deficiency by using the proposed approximation method. Besides, a solution to the question of which controller parameter values should be selected in the stability region, which provides the calculation of all stabilizing PI controllers, is proposed using the PSO algorithm.

1. Introduction

In practice, many dynamic systems cannot be satisfactorily modeled with ordinary differential equations. Actually, in many systems, the future behaviors of state variables depend on both their current values and their past values [1]. Such systems are called time delay systems. Time delay systems can occur in practice for many reasons. So, many processes contain dead time in their inner dynamics. Due to the increasing demands of dynamic performances, we need models behaving more like the real process. Therefore, the notion of time delay keeps on growing attraction for many scientific disciplines such as control engineering.

Analysis and control of systems having time delay are more complicated than integer order ones [2]. However, describing systems without using time delay component may lead to incorrect conclusions in terms of evaluating and obtaining desired control aims. Thus, analysis and control of time delay systems are very important, and considerable attention and effort have been given to the stability of linear time delay systems during the past 30 years [1, 3] (and references therein). It is still an active research area in the literature. However, to the best knowledge of the authors of this article, studies related to the stability of control systems having fractional order time delay are not extensive. Only particular cases such as a special class of distributed parameter systems have been given in [4–7].

Distributed parameters and/or delay elements described by partial differential equations can be seen in many industrial systems such as fluid lines, transmission lines, nuclear rocket engines,

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diffusion processes and chemical processes etc. [5–8]. The distributed parameter approach provides more accurate design results than lumped parameter approach [6]. Transfer functions of distributed parameter systems contain $\sqrt{s}$ or e^{-hs}, $e^{-(ks^{0.5})}$ functions, where, real $h \geq 0$ and $k \geq 0$ stand for time delay and distributed lag, respectively [6,8]. Calculating the inverse Laplace transforms of these functions is an extremely complex matter [8]. Distributed parameter systems can be considered as a special form of fractional order systems when $\sqrt{s}$ term is used. The stability analysis of systems with distributed parameters, i.e., fractional order systems has some challenges since their mathematical descriptions have irrational functions of "s" [5, 6]. The analytic methods to evaluate time response of irrational transfer functions are inadequate. Furthermore, to solve this problem, used graphical methods are inaccurate [8]. These are important shortcomings of the works have been done on this topic. In the literature, there are some important studies. In [9], necessary and sufficient conditions for the B.I.B.O. (Bounded Input Bounded Output) stability and the asymptotic stability of systems whose transfer functions are functions of s, $\sqrt{s}$, and $e^{\sqrt{Ts}}$ are established, where T is a positive constant. An algorithm for the inverse Laplace transform to obtain time response of irrational transfer functions is developed in [8] by using the Fast Fourier transform. The Hurwitz stability test is extended to lumped-distributed RC networks in [10]. An algebraic test procedure such as Routh algorithm and Hurwitz determinant is improved to a certain class of distributed parameter systems with multiple delays in [4, 5]. Two-dimensional stability criterion to a special class of distributed parameter systems has been studied in [6]. It has been showed that the conditions of stability for such systems are independent of time delay and distributed lag [6]. Similarly, a stability test independent of distributed lag and another stability test to find the intervals of distributed lag are proposed for a special class of distributed parameter systems in [7]. First order plus fractional diffusive delay is studied in [11]. However, the works in this area are mostly focused on systems containing terms e^{-hs} or $\sqrt{s}$. The current studies on $e^{-\sqrt{s}}$ consist of complex stability test procedures. What is more, the time-response analysis of such systems is quite a few [12] One of the motivations of this study is to examine the unit step responses of systems having fractional order delay. Various methods have been used in the literature to determine the parameters of PI controllers for time-delay systems [12–16]. One of these methods is the weighted geometrical center method, and the other is the centroid of the convex stability region method. In this study, a method based on the centroid of the convex stability region is presented. In this new method, the most optimum PI controller parameters are obtained by creating a triangular area under the stability curve of the system and searching this area with PSO algorithm. The first application of this method can be found in [17].

This paper is organised as follows. A brief introduction of fractional order calculus and fractional order systems are given in section 2. Fractional order systems with fractional order delay are also introduced in section 2. Brief information about particle swarm optimization is given in section 3. PI controller design for systems with fractional order delay and a stability test procedure for such systems are given in section 4. Finally, numerical examples are given in section 5.

2. Fractional order calculus and fractional order systems

Fractional order calculus, namely, non-integer order calculus of control systems is gaining more and more attention from many science disciplines. The notion of non-integer order calculus which is related to the development of regular calculus has been known for 300 years [18]. But it has mostly remained as a subject studied by prominent mathematicians owing to its complex structure. There are studies in various fields related to fractional expressions, especially in mathematics [11, 19–22]. Since it requires advanced mathematical analysis techniques, engineers and other science disciplines could not use it effectively till the development of analyzing and solution methods [18]. After obtaining important achievements in fractional calculus recently, the real order of dynamic systems can be investigated. In fractional calculus, the order of derivatives or integrals can be real or complex number [11]. Thus, the order of the fractional integrals and derivatives can be considered as a function of time or some other variable [23]. Using the fractional order models to describe real-world systems has some advantages in terms of both having more degrees of freedom in the model and having an unlimited memory which is very important to predict and influence present and future behaviors. Fractional order systems and their applications are one of the most popular research topics of today. Recently, it has been reported that fractional order representation is more accurate to describe real world systems than those of integer order models since the real-world processes are generally and/or most likely fractional order [24]. It is known that using integer order model to define a system can lead to distinction between mathematical model and actual system [24].

A fractional order system (FOS) has transfer function consisting of fractional order derivatives s^α, where $\alpha \in \mathbb{R}$. In the literature, many studies conducted on FOS use integer order approximations due to the lack of analytical solution methods. There are some integer order approximation methods of FOS such as continued fraction expansion, Oustaloup etc. (Details can be found in [25]). Stability analysis of such systems is one of the most challenging problems. To the best knowledge of the authors, there are no analytical stability test procedures such as Routh that can be applied to such systems, directly. Although using integer order approximations provide the stability analysis of such systems, time domain analysis has remained the most challenging and important problem. However, some important studies to obtain inverse Laplace transform and time response of FOS have been studied in [26, 27], recently.

2.1. Fractional order systems with fractional order delay

A system represented by a differential equation where the orders of derivatives can take any real number not necessarily integer number can be considered FOS. Thus, FOS can be defined by the fractional-order transfer function with fractional order time delay. The transfer function of the system non-integer order time delay is defined by

$$\begin{aligned} G_p(s) &= G(s)e^{-\sqrt{\tau s}} = \frac{N(s)}{D(s)}e^{-\sqrt{\tau s}} \\ &= \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\gamma_n} + a_{n-1}s^{\gamma_{n-1}} + \ldots + a_0 s^{\gamma_0}}e^{-\sqrt{\tau s}} \end{aligned} \tag{1}$$

or in general form, it can be described as follows.

$$\begin{aligned} G_p(s) &= G(s)e^{-(\tau s)^\alpha} = \frac{N(s)}{D(s)}e^{-(\tau s)^\alpha} \\ &= \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\gamma_n} + a_{n-1}s^{\gamma_{n-1}} + \ldots + a_0 s^{\gamma_0}}e^{-(\tau s)^\beta} \end{aligned} \tag{2}$$

where τ is fractional order time delay, a_k $(k = 0, ..., n)$ and b_k $(k = 0, ..., m)$ are constants, γ_k $(k = 0, ..., n)$ and β_k $(k = 0, ..., m)$ are arbitrary real numbers. And, also $\beta_m > \beta_{m-1} > .... > \beta_0$, $\gamma_n > \gamma_{n-1} > .... > \gamma_0$ without loss of generality. As stated before, the studies related to systems with fractional order time delay are not extensive. Thus, new studies need to be done on this research topic.

3. Particle swarm pptimization

Particle Swarm Optimization (PSO) is a powerful metaheuristic optimization technique based on the movement and intelligence of swarms. PSO algorithm is inspired by flocks of birds and schools of fish in nature. For instance, when birds flying and searching randomly for food, they help each other in the flock to find the best food place. In 1995, Dr. Kennedy and Dr. Eberhart have discovered PSO algorithm by examining the behavior of bird flocks [28].

The PSO algorithm can consider like a flock of birds. Particles come together to form a swarm. PSO algorithm can find problems' minimum or maximum value. In other words, it is finding the optimum value of the problem. PSO algorithm has individuals also referred to as particles. These particles are solution sets of a problem. In PSO, particles generate randomly between problems boundaries.

In the PSO algorithm, it is necessary to evaluate whether the particles are suitable for the result according to a certain criterion. This is done by the fitness function. The fitness function tests the fitness of particles. In some previous studies, performance indexes such as "ISE", "IAE", "ITSE", "ITAE" were used as fitness functions [29–31]. These fitness functions are chosen according to the problem. If it is desired to reach the minimum point in the problem, the best particle of the solution set, which gives the minimum value of the fitness function, is selected. However, if it is desired to reach the maximum point in the problem, the maximum value of the fitness function should be selected. PSO algorithm is an iterative algorithm, so it needs to be updated some parameters about the problems. There are two updates in the PSO algorithm. The first one is velocity update and the second one is position update. The velocity update formula is given in Eq. (3) [28].

$$v_{ij} = \epsilon * v_{ij} + c_1 * r_1 * (x_{ij}^{Pb} - x_{ij}) + c_2 * r_2 * (x_{ij}^{Sb} - x_{ij}) \tag{3}$$

ϵ: Coefficient of inertia
c_1: Cognitive coefficient
c_2: Social coefficient
r_1, r_2: Random coefficient(0-1)
x_{ij}: Position of particle
x_{ij}^{Pb}: Position of the best particle
x_{ij}^{Sb}: Position of the best of the swarm

The velocity equation can be handled in three different parts. In the first part, every particle has inertia and wants to maintain its motion. The expression $\epsilon * v_{ij}$ is used to express this situation. In the second part, the particle wants to reach its best position. The expression $c_1 * r_1 * (x_{ij}^{Pb} - x_{ij})$ is used to express this situation. In the last part, the particle seeks to reach the best position of the swarm. The expression $c_2 * r_2 * (x_{ij}^{Sb} - x_{ij})$ is used to express this. The combination of all these components gives us a new velocity. This process is done for all particles. After the velocity update, the position update is done.

$$\Delta v = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{1} \Rightarrow \Delta v = \Delta x \tag{4}$$

Position update formula is given in Eq. (4). In this equation, one unit time change in the PSO algorithm is equal to one iteration cycle. Thus Eq. (5) is used when updating the position [28].

$$x_{ij} = x_{ij} + v_{ij} \tag{5}$$

After the position update, the PSO algorithm tests the new positions of the swarm to avoid leaving the determined search space. For points outside the search space, a correction is made so that they fall back into the search space. Otherwise, the algorithm may give incorrect results when it leaves the search space. When the iteration is complete or the PSO algorithm satisfies the stopping condition, the outputs of the PSO algorithm are the best solutions for a problem.

4. PI controller design for systems with fractional order delay and stability analysis

PID controllers are the most common controller type in practical systems due to their simple structure. And, they have been applied to many complex systems. They are widely used in practice even today despite significant development in control theory [13]. A large number of studies have been carried out to determine appropriate parameters for these popular controllers and some methods have been developed in [32–34]. In general, the studies to obtain optimum controller parameters are still in progress and the concept of the best approach is not yet available. Thus, it is still a research topic for control engineering. In this section, PI controller design is presented for the systems having fractional order time delay. To obtain stabilizing controller parameters, the PSO algorithm has been combined with the centroid of the convex stability region concept based on the stability boundary locus method [33]. To explain PI controller design procedure, first, we need to obtain some equations. Consider the single input single output (S.I.S.O.) control system as shown in Figure 1

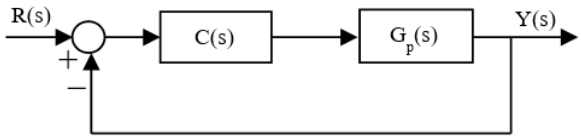

Figure 1. A S.I.S.O. control system.

where

$$G_p(s) = G(s)e^{-(\tau s)^\alpha} = \frac{N(s)}{D(s)}e^{-(\tau s)^\alpha} \tag{6}$$

is the plant to be controlled and C(s) is a PI controller of the form

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s} \tag{7}$$

The closed loop characteristic polynomial $\Delta(s)$ of the system of Figure 1, i.e. the numerator of $1 + C(s)G_p(s)$ can be written as

$$\Delta(s) = sD(s) + (k_p s + k_i)N(s)e^{-(\tau s)^\alpha} \tag{8}$$

Separating the numerator and the denominator polynomials of $G(s)$ in Eq. (6) into even and odd parts, and substituting $s = j\omega$ in the equation provides the following

$$G(j\omega) = \frac{N_e(-\omega^2) + j\omega N_o(-\omega^2)}{D_e(-\omega^2) + j\omega D_o(-\omega^2)} \tag{9}$$

In the rest of the paper, $(-\omega^2)$ notation will not be used in the following equations for the simplicity. Using Eq. (10), Eq. (12) can be obtained instead of Eq. (11).

$$(j\omega)^\alpha = \omega^\alpha(\cos\frac{\pi}{2}\alpha + j\sin\frac{\pi}{2}\alpha) \tag{10}$$

$$e^{-(s\tau)^\alpha} = e^{-(j\omega)^\alpha \tau^\alpha} \tag{11}$$

$$\begin{aligned} & e^{-\left[(\cos\frac{\pi}{2}\alpha + j\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha\right]} \\ &= e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha - j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \\ &= e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} . e^{-j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \end{aligned} \tag{12}$$

Where the first term which is a constant, and the second term can be written as follows, respectively.

$$e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \tag{13}$$

$$e^{-j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \tag{14}$$

The second term can be rearranged as in Eq. (15).

$$e^{-j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} = \cos[(\sin\tfrac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha] - j\sin[(\sin\tfrac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha] \tag{15}$$

Substituting Eqs. (16) and (17) in Eq. (11) the closed loop characteristic polynomial of Eq. (8) can be written as in Eq. (18)

$$e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} = e^{-m} \tag{16}$$

$$\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha) = n \tag{17}$$

$$\begin{aligned}\Delta(j\omega) &= -\omega^2 D_o - \omega^2 k_p N_o e^{-m}\cos(n) \\ &+k_i N_e e^{-m}\cos(n) + \omega k_p N_e e^{-m}\sin(n) \\ &+\omega k_i N_o e^{-m}\sin(n) \\ &+j[\omega k_p N_e e^{-m}\cos(n) + \omega k_i N_o e^{-m}\cos(n) \\ &+\omega^2 k_p N_o e^{-m}\sin(n) - k_i N_e e^{-m}\sin(n) + \omega D_e]\end{aligned} \tag{18}$$

Then, equating the real and imaginary parts of $\Delta(j\omega)$ to zero, one obtains

$$\begin{aligned}&k_p[-\omega^2 N_o e^{-m}\cos(n) + \omega N_e e^{-m}\sin(n)] \\ &+k_i[N_e e^{-m}\cos(n) + \omega N_o e^{-m}\sin(n)] = \omega^2 D_o\end{aligned} \tag{19}$$

and

$$\begin{aligned}&k_p[\omega^2 N_o e^{-m}\sin(n) + \omega N_e e^{-m}\cos(n)] \\ &+k_i[\omega N_o e^{-m}\cos(n) - N_e e^{-m}\sin(n)] = -\omega D_e\end{aligned} \tag{20}$$

Eqs. (19) and (20) can be rearranged as follows.

$$k_p X_3(\omega) + k_i X_4(\omega) = X_1(\omega) \tag{21}$$

$$k_p X_5(\omega) + k_i X_6(\omega) = X_2(\omega) \tag{22}$$

Where

$$X_1(\omega) = \omega^2 D_o \tag{23}$$

$$X_2(\omega) = -\omega D_e \tag{24}$$

$$\begin{aligned}X_3(\omega) &= -\omega^2 N_o e^{-m}\cos(n) + \omega N_e e^{-m}\sin(n) \\ &= e^{-m}[-\omega^2 N_o\cos(n) + \omega N_e\sin(n)]\end{aligned} \tag{25}$$

$$\begin{aligned}X_4(\omega) &= N_e e^{-m}\cos(n) + \omega N_o e^{-m}\sin(n) \\ &= e^{-m}[N_e\cos(n) + \omega N_o\sin(n)]\end{aligned} \tag{26}$$

$$\begin{aligned}X_5(\omega) &= \omega^2 N_o e^{-m}\sin(n) + \omega N_e e^{-m}\cos(n) \\ &= e^{-m}[\omega^2 N_o\sin(n) + \omega N_e\cos(n)]\end{aligned} \tag{27}$$

$$\begin{aligned}X_6(\omega) &= \omega N_o e^{-m}\cos(n) - N_e e^{-m}\sin(n) \\ &= e^{-m}[\omega N_o\cos(n) - N_e\sin(n)]\end{aligned} \tag{28}$$

From Eqs. (21) and (22), k_p and k_i can be obtained as in Eqs. (29) and (30).

$$k_p = \frac{X_1(\omega)X_6(\omega) - X_2(\omega)X_4(\omega)}{X_3(\omega)X_6(\omega) - X_5(\omega)X_4(\omega)} \tag{29}$$

$$k_i = \frac{X_2(\omega)X_3(\omega) - X_1(\omega)X_5(\omega)}{X_3(\omega)X_6(\omega) - X_5(\omega)X_4(\omega)} \tag{30}$$

The stability boundary locus represented as $l(k_p, k_i, \omega)$ can be obtained in the (k_p, k_i) plane using Eqs. (29) and (30) when the denominator $X_3(\omega)X_6(\omega) - X_5(\omega)X_4(\omega) \neq 0$. It should be noted that it is necessary to investigate whether stabilizing controllers exist or not since the stability boundary locus $l(k_p, k_i, \omega)$ and the line $k_i = 0$ can divide the (k_p, k_i) plane into sub-regions as stable and unstable [33] (Details can be found in [33, 35]).

Using Eqs. (29) and (30), PI controller parameters k_p and k_i are obtained as follows.

$$k_p = \frac{\begin{array}{c}(\omega^2 N_o D_o + N_e D_e)\cos(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha)) \\ +\omega(N_o D_e - N_e D_o)\sin(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha))\end{array}}{-e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha}({N_e}^2 + \omega^2 {N_o}^2)} \tag{31}$$

$$k_i = \frac{\begin{array}{c}\omega^2(N_o D_e - N_e D_o)\cos(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha)) \\ -\omega(N_e D_e + \omega^2 N_o D_o)\sin(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha))\end{array}}{-e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha}({N_e}^2 + \omega^2 {N_o}^2)} \tag{32}$$

Exponential functions have an infinite number of isolated roots [36]. The stability analysis of time delay systems is difficult. Moreover, when the system has a fractional order time delay, the stability analysis becomes much more complicated. Recently, an approximation method has been proposed in the literature to analyze the stability and time response of such systems in [12, 37]. Stability analysis of such systems is possible with this method. Besides, using this method time response analysis of these systems can be obtained. The 1^{st}, 2^{nd} and 3^{rd} order approximations of $e^{-(s\tau)^\alpha}$ are given in Eqs. (33), (34) and (35), respectively [12, 37].

First order approximation:

$$\frac{\frac{-(s\tau)^{\alpha}}{2}+1}{\frac{(s\tau)^{\alpha}}{2}+1} \quad (33)$$

Second order approximation:

$$\frac{\frac{(s\tau)^{2\alpha}}{12}-\frac{(s\tau)^{2\alpha}}{2}+1}{\frac{(s\tau)^{2\alpha}}{12}+\frac{(s\tau)^{2\alpha}}{2}+1} \quad (34)$$

Third order approximation:

$$\frac{-\frac{(s\tau)^{2\alpha}}{120}+\frac{(s\tau)^{2\alpha}}{10}-\frac{(s\tau)^{2\alpha}}{2}+1}{-\frac{(s\tau)^{2\alpha}}{120}+\frac{(s\tau)^{2\alpha}}{10}-\frac{(s\tau)^{2\alpha}}{2}+1} \quad (35)$$

The value of α is in the range of $0 \le \alpha \le 1$. In Figure 2, in order to see the efficiency of this approximation method, the stability regions are drawn by taking $\alpha = 0.999$ and $\alpha = 1$ in the transfer function of the system given by $(1/(s+1))e^{-(s)^{\alpha}}$. Here, the second order approximation for the fractional order time delay is used. The higher the approximation degree, the more the system's approximation model will resemble the real system model. However, the higher the degree of approximation, the more difficult the system model will be to analyze. We used the second-order approximation for the fractional order time delay in Examples 1 and 2. With the help of this approximation method, analysis related to a system with a fractional order time delay can be made easily. More detailed information can be found in [12,37].

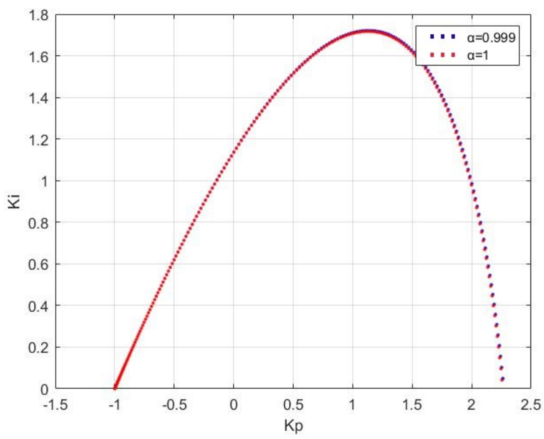

Figure 2. Stability regions for $\alpha = 0.999$ and $\alpha = 1$ using second order approximation.

5. Numerical examples

5.1. Example 1

Consider the control system of Figure 1 with the transfer function of Eq. (36)

$$G_p(s) = \frac{1}{s+1}e^{-\sqrt{s}} \quad (36)$$

The characteristic equation of the system without using PI controller is obtained as follows.

$$\Delta(s) = s + 1 + e^{-\sqrt{s}} \quad (37)$$

This equation can be rearranged as

$$\Delta(s) = (\sqrt{s})^2 + 1 + e^{-\sqrt{s}} \quad (38)$$

For the stability test of the system in Eq.(36), if the second order approximation given by Eq.(34) is substituted for the fractional order time delay in Eq.(38), the new characteristic equation will be as in Eq.(39).

$$\Delta(s) = (\sqrt{s})^2 + 1 + \frac{\frac{s}{12}-\frac{s}{2}+1}{\frac{s}{12}+\frac{s}{2}+1} \quad (39)$$

If Eq.(39) is set to zero, Eq.(40) is obtained.

$$s^2 + 6s\sqrt{s} + 14s + 24 = 0 \quad (40)$$

In Eq.(40), the $q = \sqrt{s}$ transform is performed to find the roots of the characteristic equation. Thus, the following equation is obtained.

$$q^4 + 6q^3 + 14q^2 + 24 = 0 \quad (41)$$

The roots of Eq.(41) are obtained as follows.
$q_{1,2} = -3.2937 \pm 2.3575i = 4.0504\angle \pm 144.406$
$q_{3,4} = 0.2937 \pm 1.1733i = 1.2095\angle \pm 75.9465$
The roots of the characteristic equation are shown in Figure 3. As seen from the figure, the system is stable. (Details about the stability can be found in [37]).

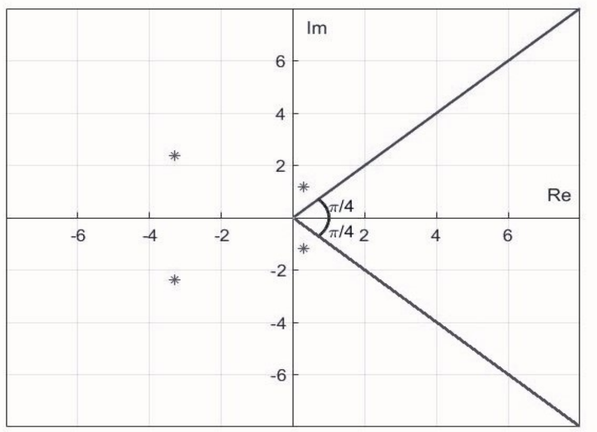

Figure 3. Roots of the characteristic equation in $q = \sqrt{s}$ plane.

As stated before, conventional stability test methods such as Routh-Hurwitz cannot be applied to the analysis of distributed systems whose transfer functions are irrational in s. However, to investigate the stability of fractional order systems, geometric techniques based on the principle of argument can be applied. These techniques provide information about the number of singularities of the function by observing the development of the function's argument. The argument principle (Nyquist diagram) is a curve surrounding the right half plane of the Riemann main sheet [38], the stability of the system can be obtained by determining the number of cycles of this curve around the origin.

The Nyquist curve of the system has been shown in Figure 4. As seen from Figure 4, the system is stable because it does not include critical point (-1, j0). The Nyquist curves for different values of α, and constant value of $\tau = 1$ are presented in Figure 5. As seen from Figure 5, while the value of α increases for constant τ, i.e., it gets closer to 1, a curve similar to the time delay (e^{-s}) in the classical calculation is obtained. The Nyquist curves for different values of τ, and constant value of $\alpha = 0.9$ are given in Figure 6. It can be seen from Figure 6, when τ increases for the constant value of α, the Nyquist curve approaches to the critical point (-1, j0).

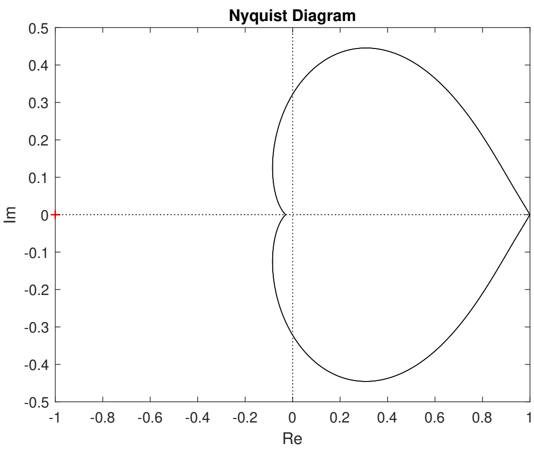

Figure 4. Nyquist diagram of Example 1.

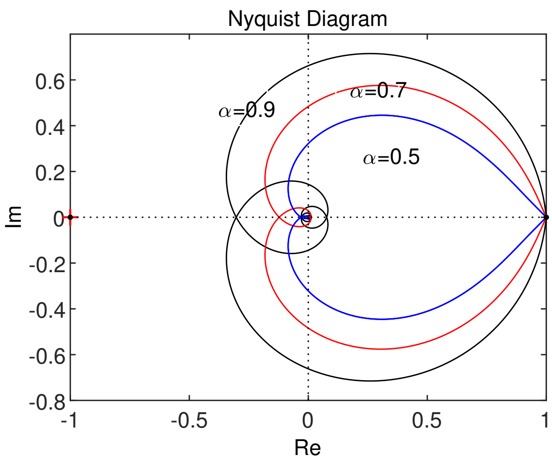

Figure 5. Nyquist diagram of Example 1 for different values of α, and fixed $\tau = 1$.

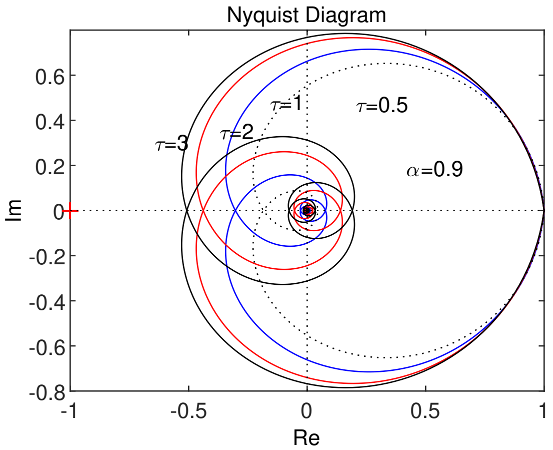

Figure 6. Nyquist diagram of Example 1 for different values of τ, and fixed $\alpha = 0.9$.

To compute all stabilizing PI controllers for the system, k_p and k_i are obtained as follows.

$$k_p = \frac{\cos(0.707\omega^{0.5}) - \omega\sin(0.707\omega^{0.5})}{-e^{-0.707\omega^{0.5}}} \tag{42}$$

$$k_i = \frac{-\omega^2\cos(0.707\omega^{0.5}) - \omega\sin(0.707\omega^{0.5})}{-e^{-0.707\omega^{0.5}}} \tag{43}$$

Stability region and unit step responses for the system can be seen in Figures 7 and 8, respectively. Any point selected within the stability region guarantees system stability. However, in order to ensure a good result in terms of system performance, it is necessary to determine new criteria to choose controller parameters from the stability region. For this purpose, a tuning method presented in [17] is used. Thus, a triangular region which is known as convex stability region has been determined in the stability curve as shown in Figure 7. In this region, the optimum point search is made with the PSO algorithm. In the PSO algorithm, the number of swarms is taken as 100 and the number of iterations is taken as 300. The number of swarms and the number of iterations are obtained by trial and error method according to the ITAE performance index.

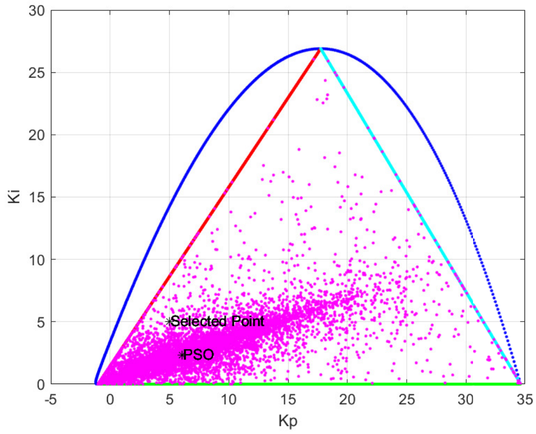

Figure 7. Stability Region and convex stability region of Example 1

Step responses of Example 1 are shown in Figure 8. As seen in Figure 8, the PSO algorithm provides a very good result.

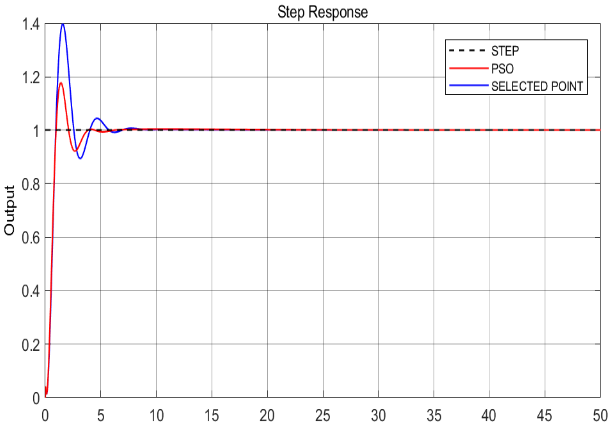

Figure 8. Unit step responses of Example 1 for $k_p = k_i = 5$ and PSO parameters $k_p = 6.0417, k_i = 2.3146$.

Using a PI controller in Example 1, the characteristic equation is given by Eq.(44) when $k_p = k_i = 5$.

$$s^3 + 6s^2\sqrt{s} + 18s^2 - 24s\sqrt{s} + 77s - 30\sqrt{s} + 60 = 0 \quad (44)$$

Substituting $q = \sqrt{s}$ in Eq.(44), Eq.(45) is found.

$$q^6 + 6q^5 + 18q^4 - 24q^3 + 77q^2 - 30q + 60 = 0 \quad (45)$$

The roots are obtained as follows.

$$q_{1,2} = -3.8401 \pm 3.6049i = 5.2670\angle \pm 136.8094$$
$$q_{3,4} = 0.8401 \pm 1.2071i = 1.4707\angle \pm 55.1634$$
$$q_{5,6} = 0 \pm i = 1\angle \pm 90$$

The roots of the system are shown in Figure 9. As seen in Figure 9, the system is stable.

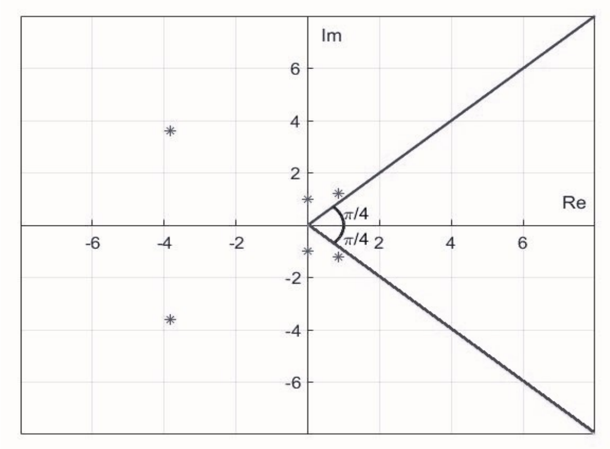

Figure 9. The roots of the PI controlled system.

5.2. Example 2

Consider the fractional order control system of Figure 1 with the transfer function of Eq.(46).

$$G_p(s) = \frac{1}{s^{1.5} + 1} e^{-\sqrt{s}} \quad (46)$$

The characteristic equation of the system without controller is obtained as

$$\Delta(s) = s^{1.5} + 1 + e^{-\sqrt{s}} \quad (47)$$

This equation can be rearranged as follows.

$$\Delta(s) = (\sqrt{s})^3 + 1 + e^{-\sqrt{s}} \quad (48)$$

The Nyquist plot of Example 2 is shown in Figure 10. As seen from Figure 10, the system is stable since it does not include critical point (-1, j0). The stability region is shown in Figure 11. The triangular region under the stability curve is also obtained for Example 2. As shown in Figure 11, the PSO algorithm has been searched for the optimum point within the triangular region. The swarm number of the PSO algorithm was taken as 100 and the number of iterations was taken as 300. ITSE performance index was used as a fitness function. The unit step changes of the system for $k_p = 1, k_i = 0.7$ and PSO parameters $k_p = 0.229, k_i = 0.4267$ are given in Figure 12. As seen from this figure, the PSO algorithm provides a better result than the selected point.

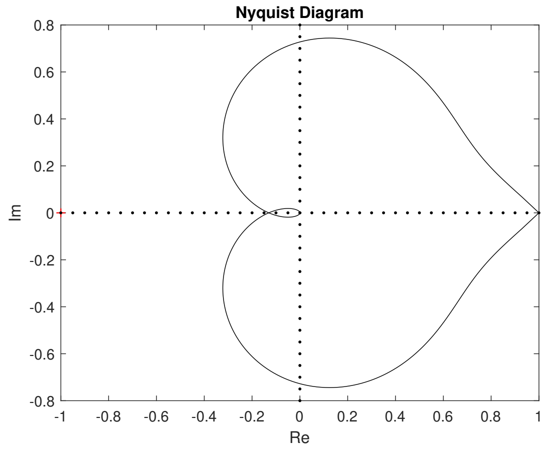

Figure 10. Nyquist Diagram for Example 2

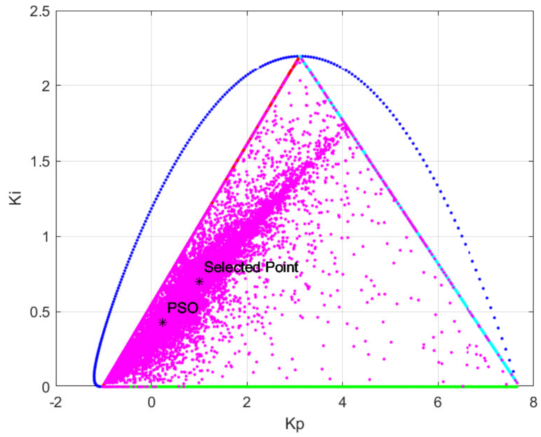

Figure 11. Stability region for Example 2

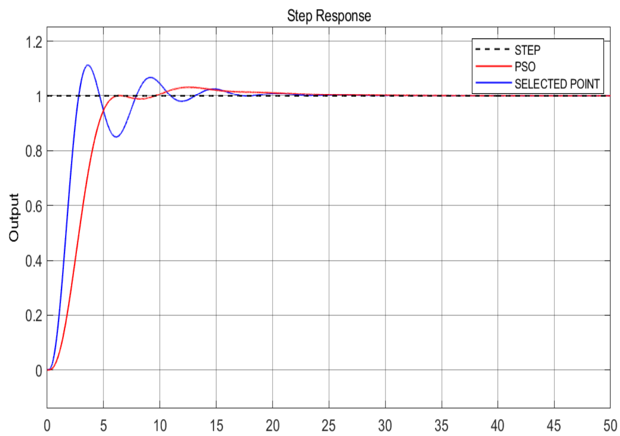

Figure 12. Unit step responses of Example 2 for $k_p = 1, k_i = 0.7$ and PSO parameters $k_p = 0.229, k_i = 0.4267$

6. Conclusion

In this paper, stability analysis and PI controller design for the systems with fractional order time delay are presented. it is known that analysis of the stability and time response of systems having fractional order delay is not possible using classical methods. To overcome this difficulty, an approximation method to investigate the stability of such systems is used. Using this approximation method, time response analysis can also be made for these systems. As for PI controller design part, a new tuning algorithm is aimed. This tuning method uses the PSO algorithm under the stability region. Thus, it has been shown that optimum PI controller parameters can be obtained with the PSO algorithm. This tuning method provides very good results as seen from the numerical examples. For future works, tuning of different controller types such as fractional order PI, PD and PID can be investigated for systems having fractional order delay. Since the reported studies are very restricted for such systems, these investigations would be very important.

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RESEARCH ARTICLE

Stability tests and solution estimates for non-linear differential equations

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ABSTRACT

This article deals with certain systems of delay differential equations (DDEs) and a system of ordinary differential equations (ODEs). Here, five new theorems are proved on the fundamental properties of solutions of these systems. The results on the properties of solutions consist of sufficient conditions and they dealt with uniformly asymptotically stability (UAS), instability and integrability of solutions of unperturbed systems of DDEs, boundedness of solutions of a perturbed system of DDEs at infinity and exponentially stability (ES) of solutions of a system of nonlinear ODEs. Here, the techniques of proofs depend upon the Lyapunov- Krasovskiĭ functional (LKF) method and Lyapunov function (LF) method. For illustrations, in particular cases, four examples are constructed as applications. Some results of this paper are given at first time in the literature, and the other results generalize and improve some related ones in the literature.

1. Introduction

Functional differential equations (FDEs) which include delay differential equations and differential integral equations have been studied for at least 200 years. However, especially, it can be seen from the relevant literature that during the last seven decades numerous qualitative behaviors of various FDEs, in particular, delay deferential equations have been studied extensively and they are still being investigated by researchers. It is known that UAS, exponential stability, instability, integrability and boundedness of solutions are the most important fundamental properties of FDEs and ODEs. There are many publications on fundamental properties of solutions of FDEs, ODEs and so on. We cite here the papers [1–6], [7–9], [10], [11–31] and the books of ([32], [33–39]) fully or partially devoted to fundamental motions of trajectories of solutions of these classes of equations. In particular, UAS and boundedness of solutions at the infinity describe long time behaviors of solutions. Additionally, during the applications of FDEs and ODEs in control theory, engineering, medicine, etc., usually it is necessary to know qualitative estimates of solutions such as instability, integrability, exponentially stability and so on.

We would like to summarize two recent works of AS, UAS and some other fundamental motions of solutions of DDEs. Recently, Tian and Ren [13] took into consideration the below system of linear DDEs:

$$\frac{dx}{dt} = Ax(t) + Bx(t - h(t)). \tag{1}$$

In [13, Theorem 1], an LKF was defined for the system (1) and based on that LKF, a theorem was proved on the AS of the zero solution of (1). In [13], the method of proof is depending upon the definition of a very interesting suitable LKF. Later, Tunç et al. [23] dealt with the nonlinear system of DDEs:

$$\begin{aligned}\frac{dx}{dt} =& A(t)x(t) + BF(x(t - h(t))) \\ &+ E(t, x(t), x(t - h(t))).\end{aligned} \tag{2}$$

In [23], three theorems, which have sufficient conditions, were proved on the UAS and integrability of solutions, when $E(.) \equiv 0$ in (2), and the boundedness of the solutions of (2), when $E(.) \neq 0$. In [35], the method used in the proofs is based on the definitions of two suitable LKFs. For some interesting recent and applicable results on the fractional mathematical models, see [40–43].

In this article, by the virtue of the systems of DDEs (1) and (2), the related ones in the references of this paper and literature, we deal with the following nonlinear system of DDEs:

$$\frac{dx}{dt} = A(t)x(t) + G(x(t)) + H(t, x(t)) + BF(x(t-h(t))) + Q(t, x(t), x(t-h(t))), \tag{3}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, $h(t) \in C^1(\mathbb{R}^+, (0, \infty))$, $A(t) \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$, $G \in C(\mathbb{R}^n, \mathbb{R}^n)$,$G(0) = 0$, $H \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $H(t, 0) = 0$,$B \in \mathbb{R}^{n\times n}$, $F \in C(\mathbb{R}^n, \mathbb{R}^n)$, $F(0) = 0$, $Q \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and the variable delay $h(t)$ of (3) fulfills the inequalities:

$$\begin{aligned} &0 \leq h_1 \leq h(t) \leq h_2, \\ &h_{12} = h_2 - h_1, \\ &0 \leq h'(t) \leq h_0 < 1. \end{aligned} \tag{4}$$

We would now like to explain the objectives of this paper.

1) In this paper, Theorem 1, Theorem 4 and Theorem 2 dealt with UAS, instability and integrability of solutions nonlinear system of DDEs (5):
$$\frac{dx}{dt} = A(t)x(t) + G(x(t)) + H(t, x(t)) + BF(x(t-h(t))). \tag{5}$$
2) The ES of the following system of ODEs was discussed by Theorem 3, when $BF(x(t-h(t))) \equiv 0$ in (5):
$$\frac{dx}{dt} = A(t)x(t) + G(x(t)) + H(t, x(t)). \tag{6}$$
3) Theorem 5 dealt with the bounded solutions of the perturbed system (3).
4) In particular cases of the considered systems, four new examples are designed to show applications of Theorems 1-5.

2. Basic information

Assume that $C_0 = C_0([-\tau, 0],\ \mathbb{R}^n)$,$\tau > 0$, is the space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$. For any $a \in \mathbb{R}$, $a \geq 0$, $\forall t_0 \geq 0$ and $x \in C_0([t_0 - \tau, t_0 + a],\ \mathbb{R}^n)$, let $x_t = x(t + \theta)$ when $-\tau \leq \theta \leq 0$ and $t \geq t_0$.

Let $x \in \mathbb{R}^n$. The norm $\|.\|$ is defined as $\|x\| = \sum\limits_{i=1}^{n} |x_i|$. Additionally, the matrix norm $\|A\|$ is defined as $\|A\| = \max\limits_{1\leq j\leq n} \left(\sum\limits_{i=1}^{n} |a_{ij}|\right)$, where $A \in \mathbb{R}^{n\times n}$.

For any $\phi \in C_0$, let

$$\|\phi\|_{C_0} = \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\| = \|\phi(\theta)\|_{[-\tau,0]}$$

and

$$C_H = \{\phi : \phi \in C_0 \ \text{ and } \ \|\phi\|_{C_0} \leq H < \infty\}.$$

In this article, without mention, let $x(t)$ represent x.

3. Stability and integrability

Let $Q(.) = 0$. Hence, we now have the nonlinear system of DDEs (5).

A. Assumptions

(H1) Let $a_A \in \mathbb{R}$, $a_A > 0$ with
$$a_{ii}(t) + \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \leq -a_A \text{ for all } \ t \in \mathbb{R}^+;$$

(H2) There exist positive constants h_0 and a_A from (3) and (H1), respectively, and f_F, g_G, h_H, $K_2 > 0$ such that
$$\|F(u) - F(v)\| \leq f_F \|u - v\|,$$
$$\forall u,\ v \in \mathbb{R}^n, F(0) = 0,$$
$$sgn x_i G_i(x) < 0$$
as
$$x_i \neq 0, \forall x \in \mathbb{R}^n, G(0) = 0,$$
$$\|G(x)\| \geq g_G \|x\| \ \text{ for all } x \in \mathbb{R}^n,$$
$$H(t, 0) = 0, sgn x_i H_i(t, x) < 0$$
as
$$x_i \neq 0, \forall t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$\|H(t, x)\| \geq h_H \|x\|, \forall t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$(a_A + g_G + h_H)(1 - h_0) - f_F \|B\| \geq K_2.$$

Theorem 1. *We suppose that conditions (H1) and (H2) are held. Then, trivial solution of (5) is UA stable.*

Proof. We define an LKF $\Delta_1 := \Delta_1(t, x_t)$ by

$$\Delta_1(t, x_t) := \|x(t)\| + \gamma \int_{t-h(t)}^{t} \|F(x(s))\| \, ds, \tag{7}$$

where $\gamma \in \mathbb{R}$, $\gamma > 0$, it will be chosen after some calculations.

From the LKF (7), we have

$$\Delta_1(t,x_t) := |x_1(t)| + ... + |x_n(t)| + \gamma \int\limits_{t-h(t)}^{t} |f_1(x(s))| ds$$

$$+ ... + \gamma \int\limits_{t-h(t)}^{t} |f_n(x(s))| ds.$$

According to the LKF of (7), it satisfies that

$$\Delta_1(t,0) = 0, \gamma_1 \|x\| \le \Delta_1(t,x_t),$$

where

$$\gamma_1 \in (0,1), \gamma_1 \in \mathbb{R},$$

Let $\gamma_2 \ge 1$, $\gamma_2 \in \mathbb{R}$, and define

$$Z_1(t,x_t) := \int\limits_{t-h(t)}^{t} \|F(x(s))\| ds.$$

Next, we have

$$\gamma_1 \|x\| + \gamma Z_1(t,x_t) \le \Delta_1(t,x_t) \le \gamma_2 \|x\| + \gamma Z_1(t,x_t).$$

Using condition (H2) and some simple evaluations, we find that

$$\begin{aligned} &\|\Delta_1(t,x_t) - \Delta_1(t,y_t)\| \\ &\le \|x(t) - y(t)\| \\ &+ \gamma F_f h_2 \sup_{t-h(t) \le s \le t} \|x(s) - y(s))\| \\ &\le M_0 \sup_{t-h(t) \le s \le t} \|x(s) - y(s)\|, \end{aligned}$$

where

$$M_0 := 1 + \gamma F_f h_2.$$

According to the above inequality, it is followed that

$$|\Delta_1(t,x_t) - \Delta_1(t,y_t)| \le M_0 \|x(s) - y(s)\|_{[t-h(t),t]}.$$

Thus, the locally Lipschitz condition in x_t is satisfied by the LKF $\Delta_1(t,x_t)$. Thus, condition (A1) of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) is held.

For the next step, by virtue of the definition of $Z_1(t,x_t)$ and condition (H2), we have

$$\begin{aligned} Z_1(t,x_t) &= \int\limits_{t-h(t)}^{t} \|F(x(s))\| ds \\ &\le f_F h(t) \sup_{t-h(t) \le s \le t} \|x(s)\| \\ &\le f_F h_2 \sup_{t-h(t) \le s \le t} \|x(s)\|. \end{aligned}$$

Using some simple calculations and condition (H2), we have

$$\begin{aligned} Z_1(t_2,x_t) - Z_1(t_1,x_t) &= \int\limits_{t_2-h(t_2)}^{t_2} \|F(x(s))\| ds \\ &- \int\limits_{t_1-h(t_1)}^{t_1} \|F(x(s))\| ds \\ &= \int\limits_{t_1}^{t_2} \|F(x(s))\| ds - \int\limits_{t_1-h(t_1)}^{t_2-h(t_2)} \|F(x(s))\| ds \\ &\le \int\limits_{t_1}^{t_2} \|F(x(s))\| ds \\ &\le f_F \sup_{t_1 \le s \le t_2} \|x(s)\| (t_2 - t_1) = M(t_2 - t_1), \end{aligned}$$

$$M_1 = f_F \sup_{t_1 \le s \le t_2} \|x(s)\|, 0 < t_1 < t_2 < \infty.$$

The obtained inequality demonstrates that the second condition, i.e., (A2), of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) is satisfied.

The differentiating the LKF $\Delta_1(t,x_t)$ of (7) and taking into account (5), we arrive that

$$\begin{aligned} \frac{d}{dt}\Delta_1(t,x_t) = &\sum_{i=1}^{n} x'_i(t) sgn x_i(t+0) + \gamma \|F(x(t))\| \\ &- \gamma(1 - h'(t)) \|F(x(t - h(t)))\|. \end{aligned} \tag{8}$$

By virtue of conditions (H1) and (H2), we obtain

$$\begin{aligned} &\sum_{i=1}^{n} sgn x_i(t+0) x'_i(t) \\ &\le \sum_{i=1}^{n} \left(a_{ii}(t) + \sum_{j=1, j \ne i}^{n} |a_{ji}(t)| \right) |x_i(t)| \\ &\quad - \|G(x(t))\| - \|H(t,x(t))\| \\ &\quad + \|B\| \ \|F(x(t - h(t)))\| \\ &\le -(a_A + g_G + h_H) \|x(t)\| \\ &\quad + \|B\| \ \|F(x(t - h(t)))\|. \end{aligned} \tag{9}$$

Thereby, putting the inequality (9) into (8) and using the condition $0 \le h'(t) \le h_0 < 1$, we have

$$\begin{aligned} \frac{d}{dt}\Delta_1(t,x_t) \le &- a_A \|x(t)\| - g_G \|x(t)\| - h_H \|x(t)\| \\ &+ \|B\| \ \|F(x(t - h(t)))\| \\ &+ \gamma \|F(x(t))\| \\ &- \gamma(1 - h'(t)) \|F(x(t - h(t)))\| \\ \le &- (a_A + g_G + h_H) \|x(t)\| \\ &+ \|F(x(t - h(t)))\| \ \|B\| \\ &+ \gamma f_F \|x(t)\| \\ &- \gamma(1 - h_0) \|F(x(t - h(t)))\|. \end{aligned}$$

Let $\gamma = \|B\|(1-h_0)^{-1}$. Then, it follows that

$$\frac{d}{dt}\Delta_1(t,x_t) \le -\Big[(a_A+g_G+h_H)-(1-h_0)^{-1}f_F\|B\|\Big]\|x(t)\| = -\frac{1}{1-h_0}\left[(a_A+g_G+h_H)(1-h_0)-f_F\|B\|\right]\|x(t)\|.$$

Using the condition (H2), clearly, we have

$$\frac{d}{dt}\Delta_1(t,x_t) \le -K_2\|x(t)\| < 0, \quad \|x(t)\| \ne 0. \tag{10}$$

Thus, it is obvious that $\frac{d}{dt}\Delta_1(t,x_t)$ is negative definite. From the inequality (10), it follows that assumption (A3) of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) is satisfied. Thus, all the assumptions of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) are held. Hence, the zero solution of (5) is UA stable. □

Theorem 2. *If the conditions (H1) and (H2) are held, then the solutions of (5) satisfies that* $\int_{t_0}^{\infty}\|x(s)\|ds < \infty$.

Proof. As in the proof of the above first theorem, we utilize the LKF $\Delta_1(t,x_t)$. According to conditions (H1) and (H2) we have

$$\frac{d}{dt}\Delta_1(t,x_t) \le -K_2\|x(t)\|.$$

This result confirms that the LKF $\Delta_1(t,x_t)$ is decreasing, i.e.,

$$\Delta_1(t,x_t) \le \Delta(t_0,\phi(t_0)) \text{ for all } t \ge t_0.$$

Integrating this inequality, it follows that

$$K_2\int_{t_0}^{t}\|x(s)\|ds \le \Delta(t_0,\phi(t_0)) - \Delta_1(t,x_t) \le K_3, \quad t \ge t_0,$$

where $K_3 = \Delta(t_0,\phi(t_0))$. Then,

$$\int_{t_0}^{t}\|x(s)\|ds \le K_2^{-1}\Delta(t_0,\phi(t_0)) \equiv K_2^{-1}K_3.$$

Let $t \to +\infty$. Hence,

$$\int_{t_0}^{\infty}\|x(s)\|ds \le K_2^{-1}K_3 < \infty.$$

Thus, the proof of Theorem 2 is finished. □

Example 1. *Let us take into account the below system of non-linear DDEs:*

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} -25-\frac{1}{1+t^4} & -\frac{1}{1+t^4} \\ -\frac{1}{1+t^4} & -25-\frac{1}{1+t^4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix} + \begin{pmatrix} -2x_1-\frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2-\frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \sin x_1(t-\frac{1}{2}|\arctan(t)|) \\ \sin x_2(t-\frac{1}{2}|\arctan(t)|) \end{pmatrix}, \tag{11}$$

where $h(t) = \frac{1}{2}|\arctan t|$ *is the delay function,* $t \ge 2^{-1}\pi$.

A comparison between the systems of DDEs (11) and DDEs (5) gives that

$$A(t) = \begin{pmatrix} -25-\frac{1}{1+t^4} & -\frac{1}{1+t^4} \\ -\frac{1}{1+t^4} & -25-\frac{1}{1+t^4} \end{pmatrix}.$$

By the virtue of the matrix $A(t)$, *we derive that*

$$a_{ii}(t) + \sum_{j=1,j\ne i}^{n}|a_{ji}(t)| = -25 < -24 = -a_A$$

because of

$$a_{11}(t) + |a_{21}(t)| = -25 - \frac{1}{1+t^4} + \frac{1}{1+t^4} = -25 < -24 = -a_A$$

and

$$a_{22}(t) + |a_{12}(t)| = -\frac{1}{1+t^4} - 25 + \frac{1}{1+t^4} = -25 < -24 = -\, a_A.$$

Hence,

$$a_{ii}(t) + \sum_{j=1,j\ne i}^{2}|a_{ji}(t)| < -a_A = -24, \forall t \in \mathbb{R}^+.$$

As for the next step, we get

$$G(x) = G(x_1,x_2) = \begin{pmatrix} G_1(x_1,x_2) \\ G_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix},$$

$$sgnx_1G_1(x) = sgnx_1G_1(x_1,x_2) = -2x_1^2 - \frac{x_1^2}{1+x_1^2} < 0, \; x_1 \ne 0,$$

$$sgnx_2G_2(x) = sgnx_2G_2(x_1,x_2) = -2x_2^2 - \frac{x_2^2}{1+x_2^2} < 0, \; x_2 \ne 0,$$

$$\|G(x)\| = \|G(x_1,x_2)\| = \left\|\begin{pmatrix} G_1(x_1,x_2) \\ G_2(x_1,x_2) \end{pmatrix}\right\| = \left\|\begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix}\right\| = \left|-2x_1 - \frac{x_1}{1+x_1^2}\right| + \left|-2x_2 - \frac{x_2}{1+x_2^2}\right|$$

$$\geq 2|x_1| - \frac{|x_1|}{1+x_1^2} + 2|x_2| - \frac{|x_2|}{1+x_2^2}$$
$$\geq |x_1| + |x_2| = \|x\|, \ g_G = 1 > 0.$$

Additionally, we have

$$H(t,x) = H(t,x_1,x_2) = \begin{pmatrix} -2x_1 - \frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2 - \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix}$$

$$sgnx_1 H_1(t,x) = sgnx_1 H_1(t,x_1,x_2) = -2x_1^2 - \frac{x_1^2}{1+\exp(t)+x_1^2} < 0, \ x_1 \neq 0,$$

$$sgnx_2 H_1(t,x) = sgnx_2 H_1(t,x_1,x_2) = -2x_2^2 - \frac{x_2^2}{1+\exp(t)+x_2^2} < 0, \ x_2 \neq 0.$$

$$\|H(t,x)\| = \|H(t,x_1,x_2)\| = \left\| \begin{pmatrix} -2x_1 - \frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2 - \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} \right\|$$

$$= \left| -2x_1 - \frac{x_1}{1+\exp(t)+x_1^2} \right| + \left| -2x_2 - \frac{x_2}{1+\exp(t)+x_2^2} \right|$$

$$\geq 2|x_1| - \frac{|x_1|}{1+\exp(t)+x_1^2} + 2|x_2| - \frac{|x_2|}{1+\exp(t)+x_2^2}$$

$$\geq |x_1| + |x_2| = \|x\|, \ h_H = 1 > 0.$$

$$B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \|B\| = 5.$$

$$F(x(t - \frac{1}{2}|arctg(t)|)$$
$$= F(x_1(t - \frac{1}{2}|arctg(t)|, x_2(t - \frac{1}{2}|arctg(t)|)$$
$$= \begin{pmatrix} \sin x_1(t - \frac{1}{2}|\arctan(t)|) \\ \sin x_2(t - \frac{1}{2}|\arctan(t)|) \end{pmatrix}$$

$$F(0) = 0, h(t) = \frac{1}{2}|\arctan(t)|.$$

Let

$$u = x(t - \frac{1}{2}|\arctan(t)|, v = y(t - \frac{1}{2}|\arctan(t)|,$$

$$u_1 = x_1(t - \frac{1}{2}|\arctan(t)|), v_1 = y_1(t - \frac{1}{2}|\arctan(t)|),$$

and

$$u_2 = x_2(t - \frac{1}{2}|\arctan(t)|)$$

$$v_2 = y_2(t - \frac{1}{2}|\arctan(t)|), t \geq \frac{\pi}{2}.$$

Then,

$$\|F(u) - F(v)\| = \|F(u_1,u_2) - F(v_1,v_2)\|$$
$$= \left\| \begin{pmatrix} \sin u_1 - \sin v_1 \\ \sin u_2 - \sin v_2 \end{pmatrix} \right\|$$
$$= |\sin u_1 - \sin v_1| + |\sin u_2 - \sin v_2|$$
$$\leq 2\left|\frac{u_1 - v_1}{2}\right| + 2\left|\frac{u_2 - u_2}{2}\right|$$
$$= \|u - v\|, f_F = 1.$$

As for the variable delay $h = h(t)$,

$$h(t) = \frac{1}{2}|\arctan(t)|,$$

$$0 < 0.001 = h_1 = \frac{1}{2}|\arctan(t)| \leq \frac{\pi}{4} = h_2,$$

$$h_{12} = h_2 - h_1 = \tfrac{\pi}{4} - 0.001,$$

$$h'(t) = \frac{1}{2+2t^2},$$

$$0 \leq h'(t) \leq \frac{1}{2} = h_0 < 1.$$

Next, we derive that

$$(a_A + g_G + h_H)(1 - h_0) - f_F\|B\|$$
$$= (24+1+1)(1-2^{-1}) - 5 = 13 - 5 = 8 \geq K_2.$$

By the virtue of the above estimates, it follows that the conditions (H1) and (H2) of Theorem 1 are held. For this reason, the solution $(x_1(t), x_2(t)) = (0,0)$ of the system of DDEs (11) is UA stable. Furthermore, $\|x(t)\|$, the norm of solutions of (11) are integrable.

B. Assumption

For the exponentially stability of the system of ODEs (6), we need the below conditions.

(H3) There exist constants h_0 from (4), a_A from (H1), and $f_F > 0$, $g_G > 0$, $H_0 > 0$, $K_2 > 0$, $e_E > 0$ such that

$$G(0) = 0, sgnx_i G_i(x) < 0 as x_i \neq 0, \text{ for all } x \in \mathbb{R}^n,$$
$$\|G(x)\| \geq g_G\|x\| \text{ for all } x \in \mathbb{R}^n,$$
$$H(t,0) = 0, sgnx_i H_i(t,x) < 0$$

as

$$x_i \neq 0, \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$\|H(t,x)\| \geq h_H\|x\| \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$(a_A + g_G + h_H) \geq e_E.$$

Theorem 3. *We suppose that conditions (H1) and (H3) are held. Then the trivial solution of the system (6) is exponentially stable.*

Proof. Define a Lyapunov function (LF) $\Delta_2 := \Delta_2(t,x)$ by

$$\Delta_2(t,x) := \|x(t)\|. \tag{12}$$

This function is equivalent to

$$\Delta_2(t,x) := |x_1(t)| + ... + |x_n(t)|.$$

From this point of view, we see that the LF $\Delta_2(t,x)$ is positive definite. The derivative of the LF $\Delta_2(t,x)$ of (12) along the system of ODEs (6) gives that

$$\frac{d}{dt}\Delta_2(t,x) = \sum_{i=1}^{n} x'_i(t) sgn x_i(t+0).$$

Using conditions (H1), (H3) and doing some simple calculations, we obtain

$$\begin{aligned}&\sum_{i=1}^{n} sgn x_i(t+0)x'_i(t)\\ &\le \sum_{i=1}^{n}\left(a_{ii}(t)+\sum_{j=1,j\ne i}^{n}|a_{ji}(t)|\right)|x_i(t)|\\ &\quad-\|G(x(t))\|-\|H(t,x(t))\|\\ &\le -\left(a_A+g_G+h_H\right)\|x(t)\|\\ &= -\left(a_A+g_G+h_H\right)\Delta_2(t,x).\end{aligned}$$

Hence,

$$\frac{d}{dt}\Delta_2(t,x) \le -\left(a_A+g_G+h_H\right)\Delta_2(t,x)$$

Integrating the last inequality, we derive that

$$\begin{aligned}&\|x(t)\| = \Delta_2(t,x(t))\\ &\le \Delta_2(t_0,x(t_0))\exp\left[-\ (a_A+g_G+h_H)\right](t-t_0).\end{aligned}$$

According to this inequality,

$$\begin{aligned}\|x(t)\| \le&\Delta_2(t_0,x(t_0))\\ &\times\exp\left[-\ (a_A+g_G+h_H)\right](t-t_0),\ t\ge t_0.\end{aligned}$$

This inequality verifies that the zero solution of (6) is exponentially stable. □

Example 2. *Consider the following two dimensional system of non-linear ODEs, which is a special case of (6):*

$$\begin{aligned}\begin{pmatrix} x'_1\\ x'_2\end{pmatrix} =& \begin{pmatrix} -25-\frac{1}{1+t^4} & -\frac{1}{1+t^4}\\ -\frac{1}{1+t^4} & -25-\frac{1}{1+t^4}\end{pmatrix}\\ &\times\begin{pmatrix} x_1\\ x_2\end{pmatrix}\\ &+\begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2}\\ -2x_2-\frac{x_2}{1+x_2^2}\end{pmatrix}\\ &+\begin{pmatrix} -2x_1-\frac{x_1}{1+\exp(t)+x_1^2}\\ -2x_2-\frac{x_2}{1+\exp(t)+x_2^2}\end{pmatrix}\quad(13)\end{aligned}$$

A comparison between the systems of ODEs (13) and ODEs (6) gives that $BF(x(t-h(t)))\equiv 0$.Next, $A(t)$, $G(x(t))$and $H(t,x(t))$ are the same as in Example 1. The estimates for the functions $A(t)$, $G(x(t))$ and $H(t,x(t))$ remain the same and correct. As for the final step for this example, it follows that

$$(a_A+g_G+h_H) = (24+1+1) = 26 > 25 = e_E.$$

According to the above discussions, it follows that conditions (H1) and (H3) of Theorem 3 are satisfied. Thus, the solution $(x_1(t),x_2(t)) = (0,0)$ of the system of ODEs (13) is exponentially stable.

4. Instability

C. Assumption

As for the instability of (5), we need the below conditions.

(H4) There exists a constant positive constant $\bar{a}_A$ such that

$$a_{ii}(t) - \sum_{j=1,j\ne i}^{n}|a_{ji}(t)| \ge \bar{a}_A \text{ for all } t\in\mathbb{R}^+.$$

(H5) There exist constants h_0 from (4), $\bar{a}_A$ from (H4) and $f_F>0$, $g_G>0$, $H_0>0$, $K_2>0$ such that

$$\begin{gathered}F(0)=0, \|F(v)\|\le f_F\|v\| \text{ for all} v\in\mathbb{R}^n,\\ G(0)=0, sgn x_i G_i(x)>0 \text{ as } x_i\ne 0, \text{ for all } x\in\mathbb{R}^n,\\ \|G(x)\|\ge g_G\|x\| \text{ for all } x\in\mathbb{R}^n,\\ H(t,0)=0, sgn x_i H_i(t,x)>0\\ \text{as}\\ x_i\ne 0, \text{ for all } t\in\mathbb{R}^+ \text{ and } x\in\mathbb{R}^n,\\ \|H(t,x)\|\ge h_H\|x\| \text{ for all } t\in\mathbb{R}^+ \text{ and } x\in\mathbb{R}^n,\\ \bar{a}_A+g_G\ +h_H-(1-h_0)^{-1}f_F\|B\|>0.\end{gathered}$$

Theorem 4. *We suppose that conditions (H4) and (H5) are held. Then, the trivial solution of the system of DDEs (5) is unstable.*

Proof. Define a new LKF $\Delta_3 := \Delta_3(t,x_t)$ by

$$\Delta_3(t,x_t) := \|x(t)\| - \gamma_1\int_{t-h(t)}^{t}\|F(x(s))\|\,ds,\quad(14)$$

where $\gamma_1\in\mathbb{R}$, $\gamma_1>0$. It will be determined at the below.

Next, the LKF (14) is equivalent to

$$\begin{aligned}\Delta_3(t,x_t) :=& |x_1(t)|+...+|x_n(t)|\\ &-\gamma_1\int_{t-h(t)}^{t}|f_1(x(s))|ds-...-\gamma_1\int_{t-h(t)}^{t}|f_n(x(s))|ds.\end{aligned}$$

From this point of view, the LKF $\Delta_3(t,x_t)$ satisfies the following relation:

$$\begin{aligned}\Delta_3(t,x_t) &\geq \|x(t)\| - \gamma_1 f_F \int_{t-h(t)}^{t} \|(x(s))\|\,ds \\ &\geq \|x(t)\| - \gamma_1 f_F h(t) \sup_{t-h(t)\leq s\leq t} \|x(s)\| \\ &\geq \|x(t)\| - \gamma_1 f_F h_1 \sup_{t-h(t)\leq s\leq t} \|x(s)\| \\ &= [1-\gamma_1 f_F h_1] \sup_{t-h(t)\leq s\leq t} \|x(s)\| > 0\end{aligned}$$

provided that $\|x(t)\| = \sup_{t-h(t)\leq s\leq t} \|x(s)\|$, $h_1 < (\gamma_1 f_F)^{-1}$ and $\|x(t)\| \neq 0$.

Next, the differentiating the LKF $\Delta_3(t,x_t)$ of (14) along (5) leads that

$$\begin{aligned}\frac{d}{dt}\Delta_3(t,x_t) =& \sum_{i=1}^{n} x'_i(t) sgn x_i(t+0) - \gamma \|F(x(t))\| \\ &+ \gamma \|F(x(t-h(t)))\| \times (1-h'(t)).\end{aligned} \tag{15}$$

For the first term of (15), using conditions (H4), (H5) and doing some elementary calculations, we obtain

$$\begin{aligned}&\sum_{i=1}^{n} sgn x_i(t+0) x'_i(t) \\ \geq& \sum_{i=1}^{n} a_{ii} |x_i(t)| - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ji}|\ |x_i(t)| \\ &+ \sum_{i=1}^{n} G_i(x(t)) sgn x_i(t+0) \\ &+ \sum_{i=1}^{n} H_i(t,x(t)) sgn x_i(t+0) \\ &- \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|\ |F_j(x(t-h(t)))| \\ =& \sum_{i=1}^{n} \left(a_{ii}(t) - \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \right) |x_i(t)| \\ &+ \|G(x(t))\| + \|H(t,x(t))\| \\ &- \|B\|\ \|F(x(t-h(t)))\| \\ \geq& \bar{a}_A \|x(t)\| + g_G\ \|x(t)\| + h_H\ \|x(t)\| \\ &- \|B\|\ \|F(x(t-h(t)))\|.\end{aligned} \tag{16}$$

Combining the inequalities (15), (16) and using the condition $0 \leq h'(t) \leq h_0 < 1$, we derive that

$$\begin{aligned}\frac{d}{dt}\Delta_3(t,x_t) \geq& \bar{a}_A \|x(t)\| + g_G\ \|x(t)\| + h_H\ \|x(t)\| \\ &- \|B\|\ \|F(x(t-h(t)))\| - \gamma_1 \|F(x(t))\| \\ &+ \gamma_1 \|F(x(t-h(t)))\| \times (1-h'(t)) \\ \geq& (\bar{a}_A + g_G\ + h_H\ ) \|x(t)\| \\ &- \|B\|\ \|F(x(t-h(t)))\| \\ &- \gamma_1 f_F \|x(t)\| + \gamma_1(1-h_0) \|F(x(t-h(t)))\|.\end{aligned}$$

Let $\gamma_1 = (1-h_0)^{-1} \|B\|$. Then,

$$\begin{aligned}\frac{d}{dt}\Delta_3(t,x_t) \geq& \left(\bar{a}_A + g_G\ + h_H - (1-h_0)^{-1} f_F \|B\|\ \right) \\ &\times \|x(t)\| > 0.\end{aligned}$$

Thus, the zero solution of the nonlinear system of DDEs (5) is unstable. □

Example 3. *Let us consider the system:*

$$\begin{aligned}\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} =& \begin{pmatrix} 25+\frac{1}{1+t^4} & \frac{1}{1+t^4} \\ \frac{1}{1+t^4} & 25+\frac{1}{1+t^4} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \\ &+ \begin{pmatrix} 2x_1 + \frac{x_1}{1+x_1^2} \\ +2x_2 + \frac{x_2}{1+x_2^2} \end{pmatrix} \\ &+ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \sin x_1(t-\frac{1}{2}|\arctan(t)|) \\ \sin x_2(t-\frac{1}{2}|\arctan(t)|) \end{pmatrix},\end{aligned} \tag{17}$$

where $h(t) = \frac{1}{2}|\arctan t|$ *is the delay function,* $t \geq 2^{-1}\pi$.

A comparison between the systems of DDEs (17) and DDEs (5) gives that

$$A(t) = \begin{pmatrix} 25+\frac{1}{1+t^4} & \frac{1}{1+t^4} \\ \frac{1}{1+t^4} & 25+\frac{1}{1+t^4} \end{pmatrix}.$$

By the virtue of the matrix $A(t)$*, we derive that*

$$a_{ii}(t) - \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \geq 25 = \bar{a}_A$$

since

$$\begin{aligned}&a_{11}(t) - |a_{21}(t)| \\ &= 25 + \frac{1}{1+t^4} - \frac{1}{1+t^4} \geq 25 = \bar{a}_A\end{aligned}$$

and

$$a_{22}(t) - |a_{12}(t)| = \frac{1}{1+t^4} + 25 - \frac{1}{1+t^4} \geq 25 = \bar{a}_A.$$

Hence,

$$a_{ii}(t) - \sum_{j=1, j\neq i}^{2} |a_{ji}(t)| \geq \bar{a}_A = 25, \forall t \in \mathbb{R}^+.$$

As for the next step, we get

$$\begin{aligned}G(x) &= G(x_1,x_2) \\ &= \begin{pmatrix} G_1(x_1,x_2) \\ G_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} 2x_1 + \frac{x_1}{1+x_1^2} \\ 2x_2 + \frac{x_2}{1+x_2^2} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}sgn x_1 G_1(x) &= sgn x_1 G_1(x_1,x_2) \\ &= 2x_1^2 + \frac{x_1^2}{1+x_1^2} > 0,\ x_1 \neq 0,\end{aligned}$$

$$sgnx_2G_2(x) = sgnx_2G_2(x_1, x_2)$$
$$= 2x_2^2 + \frac{x_2^2}{1+x_2^2} > 0,\ x_2 \neq 0,$$
$$\|G(x)\| = \|G(x_1, x_2)\|$$
$$= \left\| \begin{pmatrix} G_1(x_1,x_2) \\ G_2(x_1,x_2) \end{pmatrix} \right\| = \left\| \begin{pmatrix} 2x_1 + \frac{x_1}{1+x_1^2} \\ 2x_2 + \frac{x_2}{1+x_2^2} \end{pmatrix} \right\|$$
$$= \left| 2x_1 + \frac{x_1}{1+x_1^2} \right| + \left| 2x_2 + \frac{x_2}{1+x_2^2} \right|$$
$$\geq 2|x_1| - \frac{|x_1|}{1+x_1^2} + 2|x_2| - \frac{|x_2|}{1+x_2^2}$$
$$\geq |x_1| + |x_2| = \|x\|,\ g_G = 1 > 0.$$

Additionally, we have

$$H(t,x) = H(t, x_1, x_2)$$
$$= \begin{pmatrix} 2x_1 + \frac{x_1}{1+\exp(t)+x_1^2} \\ 2x_2 + \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix},$$
$$sgnx_1H_1(t,x) = sgnx_1H_1(t, x_1, x_2)$$
$$= 2x_1^2 + \frac{x_1^2}{1+\exp(t)+x_1^2} > 0,\ x_1 \neq 0,$$
$$sgnx_2H_1(t,x) = sgnx_2H_1(t, x_1, x_2)$$
$$= 2x_2^2 + \frac{x_2^2}{1+\exp(t)+x_2^2} > 0,\ x_2 \neq 0.$$
$$\|H(t,x)\| = \|H(t, x_1, x_2)\|$$
$$= \left\| \begin{pmatrix} 2x_1 + \frac{x_1}{1+\exp(t)+x_1^2} \\ 2x_2 + \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} \right\|,$$
$$= \left| 2x_1 + \frac{x_1}{1+\exp(t)+x_1^2} \right|$$
$$+ \left| 2x_2 + \frac{x_2}{1+\exp(t)+x_2^2} \right|$$
$$\geq 2|x_1| - \frac{|x_1|}{1+\exp(t)+x_1^2}$$
$$+ 2|x_2| - \frac{|x_2|}{1+\exp(t)+x_2^2}$$
$$\geq |x_1| + |x_2|$$
$$= \|x\|,\ h_H = 1 > 0.$$
$$B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \|B\| = 5.$$
$$F(x(t - \frac{1}{2}|arctg(t)|)$$
$$= F(x_1(t - \frac{1}{2}|arctg(t)|, x_2(t - \frac{1}{2}|arctg(t)|)$$
$$= \begin{pmatrix} \sin x_1(t - \frac{1}{2}|\arctan(t)|) \\ \sin x_2(t - \frac{1}{2}|\arctan(t)|) \end{pmatrix}$$
$$F(0) = 0, h(t) = \frac{1}{2}|\arctan(t)|.$$

Let

$$u = x(t - \frac{1}{2}|\arctan(t)|, u_1 = x_1(t - \frac{1}{2}|\arctan(t)|)$$

and

$$u_2 = x_2(t - \frac{1}{2}|\arctan(t)|), t \geq \frac{\pi}{2}.$$
$$\|F(u)\| = \|F(u_1, u_2)\| = \left\| \begin{pmatrix} \sin u_1 \\ \sin u_2 \end{pmatrix} \right\|$$
$$= |\sin u_1| + |\sin u_2|$$
$$\leq |u_1| + |u_2|$$
$$= \|u\|,$$
$$f_F = 1.$$

As for the variable delay

$$h = h(t) = \frac{1}{2}|\arctan(t)|,$$

the verifications in Example 1 for this function are the same there, too.

Finally, we have that

$$(\bar{a}_A + g_G + h_H)(1 - h_0) - f_F \|B\|$$
$$= (25 + 1 + 1)(1 - 2^{-1}) - 5$$
$$= 13.5 - 5 = 8.5 > 0.$$

By the virtue of the above estimates, it follows that the conditions (H4) and (H5) of Theorem 4 are satisfied. For this reason, the solution $(x_1(t), x_2(t)) = (0,0)$ *of the system of DDEs (17) is unstable.*

5. Boundedness

For the bounded solutions of (3), we need to modify condition (H2) as the below:

(H6) There exist positive constants h_0 and a_A from (4) and $(H1)$, respectively, f_F, g_G, h_H and a continuous function $q_Q \in C(\mathbb{R}, \mathbb{R})$ such that

$$F(0) = 0,$$
$$\|F(u) - F(v)\| \leq f_F \|u - v\| \text{ for all } u,\ v \in \mathbb{R}^n,$$
$$G(0) = 0, sgnx_iG_i(x) < 0$$

as

$$x_i \neq 0, \text{ for all } x \in \mathbb{R}^n,$$
$$\|G(x)\| \geq g_G \|x\| \text{ for all } x \in \mathbb{R}^n,$$
$$H(t,0) = 0, sgnx_iH_i(t,x) < 0$$

as

$$x_i \neq 0, \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$

$\|H(t,x)\| \geq h_H \|x\|$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$,

$$\|Q(t, x(t), x(t - h(t)))\| \leq |q_Q(t)| \ \|x(t)\|,$$
$$(a_A + g_G + h_H - |q_Q(t)|)$$
$$\times (1 - h_0) - f_F \|B\| \geq 0.$$

Theorem 5. *If conditions (H1) and (H6) are held, then the solutions of the system of DDEs (3) are bounded as $t \to +\infty$.*

Proof. By virtue of conditions (H1), (H6) and the LKF $\Delta_1(t, x_t)$, we derive that

$$\begin{aligned}\frac{d}{dt}\Delta_1(t,x_t) \le & -\frac{1}{1-h_0}\Big[(a_A+g_G+h_H)(1-h_0) \\ & -f_F\,\|B\|\Big]\,\|x(t)\| \\ & +\|Q(t,x(t),x(t-h(t)))\| \\ \le & -\frac{1}{1-h_0}\Big[(a_A+g_G+h_H-|q_Q(t)|) \\ & \times(1-h_0)-f_F\,\|B\|\Big]\,\|x(t)\|\,.\end{aligned}$$

Hence, from condition (H6), it is clear that

$$\frac{d}{dt}\Delta_1(t,x_t) \le 0.$$

Integrating this inequality, we obtain

$$\Delta_1(t,x_t) \le \Delta_1(t_0,\phi(t_0)) \equiv K_4 > 0,\ \ \phi(t_0) \ne 0. \tag{18}$$

By virtue of the LKF $\Delta_1(t, x_t)$ and (18), we derive that

$$\|x(t)\| \le K_4.$$

Next, it follows that

$$\lim_{t\to+\infty} \||x(t)|\| \le \lim_{t\to+\infty} K_4 = K_4.$$

Thus, the solutions of the system of nonlinear DDEs (3) are bounded as $t \to +\infty$. This is the end of proof of Theorem 5. □

Example 4. *Consider the following perturbed system of DDEs:*

$$\begin{aligned}\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = & \begin{pmatrix} -25-\frac{1}{1+t^4} & -\frac{1}{1+t^4} \\ -\frac{1}{1+t^4} & -25-\frac{1}{1+t^4} \end{pmatrix} \\ & \times\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ & +\begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix} \\ & +\begin{pmatrix} -2x_1-\frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2-\frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} \\ & +\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \\ & \times\begin{pmatrix} \sin x_1(t-\frac{1}{2}\,|\arctan(t)|) \\ \sin x_2(t-\frac{1}{2}\,|\arctan(t)|) \end{pmatrix} \\ & +\begin{pmatrix} \frac{4\sin x_1}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}|\arctan(t)|)} \\ \frac{4\sin x_2}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}|\arctan(t)|)} \end{pmatrix},\end{aligned} \tag{19}$$

where $h(t) = \frac{1}{2}\,|\arctan t|$ is time-varying delay, $t \ge 2^{-1}\pi$.

A comparison between the systems of DDEs (19) and DDEs (3) shows that the functions $A(t)$, $G(x(t))$, $H(t, x(t))$, $F(x(t - h(t)))$ and the constant matrix B are the same as in Example 1. From this point of view, the relations for the functions $A(t)$, $G(x(t))$, $H(t, x(t))$, $F(x(t-h(t)))$ and the matrix B remain the same and correct as in Example 1.

For the remain calculations, we consider the function

$$\begin{aligned}& Q(t,x,x(t-\frac{1}{2}\,|\arctan(t)|)) \\ & = \begin{pmatrix} \frac{4\sin x_1}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}|\arctan(t)|)} \\ \frac{4\sin x_2}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}|\arctan(t)|)} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}& \|Q(t,x,x(t-\frac{1}{2}\,|\arctan(t)|))\| \\ & = \left\|\begin{pmatrix} \frac{4\sin x_1}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}|\arctan(t)|)} \\ \frac{4\sin x_2}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}|\arctan(t)|)} \end{pmatrix}\right\| \\ & = \frac{4\,|\sin x_1|}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}\,|\arctan(t)|)} \\ & \quad + \frac{4\,|\sin x_2|}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}\,|\arctan(t)|)} \\ & \le [|x_1|+|x_2|] = |q_Q(t)|\,\|x\|\,,\end{aligned}$$

where

$$|q_Q(t)| = 1,$$
$$\|x\| = |x_1| + |x_2|\,.$$

Next,

$$\begin{aligned}& (\bar{a}_A+g_G+h_H-|q_Q(t)|)(1-h_0)-f_F\,\|B\| \\ & = (24+1+1-1)(1-2^{-1})-5 = 12.5-5 = 7.5 > 0.\end{aligned}$$

Thus, conditions (H1) and (H6) of Theorem 6 are held. By virtue of the given discussions, we conclude that all the solutions of (19) are bounded as $t \to \infty$.

6. Contributions

In this section, we make comments to the contributions of Theorems 1-5.

1) It follows that the systems of (1) and (2) are particular cases of the systems of DDEs (3) and DDEs (5). This is an improvement and a new contribution (see, [17, 23]).
2) In [13, Theorem 1], the authors proved a theorem on the AS of the linear system of DDEs (1) using a suitable LKF as basic tool. Next, in [23], the authors proved three results on the UAS, the integrability and the boundedness of the solutions

of the nonlinear system of DDEs (2) using a suitable LKF.

In this paper, we proved five new theorems related to the UAS, the instability and the integrability of solutions of the nonlinear system of DDEs (5) by Theorem 1, Theorem 4 and Theorem 2, the exponentially stability of zero solution of the system of nonlinear ODEs (6) by Theorem 3 and the boundedness of solutions of the system of nonlinear DDEs (3) by Theorem 5, respectively.

To prove Theorems 1, 2 and 5, the LKF

$$\Delta_1(t, x_t) := \|x(t)\| + \gamma \int_{t-h(t)}^{t} \|F(x(s))\| \, ds,$$

to prove Theorem 3, the LF

$$\Delta_2(t, x) := \|x(t)\|$$

and to prove Theorem 4, the LKF

$$\Delta_3(t, x_t) := \|x(t)\| - \gamma_1 \int_{t-h(t)}^{t} \|F(x(s))\| \, ds$$

were used as basic tools.

Indeed, these LKFs and LF lead very suitable conditions for Theorem 1-Theorem 5. Next, the instability and the ES results are new, the other three results are nonlinear generalizations of the former results in the literature. These are some other contributions to the topic and literature.

3) In this paper, we provide four examples, which satisfy the conditions of Theorems 1-5, and, in particular cases, we also show the applications of the Theorem 1-Theorem 5.
4) The LKF $\Delta_1(t, x_t)$ implies to eliminate the need to use the Gronwall's inequality for the boundedness of solutions at infinity. Hence, the boundedness result, Theorem 5, has weaker conditions and it is also more general as well as has simple conditions, which are more convenient for applications.

7. Conclusion

In this article, the unperturbed nonlinear system of DDEs (5) with variable delay, the perturbed nonlinear system of DDEs (3) with variable delay and the system of ODEs (6) were taken into consideration. Here, five new results, i.e., Theorem 1 –Theorem 5, which are dealt with the qualitative behaviors of trajectories of solutions called UAS, instability and integrability of solutions of the unperturbed system of DDEs (5), the boundedness of solutions of the perturbed system of DDEs (3) and the exponentially stability of solutions of the system of ODEs (6), were proved using the LKF method for the delay systems (3), (5) and the second method of Lyapunov for the system of ODEs (6), respectively. In the proof of the boundedness result, i.e., Theorem 5, it was not needed to use the Gronwall's inequality. This case allows weaker conditions. Indeed, the novelty and the contributions of the results of this paper are that the results of this article are new and they have weaker conditions than those available in the relevant literature. This idea can be seen form the items 1)-4). Finally, four examples, Example 1-Example 4, were given to make clear the applications of our results.

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RESEARCH ARTICLE

Analysing the market for digital payments in India using the predator-prey model

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ABSTRACT

Technology has revolutionized the way transactions are carried out in economies across the world. India too has witnessed the introduction of numerous modes of electronic payment in the past couple of decades, including e-banking services, National Electronic Fund Transfer (NEFT), Real Time Gross Settlement (RTGS) and most recently the Unified Payments Interface (UPI). While other payment mechanisms have witnessed a gradual and consistent increase in the volume of transactions, UPI has witnessed an exponential increase in usage and is almost on par with pre-existing technologies in the volume of transactions. This study aims to employ a modified Lotka-Volterra (LV) equations (also known as the Predator-Prey Model) to study the competition among different payment mechanisms. The market share of each platform is estimated using the LV equations and combined with the estimates of the total market size obtained using the Auto-Regressive Integrated Moving Average (ARIMA) technique. The result of the model predicts that UPI will eventually overtake the conventional digital payment mechanism in terms of market share as well as volume. Thus, the model indicates a scenario where both payment mechanisms would coexist with UPI being the dominant (or more preferred) mode of payment.

1. Introduction

The last couple of decades have witnessed technology penetrating our lives in unimaginable ways. One such area where technology has had a significant impact in is the financial sector. With the advent of technology, payment mechanisms are undergoing paradigm shifts. Electronic payment systems offer various advantages over physical currency, like speed, security, lower transaction costs for individuals, elimination of counterfeit currency, and enhanced regulation. For this reason, Central Banks are not only promoting and facilitating digital payment mechanisms, but some are also mooting the idea of completely shifting to electronic transactions by replacing physical cash with central bank digital currency (CBDC). Electronic payment mechanisms have been in vogue for a considerable period of time. By providing the aforementioned benefits to users, these mechanisms influence behaviour in very significant ways. Numerous modes of electronic payments have emerged in the past couple of decades including e-banking services, NEFT, RTGS and most recently the Unified Payments Interface or UPI.

UPI in particular, has witnessed phenomenal growth within a short span of its introduction. The UPI was launched by the National Payments Corporation of India (NPCI), a joint initiative of

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the RBI and leading banks, which has been a pioneer in developing efficient and accessible payment solutions in India. The UPI enables a set of standard application programming interface specifications to facilitate digital payments using the mobile phone [1]. It leveraged on the extensive mobile phone network and the increasing usage of smart phones, enhanced internet availability, and the growth of mobile-based payment applications in India. UPI allows for a range of financial and non-financial transactions by making mobile phones the primary payment device. The introduction of UPI coincided with two key events in the economic and business landscape, which contributed immensely to its popularity. The year 2016 saw the entry of new players like Jio, which propelled the data revolution in India, which drastically brought down the prices of internet data, thus increasing its coverage and usage. With data available at low cost, and increased availability of smart phones, UPI witnessed a consistent increase in the number of users, as well as the number of banks, live on the platform. The second factor that was significant in the initial increase in UPI usage was the demonetisation of high-value currency notes, which the Government of India announced in November 2016. This brought in noticeable changes in the perception of users regarding digital payment technologies. UPI is a significant improvement over its peers in numerous ways:

- The UPI allows for both "pull", i.e., payee initiated as well as "push", i.e., payer initiated transactions.
- UPI payments can be made using various platforms like apps, websites, etc.
- UPI eliminates the need to divulge multiple, sensitive details like bank account number, IFSC code, etc., by capturing all information in a single verifiable UPI ID.
- UPI payments are based on 2-factor authentication in which the customer only needs to enter a single MPIN, unlike other cashless payment modes where users need to enter multiple details like name, password, OTP, and others.
- UPI only requires the presence of a mobile phone and internet connection which reduces the infrastructure needed by a very large amount.
- UPI does not work in "silos" as the involved parties need not be on the same interface.

The above features have brought about an exponential increase in the usage of UPI. While this is of great importance to policymakers and lawmakers as it enhances the digitalisation of the financial sector, it is of higher significance for banks as it has a tremendous economic impact. The coexistence of UPI with similar payment technologies offers customers with a choice. When faced with a choice, the decision often depends on the opportunity cost of each alternative. As has been established, UPI outperforms its peers on important parameters like time taken to complete the transactions, cost incurred per transaction, and convenience, among others. Currently, the UPI allows for non-banking firms also to operate on the common infrastructure. This has given rise to a scenario where the market for UPI transactions is largely dominated by three technology companies, none of them being banks. If this trend were to continue, the dynamics would result in banks losing out the major portion of their revenue coming from transaction charges to these tech firms. Hence the need to study the competition between existing digital payment technologies and UPI, and whether they can coexist gains importance. While an empirical approach can be adopted to examine these questions, the predictive powers of such analyses are limited due to the fact that empirical research involves the use of past data in which variations are inherent. On the other hand, using suitable mathematical models to study the various scenarios arising out of competition can prove to be superior in describing and predicting the interaction among players in the market under study.

Mathematical models have played a central role in the understanding phenomena in various fields, including natural and applied sciences. One such model which has been widely studied is the Lotka-Volterra model or the Predator-Prey Model. Propounded to understand the dynamic nature of population growth of different species competing against each other, the model has been extended and modified extensively to mimic real life scenarios to a great extent. Though the model was initially confined to the study of evolutionary theories, it later found extensive application in economics. It was evident to researchers that competition in markets involving multiple players was not dissimilar to dynamics present among competing species. Thus the Lotka-Volterra model, and its extensions, were used in various contexts to study the different phenomena arising in economics. Some of the popular applications include the study of competition between different sectors like agriculture, industry and agriculture in a country; study of competition between different industries in the economy; competition between different technologies within an industry; competition between firms at different stages seeking investment; dynamics between websites competing for same user base; and competition between different companies within the same market to list a few.

The investigation about real-world problems is always a hot topic in the present context. The efficiency of the predictor-corrector method is effectively illustrated by researchers in [2] in order to examine the SIR model of COVID-19; in extension with this, the stability is derived in [3] for the numerical technique, which helps to solve predator-prey model, the predator-prey model associated with prey refuge was investigated in [4], the effect of a numerical method to solve the atmospheric ocean model is illustrated in [5]. In order to prove the essence and significance of mathematical modelling in connection with real-world problems, the authors in [6–8] investigated the omicron and its earlier version and presented some useful results. The current study can be extended by generalizing the integer order derivative with fractional order; for instance, the stability of the integro-differential systems within the frame of fractional order is connected by researchers in [9], the hyper-chaotic system is examined with the help of novel fractional operator in [10], the physical model with unstable cases is investigated in [11], the numerical method for higher order fractional system is proposed by researchers in [12], the chemical reaction model is investigated with the efficient numerical scheme in [13], the scholars in [14–16] investigated the fractional order models with numerical approaches. These above-cited studies can help the readers to extend the present work.

The purpose of this paper is to study one such application of the Lotka-Volterra model, i.e., in the context of the market for digital payments in India. While the estimates and forecasts of the aggregate transactions can be obtained by time series methods, the competition element among the platforms cannot be found using the same. Thus the paper uses a combination of ARIMA and LV model to analyse the dynamic between the competing platforms, i.e., Conventional Digital Payments (CDP) consisting of NEFT, RTGS and Internet Banking and the revolutionary technology UPI.

2. Literature review

2.1. Economic applications of the Lotka-Volterra model

The Lotka-Volterra model has been applied extensively to understand the competing relationships in various business ecosystems. Apedaille et al. [17] use the predator-prey mechanism to model the shares of agricultural, industrial and exospheric wealth in the open interacting economic systems. One of the earliest and most well-known applications of the predator-prey model was given by Maurer and Huberman [18] which developed a model to explain the domination of the internet by certain websites. Watanabe et al. [19] apply the Lotka-Volterra model to forecast the transition from analogue broadcasting to digital broadcasting in the context of Japan. Lee et al. [20] study the interaction between competing technologies in communication systems by inputting patent data to the Lotka-Volterra model. Tsenf et al. use the Lotka-Volterra model to analyse competition between smartphone operating systems and thus attempt to forecast sales volumes. Lee and Oh [21] use the Lotka-Volterra model to analyse the competition between two rival markets namely the Korean Stock Exchange and the Korean Securities Dealers Automated Quotation. Ren et al. [22] studies competition among websites by dividing consumers into 'users' and 'visitors' and formulating a two-competitor model to find a situation (represented by a stable solution) where the competing website can coexist. Brander and de Bettignies [23] use the predator-prey model to provide a contributing explanation for both high-venture capital concentration by industry and 'boom and bust' industry-level investment dynamics. Kreng and Wang [24] use the Lotka-Volterra equations to model the competition between LCD and Plasma Display televisions. Chiang & Wong [25] considered the LV-model to estimate market diffusion by considering the competition between desktops and notebook computers. A similar study in the Indian context by Pant and Bagai [26] looks at the coexistence of the organised and unorganised sectors in the retail industry where use a modified Lotka-Volterra model was used to describe the competition between the two sectors. Crookes and Blignaut [27] use the predator prey model to stimulate the intersectoral dynamics of the steel sector. Hung et al. [28] apply an enhanced Lotka-Volterra model to study the competition between convenience-oriented and budget-oriented retail stores in Taiwan by decomposing data into three components and hence obtaining more efficient estimates as a result. Nikolaieva and Bochko [29] have studied the behaviour of the market share of operating systems using the Lotka-Verra model and subsequently tried to predict the market share for Android and iOS operating systems using numerical integration. Evidently, the Lotka-Volterra model is a reliable forecasting method for two or more competing species.

2.2. UPI technology in India

Gochhwal [1] point out that penetration of telecommunication, increase in bank coverage, elimination of the need to share sensitive bank details, and reduction in time and cost compared to pre-existing electronic payment services are factors favouring enhanced usage of UPI. Mohapatra [30] emphasises the proliferation of smartphones, availability of an online individual identity, universal access to banking and the introduction of biometric sensors in smartphones as some trends which would aid in further developing cashless payment technologies. Kakade and Veshne [31] establish that among the reasons for the widespread use of UPI is its 24x7 availability and emphasise its role in enhancing transaction efficiency and making India a cashless economy. Vipin and Sumathy [32] found that habitual use of cash and complexity in using digital payments was the main barriers for trying digital payments cited by the users. Patil [33] analyses the adoption of UPI and studies the demographic factors affecting UPI perception among consumers using primary data. They found that while the age of consumers did not influence the perceptions regarding usefulness and cost, it did influence the perception regarding ease of use. There was no significant difference in perception among different educational groups and income categories. Philip [34] analysed the impact of UPI on customers' satisfaction using primary data and found that UPI had a significant positive impact on customers, and perceptions of UPI and traditional payment methods varied significantly among consumers. Kumar et al. [35] analysed the security dimension of UPI and other payment apps in India and discovered unreported multi-factor flaws in the authentication design, making the interface vulnerable to significant potential attacks.

3. Theoritical Framework

3.1. ARIMA

Auto-Regressive Integrated Moving Average is a forecasting technique used to analyse time series data. This model is applicable in cases where data displays non-stationary behaviour (i.e., non-constancy with respect to mean but not with respect to variance). Non-stationarity thus arising can be dealt with using differencing techniques, i.e., differencing the data with itself one or more times.

An ARIMA(p, d, q) implies

$$\begin{aligned} {y'}_t &= c + \varphi_1 {y'}_{t-1} + \varphi_2 {y'}_{t-2} \cdots \varphi_p {y'}_{t-p} \\ &+\varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} \cdots \theta_q \varepsilon_{t-q}, \end{aligned} \tag{1}$$

where ${y'}_t$ is the differenced series with order of differencing d; p is the order of the Auto-regressive part and q is the order of the Moving-Average part.

3.2. Lotka-Volterra equations

The simple Lotka-Volterra model or the predator-prey model is a system of non-linear ordinary differential equations that describe the trajectories of the population of two interacting species, namely predator and prey, over a time period. It is given by

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta xy, \\ \frac{dy}{dt} &= \delta xy - \gamma y, \end{aligned} \tag{2}$$

where

- x is the number of preys,
- y is the number of predators,
- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the instantaneous growth rates of the two populations,
- t represents time,
- α, β, γ, δ are positive real parameters describing the interaction of the two species.

While the equations in system 1 represent the dynamics where species preys on another, the same can be extended to represent the dynamic where both the species prey on each other. Thus the growth in one both species influences the population of the other species negatively. Such a system is given by

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x^2 - \gamma xy, \\ \frac{dy}{dt} &= \varphi y - \psi y^2 - \mu xy, \end{aligned} \tag{3}$$

where

- x is the number of species x,
- y is the number of species y,
- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the instantaneous growth rates of the two populations,
- t represents time,
- α, β, γ, φ, ψ, and μ are positive real parameters describing the interaction of the two species.

In the above system (3), α and φ are the per-capita birth rates (we may also consider them as overall per-capita growth rates) of x and y respectively and incorporate deaths (independent of the other species) as well as births. Thus they are per-capita growth rates, or per-capita reproduction rates, while the parameters β and ψ are the self-interaction parameters, which denote the decline in x and y in the absence of the other species. The parameters γ and μ are the interaction parameters and describe the competition between the species.

While the ARIMA process provides parameters that fit the behaviour of time series data, the estimates and the subsequent forecasts obtained from the model are valid only when certain assumptions made regarding the error terms are satisfied. The ARIMA estimation technique makes two major assumptions regarding the errors:

i There is no serial correlation among the error terms.
ii The error terms are normally distributed with constant mean and finite variance, i.e., $a_t \sim N(\mu, \sigma^2)$

Upon fitting the model and obtaining the best fit parameter values, various tests can be performed to check if the assumptions are satisfied in order to validate the results of the model. The first assumption can be checked by Box-Pierce Test, Box-Ljung Test among others. With respect to testing normality, standard testing procedures like the Shapiro-Wilk test can be used.

The last couple of decades have witnessed technology penetrating our lives in unimaginable ways. One such area where technology has had a significant impact in is the financial sector. With the advent of technology, payment mechanisms are undergoing paradigm shifts. Electronic payment systems offer various advantages over physical currency, like speed, security, lower transaction costs for individuals, elimination of counterfeit currency, and enhanced regulation. For this reason, Central Banks are not only promoting and facilitating digital payment mechanisms, but some are also mooting the idea of completely shifting to electronic transactions by replacing physical cash with central bank digital currency (CBDC). Based on the literature review, it can be understood that in a market where two or more firms compete against each other, the growth in the market share of one firm affects the market share of the other. In such cases, it is not possible to draw a clear distinction as to which firm is the predator and which firm is the prey. Thus, using a model in which both populations compete against each other, as in the system, would be more appropriate to analyse such a market. This leads us to the proposed model to describe the dynamics in the market for digital payments in India. Consider the system of equations

$$\begin{aligned} \frac{dU}{dt} &= \alpha_1 U - \beta_1 U^2 - \gamma_1 UC, \\ \frac{dC}{dt} &= \alpha_2 C - \beta_2 C^2 - \gamma_2 UC, \end{aligned} \tag{4}$$

where

- U is the market share of the UPI platform,
- C is market share of the Conventional Digital Payment Mechanisms like NEFT, RTGS, and Internet Banking,
- $\frac{dU}{dt}$ and $\frac{dC}{dt}$ represent the instantaneous growth rates of the two competing platforms,
- t represents time,
- α_1, $\beta_1\, \gamma_1, \alpha_2$, β_2, and γ_2 are real parameters describing the interaction of the two technologies
- no new technologies are introduced in subsequent periods.

As in system (3), α_1 and α_2 are the growth rate of the market shares of UPI and CDP platforms simultaneously. While the per capita growth rate in a biological context signifies the reproductive capacity of the species, in the context of market competition, they signify the ability of the concerned player to attract new customers. In this case, α_1 represents the ability of the UPI as a platform to induce existing users to repeat transactions in the successive time period as well as attract new users to perform transactions using this mode. A similar explanation follows for α_2. Intuitively, α_1 and α_2 have a positive impact on U and C respectively.

In system (4), β_1 and β_2 give the respective death rates of the population. It is technically the internal interaction within the species. In this context, β_1 for example, describes the rate at which users of the UPI platform withdraw from using it. A similar explanation follows for β_2.

On the other hand, the interaction parameters. γ_1 and γ_2 capture the competition between U and C in a given time period. In particular, γ_1 specifies the rate at which the UPI platform loses its users to Conventional Digital Payments. Similarly, γ_2 is the rate at which Conventional Digital Payments lose users to the UPI platform. The model makes some generalising assumptions. They are as follows:

- We assume the total number of users to be sufficiently large so that random fluctuations can be ignored without consequence

- We assume that the two system model reflects the market sufficiently accurately
- We assume each population grows exponentially in the absence of the other competitor
- We assume that access to both the platforms, level of awareness regarding both platforms, access to the internet, etc. are uniform across geographies and time periods
- Literature suggests that users of UPI have a great experience using the platform, hence reducing the chance of customer withdrawal which brings us to the assumption

$$0 < \beta_1 < \beta_2 < 1.$$

- Given that UPI outperforms its competitors we expect it to behave more predatorily. Hence we assume that

$$0 < \gamma_1 < \gamma_2 < 1.$$

- Most importantly we assume that there is no limit on the growth (i.e., carrying capacity) on the number of transactions in a platform

3.3. Stability analysis

In order for the system to be at equilibrium, the rate of change with respect to time must be zero, i.e., $\frac{dU}{dt}$ and $\frac{dC}{dt}$ must be equal to zero. We obtain the solutions for these by equating the right hand side of the respective equations to zero. By solving these we get two points where the slopes are equal to zero $P_1(0,0)$ and $P_2(\frac{\alpha_1\beta_2-\alpha_2\gamma_1}{\beta_1\beta_2-\gamma_1\gamma_2}, \frac{\alpha_2\beta_1-\alpha_1\gamma_2}{\beta_1\beta_2-\gamma_1\gamma_2})$.

Clearly, P_1 is a trivial solution as it indicates a situation where both the platforms have zero transactions and hence is of no interest to us. On the other hand, P_2 describes a situation where both platforms have a positive number of transactions and are of special interest to us.

The stability of this fixed point can be analysed using the Jacobian matrix:

$$J = \begin{vmatrix} \alpha_1 - 2\beta_1 U - \gamma_1 C & -\gamma_1 U \\ -\gamma_2 C & \alpha_2 - 2\beta_2 C - \gamma_2 U \end{vmatrix},$$

$$J\ (P_2) = \begin{vmatrix} \beta_1 \frac{\alpha_2\gamma_1-\alpha_1\beta_2}{\beta_1\beta_2-\gamma_1\gamma_2} & -\gamma_1 \frac{\alpha_1\beta_2-\alpha_2\gamma_1}{\beta_1\beta_2-\gamma_1\gamma_2} \\ -\gamma_2 \frac{\alpha_2\beta_1-\alpha_1\gamma_2}{\beta_1\beta_2-\gamma_1\gamma_2} & \beta_2 \frac{\alpha_1\gamma_2-\alpha_2\beta_1}{\beta_1\beta_2-\gamma_1\gamma_2} \end{vmatrix}.$$

The eigenvalues of the above matrix are calculated to determine the stability of the system at the above point. Since one eigenvalue is positive and one eigenvalue is negative, we infer that the fixed point P_2 is a saddle point.

3.4. Existence and uniqueness of Ssolution

Let $\mathcal{G}(J)$ be the Banach space with the maximal norm given by $\|x\| = \max_{t\in\mathcal{J}} \mid x(t) \mid$ where $\mathcal{J} = [0, \mathcal{T}_1]$ and $\mathcal{T}_1 = \mathcal{G}(J) \times \mathcal{G}(J)$.
Let us consider

$$F_1(t,U) = \alpha_1 U - \beta_1 U^2 - \gamma_1 UC,$$
$$F_2(t,C) = \alpha_2 C - \beta_2 C^2 - \gamma_2 UC.$$

Theorem 1. *The kernel F_1 and F_2 admit the Lipschitz condition and contraction when $0 \le \Lambda_1, \Lambda_2 < 1$, where $\lambda_1 = \alpha_1 - \beta_1(\epsilon_1 + \kappa_1) - \gamma_1\epsilon_2$, $\lambda_2 = \alpha_2 - \beta_2(\epsilon_2 + \kappa_2) - \gamma_2\epsilon_1$.*

Proof. We assume that the solution of the system is bounded, such that $\|U\| \le \epsilon_1$ and $\|C\| \le \epsilon_2$.
Consider two functions U and U^*, such that

$$\|F_1(t,U) - F_1(t,U^*)\|$$
$$= \|(\alpha_1 U - \beta_1 U^2 - \gamma_1 UC)$$
$$- (\alpha_1 U^* - \beta_1 U^{2^*} - \gamma_1 CU^*)\|$$
$$= \|(\alpha_1 - \gamma_1 C)(U - U^*) - \beta_1(U + U^*)(U - U^*)\|$$
$$\le (\alpha_1 - \beta_1(\epsilon_1 + \kappa_1) - \gamma_1\epsilon_2)\|(U - U^*)\|$$
$$\le \lambda_1\|U - U^*\|,$$

where $\|U^*\| = \kappa_1$ and $\|C^*\| = \kappa_2$.
Consider two functions C and C^*, such that

$$\|F_3(t,C) - F_3(t,C^*)\|$$
$$= \|(\alpha_2 C - \beta_2 C^2 - \gamma_2 UC)$$
$$- (\alpha_2 C^* - \beta_2 C^{2^*} - \gamma_2 UC^*)\|$$
$$= \|(\alpha_2 - \gamma_2 U)(C - C^*) - \beta_2(C + C^*)(C - C^*)\|$$
$$\le (\alpha_2 - \beta_2(\epsilon_2 + \kappa_2) - \gamma_2\epsilon_1)\|(C - C^*)\|$$
$$\le \lambda_2\|C - C^*\|.$$

□

Theorem 2. *The solution of the model exists and is unique.*

Proof. Let, $\mathcal{K} = \max_{(U,C)\in\Lambda}\{\|F_1(U)\|, \|F_2(C)\|\}$.
The integral form of the system is given by

$$U(t) = U_0 + \int_0^t F_1(U(\tau))d\tau,$$
$$C(t) = C_0 + \int_0^t F_2(C(\tau))d\tau.$$

Using the successive approximations of the solution of the integral equations, we get

$$U_{n+1}(t) = U_0 + \int_0^t F_1(U_n(\tau))d\tau,$$
$$C_{n+1}(t) = C_0 + \int_0^t F_2(C_n(\tau))d\tau.$$

The solutions are continuous and satisfy

$$\begin{aligned}\|U_{n+1}(t) - T_0\| &= \|\int_0^t F_1(U_n(\tau))d\tau\| \\ &\leq \int_0^t \|F_1(U_n(\tau))\|d\tau \\ &\leq \mathcal{K}t.\end{aligned}$$

Let $\max\|U_1(t) - U_0\| \leq b$. We show that $\|U_{n+1}(t) - U_n(t)\| \leq (a_1 t)^{k-1} b$ using principal of mathematical induction. For $n = 1$ consider

$$\begin{aligned}&\|U_2(t) - U_1(t)\| \\ &= \|U_0 + \int_0^t F_1(U_1(\tau))d\tau - U_0 - \int_0^t F_1(U_0(\tau))d\tau\| \\ &= \|\int_0^t (F_1(U_1(\tau)) - F_1(U_0(\tau)))\, d\tau\| \\ &\leq \int_0^t \|F_1(U_1(\tau)) - F_1(U_0(\tau))\|d\tau \\ &\leq a_1 \int_0^t \|U_1(\tau) - U_0(\tau)\|d\tau \\ &\leq a_1 \max\|U_1(t) - U_0\|t \\ &\leq a_1 bt.\end{aligned}$$

Assume that the inequality holds for some $k \in \mathbb{N}$, i.e., $\|U_k(t) - U_{k-1}(t)\| \leq (a_1 t)^{k-1} b$.
Then for some integer $k \geq 2$, it follows that,

$$\begin{aligned}&\|U_{k+1}(t) - U_k(t)\| \\ &= \|U_0 + \int_0^t F_1(f_k(\tau))d\tau - U_0 - \int_0^t F_1(U_{k-1}(\tau))d\tau\| \\ &= \|\int_0^t F_1(U_k(\tau)) - F_1(U_{k-1}(\tau))d\tau\| \\ &\leq \int_0^t \|F_1(U_k(\tau)) - F_1(U_{k-1}(\tau))\|d\tau \\ &\leq a_1 \int_0^t \|U_k(\tau) - U_{k-1}(\tau)\|d\tau \\ &\leq (a_1 t)^k b.\end{aligned}$$

Let $at = \gamma$. For some $m, n \geq N$ we get,

$$\begin{aligned}\|U_m(t) - U_n(t)\| &\leq \sum_{k=n}^{m-1} \|U_{k+1}(t) - UU_k(t)\| \\ &\leq \sum_{k=N}^{\infty} \|U_{k+1}(t) - U_k(t)\| \\ &\leq \sum_{k=N}^{\infty} (a_1 t)^k b \\ &= \sum_{k=N}^{\infty} \gamma^k b \\ &= \frac{\gamma^N}{1-\gamma} b.\end{aligned}$$

This tends to 0 as $N \to \infty$. Therefore, for all $\epsilon > 0$ there exists N such that for $m, k \geq N$,

$$\|U_m(t) - U_n(t)\| \leq \epsilon,$$

i.e., $\{U_n\}$ is a Cauchy sequence in $\mathcal{G}(J)$ and therefore converges uniformly to a function U. Taking the limit as $n \to \infty$ on both sides of the definition of successive approximation we see that the function

$$U(t) = \lim_{n\to\infty} U_n(t),$$

admits

$$U(t) = U_0 + \int_0^t F_1(U(\tau))d\tau.$$

Since $f(t)$ is continuous, $F_1(U(t))$ is also continuous and using the Fundamental theorem of Integral Calculus, we get $U'(t) = F_1(U(t))$. Similarly, we can show that C_n is a Cauchy sequence that converges uniformly $C(t)$, and we can obatine $C'(t) = F_2(C(t))$. Furthermore, $U(0) = U_0$ and $C(0) = C_0$. Therefore $U(t), C(t)$ is a solution of the system.
Suppose $\bar{U}(t), \bar{C}(t)$ is another set of solution for the system. Now, consider

$$\begin{aligned}\|U - \bar{U}\| &= \|U_0 + \int_0^t F_1(U(x))dx - f_0 \\ &\quad - \int_0^t F_1(\bar{f}(\tau))d\tau\| \\ &= \|\int_0^t F_1(U(\tau)) - \int_0^t F_1(\bar{U}(\tau))d\tau\| \\ &\leq \int_0^t \|F_1(U(\tau)) - F_1(\bar{U}(\tau))\|d\tau \\ &\leq a_1 \int_0^t \|U(\tau) - \bar{U}(\tau)\|d\tau \\ &\leq a_1 t\|U - \bar{U}\|.\end{aligned}$$

Since $a_1 t < 1$, the inequality is satisfied only when $\|U - \bar{U}\| = 0$. Thus, $U(t) = \bar{U}(t)$. Similarly we can show $C(t) = \bar{C}(t)$. Therefore, the system has a unique solution. □

3.5. Boundedness

Theorem 3. *The solution of the model is uniformly bounded.*

Proof. Let $P(t) = U(t) + C(t)$. Taking the derivative along with the control parameter, we get

$$\begin{aligned}&\left(\frac{d}{dt} + \phi_1(t)\right)(P(t)) \\ &= \frac{d}{dt}[U(t) + C(t)] + \mu_1(t)[U(t) + C(t)] \\ &= \alpha_1 U - \beta_1 U^2 - \gamma_1 UC + \alpha_2 C - \beta_2 C^2\end{aligned}$$

$$- \gamma_2 UC + \phi_1(t)[U(t) + C(t)]$$
$$\leq \alpha_1 U + \alpha_2 C + \phi_1(t)[U(t) + C(t)].$$

The solution exists and is unique in

$$\Lambda = \{U, C) \in \mathbb{R}^2 : max(\mid U \mid, \mid C \mid) \leq \epsilon\}.$$

The previous inequality yields

$$\left(\frac{d}{dt} + \phi_1(t)\right)(P(t)) \leq \epsilon\left[\alpha_1 + \alpha_2 + 2\phi_1(t)\right].$$

Therefore, the solution of the system is bounded.

4. Methodology

4.1. Data

For the purpose of this study, real-time data regarding transactions facilitated by the different transforms are considered. In order to measure the activity happening on each platform, the total volume of transactions in each month is considered. Data was collected for two variables: Conventional Digital Payments (CDP) which is the sum of all transactions happening through NEFT, RTGS and Internet Banking platforms and UPI which is the volume of transactions happening through UPI platforms. The data for CDP was collected from the RBI, while data for UPI was sourced from the NCPI. The data was collected for a period of 62 months starting from January 2017 to February 2022.

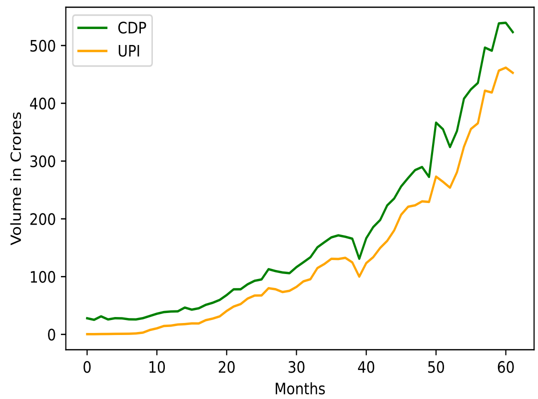

Figure 1. Volume of UPI and CDP Transactions 2017-22.

It is evident from Figure 1 that there is an explicitly increasing trend in the volume of transactions of both conventional digital payments as well as the UPI platforms. However, when the market share of each platforms is considered, there is a clear indication of competition among the two platforms. As seen in Figure 2, the UPI platform has witnessed a phenomenal increase in market share whereas the former has seen a consistent decline.

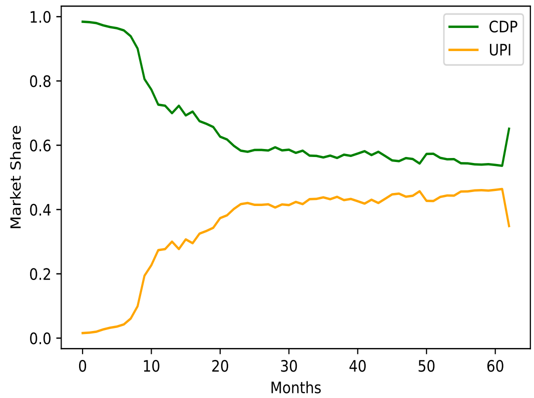

Figure 2. Market Share of UPI and CDP 2017-22.

4.2. Forecasting Method

The proposed methodology for estimating the market share of the respective platforms are captured in Figure 3. The estimation procedure consists of two modules. The first module involves estimating the total volume of transactions, $\hat{V}$, using time series techniques. The time series technique suitable for this purpose would be ARIMA as the volume of the transaction contains no significant seasonal or cyclical component. The second module is concerned with estimating the market share of each of the platforms i.e., U and C, which is determined by their respective competitive natures, using the Lotka-Volterra equations. The particular values of the parameters are obtained from real data and plugged into system 3. The above system of equations is of a non-linear kind and cannot be solved using known methods. Hence we need to use some numerical methods to obtain an approximate solution. For the purpose of estimating the market shares in different time periods, we propose to use the fourth order Runge-Kutta method. Once the estimates of the market share are obtained, it is combined with the ARIMA estimate to obtain the estimates of the volume of transactions in individual platforms, $\widehat{V_U}$ and $\widehat{V_C}$.

5. Results and Discussion

The fourth order Runge-Kutta method is employed to obtain a numerical solution of the market share. The iterative method is employed after using real data to estimate the value of the required parameters. Based on the collected data, the following values of the parameters in system 2 is chosen

$$\alpha_1 = 0.0664, \quad \beta_1 = 0.0005, \quad \gamma_1 = \ 0.02,$$

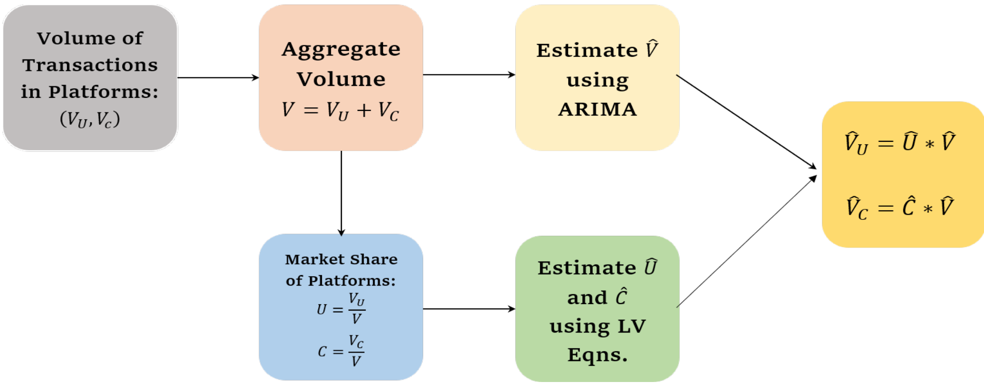

Figure 3. Proposed Forecasting Procedure

$$\alpha_2 = 0.0096, \quad \beta_2 = 0.0009, \quad \gamma_2 = \ 0.1,$$

and substituted in system 3. The results thus obtained are presented are in Figure 4. It can be observed that there is a progressive decline in the market share of CDP which is consistent with the trend established by the real data in Figure 2. Similarly, the market share of UPI is seen to witness continuously, which is again consistent with the trend established by real data. The results of the LV equations also establish that the growth in market share for UPI, and the decline in market share of CDP, reduces gradually. This is made evident by the plateauing and the stabilizing of the respective curves. The stability analysis of the same has been attached with the Appendix.

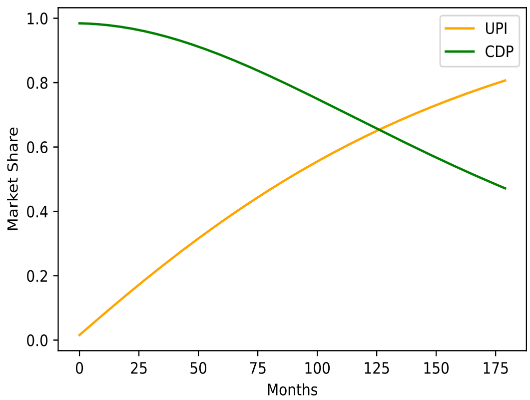

Figure 4. Estimated Market Share of UPI and CDP.

The ARIMA procedure requires that the errors of the fitted model are distributed normally. This assumption is not satisfied by the variable under consideration. Hence a log transformation is employed to ensure that error terms are normally distributed. The diagnostic tests of the new fitted model is added in the appendix. The new variable provides the best fit model to be ARIMA (0, 1, 0). The estimated equation is given as

$$\hat{V}_t = \ 0.0580 + \hat{V}_{t-1} + \ \varepsilon_t.$$

This signifies a positive association between the current values and the previously estimated terms. The values forecasted using the above equation are presented in Figure 5. The forecast is in line with the behaviour observed in the real data which shows a continuously increasing trend.

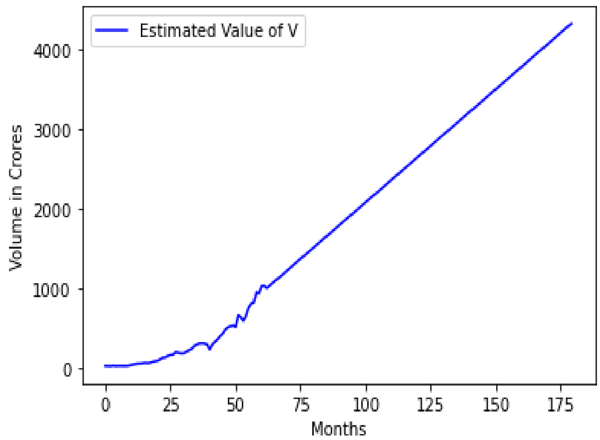

Figure 5. ARIMA estimate of aggregate volume.

The estimated value of the total number of transactions $(\hat{V})$ and the estimated market share of each platform (U and C) when combined, give the estimates of the volume of transactions in individual platforms, $\widehat{V_U}$ and $\widehat{V_C}$. The result of the same can be seen in Figure 6 which shows an almost exponential increase in the volume of transactions using the UPI platform. The volume of transactions on the CDP platforms on the other hand witnessed a steady increase before stabilizing after a given time period at a certain level. Just as in the case of the market share estimates, the volume of the UPI platform overtakes the volume of the CDP platform at a particular point in time.

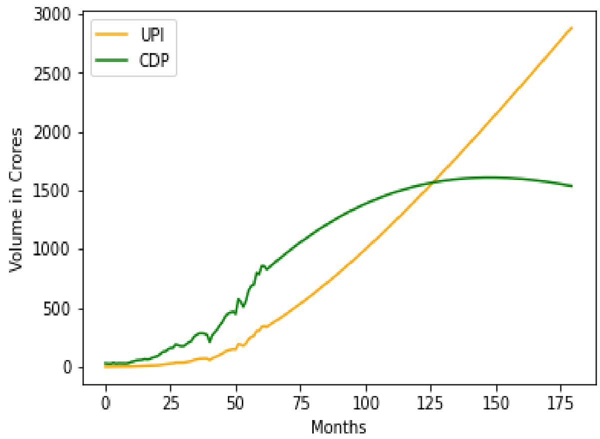

Figure 6. Estimated Volumes of UPI and CDP.

The above results establish certain phenomena explicitly. The first clear trend that emerges is that digital payment transactions in general witness a near exponential growth. The increase in the volume of digital payments is driven by an increase in both UPI and CDP platforms. This fits well with economic intuition and empirical evidence. As the Indian economy grows, and as the greater portion of it gets formalised, it would lead to greater adoption of digital payment mechanisms. This would in turn lead to an increase in the volume as well as the value of transactions being processed in each platform as suggested by the forecasts.

The second trend, and the one which is of more interest to us, is the coexistence of the two platforms in current as well as future time periods. This can be explained by some of the economic and policy related features of the market for digital payments. For example, there exists an upper limit on the value of the transaction that can be carried out using the UPI platform. This naturally shifts a finite portion of the market to conventional digital payment platforms which enable the transfer of money above a certain limit. Literature also suggests the existence of concerns among users regarding security, veracity and accessibility with regard to the UPI platform. Such concern may result in the CDP platforms retaining a certain market share despite the phenomenal growth of UPI. This seeming anomaly of decreasing market share but the increasing volume of the CDP platform can be understood in the perspective of the first trend. While the volume of total digital transactions increases, this causes an increase in the volume of transactions on the CDP platforms owing to the expansionary nature of the economy. The enormous increase in the volume of transactions on the UPI platform thus does not necessarily imply a shift in the user ship from one platform to another. While the rise in user ship of UPI up to a certain point (represented by the point where the two curves intersect) can be attributed to a shift from the CDP platforms, the volume of UPI transactions continues to rise beyond this point while the volume of transactions on CDP platforms stabilises. One possible explanation for this could be that while CDP platforms retain the high value transactions, UPI platforms gain popularity among low value transactions, replacing cash.

6. Concluding remarks

The purpose of this paper was to analyse the dynamics in the market for digital payments in India. The interaction between two competing platforms, conventional digital payments (which consist of NEFT, RTGS and Internet Banking) and the Unified Payments Interface, was examined using the Lotka-Volterra system of equations. The estimates of the competition element were combined with the estimate of the volume of transactions obtained using the ARIMA procedure to forecast the trends in the volume of transactions of the two platforms. The forecasts revealed that the volume of transactions in such platforms would increase manifold, thus highlighting the trend of digitalization of the economy. The results also suggest that the market share occupied by UPI would eventually overtake the market share of other platforms. However, the former would later exhibit a lack of growth and the latter a lack of decline, thus hinting at coexistence. While the results of the model do not indicate the extinction of services by banks, it asserts the supremacy of technological innovation by predicting that the technologically advanced UPI platform will dominate the market. This is a clarion call to banks and other financial institutions to explore, adopt and invest in new technologies if they seek to maintain their dominance over the financial sector.

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RESEARCH ARTICLE

The null boundary controllability for the Mullins equation with periodic boundary conditions

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ABSTRACT

This paper studies the null controllability of the Mullins equation with a control acting on the periodic boundary. The main result proves that the system is controllable for a specific class of initial conditions and also identifies uncontrollable states. Additionally, the existence and uniqueness theorem for the solution of the backward adjoint system is provided.

1. Introduction

In this paper, we study the null controllability problem for the Mullins equation [1] with periodic boundary conditions. This equation is a linear analog of the Kuramoto-Sivashinsky equation and has the form

$$y_t + By_{xxxx} = 0, \tag{1}$$

where B is a positive constant known as the Mullins coefficient. The Mullins equation is a linear parabolic partial differential equation that is often used to model the evolution of thin films in materials science and engineering.

The controllability problems for parabolic equations have received considerable attention in the literature (see [2–12]). However, the null controllability of fourth-order parabolic equations has been studied in a few papers. Firstly, Y.L. Guo [13] used two well-posed problems to solve the null boundary controllability problem for a fourth-order parabolic equation. Later, the null interior controllability problem for a fourth-order parabolic equation was solved by Han Yu [14] using the method based on Lebeau-Rabbino Inequality. Also, Z. Zhou [15] derived the observability inequalities for a one-dimensional linear fourth-order parabolic equation with potential using establishing global Carleman estimates and presented null controllability results for the one-dimensional fourth-order semilinear equation. More recently, S. Guerrero and K. Kassab obtained the null controllability results for the higher dimensional fourth-order parabolic equation in [16]. These studies have mostly focused on the case of Dirichlet boundary conditions. This paper, however, explores the null controllability problem with periodic boundary conditions. There have been some works on null controllability for different types of systems using periodic boundary conditions. For example, Imanuvilov considered the controllability problem for the Boussinesq system with periodic boundary conditions [17], Beauchard and Zuazua studied the null controllability problem of the Kolmogorov equation under periodic boundary conditions [18], and Chowdhury and Mitra proved that the linearized compressible Navier-Stokes equations with periodic boundary conditions are null controllable [19]. More recently, Oner obtained null controllability results for a heat equation with periodic boundary conditions [20]. However, to the best of our knowledge, no work on the controllability problem for fourth-order parabolic equations with periodic boundary conditions has been published in the literature. This observation motivated us to consider this problem.

In addition, the above-aforementioned studies generally preferred the Carleman method to solve this problem and this method is quite technical. Here, we used duality and the moment method. The moment method was developed by Fattorini and Russell (see [3, 21]), and it allows obtaining the solution of the problem using the spectral properties of the system.

The main contributions of this article are as follows. First of all, the existence and uniqueness of the solution of the adjoint system have been proven. Then, with periodic boundary conditions, it is shown that the system is not always controllable for every initial condition, and a class containing controllable initial conditions is determined. Finally, for this admissible initial data class, the null boundary controllability problem of the Mullins equation with periodic boundary conditions has been solved by using the moment method.

The paper is organized as follows. In Section 2, we define the problem and give some initial results by using duality between controllability and observability. Subsequently, in Section 3, we provide some spectral results to reduce the null controllability problem to a moment problem. In Section 4, we focus on the null boundary controllability problem for the Mullins equation with periodic boundary conditions. Since the null controllability of the system is not always possible, we first determine the restricted initial data class and then show that the system is null controllable for this initial data class. Finally, in Section 4, we indicate the conclusion.

2. Problem Formulation

In the present work, we consider the null controllability of the following system:

$$\begin{cases} u_t + u_{xxxx} + cu = 0, & \text{in } D \\ u(\pi,t) - u(-\pi,t) = v(t), & \text{in } (0,T) \\ u_x(\pi,t) - u_x(-\pi,t) = 0, & \text{in } (0,T) \\ u_{xx}(\pi,t) - u_{xx}(-\pi,t) = 0, & \text{in } (0,T) \\ u_{xxx}(\pi,t) - u_{xxx}(-\pi,t) = 0, & \text{in } (0,T) \\ u(x,0) = u^0(x), & \text{in } \Omega \end{cases} \tag{2}$$

where $D = \Omega \times (0,T)$, $\Omega = (-\pi,\pi)$, $u^0(x) \in L^2(\Omega)$, $v(t) \in L^2(0,T)$, and c is any positive number. The system we are considering is not always controllable. Therefore, we will first identify the uncontrollable cases and then determine the conditions under which the system is controllable. Let us call this class $\mathcal{F}$, which will be determined later. Now, we can define the null controllability.

Definition 1. *System (2) is null controllable at time T if for every initial condition $u^0 \in \mathcal{F}$, there exists a control $v(t) \in L^2(0,T)$ such that $u(x,T) = 0$ for all $x \in \Omega$.*

Now, we can present a lemma that will be used in the proof of our main result.

Lemma 1. *The system (2) is null controllable in time $T > 0$ if and only if for any $u^0 \in \mathcal{F}$ there exists $v(t) \in L^2(0,T)$ such that*

$$\int_{-\pi}^{\pi} u^0(x)\varphi(x,0)dx + \int_0^T v(t)\varphi_{xxx}(\pi,t)dt = 0 \tag{3}$$

holds for any $\varphi^0 \in L^2(\Omega)$, where $\varphi(x,t)$ is a solution of the backward adjoint problem given in as follows.

$$\begin{aligned} \varphi_t - \varphi_{xxxx} - c\varphi &= 0, \ \textit{in } D && \text{(4a)}\\ \varphi(\pi,t) - \varphi(-\pi,t) &= 0, \ \textit{in } (0,T) && \text{(4b)}\\ \varphi_x(\pi,t) - \varphi_x(-\pi,t) &= 0, \ \textit{in } (0,T) && \text{(4c)}\\ \varphi_{xx}(\pi,t) - \varphi_{xx}(-\pi,t) &= 0, \ \textit{in } (0,T) && \text{(4d)}\\ \varphi_{xxx}(\pi,t) - \varphi_{xxx}(-\pi,t) &= 0, \ \textit{in } (0,T) && \text{(4e)}\\ \varphi(x,T) &= \varphi^0(x), \ \textit{in } \Omega && \text{(4f)} \end{aligned}$$

Proof. Let v be an arbitrary element of $L^2(0,T)$, and let φ be the solution of (4). By multiplying (2) by φ and integrating the resulting expression over D using integration by parts, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_{-\pi}^{\pi} (u_t + u_{xxxx} + cu)\varphi dxdt \\ &= \int_0^T \int_{-\pi}^{\pi} u(-\varphi_t + \varphi_{xxxx} + c\varphi)dxdt \\ &+ \int_0^1 u\varphi\Big|_0^T dx \\ &+ \int_0^T [\varphi u_{xxx} - \varphi_x u_{xx} + \varphi_{xx}u_x - u\varphi_{xxx}]\Big|_{-\pi}^{\pi} dt. \end{aligned}$$

Using the given initial condition and boundary conditions, we have

$$\int_{-\pi}^{\pi} u(x,T)\varphi^0(x)dx - \int_{-\pi}^{\pi} u^0(x)\varphi(x,0)dx - \int_0^T v(t)\varphi_{xxx}(\pi,t)dt = 0. \tag{5}$$

If equation (3) holds, then it follows that $\int_{-\pi}^{\pi} u(x,T)\varphi^0(x)dx = 0$ for all $\varphi^0(x) \in L^2(\Omega)$ which means that $u(x,T) = 0$ for all $x \in \Omega$. As a result, system (2) is null-controllable. On the contrary, suppose that system (2) is null controllable at time T , that is, for every initial condition $u^0 \in \mathcal{F}$, there exists a control $v(t) \in L^2(0,T)$ such that $u(x,T) = 0$ for all $x \in \Omega$. Substituting $u(x,T) = 0$ into (5), we conclude that (3) holds. □

Above Lemma shows that the system (2) is controllable if and only if equation (3) holds. Therefore, we need to find a solution for system (4). In the following section, we will first prove the existence and uniqueness of the solution for system (4).

3. Fourier Series representation of adjoint system

To find solution of system in equation (4), we will apply the method of separation of variables by letting $\varphi(x,t) = X(x)T(t)$. This gives us:

$$\begin{cases} X''''(x) = (\lambda - c)X, & -\pi < x < \pi \\ X(\pi) - X(-\pi) = 0, & \text{in } (0,T) \\ X_x(\pi) - X_x(-\pi) = 0, & \text{in } (0,T) \\ X_{xx}(\pi) - X_{xx}(-\pi) = 0, & \text{in } (0,T) \\ X_{xxx}(\pi) - X_{xxx}(-\pi) = 0, & \text{in } (0,T) \end{cases}$$

which is self adjoint in $L^2(\Omega)$. Now, we will find a basis for $L^2(\Omega)$ formed by the eigenfunctions of this auxiliary problem. The eigenvalues and normalized eigenfunctions of this auxiliary spectral problem are $\lambda_n = n^4 + c,\ n = 0, 1, \dots$ and

$$X_0(x) = \frac{1}{\sqrt{2\pi}},$$

$$X_{2n-1}(x) = \frac{\cos(nx)}{\sqrt{\pi}},$$

$$X_{2n}(x) = \frac{\sin(nx)}{\sqrt{\pi}}$$

for $n = 1, 2, \dots$ Then, the solution of (4) can be expressed as a Fourier series expansion as follows:

$$\varphi(x,t) = \frac{\beta_0 e^{-\lambda_0(T-t)}}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{e^{-\lambda_n(T-t)}[\beta_{2n-1}\cos(nx) + \beta_{2n}\sin(nx)]}{\sqrt{\pi}} \tag{6}$$

where $\beta_n = (\varphi(x,T), X_n(x))$ for $n = 0, 1, 2, \dots$

To prove the existence and uniqueness of the solution of system (4), we need an auxiliary result that will be presented in the following lemma.

Lemma 2. *Assume that function $\varphi^0(x) \in C^4[-\pi,\pi]$ satisfies the following conditions:*

$$\varphi^0(\pi) - \varphi^0(-\pi) = 0,$$
$$\varphi^0_x(\pi) - \varphi^0_x(-\pi) = 0,$$
$$\varphi^0_{xx}(\pi) - \varphi^0_{xx}(-\pi) = 0,$$
$$\varphi^0_{xxx}(\pi) - \varphi^0_{xxx}(-\pi) = 0.$$

Then, the following inequality holds.

$$\sum_{n=1}^{\infty} n^3(|\beta_{2n-1}| + |\beta_{2n}|) \le 2C\|(\varphi^0)''''\|_{L_2(-\pi,\pi)} \tag{7}$$

where $\beta_{2n-1} = (\varphi^0, X_{2n-1})$, $\beta_{2n} = (\varphi^0, X_{2n})$, and $C = \dfrac{\pi}{\sqrt{6}}$.

Proof. Let $\varphi^0(x) \in C^4[-\pi,\pi]$ satisfy the assumption of lemma. From equation (6), it is seen that

$\beta_{2n-1} = (\varphi^0(x), X_{2n-1})$ and $\beta_{2n} = (\varphi^0(x), X_{2n})$.

Then, we have

$$n^3(\beta_{2n-1}+\beta_{2n}) = \frac{1}{n}(\varphi^0, n^4 X_{2n-1}) + \frac{1}{n}(\varphi^0, n^4 X_{2n}).$$

Since

$$X''''_{2n-1} = n^4 X_{2n-1} \text{ and } X''''_{2n} = n^4 X_{2n},$$

we can rewrite the equation as follows.

$$n^3(\beta_{2n-1} + \beta_{2n}) = \frac{1}{n}(\varphi^0, X''''_{2n-1}) + \frac{1}{n}(\varphi^0, X''''_{2n}).$$

Applying integration by part, we obtain

$$= \frac{1}{n}((\varphi^0)'''', X_{2n-1}) + \frac{1}{n}((\varphi^0)'''', X_{2n}).$$

Using this equation, we get

$$\sum_{n=1}^{\infty} n^3(|\beta_{2n-1}| + |\beta_{2n}|) = \sum_{n=1}^{\infty} \frac{1}{n}|(\varphi^0)'''', X_n)|$$

By using Cauchy -Schwartz and Bessel inequalities, we obtain

$$\sum_{n=1}^{\infty} n^3(|\beta_{2n-1}| + |\beta_{2n}|) = \sum_{n=1}^{\infty} \frac{1}{n}|((\varphi^0)'''', X_n)|$$
$$\le \Big(\sum_{n=1}^{\infty} \frac{1}{n^2}\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \big[|((\varphi^0)'''', X_n)|^2\Big)^{\frac{1}{2}}$$
$$\le C\|(\varphi^0)''''\|_{L_2(-\pi,\pi)}$$

with $C = \big(\sum_{n=1}^{\infty} \frac{1}{n^2}\big)^{\frac{1}{2}} = \dfrac{\pi}{\sqrt{6}}$. □

Now, we can prove the existence and uniqueness of the solution.

Lemma 3. *Let $\varphi^0(x)$ satisfy the conditions of Lemma (2). Then, the system (4) has a unique solution $\varphi(x,t) \in (C^{4,1}(D) \cap C^{3,0}(\bar{D}))$ of the form (6).*

Proof. Since $X_n(x)_{n\geq 0}$ are bases in $L^2(\Omega)$, $\varphi(x,t)$ can be represented by equation (6). To prove that $\varphi(x,t)$ given in (6) is solution of system (4), we need to show that the first partial derivative of $\varphi(x,t)$ with respect to t and the fourth partial derivative of $\varphi(x,t)$ with respect to x are continuous and it satisfies (4a) in Ω for $t > 0$. Additionally, the function in equation (6) and its first, second, and third partial derivatives with respect to spatial variable, as well as its first partial derivative with respect to time, must be continuous at boundary points. We need to show that the series

$$\varphi_t(x,t) \sim \frac{\beta_0\lambda_0 e^{-\lambda_0(T-t)}}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{\lambda_n e^{-\lambda_n(T-t)}[\beta_{2n-1}\cos(nx) + \beta_{2n}\sin(nx)]}{\sqrt{\pi}} \tag{8}$$

and

$$\varphi_{xxxx}(x,t) \sim \sum_{n=1}^{\infty} \frac{n^4 e^{-\lambda_n(T-t)}[\beta_{2n-1}\cos(nx) + \beta_{2n}\sin(nx)]}{\sqrt{\pi}}. \tag{9}$$

converge uniformly for $T-t \geq \epsilon$, where ϵ is an arbitrary positive number. The majorants of these series are

$$\sum_{n=1}^{\infty} \frac{\lambda_n e^{-\lambda_n \epsilon}(|\beta_{2n-1}| + |\beta_{2n}|)}{\sqrt{\pi}}$$

and

$$\sum_{n=1}^{\infty} \frac{n^4\lambda_n e^{-\lambda_n \epsilon}(|\beta_{2n-1}| + |\beta_{2n}|)}{\sqrt{\pi}}.$$

By using Lemma (3) and D'Alembert criterion, it is seen that these two majorant series are convergent. Therefore, the series in equations (8) and (9) are uniformly convergent for $T-t \geq \epsilon > 0$. Also, we conclude from superposition principle that the function defined by (6) satisfies equation (4a) for all $T > t$ because t is arbitrary. The function in equation (6) and its first, second, and third partial derivatives with respect to spatial variable and first partial derivative with respect to time must be continuous at boundary points. Namely, the series in equation (6) must be continuous at $t = T$,

$$\varphi(x,T) = \frac{\beta_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{[\beta_{2n-1}\cos(nx) + \beta_{2n}\sin(nx)]}{\sqrt{\pi}}$$

and the following functions must be continuous at boundary points $x = -\pi$ and $x = \pi$:

$$\varphi_{xxx}(x,t) \sim \sum_{n=1}^{\infty} \frac{n^3 e^{-\lambda_n(T-t)}[\beta_{2n-1}\sin(nx) - \beta_{2n}\cos(nx)]}{\sqrt{\pi}},$$

$$\varphi_{xx}(x,t) \sim \sum_{n=1}^{\infty} \frac{n^2 e^{-\lambda_n(T-t)}[-\beta_{2n-1}\cos(nx) - \beta_{2n}\sin(nx)]}{\sqrt{\pi}}.$$

$$\varphi_{x}(x,t) \sim \sum_{n=1}^{\infty} \frac{n e^{-\lambda_n(T-t)}[-\beta_{2n-1}\sin(nx) + \beta_{2n}\cos(nx)]}{\sqrt{\pi}}.$$

By using Weierstrass M-test and Lemma (2), we see that the following majorant series are uniformly convergent.

$$\sum_{n=1}^{\infty} \frac{|\beta_{2n-1}| + |\beta_{2n}|}{\sqrt{\pi}}, \quad \sum_{n=1}^{\infty} \frac{n^3|\beta_{2n}|}{\sqrt{\pi}}$$

$$\sum_{n=1}^{\infty} \frac{n^2|\beta_{2n-1}|}{\sqrt{\pi}} \text{ and } \sum_{n=1}^{\infty} \frac{n|\beta_{2n}|}{\sqrt{\pi}}.$$

Therefore, the above series are continuous at the boundary points. Finally, we obtain a function $\varphi(x,t) \in (C^{4,1}(D) \cap C^{3,0}(\bar{D}))$ which is a solution of system (4) given by the Fourier series in equation (6). This solution is also unique due to the uniqueness of the Fourier representation of functions. □

4. Null boundary controllability of Mullins equation

In this section, we will reduce the null controllability problem to a moment problem using the spectral properties of the problem. Since $\{X_n(x)\}_{n\geq 0}$ is a basis in $L_2(\Omega)$, any initial data $u^0 \in L_2(\Omega)$ can be represented as follows.

$$u^0(x) = \frac{\eta_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{[\eta_{2n-1}\cos(nx) + \eta_{2n}\sin(nx)]}{\sqrt{\pi}}, \tag{10}$$

where $\eta_n = (u^0(x), X_n(x))$ for $n = 0, 1, 2, \ldots$ Substituting equations (6) and (10) into (3), we get

$$\beta_0\eta_0 e^{-\lambda_0 T} + \sum_{n=1}^{\infty} e^{-\lambda_n T}[\beta_{2n-1}\eta_{2n-1} + \beta_{2n}\eta_{2n}] - \int_0^T \frac{v(t)}{\sqrt{\pi}} \sum_{n=1}^{\infty} n^3 e^{\lambda_n(T-t)}\beta_{2n}(-1)^n dt = 0 \tag{11}$$

According to Lemma 1, system (2) is null controllable in time $T > 0$ if and only if for any $u^0 \in \mathcal{F}$ there exists $v(t) \in L^2(0,T)$ such that (3) is satisfied. Since $\{X_n(x)\}_{n\geq 0}$ is an orthonormal basis for $L^2(\Omega)$, equation (3) is verified if and only if it is verified by $\varphi_m^0(x) = X_m(x), m = 0, 1, \ldots$. Therefore, if in particular $\varphi_m^0(x) = X_m(x)$, then $\beta_n = \delta_{m,n}$, and $\eta_0 = 0$, $\eta_{2m-1} = 0$ and

$$\int_0^T \frac{v(t)}{\sqrt{\pi}} e^{\lambda_m(T-t)} m^3 (-1)^m dt = e^{-\lambda_m T} \eta_{2m}$$

for $m = 1, 2, \ldots$. Taking $v(t) = f(T-t)$ in the last equation, we have proven the following theorem, which is the main result of this article. From above, it is clear that system (2) is not always controllable for all initial data classes. That is why we need to define the following admissible initial data classes to make the system null controllable.

$$\mathcal{F} = \{u^0(x) \in L^2(\Omega) \big| \eta_0 = 0 \text{ and } \eta_{2m-1} = 0\}.$$

Now, we are in a position to state the main theorem of this article.

Theorem 1. *The system* (2) *is null controllable in time* $T > 0$ *if and only if for any* $u^0 \in \mathcal{F}$ *with Fourier expansion*

$$u^0(x) = \sum_{n=1}^{\infty} \eta_{2n} \frac{\sin(nx)}{\sqrt{\pi}},$$

there exists a function $f \in L^2(0,T)$ *such that*

$$\int_0^T f(t) e^{-\lambda_m t} dt = \frac{(-1)^m \eta_{2m} \sqrt{\pi} e^{-\lambda_m T}}{m^3} \tag{12}$$

for $m = 1, 2, \cdots$.

To have a precise understanding of Theorem 1, we provide the following example.

Example 1. *It is seen that*

$$\varphi_n(x,t) = \frac{\cos(nx)}{\sqrt{\pi}} e^{-\lambda_n(T-t)}$$

is a solution of (4) *with the initial data* $\frac{\cos(nx)}{\sqrt{\pi}}$ *for arbitrary fixed positive integer* n*. Taking into consideration these values in* (5)*, we have*

$$\int_{-\pi}^{\pi} u(x,T) \frac{\cos(nx)}{\sqrt{\pi}} dx - \int_{-\pi}^{\pi} u^0(x) \frac{e^{-\lambda_n T} \cos(nx)}{\sqrt{\pi}} dx = 0$$

the second term of equation is independent of the control and non-zero unless $\eta_{2n-1} = 0$*. This explains how we choose the initial data classes.*

4.1. Moment Problem

We need to find $f(t)$ that satisfies (12) to find control $v(t)$. This is a moment problem in $L^2(0,T)$ with respect to the family $\Lambda = \{e^{-\lambda_m t}\}_{m \geq 0}$. From Theorem (1), we see that controllability holds if and only if the moment problem (12) is solvable. To solve this moment problem, we can apply the general theory developed in [21] by Fattorini and Russell. Suppose that we can construct a sequence of functions $\{\Psi_n\}_{n \geq 0}$ that are biorthogonal to the set Λ in $L^2(0,T)$, such that

$$\int_0^T e^{-\lambda_m t} \Psi_n(t) dt = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \tag{13}$$

for all $m, n = 0, 1, 2 \ldots$. Then, moment problems (12) have solutions by setting

$$f(t) = \sum_{n=1}^{\infty} \frac{\eta_{2n} e^{-\lambda_n T} (-1)^n \sqrt{\pi}}{n^3} \Psi_n(t)$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{n^4 + c} < \infty, \tag{14}$$

Muntz's Theorem shows that biorthogonal sequence $\{\Psi_n\}_{n \geq 0}$ exists. In addition, the general estimations of $\|\Psi_n\|_{L^2(0,\infty)}$ was calculated by H.O. Fattoroni and D. L. Russell. They showed in [3] that if the λ_n are real and satisfy the following asymptotic relationship

$$\lambda_n = K(n+\alpha)^\zeta + o(n^{\zeta-1}) \quad (n \to \infty)$$

where $K > 0, \zeta > 1$ and α is real, then there exists constants $\hat{K}, K_\zeta$ such that

$$\|\Psi_n(t)\|_{L^2(0,\infty)} \leq \hat{K} \exp[(K_\zeta + o(1))\lambda_n^{1/\zeta}] \ (n \geq 1)$$

where $o(1)$ indicates a term tending to zero as n goes to infinity. The computation of the constant K_ζ is given in [21]. To relate the interval $[0, \infty]$ with the finite interval [0, T], they used results given in [22]. Since $\lambda_n = n^4 + c$, using these results it can be seen that

$$\|\Psi_n(t)\|_{L^2(0,T)} \leq K e^\rho \text{ for } n \geq 0$$

where K and ρ are some positive constants. Now, we can state the following results.

Corollary 1. *Given any* $T > 0$*, suppose that there exists a sequence* $\{\Psi_n(t)\}_{n \geq 0}$ *in* $L^2(0,T)$ *biorthogonal to the set* Λ *such that*

$$\|\Psi_n\|_{L^2(0,T)} \leq K e^{n\rho}, \quad \forall n \geq 0 \tag{15}$$

holds, where K *and* ρ *are two positive constants. Then, system* (2) *is null-controllable in time* T.

Proof. According to Theorem (1), the system (2) is null controllable in time T if for any $u^0 \in \mathcal{F}$ with Fourier expansion

$$u^0(x) = \sum_{n=1}^{\infty} \frac{\eta_{2n} \sin(nx)}{\sqrt{\pi}},$$

there exists a function $f \in L^2(0,T)$ which holds (12). Choose

$$f(t) = \sum_{n=1}^{\infty} \frac{\eta_{2n} e^{-\lambda_n T} (-1)^n \sqrt{\pi}}{n^3} \Psi_n(t) \tag{16}$$

Since $\|\Psi_n\|_{L^2(0,T)} \le Ke^{n\rho}$, for all $n \ge 0$, we deduce that

$$\begin{aligned}
&\|\sum_{n=1}^{\infty} \frac{\eta_{2n} e^{-\lambda_n T}(-1)^n\sqrt{\pi}}{n^3}\Psi_n\|_{L^2(0,T)} \\
&\le \sum_{n=1}^{\infty} \frac{|\eta_{2n}|\sqrt{\pi}}{n^3} e^{-\lambda_n T}\|\Psi_n\|_{L^2(0,T)} \\
&\le K\sum_{n=1}^{\infty} \frac{|\eta_{2n}|\sqrt{\pi}}{n^3} e^{-\lambda_n T+n\rho} < \infty
\end{aligned}$$

i.e., $f(t)$ converges in $L^2(0,T)$. Hence, (16) implies that f satisfies (12) and the proof finishes. □

5. Conclusion

In this paper, we studied the null boundary controllability of the Mullins equation with periodic boundary conditions. We demonstrated that the system is controllable on a specific admissible initial data class and solved the null boundary controllability problem by reducing it to a moment problem using the spectral properties of the system. Additionally, we established the existence and uniqueness of the solution of the system.

As a future direction, we will consider the system with nonlocal boundary conditions. In this case, the system is not self-adjoint and will require a different approach.

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RESEARCH ARTICLE

M-truncated soliton solutions of the fractional (4+1)-dimensional Fokas equation

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ABSTRACT

This article aims to examine M-truncated soliton solutions of the fractional $(4+1)$-dimensional Fokas equation (FE), which is a generalization of the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations. The fractional $(4+1)$-dimensional Fokas equation with the M-truncated derivative is also studied first time in this study. The generalized projective Riccati equations method (GPREM) is successfully implemented. In the application of the presented method, a suitable fractional wave transformation is chosen to convert the proposed model into a nonlinear ordinary differential equation. Then, a linear equation system is acquired utilizing the GPREM, the system is solved, and the suitable solution sets are obtained. Dark and singular soliton solutions are successfully derived. Under the selection of appropriate values of the parameters, 2D, 3D, and contour plots are also displayed for some solutions.

1. Introduction

A variety of real-life problems have been modeled using the nonlinear partial differential equations (NLPDEs) with integer or fractional order. Moreover, NLPDEs with integer or fractional order have main roles in area of quantum mechanics, fluid dynamics, nonlinear optics, plasma physics as well as biology, chemistry, and finance. Because of its wide application, investigation of the analytical and soliton solutions of the NLPDEs with integer or fractional order has been very popular among authors over the past few decades. Numerous techniques consisting of the analytical and numerical methods have been improved to gain the soliton and analytical solutions of the PDEs such as the combined improved Kudryashov-new extended auxiliary sub equation method [1], the enhanced modified extended tanh-expansion approach [2,3], the sine-Gordon equation approach [4], F-expansion method [5], the tanh–coth function, the modified kudryashov expansion and rational sine–cosine approaches [6], the Riccati equation method [7], the $tan(\Theta/2)$ expansion approach [8], the Jacobi elliptic functions methodology [9], the generalized Bernoulli sub-ODE scheme [10], the extended $(\frac{G'}{G^2})$-expansion scheme [11], Nucci's reduction method [12], the new Kudryashov method [13–15], the sub-equation method based on Riccati equation [16], and the modified Sardar sub-equation method [17].

Fractional differential equations have been utilized the modeling many phenomena in a variety of branches of engineering and science [18,19]. Thus, various and substantial definitions of fractional derivatives types have been enhanced in the literature such as: Grunwald– Letnikov, Riemann–Liouville, Caputo [20], Caputo–Fabrizio [21], Atangana–Baleanu [22,23], the conformable fractional derivative [24], and the M-truncated derivative [11].

The $(4+1)$-dimensional Fokas equation is expressed by the following structure [25]:

$$4\vartheta_{tx}-\vartheta_{xxxy}+\vartheta_{xyyy}+12\vartheta_x\vartheta_y+12\vartheta\vartheta_{xy}-6\vartheta_{zs}=0, \quad (1)$$

in which $\vartheta(x,y,z,s,t)$. The model was firstly presented by A. S. Fokas [25] and the FE is the extension form of Davey–Stewartson and Kadomtsev–Petviashvili equations to some higher-dimensional nonlinear wave equations [25]. So, the $(4+1)$-dimensional FE is taken into account as a higher dimensional integrable model in mathematical physics. The $(4+1)$-dimensional FE models the finite-amplitude wave packet in fluid dynamics.

Up to now, soliton and analytical solutions of the FE have been investigated by utilizing various significant approaches. In [26], Yinghui He discussed the analytical solutions using the extended F-expansion scheme. Hirota's bilinear methodology was used to examine the FE in [27]. Kim and Sakthival obtained some traveling wave solutions of the (4+1)-dimensional FE by applying the $(\frac{G'}{G})$-expansion scheme in [28]. In [29], the Sardar subequation scheme and new extended hyperbolic function approach were utilized to build the soliton solutions of the (4+1)-dimensional fractional-order FE. Bo and Sheng used the generalized F-expansion method to construct some exact solutions with arbitrary functions in [30]. Wazwaz implemented the simplified Hirota's approach to gain a variety of soliton solutions of the presented model in [31]. In [32], the (4+1)-dimensional FE was studied using the modified simple equation and the extended simplest equation schemes by Al-Amr and El-Ganaini. Baskonus et.al. constructed various soliton solutions by performing sine-Gordon expansion method in [33].

In this paper, we intend to achieve the soliton solutions of the space-time fractional $(4+1)$-dimensional FE involving M-truncated derivative in the form:

$$\begin{aligned}4D_{M,t}^{\alpha,\gamma}D_{M,x}^{\alpha,\gamma}\vartheta - D_{M,x}^{3\alpha,\gamma}D_{M,y}^{\alpha,\gamma}\vartheta + D_{M,x}^{\alpha,\gamma}D_{M,y}^{3\alpha,\gamma}\vartheta \\ +12D_{M,x}^{\alpha,\gamma}\vartheta D_{M,y}^{\alpha,\gamma}\vartheta + 12\vartheta D_{M,x}^{\alpha,\gamma}D_{M,y}^{\alpha,\gamma}\vartheta \\ -6D_{M,z}^{\alpha,\gamma}D_{M,s}^{\alpha,\gamma}\vartheta = 0.\end{aligned} \quad (2)$$

Herein, $D_{M,x}^{\alpha,\gamma}\vartheta(x,y,z,s,t)$ represents the M-truncated derivative of ϑ with respect to x, $0 < \alpha \leq 1$. The space-time fractional form including the M-truncated derivative of $(4+1)$-dimensional FE has been examined utilizing the GPREM for the first time in this study.

The remain of this study is arranged as follows: The definition and properties of the M-truncated derivative are expressed in section 2. Mathematical analysis of the presented model is offered in section 3. We also submit the description and enforcement of GPREM in section 4. To observe the physical explanations of the derived results, we present the graphical potraits in section 5. Finally, we give the conclusion in the last section.

2. The M-truncated derivative

Definition 1. *The truncated Mittag-Leffler function [11] is identified as:*

$$_{i}E_{\gamma}(c) = \sum_{m=0}^{i} \frac{c^m}{\Gamma(m\gamma+1)},$$

for $\gamma > 0$, and $c \in \mathrm{C}$.

Definition 2. *Presume that $\delta : [0,\infty) \to \mathrm{R}$, the M-truncated derivative of δ with order α is defined by [11]*

$$D_M^{\alpha,\gamma}(\delta(x)) = \lim_{\varepsilon\to 0} \frac{\delta(x\ _{i}E_{\gamma}(\varepsilon x^{-\alpha})) - \delta(x)}{\varepsilon},$$

where $x > 0$ and $\alpha \in (0,1)$.

Theorem 1. *Consider if $0 < \alpha \leq 1, \gamma > 0$ and considering $\delta(t)$ and $\theta(t)$ are differentiable of α's order at $x > 0$, then*

(1) $D_M^{\alpha,\gamma}(a\delta(x) + b\theta(x)) = aD_M^{\alpha,\gamma}(\delta(x)) + bD_M^{\alpha,\gamma}(\theta(x))$, *for all* $a, b \in \mathrm{R}$,
(2) $D_M^{\alpha,\gamma}(\delta(x)\theta(x)) = \theta(x)D_M^{\alpha,\beta}(\delta(x)) + \delta(x)D_M^{\alpha,\gamma}(\theta(x))$,
(3) $D_M^{\alpha,\gamma}\left(\frac{\delta(x)}{\theta(x)}\right) = \frac{\theta(x)D_M^{\alpha,\gamma}(\delta(x)) - \delta(x)D_M^{\alpha,\gamma}(\theta(x))}{\theta^2(x)}$,
(4) $D_M^{\alpha,\gamma}\delta(x) = \frac{x^{1-\alpha}}{\Gamma(\gamma+1)}\frac{d\delta}{dx}$.

The truncated M-fractional derivative is an extension structure of the conformable fractional derivative.

3. Nonlinear ordinary differential form of the fractional $(4+1)$-dimensional FE

In order to gain the NODE form of Eq. (2), we should firstly determine wave transformation with M-truncated derivative as follows:

$$\vartheta(x,y,z,s,t) = V(\zeta), \quad \zeta = \frac{\Gamma(1+\gamma)(\rho_1 x^\alpha + \rho_2 y^\alpha + \rho_3 z^\alpha + \rho_4 s^\alpha + \rho_5 t^\alpha)}{\alpha}. \quad (3)$$

Herein, $\rho_1, \rho_2, \rho_3, \rho_4$ and ρ_5 are nonzero real numbers. Using the wave transformation in Eq.(3), Eq. (2) transform into the following NODE:

$$\begin{aligned}\left(\rho_1{\rho_2}^3 - {\rho_1}^3\rho_2\right)V^{(iv)} + (4\rho_1\rho_5 - 6\rho_3\rho_4)V'' \\ +\ 12\rho_1\rho_2\left(VV'\right)' = 0.\end{aligned} \quad (4)$$

Integrating Eq.(4) twice with respect to ζ and presuming the integration constants to zero, we achieve the following equation:

$$\begin{aligned}\left(\rho_1{\rho_2}^3 - {\rho_1}^3\rho_2\right)V'' + (4\rho_1\rho_5 - 6\rho_3\rho_4)V \\ +12\rho_1\rho_2V^2 = 0.\end{aligned} \quad (5)$$

4. A brief sketch of the GPREM and its application

4.1. Outline of the GPREM

According to the GPREM [14], the solution of the Eq.(5) has the following structure:

$$V(\zeta) = A_0 + \sum_{k=1}^{M} \kappa^{k-1}(\zeta)\left[A_k\kappa(\zeta) + B_k\sigma(\zeta)\right], \quad (6)$$

in which A_0, A_k and B_k $(1,2,...,M)$ are real contants to be computed, M is the balance number, and $\kappa(\zeta)$ and $\sigma(\zeta)$ satisfy the following equations:

$$\kappa'(\zeta) = \varepsilon\kappa(\zeta)\sigma(\zeta), \quad (7)$$

$$\sigma'(\zeta) = \lambda + \varepsilon\sigma^2(\zeta) - \chi\kappa(\zeta), \quad \varepsilon = \mp 1, \quad (8)$$

in which

$$\sigma^2(\zeta) = -\varepsilon\left(\lambda - 2\chi\kappa(\zeta) + \frac{\chi^2 + c}{\lambda}\kappa^2(\zeta)\right), \quad (9)$$

$c = \mp 1, \lambda$ and χ are nonzero real constants. Assuming as $\lambda = \chi = 0$, Eq.(5) has the following solution structure:

$$V(\zeta) = \sum_{k=1}^{M} A_k\sigma^k(\zeta), \quad (10)$$

in which $\sigma(\zeta)$ satisfies the following relation:

$$\sigma'(\zeta) = \sigma^2(\zeta). \quad (11)$$

Utilizing the Eqs.(7) and (8), the following solution functions are gained:

Case 1: If $\varepsilon = -1, c = -1$ and $\lambda > 0$, we get,

$$\kappa_1(\zeta) = \frac{\lambda\,\mathrm{sech}(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{sech}(\sqrt{\lambda}\zeta)+1}, \quad \sigma_1(\zeta) = \frac{\sqrt{\lambda}\tanh(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{sech}(\sqrt{\lambda}\zeta)+1}. \quad (12)$$

Case 2: If $\varepsilon = -1, c = 1$ and $\lambda > 0$, we get,

$$\kappa_2(\zeta) = \frac{\lambda\,\mathrm{csch}(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{csch}(\sqrt{\lambda}\zeta)+1}, \quad \sigma_2(\zeta) = \frac{\sqrt{\lambda}\coth(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{csch}(\sqrt{\lambda}\zeta)+1}. \quad (13)$$

Case 3: If $\varepsilon = 1, c = -1$ and $\lambda > 0$,we get,

$$\kappa_3(\zeta) = \frac{\lambda\sec(\sqrt{\lambda}\zeta)}{\chi\sec(\sqrt{\lambda}\zeta)+1}, \quad \sigma_3(\zeta) = \frac{\sqrt{\lambda}\tan(\sqrt{\lambda}\zeta)}{\chi\sec(\sqrt{\lambda}\zeta)+1},$$
$$\kappa_4(\zeta) = \frac{\lambda\csc(\sqrt{\lambda}\zeta)}{\chi\csc(\sqrt{\lambda}\zeta)+1}, \quad \sigma_4(\zeta) = -\frac{\sqrt{\lambda}\cot(\sqrt{\lambda}\zeta)}{\chi\csc(\sqrt{\lambda}\zeta)+1}. \quad (14)$$

Case 4: If $\lambda = \chi = 0$,

$$\kappa_5(\zeta) = \frac{C}{\zeta}, \quad \sigma_5(\zeta) = \frac{1}{\varepsilon\zeta}. \quad (15)$$

Herein, C is a nonzero constant. Insert Eq. (6) and its derivatives to Eq. (5) and taking into account Eqs. (7)-(9), we acquire a polynomial consisting of $\kappa^k(\zeta)\sigma^l(\zeta), (k,l = 0,1.2.3,...,M)$. If we collect the coefficients of $\kappa^k(\zeta)\sigma^l(\zeta)$ involving the same power and equal each coefficient to zero, we gain a system of algebraic equations which consist of $A_0, A_k, B_k, \chi, \lambda, \rho_1, \rho_2, \rho_3, \rho_4$, and ρ_5. Solving this system, then inserting these parameter values into Eq. (5), afterwards utilizing the solutions Eqs.(12)-(15) and Eq. (3), we achieve the solutions to Eq. (2).

4.2. Implementation of GPREM to the fractional $(4+1)$-dimensional FE

Considering Eq. (5) and applying the balance rule, we get $M = 2$. Eq. (6) is rewritten in the following structure:

$$V(\zeta) = A_0 + A_1\kappa(\zeta) + B_1\sigma(\zeta) + A_2\kappa(\zeta) + B_2\sigma(\zeta), \quad (16)$$

in which A_0, A_1, A_2, B_1, and B_2 are constants. Inserting Eq. (16) into Eq.(5) taking into account Eqs. (7)-(9), we get a system of algebraic equations. Considering the coefficients of $\kappa^k(\zeta)\sigma^l(\zeta)$ as zero, then solving the system, we derive the solution functions as:

Case 1: If $\varepsilon = -1, c = -1$ and $\lambda > 0$, we get the following results:

Result$_1$:

$$\left\{\begin{array}{l} \lambda = \frac{2(2\rho_1\rho_5 - 3\rho_3\rho_4)}{(\rho_1{}^2 - \rho_2{}^2)\rho_1\rho_2},\ A_0 = 0, A_1 = -\frac{1}{4}\chi\left(\rho_1{}^2 - \rho_2{}^2\right), \\ A_2 = \frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5 - 24\rho_3\rho_4}, B_1 = 0, \\ B_2 = -\frac{\left(\rho_1{}^2-\rho_2{}^2\right)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)(\rho_1{}^2-\rho_2{}^2)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4}, \end{array}\right\} \quad (17)$$

Result$_2$:

$$\left\{\begin{array}{l} \lambda = \frac{2(2\rho_1\rho_5 - 3\rho_3\rho_4)}{(\rho_1{}^2 - \rho_2{}^2)\rho_1\rho_2},\ A_0 = 0, A_1 = -\frac{1}{4}\chi\left(\rho_1{}^2 - \rho_2{}^2\right), \\ A_2 = \frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5 - 24\rho_3\rho_4}, B_1 = 0, \\ B_2 = \frac{\left(\rho_1{}^2-\rho_2{}^2\right)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)(\rho_1{}^2-\rho_2{}^2)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4}, \end{array}\right\} \quad (18)$$

and subsequently, we get the following solution functions:

$$\vartheta_{1,1}(x,y,z,s,t) = \left(\frac{(3\chi\rho_3\rho_4 - 2\chi\rho_1\rho_5)cosh(\sqrt{\lambda}\zeta)}{2\rho_1\rho_2\left(\chi + cosh(\sqrt{\lambda}\zeta)\right)}\right) - \left(\frac{\sqrt{\lambda}\omega_1\sinh\left(\frac{1}{\alpha}(\sqrt{\lambda}\zeta)\right) - 3\rho_3\rho_4 + 2\rho_1\rho_5}{2\rho_1\rho_2\left(\chi + cosh(\sqrt{\lambda}\zeta)\right)^2}\right), \quad (19)$$

$$\vartheta_{1,2}(x,y,z,s,t) = \left(\frac{(3\chi\rho_3\rho_4 - 2\chi\rho_1\rho_5)cosh(\sqrt{\lambda}\zeta)}{2\rho_1\rho_2\left(\chi + cosh(\sqrt{\lambda}\zeta)\right)}\right) + \left(\frac{\sqrt{\lambda}\omega_1\sinh\left(\frac{1}{\alpha}(\sqrt{\lambda}\zeta)\right) + 3\rho_3\rho_4 - 2\rho_1\rho_5}{2\rho_1\rho_2\left(\chi + \left(cosh(\sqrt{2}\lambda\zeta)\right)\right)^2}\right), \quad (20)$$

in which

$$\omega_1 = \sqrt{\rho_1\rho_2\left(\rho_1{}^2 - \rho_2{}^2\right)\left(\chi^2 - 1\right)\frac{(2\rho_1\rho_5 - 3\rho_3\rho_4)}{2}}$$

and

$$\zeta = \frac{\Gamma(1+\gamma)\left(\rho_1 x^\alpha + \rho_2 y^\alpha + \rho_3 z^\alpha + \rho_4 s^\alpha + \rho_5 t^\alpha\right)}{\alpha}.$$

Case 2: If $\varepsilon = -1, c = 1$ and $\lambda > 0$,we get the following results:

Result$_3$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{({\rho_1}^2-{\rho_2}^2)\rho_1\rho_2},\ A_0=0, A_1=-\frac{1}{4}\chi\left({\rho_1}^2-{\rho_2}^2\right),\\ A_2=\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2+1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=-\frac{\left({\rho_1}^2-{\rho_2}^2\right)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)({\rho_1}^2-{\rho_2}^2)(\chi^2+1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{21}$$

Result$_4$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{({\rho_1}^2-{\rho_2}^2)\rho_1\rho_2},\ A_0=0, A_1=-\frac{1}{4}\chi\left({\rho_1}^2-{\rho_2}^2\right),\\ A_2=\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2+1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=\frac{\left({\rho_1}^2-{\rho_2}^2\right)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)({\rho_1}^2-{\rho_2}^2)(\chi^2+1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{22}$$

and subsequently, we get the following solution functions:

$$\vartheta_{2,1}(x,y,z,s,t)=\left(\frac{-\sqrt{\lambda}\omega_2 cosh\left(\sqrt{\lambda}\zeta\right)+(3\rho_3\rho_4-2\rho_1\rho_5)\left(\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}{2\rho_1\rho_2\left(\left(cosh\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+2\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}\right), \tag{23}$$

$$\vartheta_{2,2}(x,y,z,s,t)=\left(\frac{\sqrt{\lambda}\omega_2 cosh\left(\sqrt{\lambda}\zeta\right)+(3\rho_3\rho_4-2\rho_1\rho_5)\left(\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}{2\rho_1\rho_2\left(\left(cosh\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+2\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}\right), \tag{24}$$

in which

$$\omega_2=-\sqrt{\rho_1\rho_2\left({\rho_1}^2-{\rho_2}^2\right)(\chi^2+1)\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{2}}$$

and

$$\zeta=\frac{\Gamma(1+\gamma)\left(\rho_1x^\alpha+\rho_2y^\alpha+\rho_3z^\alpha+\rho_4s^\alpha+\rho_5t^\alpha\right)}{\alpha}.$$

Case 3: If $\varepsilon=1, c=-1$ and $\lambda>0$,we get the following results:

Result$_5$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{\left({\rho_1}^2-{\rho_2}^2\right)\rho_1\rho_2},\ A_0=-\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{6\rho_1\rho_2},\\ A_1=\frac{1}{4}\chi\left({\rho_1}^2-{\rho_2}^2\right),\\ A_2=-\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=-\frac{\left({\rho_1}^2-{\rho_2}^2\right)\sqrt{2}\sqrt{-(2\rho_1\rho_5-3\rho_3\rho_4)\left({\rho_1}^2-{\rho_2}^2\right)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{25}$$

Result$_6$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{\left({\rho_1}^2-{\rho_2}^2\right)\rho_1\rho_2},\ A_0=-\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{6\rho_1\rho_2},\\ A_1=\frac{1}{4}\chi\left({\rho_1}^2-{\rho_2}^2\right),\\ A_2=-\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=\frac{\left({\rho_1}^2-{\rho_2}^2\right)\sqrt{2}\sqrt{-(2\rho_1\rho_5-3\rho_3\rho_4)\left({\rho_1}^2-{\rho_2}^2\right)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{26}$$

and subsequently, we get the following solution function

$$\vartheta_{3,1}(x,y,z,s,t)=\left(\frac{3\sqrt{\lambda}\sqrt{-\omega_1}\sin\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2-\chi\cos\left(\sqrt{\lambda}\zeta\right)+\chi^2-3\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right), \tag{27}$$

$$\vartheta_{3,2}(x,y,z,s,t)=\left(\frac{-3\sqrt{\lambda}\sqrt{-\omega_1}\sin\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2-\chi\cos\left(\sqrt{\lambda}\zeta\right)+\chi^2-3\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right), \tag{28}$$

$$\vartheta_{4,1}(x,y,z,s,t)=\left(\frac{3\sqrt{\lambda}\sqrt{-\omega_1}\cos\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi\sin\left(\sqrt{\lambda}\zeta\right)-\chi^2+2\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(2\chi\sin\left(\sqrt{\lambda}\zeta\right)-2\left(cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+1\right)}\right), \tag{29}$$

$$\vartheta_{4,2}(x,y,z,s,t)=\left(\frac{-3\sqrt{\lambda}\sqrt{-\omega_1}\cos\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi\sin\left(\sqrt{\lambda}\zeta\right)-\chi^2+2\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(2\chi\sin\left(\sqrt{\lambda}\zeta\right)-2\left(cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+1\right)}\right), \tag{30}$$

in which

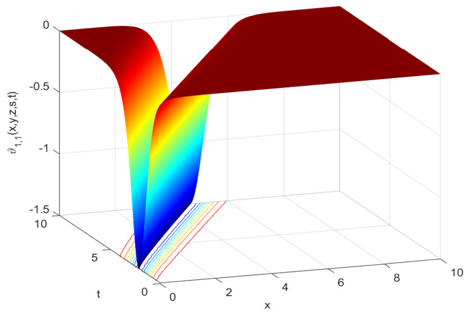
$$\omega_1=\sqrt{\rho_1\rho_2\left({\rho_1}^2-{\rho_2}^2\right)(\chi^2-1)\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{2}}$$

and $\zeta=\frac{\Gamma(1+\gamma)(\rho_1x^\alpha+\rho_2y^\alpha+\rho_3z^\alpha+\rho_4s^\alpha+\rho_5t^\alpha)}{\alpha}$.

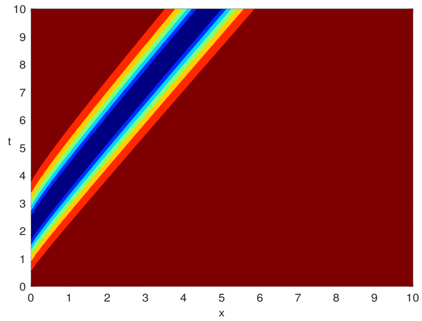
5. Results and discussion

In this section, we present some graphical portraits and physical interpretations of the resulted solutions. For appropriate variables of unknown parameters, we depict various graphs with 3D, 2D and contour plots. We get the dark and singular soliton solutions for the model. Figs. (1-2) show some of the obtained solutions. We display 3D and contour graphs of $\vartheta_{1,1}(x,y,z,s,t)$ in Eq. (19) for the parameters $\rho_1=3$, $\rho_2=-1$, $\rho_3=2, \rho_4=1$, $\rho_5=-2$,$\gamma=0.5$, y = 1, z = 1, $s=1, \chi=5$, and α= 0.8. Fig.1-(a) and fig.1-(b) show the dark soliton. Fig. 1-(c) also demonstrates 2D soliton profile for $t=5,7,9$. It can be seen that the amplitude and the shape of the dark soliton remain same. Moreover, as t increases, soliton goes to the right. Fig. 1-(d) is the 2D graphical portrait to demonstrate the effect of the α when α takes the values as 0.7, 0.8, 0.9 and 1.0, respectively. Soliton keeps its dark soliton view but if we pay attention to the peaks of the soliton, as if soliton moves to the right. Thus, we can say that this situation is not the various structures of the soliton resting on the fractional orders.

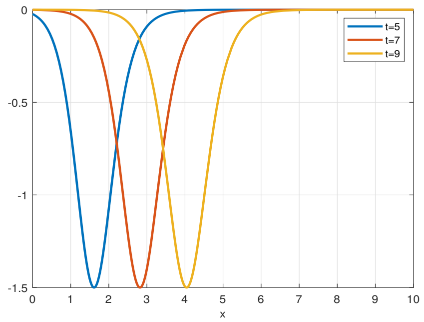
Fig.2-(a) and Fig.2-(b) are 3D and contour graphs of $\vartheta_{2,1}(x,y,z,s,t)$ given in Eq.(23) and these graphs demonstrate the singular soliton for the parameters $\rho_1=3$,$\rho_2=-1$,$\rho_3=3$,$\rho_4=1$,$\rho_5=-3$,$\gamma=0.5$, $y=1, z=1$, s = 1, $\chi=-5$, and α= 1. Fig. 2-(c) also demonstrates 2D soliton profile for $t=3,5,7$. It can be seen that the shape of the singular soliton remains the same. Moreover, while t decrases, soliton moves to the left. Fig. 2-(d) is the 2D graphical portrait to demonstrate the



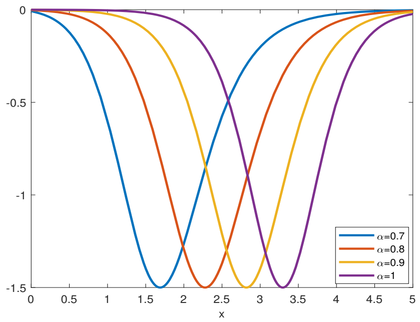
(a) 3D portrait of $\vartheta_{1,1}(x, 1, 1, 1, t)$



(b) Contour plot of $\vartheta_{1,1}(x, 1, 1, 1, t)$

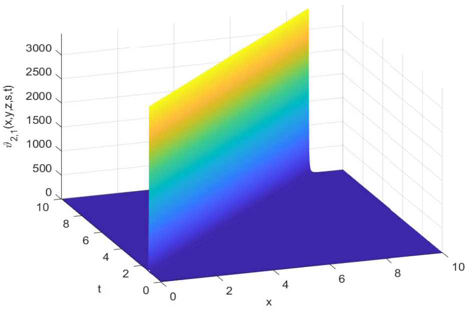


(c) 2D portrait of $\vartheta_{1,1}(x, 1, 1, 1, t)$

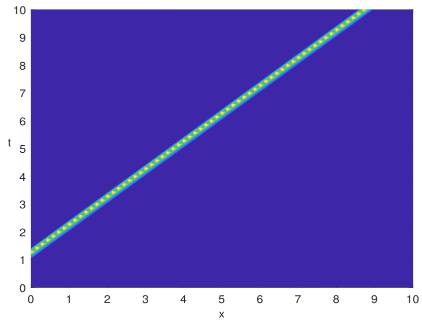


(d) 2D portraits of $\vartheta_{1,1}(x, 1, 1, 1, 7)$

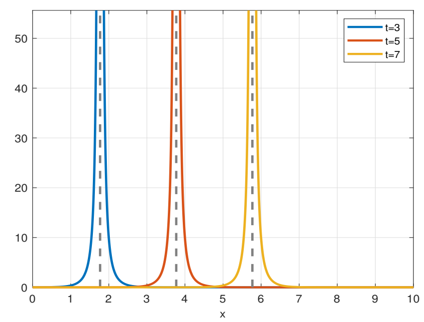
Figure 1. The dark soliton portraits of $\vartheta_{1,1}(x, 1, 1, 1, t)$ in Eq. (19) for the parameters $\rho_1 = 3, \rho_2 = -1, \rho_3 = 3, \rho_4 = 2, \rho_5 = -3, \gamma = 0.5$, $y = 1, z = 1$, s $= 1$, $\chi = 5$, and $\alpha = 0.8$.



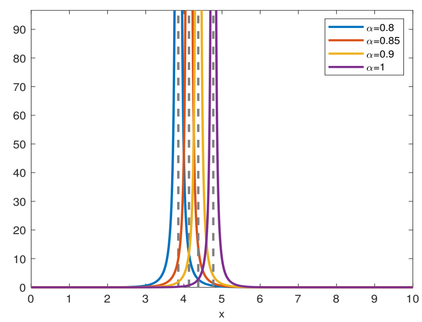
(a) 3D portrait of $\vartheta_{2,1}(x, 1, 1, 1, t)$



(b) Contour plot of $\vartheta_{2,1}(x, 1, 1, 1, t)$



(c) 2D portrait of $\vartheta_{2,1}(x, 1, 1, 1, t)$



(d) 2D portraits of $\vartheta_{2,1}(x, 1, 1, 1, 7)$ at $t = 7$

Figure 2. The singular soliton portraits of $\vartheta_{2,1}(x, 1, 1, 1, t)$ in Eq. (23) for the parameters $\rho_1 = 3, \rho_2 = -1, \rho_3 = 3, \rho_4 = 2, \rho_5 = -3, \gamma = 0.5, y = 1, z = 1, s = 1, \chi = -5$, and $\alpha = 1$.

effect of the α when α takes the values as 0.8, 0.85, 0.9 and 1.0, respectively. Thus, we can say that this situation is not the different structures of the soliton resting on the fractional orders.

6. Conclusion

In this study, for the first time, the generalized projective Riccati equations method was efficaciously employed to scrutinize analytical solutions for the fractional (4+1)-dimensional Fokas equation with M-truncated derivative. Some analytical solutions and singular and dark soliton solutions are acquired. 3D, contour, and 2D graphs were added to exhibit the physical illustrations of some resulted solutions. The GPREM can be successfully implemented to the different fractional forms of (4+1)-dimensional Fokas equation. Hence, the results show that the GPREM is a very effectual and profitable tool for solving such higher order NLPDEs occurring in region associated with physics, chemical, biology, and mathematics along with engineering.

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RESEARCH ARTICLE

A new approach on approximate controllability of Sobolev-type Hilfer fractional differential equations

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ABSTRACT

The approximate controllability of Sobolev-type Hilfer fractional control differential systems is the main emphasis of this paper. We use fractional calculus, Gronwall's inequality, semigroup theory, and the Cauchy sequence to examine the main results for the proposed system. The application of well-known fixed point theorem methodologies is avoided in this paper. Finally, a fractional heat equation is discussed as an example.

1. Introduction

Differential systems of fractional order are found to be useful models for a variety of physical, biological, and engineering challenges. As a result, they have gotten greater attention from researchers in the last two decades. Fractional derivatives are a stronger tool for illustrating memory and hereditary features. As a result, they've found widespread use in physics, electrodynamics, economics, aerodynamics, control theory, viscoelasticity, and heat conduction. In recent years, significant advances in the theory and applications of fractional systems have been made, one can review the books [1–4]. The notation of exact and approximate controllability is useful in analysis and design control frameworks. In [5] authors studied the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness using fixed point theory approach. In [6–13] Anurag et al. studied the controllability of semilinear deterministic and stochastic systems of integral and fractional order with several important extensions using different approaches. The numerical model of numerous physical phenomena, such as the movement of liquid through split rocks, thermodynamics, and so on, is usually Sobolev-type. (see [14–17]).

Another type of fractional order derivative introduced by Hilfer [18] is the Caputo fractional and Riemann-Liouville derivative. Several authors have focused on the Hilfer fractional derivative including [19–27] for the existence and controllability of deterministic and stochastic fractional order systems. Many academics have recently considered the exact and approximate controllability of

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systems characterized by impulsive functional inclusions, integro-differential equations, semilinear functional equations, neutral functional differential equations, and evolution inclusions, to name a few examples, see [23, 24, 27] and references in that. In [28–34] Ravi et. al. studied the existence, uniqueness, controllability, and optimal control of fractional differential control systems and their real-life mathematical applications using various types of approaches.

Consider the following Sobolev-type Hilfer fractional control system as below.

$$yD^{\wp,\varpi}_{0+}[Lz(\sigma)] = Az(\sigma) + Bv(\sigma) + F(\sigma, z(\sigma), v(\sigma)),\ \sigma \in J = (0, c], \tag{1}$$

$$J^{(1-\wp)(1-\varpi)}_{0+} z(0) = z_0, \tag{2}$$

$D^{\wp,\varpi}_{0+}$is the Hilfer fractional derivative, $0 \leq \wp \leq 1$; $\frac{1}{2} < \varpi < 1$;is the Banach Space X with $\|\cdot\|$, and $z(\cdot)$ is the Banach Space X with $\|\cdot\|$. The non densely defined closed linear operator $A : D(A) \subseteq X \to X$ yields an integrated semigroup $\{T(t)\}_{t\geq 0}$ in Banach Space X with $\|\cdot\|$. The function $F : J \times X \times U \to X$ is a purely nonlinear function and $B : X \to U$ is a bounded linear operator.

This article makes the following major contributions:

- Using two separate situations, investigate the sufficient conditions for the approximate controllability of the suggested systems (1)-(2).
- In the first case, we assume that $B = I$ (where I is an identity operator) and in the second case, we assume that $B \neq I$.
- controllability results are achieved using Gronwall's inequality and the Cauchy sequence.
- Results are obtained with weaker conditions (Lipschitz) on nonlinearity and can be extended for the delay differential equations.
- The suggested method is simple in terms of hefty estimations as compared to standard ways such as the fixed point theory approach.
- The results are advanced and weighted enough as contribution in control differential equations.

We have divided this paper into the following sections: Section (1) provides a review of some essential concepts and preparatory outcomes Section (2). The main discussion of our manuscript is presented in Section (3). Finally, in Section (4), an application for drawing the theory of the primary outcomes is discussed.

2. Preliminaries

Let the spaces of all continuous functions is denoted by $C(J, X)$. Take $\eta = \wp + \varpi - \wp\varpi$, then $(1 - \eta) = (1 - \wp)(1 - \varpi)$. We now define $C_{1-\eta}(J, X) = \{z : \sigma^{1-\eta}z(\sigma) \in C(J, X)\}$ along $\|\cdot\|_\eta$ by $\|z\|_C = \sup\{\sigma^{1-\eta}\|z(\sigma)\|, \sigma \in J, \eta = (\wp + \varpi - \wp\varpi)\}$. It is clear that $C_{1-\eta}(J, X)$ is a Banach space.

The linear operators $A : D(A) \subset X \to X$ and $L : D(A) \subset X \to X$ satisfies the properties discussed in $A : D(A) \subset X \to X$. [17]:

- **(P₁)** A and L are closed linear operators.
- **(P₂)** $D(L) \subset D(A)$ and L is bijective.
- **(P₃)** $L^{-1} : X \to D(L)$ is continuous.

Additionally, because $(\mathbf{P_1})$ and $(\mathbf{P_2})$, L^{-1} is closed, by $(\mathbf{P_3})$ and from closed graph theorem, we have the boundedness of $AL^{-1} : X \to X$. Define $||L^{-1}|| = \widetilde{L}_1$ and $\|L\| = \widetilde{L}_2$.

Introducing acquaint essential facts relevant to fractional theory. (The readers can check [18,35]).

Definition 1. *[3] "The left sided Riemann-Liouville fractional integral of order ϖ having lower limit c for $F : [c, +\infty) \to \mathbb{R}$ is presented as*

$$J^{\varpi}_{c^+} F(\varrho) = \frac{1}{\Gamma(\varpi)} \int_c^{\varrho} \frac{F(\tau)}{(\varrho - \tau)^{1-\varpi}} d\tau,\ \varrho > c;\ \varpi > 0,$$

if the right side is pointwise determined on $[c, +\infty)$, where $\Gamma(\cdot)$ denotes gamma function."

Definition 2. *[3] "The left-sided Riemann-Liouville fractional derivative of order $\varpi \in [k-1, k)$, $k \in X$ for $F : [c, +\infty) \to \mathbb{R}$ is given by*

$${}^L D^{\varpi}{}_{c^+}\ F(\varrho) = \frac{1}{\Gamma(k - \varpi)} \frac{d^k}{d\varrho^k} \int_c^{\varrho} \frac{F(\tau)}{(\varrho - \tau)^{\varpi+1-k}} d\tau, \quad \varrho > c,\ k - 1 < \varpi < k."$$

Definition 3. *[3] "The left-sided Hilfer fractional derivative of order $0 \leq \wp \leq 1$ and $0 < \varpi < 1$ function of $F(\varrho)$ is given by*

$$D^{\wp,\varpi}_{c^+} F(\varrho) = (J^{\wp(1-\varpi)}_{c^+} D(J^{(1-\wp)(1-\varpi)}_{d^+} F))(\varrho)."$$

Remark 1. *[3] "*

(i) *Given $\varpi = 0$, $0 < \wp < 1$ also $c = 0$, the Hilfer fractional derivative identical with standard Riemann-Liouville fractional derivative:*

$$D^{0,\varpi}_{0^+} F(\varrho) = \frac{d}{d\varrho} J^{1-\varpi}_{0^+}\ F(\varrho) = {}^L D^{\varpi}{}_{0^+}\ F(\varrho).$$

(ii) *Given $\varpi = 1$, $0 < \wp < 1$ also $c = 0$, the Hilfer fractional derivative identical with standard Caputo derivative:*

$$D_{0^+}^{1,\varpi} F(\varrho) = J_{0^+}^{1-\varpi} \frac{d}{d\varrho} F(\varrho) = {}^c D^{\varpi}{}_{0^+} \; F(\varrho)."$$

Remark 2. *We show the mild solution of* (1)-(2)*in the following way using the Wright function $M_\varpi(s)$.*

$$M_\varpi(s) = \sum_{k=1}^{\infty} \frac{(-s)^{k-1}}{(k-1)!\Gamma(1-k\varpi)}, \; 0 < \varpi < 1, \; s \in C,$$

and satisfies

$$\int_0^\infty s^\zeta M_\varpi(s) ds = \frac{\Gamma(1+\zeta)}{\Gamma(1+\varpi\zeta)}, \; \textit{for } s \geq 0.$$

Lemma 1. *There exists $F : J \times X \times U \to X$ such that the system* (1)-(2)*is satisfied.*

$$\begin{aligned} z(\sigma) &= L^{-1} \mathscr{P}_{\wp,\varpi}(\sigma) L z_0 \\ &+ \int_0^\sigma L^{-1} \mathscr{R}_\varpi(\sigma) F(\zeta, z(\zeta), v(\zeta)) d\zeta \\ &+ \int_0^\sigma L^{-1} \mathscr{R}_\varpi(\sigma) B v(\zeta) d\zeta, \quad \sigma \in J, \end{aligned}$$

where

$$\mathscr{P}_{\wp,\varpi}(\sigma) = J_{0^+}^{v(1-\varpi)}(\sigma)^{\varpi-1} \mathscr{S}_\varpi(\sigma);$$
$$\mathscr{R}_\varpi(\sigma) = \sigma^{\varpi-1} \mathscr{S}_\varpi(\sigma);$$
$$\mathscr{S}_\varpi(\sigma) = \int_0^\infty \varpi\omega M_\varpi(\omega) S(\sigma^\varpi \omega) d\omega.$$

Definition 4. *([36]) A function $z : [0, c] \to X$ is called the mild solution of* (1)-(2) *provided that $z(0) = z_0 \in X$ and fulfills*

$$\begin{aligned} z(\sigma) &= L^{-1} \mathscr{P}_{\wp,\varpi}(\sigma) L z_0 \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1} \mathscr{S}_\varpi(\sigma-\zeta) B v(\zeta) d\zeta \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1} \mathscr{S}_\varpi(\sigma-\zeta) \\ &\quad F(\zeta, z(\zeta), v(\zeta)) d\zeta, \; \sigma \in J, \end{aligned} \tag{3}$$

where

$$\mathscr{P}_{\wp,\varpi}(\sigma) = \int_0^\infty \xi_\varpi(\omega) M(\sigma^\varpi \omega) d\omega,$$
$$\mathscr{S}_\varpi = \varpi \int_0^\infty \omega \xi_\varpi(\omega) M(\sigma^\varpi \omega) d\omega,$$

and for $\omega \in (0, \infty)$

$$\xi_\varpi(\omega) = \frac{1}{\varpi} \omega^{-1-\frac{1}{\varpi}} \overline{z}_\varpi(\omega^{-\frac{1}{\varpi}}) \geq 0,$$

$$\overline{z}_\varpi(\omega) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \sigma^{-n\varpi-1} \frac{\Gamma(n\varpi+1)}{n!} sin(n\pi\varpi),$$

where ξ_ϖ is a probability density function defined on $(0, \infty)$, i.e,

$$\xi_\varpi(\omega) \geq 0, \quad \omega \in (0, \infty) \quad \textit{also} \quad \int_0^\infty \xi_\varpi(\omega) d\omega = 1.$$

Lemma 2. *([36]) "The operators $\mathscr{P}_{\wp,\varpi}$ and $\mathscr{S}_\varpi$ fulfills:*

(i) *For $\sigma \geq 0$, $\mathscr{P}_{\wp,\varpi}$ and $\mathscr{S}_\varpi$ are linear and bounded, that is, for every $z \in X$, $\|\mathscr{P}_{\wp,\varpi}(\sigma) z\| \leq \frac{M\sigma^{\eta-1}}{\Gamma(\wp(1-\varpi)+\varpi)} \|z\|$ and $\|\mathscr{S}_\varpi(\sigma) z\| \leq \frac{M}{\Gamma(\varpi)} \|z\|$, where $\mathscr{P}_{\wp,\varpi}(\sigma) = J_{0^+}^{\wp(1-\varpi)} \mathscr{R}_\varpi(\sigma)$, $\mathscr{R}_\varpi(\sigma) = \sigma^{\varpi-1} \mathscr{S}_\varpi(\sigma)$.*

(ii) *The operators $\{\mathscr{P}_{\wp,\varpi}(\sigma)\}_{\sigma \geq 0}$ and $\{\mathscr{S}_\varpi(\sigma)\}_{\geq 0}$ are strongly continuous.*

(iii) *For every $z \in X$, $\mu, \varpi \in (0, 1]$, one can get*

$$A \mathscr{S}_\varpi(\sigma) z = A^{1-\mu} \mathscr{S}_\varpi(\sigma) A^\mu z, \; 0 \leq \sigma \leq c;$$

$$\|A^\mu \mathscr{S}_\varpi(\sigma)\| \leq \frac{\varpi C_\mu \Gamma(2-\mu)}{\sigma^{\varpi\mu} \Gamma(1+\varpi(1-\mu))}, \; 0 < \sigma \leq c."$$

Definition 5. *[6] "The reachable set of (1)-(2) is given by*
$K_c(F) = \{z(c) \in X : z(\sigma)$ represents mild solution of (1)-(2)$\}$.
In case $F \equiv 0$, then the system (1)-(2) reduces to the corresponding linear system. The reachable set in this case is denoted by $K_c(0)$."

Definition 6. *[6] "If $\overline{K_c(F)} = X$, then the semilinear control system is approximately controllable on $[0, c]$. Here $\overline{K_c(F)}$ represents the closure of $K_c(F)$. It is clear that, if $\overline{K_c(0)} = X$, then linear system is approximately controllable."*

3. Controllability results

3.1. Controllability of semilinear system: when $B = I$

The linear system has an approximate controllability is proven to reach from the semilinear system under specified nonlinear term constraints in this study. Clearly, $X = U$.

Let us consider the subsequent linear system

$$D_{0^+}^{\wp,\varpi}[Lw(\sigma)] = Aw(\sigma) + u(\sigma), \; \sigma \in J = (0, c], \tag{4}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} w(0) = z_0, \tag{5}$$

and the semilinear system

$$\begin{aligned} D_{0^+}^{\wp,\varpi}[Lz(\sigma)] &= Az(\sigma) + v(\sigma) \\ &+ F(\sigma, z(\sigma), v(\sigma)), \; \sigma \in J, \end{aligned} \tag{6}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} z(0) = z_0, \tag{7}$$

We need to present the following assumptions to prove the primary aim of this section, which is the approximate controllability of (6)-(7):

Assumption 1. *The linear system (4)-(5) is approximately controllable.*

Assumption 2. *$F(\sigma, z(\sigma), v(\sigma))$ is a nonlinear function that, in z and v, satisfies the Lipschitz condition.*

$\|F(\sigma, z, v) - F(\sigma, w, u)\| \leq l(\|z - w\| + \|v - u\|)$,

where $l > 0$, $\forall\, z, w \in X$, $\sigma \in [0, c]$.

Theorem 1. *Under the assumptions (1)-(2), the system (6)-(7) is approximately controllable provided that $l < 1$.*

Proof. Assume that $w(\sigma)$, along with the control u, is the mild solution of (4)-(5). Assume that the semilinear system of the following kind:

$$D_{0+}^{\wp,\varpi} z(\sigma) = Az(\sigma) + F(\sigma, z(\sigma), v(\sigma)) + u(\sigma) - F(\sigma, w(\sigma), v(\sigma)), \quad (8)$$

$$J_{0+}^{(1-\wp)(1-\varpi)} z(0) = z_0, \quad (9)$$

Compare (6)-(7) and (8)-(9), the control function $v(\sigma)$ is chosen in such a way that

$$v(\sigma) = u(\sigma) - F(\sigma, w(\sigma), v(\sigma)). \quad (10)$$

We consider for the given $u(\sigma)$ and $w(\sigma)$, there exists $v(\varrho)$ fulfilling (10) (We need to verify the existence and uniqueness of v).

The mild solution of (4)-(5) is given by

$$w(\sigma) = L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 + \int_0^\sigma (\sigma - \zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma - \zeta)u(\zeta)d\zeta \quad (11)$$

and for the system (8)-(9) is given by

$$\begin{aligned} z(\sigma) &= L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 \\ &+ \int_0^\sigma (\sigma - \zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma - \zeta)F(\zeta, z(\zeta), v(\zeta))d\zeta \\ &+ \int_0^\sigma (\sigma - \zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma - \zeta)u(\zeta)d\zeta \\ &- \int_0^\sigma (\sigma - \zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma - \zeta)F(\zeta, w(\zeta), v(\zeta))d\zeta \end{aligned} \quad (12)$$

From (11) and (12), we get

$$w(\sigma) - z(\sigma) = \int_0^\sigma (\sigma - \zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma - \zeta) \{F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\}d\zeta \quad (13)$$

Applying norm on both sides, one can get

$$\begin{aligned} &\|w(\sigma) - z(\sigma)\|_X \\ &\leq \int_0^\sigma (\sigma - \zeta)^{\varpi-1} \|L^{-1}\mathscr{S}_\varpi(\sigma - \zeta)\| \\ &\|F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\|d\zeta \\ &\leq \frac{M\widetilde{L}_1}{\Gamma(\varpi)} \int_0^\sigma (\sigma - \zeta)^{\varpi-1} \\ &\|F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\|d\zeta \end{aligned} \quad (14)$$

Using assumption (2), we get

$$\|w(\sigma) - z(\sigma)\|_X \leq \frac{M\widetilde{L}_1 l}{\Gamma(\varpi)} \int_0^\sigma (\sigma - \zeta)^{\varpi-1} \|w(\zeta) - z(\zeta)\|d\zeta$$

By referring the Gronwall's inequality, $w(\sigma) = z(\sigma)$, $\forall\ \sigma \in [0, c]$. As a result, the linear system's solution w along the control u is a semilinear system's solution z along the control v, i.e., $K_c(F) \supset K_c(0)$. Because $K_c(0)$ is dense in X (according to assumption 1), $K_c(F)$ is dense in X as well, implying that system (6)-(7) is approximate controllable. The proof is finished.

We need to verify that there exists a $v(\sigma) \in X$ such that $v(\sigma) = u(\sigma) - F(\sigma, w(\sigma), v(\sigma))$, $\forall\ \sigma \in [0, c]$.

Assume that $v_0 \in X$ and $v_{n+1} = u - F(\sigma, w(\sigma), v_n)$: $n = 0, 1, 2, ...$ Thus, one can get

$$v_{n+1} - v_n = F(\sigma, w(\sigma), v_{n-1}) - F(\sigma, w(\sigma), v_n).$$

Hence, by referring assumption (2),

$$\|v_{n+1} - v_n\|_X = l\|v_n - v_{n-1}\|_X \leq l^n\|v_1 - v_0\|_X. \quad (15)$$

When $n \to \infty$ (since $l < 1$), the R.H.S of (15)goes to zero. As a result, $\{v_n\}$ is a Cauchy sequence in X that converges to $v \in X$.

Next,

$$\begin{aligned} &\|u - v_{n+1} - F(\sigma, w(\sigma), v)\|_X = \\ &\quad \|F(\sigma, w(\sigma), v_n) - F(\sigma, w(\sigma), v)\|_X \\ &\quad \leq l\|v_n - v\|. \end{aligned} \quad (16)$$

Because, R.H.S of (16) approaches to zero when $n \to \infty$, one can obtain

$$F(\sigma, w(\sigma), v) = \lim_{n\to\infty} (u - v_{n+1}) = u - v \Rightarrow v = u - F(\sigma, w(\sigma), v).$$

Now, we will show that v is unique. For proving it let $v_1 = u - F(\sigma, w(\sigma), v_1)$ and $v_2 = u - F(\sigma, w(\sigma), v_2)$. Then using assumption (2), we get

$$\|v_2 - v_1\| = \|F(\sigma, w(\sigma), v_1) - F(\sigma, w(\sigma), v_2)\| \leq l\|v_2 - v_1\| \Rightarrow (1 - l)\|v_2 - v_1\| \leq 0.$$

But $0 < l < 1$ therefore $\|v_2 - v_1\| = 0 \Rightarrow v_2 = v_1$. Hence v is unique. □

3.2. Controllability of semilinear system: when $B \neq I$

The approximate controllability of the semilinear system under simple conditions B and F as indicated by assumptions (3)-(6). Let us consider the subsequent linear system

$$D_{0^+}^{\wp,\varpi}[Lw(\sigma)] = Aw(\sigma) + Bu(\sigma),\ \sigma \in J, \tag{17}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} w(0) = z_0, \tag{18}$$

and the semilinear system

$$D_{0^+}^{\wp,\varpi}[Lz(\sigma)] = Az(\sigma) + Bv(\sigma) + F(\sigma, z(\sigma), v(\sigma)),\ \sigma \in J = (0, c], \tag{19}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} z(0) = z_0, \tag{20}$$

We must make the following assumptions to prove the fundamental aim of this section, namely, the approximate controllability of (19)-(20):

Assumption 3. *The linear system (17)-(18) is approximately controllable.*

Assumption 4. *Assumption (2) is fulfilled.*

Assumption 5. $Range(F) \subseteq \overline{Range(B)}$.

Assumption 6. *There exists* $\xi > 0$ *such that* $\|Bv\| \geq \xi\|v\|,\ \forall\, v \in U$

Theorem 2. *Under the assumptions (3)-(6), the system (19)-(20) is approximately controllable, provided that* l *fulfills* $l < \xi$.

Proof. Assume that $w(\sigma)$ and the control u are the mild solution of (17)-(18). Assume that the semilinear system that follows is

$$D_{0^+}^{\wp,\varpi}[Lz(\sigma)] = Az(\sigma) + F(\sigma, z(\sigma), v(\sigma)) + Bu(\sigma) - F(\sigma, w(\sigma), v(\sigma)), \tag{21}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} z(0) = z_0, \tag{22}$$

In the above, the control function v in (21)-(22) fulfills $Bv(\sigma) = Bu(\sigma) - F(\sigma, w(\sigma), v(\sigma))$, and assumption (5), concludes that the considered equation is well defined. By employing assumption (6) and the way of approached followed in Theorem 2, we can easily prove that provided that $l < \xi$, $\exists\ v(\sigma) \in U$ such that $Bv(\sigma) = Bu(\sigma) - F(\sigma, w(\sigma), v(\sigma))$.

The mild solutions for (17)-(18) and (21)-(22) are given by

$$w(\sigma) = L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 + \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)Bu(\zeta)d\zeta \tag{23}$$

and

$$\begin{aligned} z(\sigma) &= L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)F(\zeta, z(\zeta), v(\zeta))d\zeta \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)Bu(\zeta)d\zeta \\ &- \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)F(\zeta, w(\zeta), v(\zeta))d\zeta \end{aligned} \tag{24}$$

From equations (23) and (24), one can get

$$w(\sigma) - z(\sigma) = \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta) \times \{F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\}d\zeta. \tag{25}$$

Equation (25) is the same when compared with (13). From Theorem 2, one can easily verify $w(\sigma) = z(\sigma)$, $\forall\ \sigma \in [0, c]$, i.e., the reachable set of (17)-(18) is dense in the reachable set of (19)-(20), which is dense in X, by referring assumption (3) and which concludes the proof. □

4. Example

Consider $U = L_2[0, \pi]$. Also, define the operator $B : D(B) \subset U \to U$ as

$$Bx = x'',\ \ x \in D(B),$$

$D(B) = \{x \in U :\ x,\ x'$ are absolutely continuous, $x" \in U,\ x(0) = x(\pi) = 0\}$. Assume that $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$, and $Lx = x - x''$ are the operators determined by $Ax = x''$ and $Lx = x - x''$, respectively, and that $D(A)$ and $D(L)$ are presented by

$$\{x \in X : x, x' \text{ are absolutely continuous, } x(0) = x(\pi) = 0\}.$$

Additionally, A and L are given by

$$Ax = \sum_{m=1}^{\infty} m^2 \langle x, u_m\rangle u_m,\ x \in D(A),$$

$$Lx = \sum_{m=1}^{\infty} (1 + m^2)\langle x, u_m\rangle u_m, x \in D(L),$$

where $u_m(y) = \sqrt{\frac{2}{\pi}}\sin(my)$, $m = 1, 2, 3, \cdots$ is the orthonormal of vectors of A. Additionally, for $z \in X$, we have

$$L^{-1}z = \sum_{m=1}^{\infty} \frac{1}{(1 + m^2)}\langle z, u_m\rangle u_m,$$

and

$$AL^{-1}z = \sum_{m=1}^{\infty} \frac{m^2}{(1+m^2)} \langle z, u_m \rangle u_m.$$

The operator B has eigen values $\lambda_m = -m^2 \ \ m \in \mathbb{N}$ and corresponding eigenfunction is given by u_n. Hence the spectral representation of B is presented as

$$Bz = \sum_{m=1}^{\infty} -m^2 \langle x, u_n \rangle u_n, \ \ x \in D(B).$$

Further, $\mathcal{S}(\varrho)$ which is a C_0-semigroup generated by B has e_n as the eigenfunctions corresponding to eigenvalues $\exp(\lambda_m t)$, that is

$$\mathcal{S}(\varrho)x = \sum_{m=1}^{\infty} \exp(-m^2\varrho) \langle x, u_n \rangle u_n, \ \ x \in U.$$

Define by

$$\widehat{U} = \left\{ v | v = \sum_{m=2}^{\infty} v_m u_m, \ with \ \sum_{m=2}^{\infty} v_m^2 < \infty \right\},$$

where $\widehat{U}$ is an infinite dimensional space with a norm of

$$\|v\|_{\widehat{U}} = \left(\sum_{m=2}^{\infty} v_m^2 \right)^{\frac{1}{2}}$$

Define $B : \widehat{U} \to U$ by

$$Bv = 2v_2 e_1 + \sum_{m=2}^{\infty} v_m u_m, \quad v = \sum_{m=2}^{\infty} v_m u_m \in \widehat{U}.$$

where B is a linear continuous map.

Assume that the Hilfer fractional semilinear control heat system is as follows:

$$D_{0^+}^{\wp,\varpi}\Big[z(\varrho,x) - \frac{\partial^2 z(\varrho,x)}{\partial z^2}\Big]$$
$$= \frac{\partial^2 z(\varrho,x)}{\partial z^2} + Bu(\varrho,x) + \gamma(\varrho, z(\varrho,x)); \quad 0 < \varrho \le \iota, \tag{26}$$

$$z(\varrho,0) = z(\varrho,\pi) = 0, \quad \varrho > 0;$$
$$J_{0^+}^{(1-\gamma)}(z(0,x)) = z_0(x), \quad 0 \le x \le \pi, \tag{27}$$

The Hilfer fractional derivative of order $\wp \in (0,1)$ and type $\varpi \in [0,1]$ is denoted by $D_{0^+}^{\alpha,\eta}$. If the assumptions (1)-(6) hold, the above system (26)-(27) is approximate controllable.

5. Conclusion

The focus of this study is on the Sobolev-type approximate controllability of Hilfer fractional semilinear control systems. The results were obtained using Gronwall's inequality, the Cauchy sequence, and the fixed point technique was avoided. With appropriate changes, these conclusions may be extended to include many types of delay for both deterministic and stochastic systems.

Remark 3. *One can replace the Lipschitz condition on the nonlinearity by monotonic nonlinearity or integral contractor type nonlinearity and obtained a different set of sufficient conditions for the approximate controllability of the proposed system.*

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Book

Author, A. (Year). Title of book. Publisher, Place of Publication.

Mercer, P.A., & Smith, G. (1993). Private Viewdata in the UK. 2nd ed. Longman, London.

Chapter

Author, A. (Year). Title of chapter. In: A. Editor and B. Editor, eds. Title of book. Publisher, Place of publication, pages.

Bantz, C.R. (1995). Social dimensions of software development. In: J.A. Anderson, ed. Annual review of software management and development. CA: Sage, Newbury Park, 502–510.

Internet document

Author, A. (Year). Title of document [online]. Source. Available from: URL [Accessed (date)].

Holland, M. (2004). Guide to citing Internet sources [online]. Poole, Bournemouth University. Available from: http://www.bournemouth.ac.uk/library/using/guide_to_citing_internet_sourc.html [Accessed 4 November 2004].

Newspaper article

Author, A. (or Title of Newspaper) (Year). Title of article. Title of Newspaper, day Month, page, column.

Independent (1992). Picking up the bills. Independent, 4 June, p. 28a.

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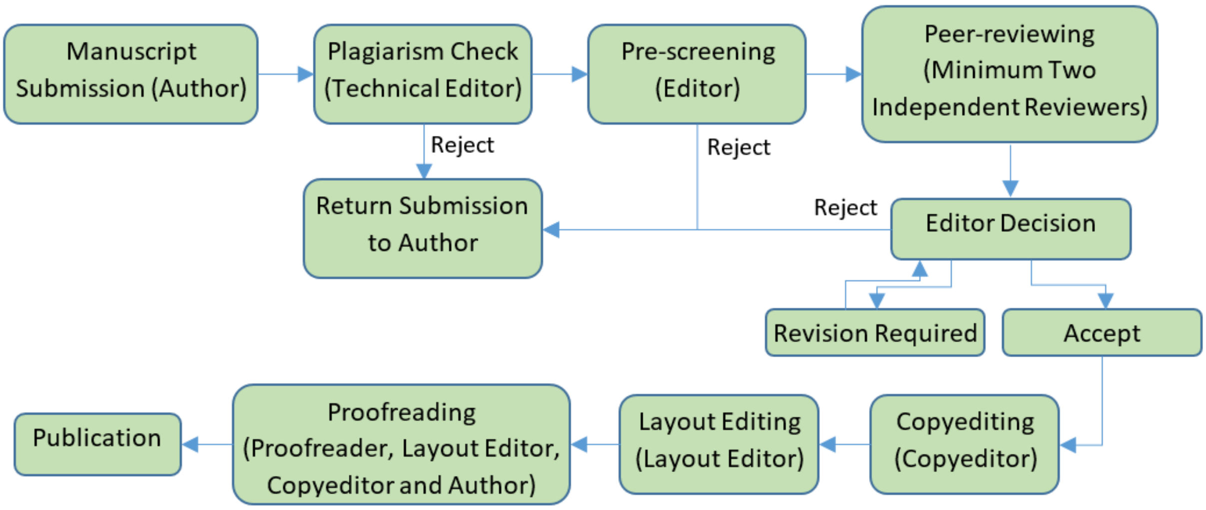

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