

RESEARCH ARTICLE

Dynamics of malaria-dengue fever and its optimal control

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ABSTRACT

The mosquito-borne infectious diseases like malaria and dengue are putative as important tropical infections and cause high morbidity and mortality around the world. In some cases, simultaneous coexistence of both the infections in one individual is seen which is very hard to distinguish as both diseases have almost similar symptoms. In this proposed article, dynamical system of non-linear differential equations is constructed with the help of mathematical modeling, which describe dynamics of the spread of these infectious diseases separately and concurrently. Basic reproduction number is evaluated to understand dynamical behaviour of the model. Local and global stability criteria have been deliberated rigorously. Control parameters are used to perceive effect of medication on these prevalent tropical diseases. Numerical simulations are used to observe effect of control parameters graphically.



1. Introduction

The present era witnesses the globalization of infectious diseases that occurs frequently by an unprecedented level. In this “globalized” environment of interdependent trade, travel, migration, and international economic markets, many factors now play an important role in the emergence and spread of infectious disease, which necessitates a coordinated, global response [17]. Mosquitoes are one of the deadliest insects in the world, with their ability to carry and spread disease to humans causes millions of deaths every year. Mosquito-borne infectious disease is accepted as one of the important tropical infections and is the focused topic in tropical medicine [23]. There are several tropical mosquito borne infections. Malaria and dengue are the two common mosquito infections that are easily spread and cause high morbidity and mortality for many patients around the world. Malaria is caused by Plasmodium parasites, which spreads through the bites of infected female Anopheles mosquitoes, called ‘malaria vectors’ [18]. Dengue is single positive-stranded RNA virus of the family Flaviviridae which is ingested by female mosquitoes (Aedes mosquito) during feeding [22]. The virus then infects the other mosquito and humans over its incubation period. Due to tremendous progress in malaria and dengue infection, the disease burden

remains high mostly in subtropical and tropical areas [21].

Presence of infection in the body results in weakness in immune system, it increases the probability that individual gets infected by another infections. Hence there is a possibility that both malaria and dengue infection can be present in the individual at the same time (e.g., [4], [6], [8], [13], [21], [24], [30] or [13]). This scenario is called concurrent malaria-dengue infection. This overlapping of two different infections can result in more severe situations where both diagnosis and treatment of a patient may become difficult [10]. Initially, two cases of concurrent malaria and dengue infection were identified in July, 2005 and November, 2006 [4]. Malaria and dengue fever represent 2 major public health concerns in South America, whose 92% of area is covered by Amazon rain forest. According to the report in a French territory in South America, 0.99% from overall febrile patients are infected by malaria and dengue concurrently [4]. Malaria vectors and dengue vectors are habited in the forest [20] and in the city [7] respectively. Hence, overlapping of the habitat cannot be easily available and therefore concurrent malaria dengue infection cases are less in number.

Mathematical models relevant to the concurrent infections helps the researchers, biologists and public

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health personnel to adopt improved and most effective strategies to control the diseases. Aldila D. and Agustin M. R. developed a nine-dimensional mathematical model to understand the spread of dengue and chikungunya in a closed population [1]. Isea R. & Lonngren K. E. presented two preliminary models that consist of the individual transmission dynamics of dengue, Chikungunya or Zika, and any possible co-infection between two diseases in the same population [12]. Sharomi et. al. developed a deterministic model which incorporates many of the essential biological and epidemiological features of HIV and tuberculosis and the synergistic interaction between them [25]. Silva C. J. & Torres D. F. proposed a population and introduced optimal treatment strategies for co-infection transmission dynamics of TB and HIV [26].

Some cases are reported where patients have symptoms of malaria and dengue both at the same time. In such situations, higher mortality rate is observed. On the basis of this observation, a mathematical model is constructed in the present work. Also two optimal controls are applied in the model in such a way that it helps to analyse malaria-dengue concurrent case and effect of recovery rate on the disease transmission. The paper is organized as follows. The malaria-dengue model construction will be discussed in section 2. Section 3 focuses on formulating basic reproduction number for concurrent malaria-dengue infection, moreover the equilibrium points of the given model are calculated. Local and global stability of all four equilibrium points are proved in section 4. Optimal control theory is introduced and applied to the model in section 5. The model is analysed numerically and graphically in the next section which provides better explanation of the analytic results.

2. Mathematical modeling

The environmental stress also damages the immune system and makes the individual weak to resist various kinds of infections. Motivated from this concurrent disease problem, we have proposed a compartmental model to analyze the spread of malaria and dengue infections individually and concurrently. The model subdivides the human population (N) into four mutually-exclusive compartments, namely susceptible individuals (S), malaria infected individuals (M), dengue infected individuals (D) and corresponding to two infectious agent class of recovered individuals is (R). Total recruitment rate in class of susceptible at time t is B . Susceptible individuals are infected by malaria infection with transmission rate α_1 . The disease transmission from the class of susceptible individuals to the class of dengue infected individuals is taken as a saturated form with disease transmission rate α_2 and α_3 be the reciprocal of the half saturation constant. Therefore, from compartment S to D the disease

transmission form is taken as $\frac{\alpha_2SD}{1+\alpha_3D}$. The parameter

α_4 represents the rate of the malaria infection giving rise to the dengue infection due to weak immunity. α_5 and α_6 are the rates at which the population infected by malaria and dengue are recovered respectively. μ is assumed as a natural death rate and μ_D be the dengue infection related death rate.

In Figure 1 the schematic diagram of the transmission of disease is shown. Here a concurrent disease case in which individual first get affected by dengue and then by malaria is ignored.

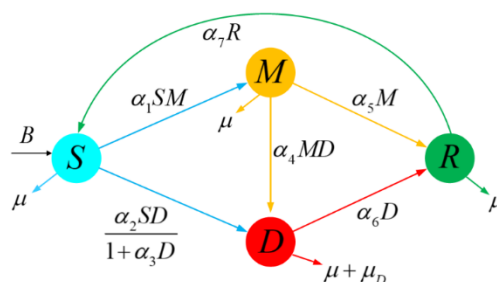


Figure 1. Schematic diagram of malaria-dengue model

On the basis these assumptions and figure 1, we formulate our model as:

$$\begin{aligned}
 \frac{dS}{dt} &= B + \alpha_7 R - \alpha_1 SM - \frac{\alpha_2 SD}{1 + \alpha_3 D} - \mu S \\
 \frac{dM}{dt} &= \alpha_1 SM - \alpha_4 MD - \alpha_5 M - \mu M \\
 \frac{dD}{dt} &= \frac{\alpha_2 SD}{1 + \alpha_3 D} + \alpha_4 MD - \alpha_6 D - (\mu + \mu_D) D \\
 \frac{dR}{dt} &= \alpha_6 D + \alpha_5 M - \alpha_7 R - \mu R
 \end{aligned}
 \tag{1}$$

The initial conditions of the system (1) are $S(0) > 0$, $M(0) > 0$, $D(0) > 0$, $R(0) > 0$.

3. Basic reproduction number (R_0) and equilibrium

Environments Note that $S + M + D + R = N$ and all the compartments are taken positive. Summing all the equations of the system (1) gives,

$$\frac{d}{dt}(S + M + D + R) = B - \mu(S + M + D + R) - \mu_D D \geq 0$$

Hence, $\limsup_{t \rightarrow \infty} (S + M + D + R) \leq \frac{B}{\mu}$

Therefore, the feasible region for system (1) is:

$$\Lambda = \left\{ \begin{aligned} &(S + M + D + R) / (S + M + D + R) \leq \frac{B}{\mu}; \\ &S \geq 0, M \geq 0, D \geq 0, R \geq 0 \end{aligned} \right\}
 \tag{2}$$

Clearly the point $E_0 = (S_0, 0, 0, 0)$, where $S_0 = \frac{B}{\mu}$ is an equilibrium point of the system (1), which is called a disease free equilibrium point. The model has three more equilibrium points as follows,

I. Dengue free equilibrium point $E_1 = (S_1, M_1, 0, R_1)$

$$\text{where, } S_1 = \frac{\alpha_5 + \mu}{\alpha_1}, M_1 = \frac{(\alpha_7 + \mu)k_1}{\alpha_1\mu}$$

$$R_1 = \frac{\alpha_5 k_1}{\alpha_1 \mu} \text{ and } k_1 = \frac{(B\alpha_1 - \alpha_5\mu - \mu^2)}{\mu(\alpha_5 + \alpha_7 + \mu)}$$

II. Malaria free equilibrium point

$$E_2 = (S_2, 0, M_2, R_2) \text{ where, } M_2 = k_2(\alpha_7 + \mu),$$

$$R_2 = \alpha_6 k_2, \text{ where}$$

$$S_2 = \frac{k_2 \left((\alpha_7 + \mu) ((\mu_D + \alpha_6)(B\alpha_3 + \mu_D + \mu) + \mu(\mu_D\mu + B\alpha_3 + \mu_D\alpha_6)) + \alpha_6\mu(\alpha_6 + \mu) \right)}{\mu(\mu_D + \mu) + \alpha_6\mu - B\alpha_2} \text{ and}$$

$$k_2 = \frac{\mu(\mu_D + \mu) + \alpha_6\mu - B\alpha_2}{\left((\alpha_7 + \mu)(\mu_D\alpha_3\mu + \alpha_3\mu(\alpha_6 + 1) + \alpha_2(\mu_D + \mu)) + \alpha_2\alpha_6\mu \right)}$$

III. Endemic equilibrium point

$$E^* = (S^*, M^*, D^*, R^*), \text{ where}$$

$$S^* = \frac{\alpha_4 k_3 + \alpha_5 + \mu}{\alpha_1}, D^* = k_3,$$

$$M^* = -\frac{(\alpha_7 + \mu)(\alpha_1 k_3 (\mu_D + \mu) + \mu(\alpha_4 k_3 + \alpha_5 + \mu) - B\alpha_1)}{\alpha_1 \mu (\alpha_5 + \alpha_7 + \mu)}$$

$$R^* = -\frac{\left(\alpha_1 \alpha_5 k_3 (\mu_D + \mu) - k_3 \mu (\alpha_4 \alpha_5 - \alpha_1 \alpha_6) \right)}{\alpha_1 \mu (\alpha_5 + \alpha_7 + \mu)},$$

$$k_3 = \text{Root of } \left\{ ((\alpha_7 + \mu)(\alpha_3\alpha_4(\mu_D\alpha_1 + \alpha_1\mu + \alpha_4\mu)) + \alpha_1\alpha_3\alpha_4\alpha_6\mu)z^2 + ((\alpha_7 + \mu)((\alpha_1\alpha_3\mu + \alpha_1\alpha_4)(\mu_D + \mu) + \alpha_3\alpha_4\mu(\alpha_5 + \mu) + \alpha_1\alpha_3(B\alpha_4 + \alpha_6\mu) + \alpha_4\mu(\alpha_4 - \alpha_2)) + \alpha_1\alpha_3\alpha_5\mu(\alpha_6 + \mu) + \mu_D\alpha_1\alpha_3\alpha_5\mu + \alpha_1\alpha_4\alpha_6\mu - \alpha_2\alpha_4\alpha_5\mu)z + (\alpha_7 + \mu)(\alpha_1\mu(\alpha_6 + \mu) + \alpha_1(\mu_D\mu - B\alpha_4) + \mu(\alpha_4 - \alpha_2)(\alpha_5 + \mu)) + \alpha_1\alpha_5\mu(\alpha_6 + \mu) - \alpha_2\alpha_5\mu(\alpha_5 + \mu) + \mu_D\alpha_1\alpha_5\mu \right\}$$

Since the threshold parameter is useful in characterizing the spread of an infectious disease. Here, we use the next generation matrix ([9], [28], [3]) to obtain the expression of basic reproduction number R_0 for concurrent malaria-dengue infection.

Let $X = (S + M + D + R)$, then system (1) can be written as $X' = \mathcal{F}(X) - \mathcal{V}(X)$ such that,

$$\mathcal{F} = \begin{bmatrix} \alpha_1 SM \\ \frac{\alpha_2 SD}{1 + \alpha_3 D} + \alpha_4 MD \\ 0 \\ 0 \end{bmatrix},$$

$$\mathcal{V} = \begin{bmatrix} \alpha_4 MD + \alpha_5 M + \mu M \\ \alpha_6 D + \mu D + \mu_D D \\ -\alpha_6 D - \alpha_5 M + \alpha_7 R + \mu R \\ -B - \alpha_7 R + \alpha_1 SM + \frac{\alpha_2 SD}{1 + \alpha_3 D} + \mu S \end{bmatrix}$$

Let matrices \mathcal{F} and \mathcal{V} are be the Jacobian of \mathcal{F} and \mathcal{V} respectively around disease free equilibrium point (E_0):

$$\mathcal{F} = \begin{bmatrix} \frac{\alpha_1 B}{\mu} & 0 & 0 & 0 \\ 0 & \frac{\alpha_2 B}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{V} = \begin{bmatrix} \alpha_5 + \mu & 0 & 0 & 0 \\ 0 & \mu + \mu_D + \alpha_6 & 0 & 0 \\ -\alpha_5 & -\alpha_6 & \alpha_7 + \mu & 0 \\ \frac{\alpha_1 B}{\mu} & \frac{\alpha_2 B}{\mu} & -\alpha_7 & \mu \end{bmatrix}$$

Here, the matrix \mathcal{F} is related to the rate of increase of new individual in compartment and \mathcal{V} to the rate of the diseases transmission in compartments.

The next generation matrix $K = (\mathcal{F}\mathcal{V}^{-1})$ have non negative eigenvalues. The basic reproduction number R_0 for the model is the spectral radius of $K = (\mathcal{F}\mathcal{V}^{-1})$, which is:

$$R_0 = \frac{\alpha_1 B}{\mu(\alpha_5 + \mu)} + \frac{\alpha_2 B}{\mu(\mu_D + \alpha_6 + \mu)} \tag{3}$$

4. Stability analysis

This section includes stability results of all the equilibrium points of the proposed malaria-dengue model.

4.1. Local stability

Local stability of all the equilibrium points has been established by following theorems.

Theorem 1. *The disease free equilibrium point E_0 of model is locally asymptotically stable if it satisfy following two conditions.*

- I. $\frac{\alpha_1 B}{\mu} > \alpha_5 + \mu$

$$II. \frac{\alpha_2 B}{\mu} > \mu_D + \alpha_6 + \mu$$

Proof. Jacobian matrix of the model around point E_0 is:

$$J(E_0) = \begin{bmatrix} -\mu & \frac{-\alpha_1 B}{\mu} & \frac{-\alpha_2 B}{\mu} & \alpha_7 \\ 0 & \frac{\alpha_1 B}{\mu} - \alpha_5 - \mu & 0 & 0 \\ 0 & 0 & \frac{\alpha_2 B}{\mu} - \mu_D - \alpha_6 - \mu & 0 \\ 0 & \alpha_5 & \alpha_6 & -\alpha_7 - \mu \end{bmatrix}$$

Eigenvalues of the matrix $J(E_0)$ are $\lambda_1^0 = -\mu$,

$$\lambda_2^0 = \frac{\alpha_1 B}{\mu} - \alpha_5 - \mu, \lambda_3^0 = \frac{\alpha_2 B}{\mu} - \mu_D - \alpha_6 - \mu,$$

$$\lambda_4^0 = -(\alpha_7 + \mu).$$

Clearly all the eigenvalues are negative if $\frac{\alpha_1 B}{\mu} > \alpha_5 + \mu$ and $\frac{\alpha_2 B}{\mu} > \mu_D + \alpha_6 + \mu$, hence disease free equilibrium point is locally asymptotically stable under these conditions. \square

Theorem 2. The dengue free equilibrium point E_1 is locally asymptotically stable if it satisfy following two conditions.

I. $B\alpha_1 > \mu(\alpha_5 + \mu)$

II. $\alpha_2(\alpha_5 + \mu) < \alpha_1(\mu_D + \mu)$ and $\alpha_6\mu > B\alpha_5$

Proof. Jacobian matrix of the model around point E_1 is:

$$J(E_1) = \begin{bmatrix} -n_1 - \mu & -n_5 - \mu & -n_2 & \alpha_7 \\ n_1 & 0 & -\frac{\alpha_4 n_1}{\alpha_1} & 0 \\ 0 & 0 & n_2 + \frac{\alpha_4 n_1}{\alpha_1} - \mu_D - \alpha_6 - \mu & 0 \\ 0 & \alpha_5 & \alpha_6 & -(\alpha_7 + \mu) \end{bmatrix}$$

Where, $n_1 = \frac{(B\alpha_1 - \alpha_5\mu - \mu^2)(\alpha_7 + \mu)}{\mu(\alpha_5 + \alpha_7 + \mu)}$ and

$$n_2 = \frac{\alpha_2(\alpha_5 + \mu)}{\alpha_1}.$$

Eigenvalues of the Jacobian matrix $J(E_1)$ are:

$$\lambda_1^1 = -\mu, \lambda_2^1 = -\frac{1}{2\mu(\alpha_5 + \alpha_7 + \mu)}(\beta_1 - \sqrt{\beta_2}),$$

$$\lambda_3^1 = -\frac{(\beta_1 + \sqrt{\beta_2})}{2\mu(\alpha_5 + \alpha_7 + \mu)},$$

$$\lambda_4^1 = -\frac{((\alpha_7 + \mu)(\alpha_1\mu(\mu_D + \mu)(1 + \alpha_5) - B\alpha_1\alpha_5 + \alpha_1\alpha_6\mu + (\alpha_5 + \mu)(\alpha_4\mu - \alpha_2\mu(1 + \alpha_5))))}{\alpha_1\mu(\alpha_5 + \alpha_7 + \mu)}$$

Where, $\beta_1 = (\alpha_7 + \mu)(B\alpha_1 + \mu\alpha_7)$,

$$\beta_2 = (\alpha_7 + \mu)((\alpha_7 + \mu)((\alpha_7 + \mu)(4\mu^2(\alpha_7 + \mu) - 2B\alpha_1\mu) + 2\mu(\alpha_5 + \mu)(4\alpha_5\mu - B\alpha_1) + B\alpha_1(B\alpha_1 - 6\alpha_5\mu) + \alpha_7^2\mu^2) + 4\alpha_5^2\mu(-B\alpha_1 + \alpha_5 + \mu))$$

Eigenvalues λ_2^1 and λ_3^1 are complex when β_2 is negative, real part of both these eigenvalues are negative and when β_2 is positive, real part of both the eigenvalues λ_2^1 and λ_3^1 are negative when $(\beta_1 - \sqrt{\beta_2}) > 0$,

i.e $(\beta_1^2 - \beta_2) > 0$

$$\beta_1^2 - \beta_2 = (\alpha_7 + \mu)((\alpha_7 + \mu)((\alpha_7 + \mu)4\mu(B\alpha_1 - \alpha_5\mu - \mu^2) + 8\alpha_5^2\mu(B\alpha_1 - \alpha_5\mu - \mu^2)) + 4\alpha_5^2\mu(B\alpha_1 - \alpha_5\mu - \mu^2))$$

Hence, real part of eigenvalues λ_2^1 and λ_3^1 are negative when $B\alpha_1 > \mu(\alpha_5 + \mu)$.

$$\lambda_4^1 = \frac{((\alpha_7 + \mu)(\mu(\alpha_4 - \alpha_2)(\alpha_5 + \mu) + \alpha_1\mu(\mu_D + \mu) - B\alpha_1\alpha_5 + \alpha_1\alpha_6\mu + \alpha_1\alpha_5\mu(\mu_D + \mu) - \alpha_2\alpha_5\mu(\alpha_5 + \mu))}{-\alpha_1\mu(\alpha_5 + \alpha_7 + \mu)}$$

$\lambda_4^1 < 0$ when $\alpha_2(\alpha_5 + \mu) < \alpha_1(\mu_D + \mu)$ and $\alpha_6\mu > B\alpha_5$. Clearly, all the eigenvalues are negative under these conditions. Hence, the theorem. \square

Theorem 3. The malaria free equilibrium point E_2 is locally asymptotically stable if it satisfy following two conditions.

I. $k_2(\alpha_7 + \mu)\alpha_4 + \frac{\alpha_1 n_3 k_2}{n_4} > \alpha_5 + \mu$

II. $\mu > n_6$, $\alpha_6 > n_5$ and $n_5^2 + n_6^2 > 2\mu n_6$

Proof. Jacobian matrix of the model around point E_2 is:

$$J(E_2) = \begin{bmatrix} n_6 - \mu & -\frac{\alpha_1 n_3 k_2}{n_4} & -n_5 & \alpha_7 \\ 0 & k_2(\alpha_7 + \mu)\alpha_4 + \frac{\alpha_1 n_3 k_2}{n_4} - \alpha_5 - \mu & 0 & 0 \\ -n_6 & -k_2(\alpha_7 + \mu)\alpha_4 & n_5 - \mu_D - \alpha_6 - \mu & 0 \\ 0 & \alpha_5 & \alpha_6 & -\alpha_7 - \mu \end{bmatrix}$$

Where, $n_3 = \mu(\mu_D + \mu) + \alpha_6\mu - B\alpha_2$,

$$n_4 = (\alpha_7 + \mu)((\mu_D + \alpha_6)(B\alpha_3 + \mu_D + \mu) + \mu(\mu_D\mu + B\alpha_3 + \mu_D\alpha_6)) + \alpha_6\mu(\alpha_6 + \mu),$$

$$n_5 = \frac{-\alpha_2 n_2 k_2}{n_4(-k_2(\alpha_7 + \mu)\alpha_3 + 1)^2}, n_6 = \frac{\alpha_2 k_2(\alpha_7 + \mu)}{-n_5(\alpha_7 + \mu)\alpha_3 + 1}.$$

Clearly, $\lambda_1^2 = k_2(\alpha_7 + \mu)\alpha_4 + \frac{\alpha_1 n_3 k_2}{n_4} - \alpha_5 - \mu$ is one of the eigen value of $J(E_2)$, hence characteristic equation is given by:

$$Ch(x) = \left(x - k_2(\alpha_7 + \mu)\alpha_4 - \frac{\alpha_1 n_3 k_2}{n_4} + \alpha_5 + \mu \right) \phi(x)$$

Where, $\phi(x) = x^3 + a_1x^2 + a_2x + a_3$

$$a_1 = \alpha_7 + 3\mu - n_5 + \mu_D + \alpha_6 - n_6,$$

$$\begin{aligned}
 a_2 &= (\alpha_7 + \mu)(\alpha_6 + \mu) + (\mu - n_6)(\mu_D + \alpha_6 + \alpha_7 + \mu) \\
 &\quad + (\mu - n_5)(\alpha_7 + 2\mu) - \mu n_6, \\
 a_3 &= (\mu - n_6)(\mu_D(\alpha_7 + \mu) + \mu(\alpha_6 + \alpha_7)) + \alpha_7\mu(\alpha_6 - n_5) \\
 &\quad + \mu^2(\mu - n_5) \\
 a_1 a_2 - a_3 &= (\alpha_7 + \mu)(\mu_D(\mu_D\mu + \alpha_6) + (\alpha_6 + \alpha_7)(\mu_D + \alpha_6)) \\
 &\quad + (\mu - n_6)((\mu_D + 2\mu)(\mu_D + \alpha_6) + (\mu_D + \alpha_6)(\alpha_6 + \alpha_7)) \\
 &\quad + (\alpha_7 + \mu)(\mu_D + 2\alpha_6 + \alpha_7) + \mu(\mu_D + \alpha_7) + \mu_D\alpha_7\mu \\
 &\quad + (\mu - n_5)((2\mu_D + 2\alpha_6 + \alpha_7)(\alpha_7 + \mu) + 2\mu(\mu_D + \alpha_6)) \\
 &\quad + \mu(5\alpha_7 + 8\mu) + n_6(\mu_D + \alpha_6)(n_5 + n_6 - 2\mu) \\
 &\quad + 4\mu n_6(n_5 - n_6) + (n_5 + n_6)(\alpha_7 n_5 + \alpha_7 n_6) \\
 &\quad + 2\mu(n_5^2 + n_6^2 - 2\mu n_6)
 \end{aligned}$$

$a_1, a_3 > 0$ and $a_1 a_2 > a_3$ if $\mu > n_6$, $\alpha_6 > n_5$ and $n_5^2 + n_6^2 > 2\mu n_6$. Hence, by applying Routh-Hurwitz criteria we can say all real roots of $Ch(x)$ are negative under these conditions. \square

Theorem 4. *The endemic equilibrium point E^* is locally asymptotically stable if it satisfy following two conditions.*

I. $k_2(\alpha_7 + \mu)\alpha_4 + \frac{\alpha_1 n_3 k_2}{n_4} > \alpha_5 + \mu$

II. $\mu > n_6$, $\alpha_6 > n_5$ and $n_5^2 + n_6^2 > 2\mu n_6$

Proof. Jacobian matrix of the model around point E^* is $J(E^*) = [x_{ij}]$.

Where, $x_{11} = -\alpha_1 M^* - \frac{\alpha_2 D^*}{D^* \alpha_3 + 1} - \mu$, $x_{12} = -\alpha_1 S^*$,

$x_{13} = \frac{-\alpha_2 S^*}{(D^* \alpha_3 + 1)^2}$, $x_{14} = \alpha_7$, $x_{21} = \alpha_1 M^*$,

$x_{22} = -D^* \alpha_4 + S^* \alpha_1 - \alpha_5 - \mu$, $x_{23} = -\alpha_4 M^*$, $x_{24} = 0$,

$x_{31} = \frac{\alpha_2 D^*}{D^* \alpha_3 + 1}$, $x_{32} = \alpha_4 D^*$, $x_{34} = 0$, $x_{41} = 0$, $x_{42} = \alpha_5$

$x_{33} = \frac{\alpha_2 S^*}{(D^* \alpha_3 + 1)^2} + \alpha_4 M^* - \mu_D - \alpha_6 - \mu$, $x_{43} = \alpha_6$,

$x_{44} = -\alpha_7 - \mu$

The characteristic equation of matrix $J(E^*)$ is $Ch^*(x) = x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4 = 0$.

Where, $b_1 = -x_{44} - x_{33} - x_{22} - x_{11}$,

$b_2 = x_{11}x_{22} + x_{11}x_{33} + x_{11}x_{44} - x_{12}x_{21} - x_{13}x_{31} + x_{22}x_{33}$
 $+ x_{22}x_{44} - x_{32}x_{23} + x_{33}x_{44}$

$b_3 = -x_{11}x_{22}x_{33} - x_{11}x_{22}x_{44} + x_{11}x_{23}x_{32} - x_{11}x_{33}x_{44}$
 $+ x_{12}x_{21}x_{33} + x_{12}(x_{21}x_{44} - x_{23}x_{31}) - x_{21}(x_{13}x_{32} - x_{14}x_{42})$
 $+ x_{31}x_{13}x_{22} - x_{31}(x_{14}x_{43} - x_{13}x_{44}) - x_{22}x_{33}x_{44} + x_{23}x_{32}x_{44}$

$b_4 = x_{11}x_{44}(x_{22}x_{33} - x_{23}x_{32}) + x_{13}x_{44}(x_{12}x_{23} - x_{13}x_{22})$
 $+ x_{21}x_{32}(x_{13}x_{44} - x_{14}x_{43}) + x_{21}x_{33}(x_{14}x_{42} - x_{12}x_{44})$
 $+ x_{14}x_{31}(x_{22}x_{43} - x_{23}x_{42})$.

$b_2 > 0$ when x_{22} , x_{33} are negative.

$x_{22} < 0 \Leftrightarrow S^* \alpha_1 < \alpha_5 + \mu + D^* \alpha_4$ and

$x_{33} < 0 \Leftrightarrow \frac{\alpha_2 S^*}{(D^* \alpha_3 + 1)^2} + \alpha_4 M^* < \mu_D + \alpha_6 + \mu$.

$b_3 > 0$ when $\alpha_1 M^*(\alpha_7 - \mu) > \frac{\alpha_4 \alpha_2 D^* M^*}{D^* \alpha_3 + 1}$,

$\frac{\alpha_2 \alpha_4 S^* D^*}{(D^* \alpha_3 + 1)^2} > \alpha_7 \alpha_5$ and $\frac{\alpha_2 S^*(\alpha_7 + \mu)}{(D^* \alpha_3 + 1)^2} > \alpha_7 \alpha_6$.

$b_4 = x_{11}x_{44}(x_{22}x_{33} - x_{23}x_{32}) + x_{13}x_{44}(x_{12}x_{23} - x_{13}x_{22})$
 $+ x_{21}x_{32}(x_{13}x_{44} - x_{14}x_{43}) + x_{21}x_{33}(x_{14}x_{42} - x_{12}x_{44})$
 $+ x_{14}x_{31}(x_{22}x_{43} - x_{23}x_{42})$

and $b_4 > 0$ when,

$\alpha_1 \alpha_4 S^* M^* > \frac{\alpha_2 S^*}{(D^* \alpha_3 + 1)^2} (D^* \alpha_4 - S^* \alpha_1 + \alpha_5 + \mu)$,

$\frac{\alpha_2 S^*(\alpha_7 + \mu)}{(D^* \alpha_3 + 1)^2} > \alpha_7 \alpha_6$, $\alpha_7 \alpha_5 > \alpha_1 S^*(\alpha_7 + \mu)$,

$\alpha_4 \alpha_5 M^* > (D^* \alpha_4 - S^* \alpha_1 + \alpha_5 + \mu) \alpha_6$. \square

4.2. Global stability

To perform the global stability analysis of the disease free equilibrium we use the method developed by [5].

4.2.1. Global stability of disease-free equilibrium point (E_0)

The model system can be written as follows:

$$\begin{aligned}
 \frac{dX_{(0)}}{dt} &= F_0(X_{(0)}, Z_{(0)}) \\
 \frac{dZ_{(0)}}{dt} &= G_0(X_{(0)}, Z_{(0)}), \quad G_0(X_{(0)}, 0) = 0
 \end{aligned} \tag{4}$$

Here $X_{(0)} = X_{(0)}(X_1^0) \in \mathbb{R}$ represents the number of uninfected individuals and $Z_{(0)} = Z_0(Y_1^0, Y_2^0, Y_3^0) \in \mathbb{R}^3$ denotes the number of infected individuals. According to this notation the disease-free equilibrium point is denoted by $E_0 = (S_0, 0)$.

Now as per the method given in [5], following two conditions will ensure global stability of the disease-free equilibrium point.

[H1] $\frac{dX_{(0)}}{dt} = F_0(X_{(0)}, 0)$, $E_0 = (X_0, 0)$ is globally asymptotically stable.

[H2] $G_0(X_{(0)}, Z_{(0)}) = B_1 M + \left[B_2 - \frac{\alpha_2 \alpha_3 D S^0}{1 + \alpha_3 D} \right] D + B_3 R - \hat{G}_0(X_{(0)}, Z_{(0)})$
 , where $\hat{G}_0(X_{(0)}, Z_{(0)}) \geq 0$ for $(X_{(0)}, Z_{(0)}) \in \Lambda$.

Here, $B_1 = D_M G_0(X_0, 0)$, $B_2 = D_D G_0(X_0, 0)$ and $B_3 = D_R G_0(X_0, 0)$ are matrix with non-negative off diagonal entries.

Lemma 1. *The fixed point $E_0 = (S_0, 0)$ is a globally*

asymptotically stable equilibrium of the system, provided $R_0 < 1$ and assumptions [H1] and [H2] are satisfied.

Theorem 5. For $R_0 < 1$, the disease free equilibrium point is globally asymptotically stable.

Proof. we begin by showing [H1] as $F_0(X_{(0)}, 0) = [B - \mu S]$ and $Ch_{(0)}(\lambda) = (\lambda + \mu) = 0$ is the characteristic polynomial of its Jacobian matrix. Since the polynomial have a negative root, $E_0 = (S_0, 0)$ is globally asymptotically stable.

Now, we have

$$G_0(X_{(0)}, Z_{(0)}) = (\alpha_1 S_0 - \mu)M + \left(\frac{\alpha_2 S_0}{1 + \alpha_3 D} - (\mu + \mu_D) \right) D - (\alpha_7 + \mu)R - \left(\alpha_1 M(S_0 - S) + \frac{\alpha_2 D}{1 + \alpha_3 D} - (S_0 - S) \right) = B_1 M + \left[B_2 - \frac{\alpha_2 \alpha_3 D S_0}{1 + \alpha_3 D} \right] D + B_3 R - \hat{G}_0(X_{(0)}, Z_{(0)})$$

Here, $\hat{G}_0(X, Z) \geq 0$ hence, the conditions (H1) and (H2) stated above are satisfied. \square

4.2.2. Global stability of dengue-free equilibrium point (E_1)

The model system can be written as

$$\begin{aligned} \frac{dX_{(1)}}{dt} &= F_1(X_{(1)}, Z_{(1)}) \\ \frac{dZ_{(1)}}{dt} &= G_1(X_{(1)}, Z_{(1)}), \quad G_1(X_{(1)}, 0) = 0 \end{aligned} \tag{5}$$

Here $X_{(1)} = X_{(1)}(X_1^1, X_2^1, X_3^1) \in \mathbb{R}^3$ represents the number of uninfected individuals and $Z_{(1)} = Z_1(Y_1^1) \in \mathbb{R}$ denotes the number of infected individuals. According to this notation the Dengue free equilibrium point is denoted by $E_1 = (X_{(1)}^1, 0)$, where $X_{(1)}^1 = (S_1, M_1, R_1)$.

The following two conditions will ensure global stability of the dengue-free equilibrium point:

[H3] $\frac{dX_{(1)}}{dt} = F_1(X_{(1)}, 0)$, $E_1 = (X_{(1)}^1, 0)$ is globally asymptotically stable.

[H4] $G_1(X_{(1)}, Z_{(1)}) = B_4 D - \frac{\alpha_2 \alpha_3 D S_1}{1 + \alpha_3 D} - \hat{G}_1(X_{(1)}, Z_{(1)})$,

where $\hat{G}_1(X_{(1)}, Z_{(1)}) \geq 0$ for $(X_{(1)}, Z_{(1)}) \in \Lambda$.

Here $B_4 = D_D G_1(X_{(1)}^1, 0)$ is a M-matrix.

Lemma 2. The fixed point $E_1 = (X_{(1)}^1, 0)$ is a globally asymptotically stable equilibrium of the system, provided $R_0 < 1$ and assumptions [H3] and [H4] are satisfied.

Theorem 6. For $R_0 < 1$, the disease-free equilibrium point is globally asymptotically stable when

$$D\alpha_4 > S\alpha_1.$$

Proof. we begin by showing [H3] as

$$F_1(X_{(1)}, 0) = \begin{bmatrix} B + \alpha_7 R - \alpha_1 S M - \mu S \\ \alpha_1 S M - \alpha_5 M - \mu M \\ \alpha_5 M - \alpha_7 R - \mu R \end{bmatrix} \text{ and}$$

$Ch_{(1)}(\lambda) = (\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3) = 0$ is the characteristic polynomial of its Jacobian matrix.

Where, $c_1 = (D\alpha_4 - S\alpha_1) + M\alpha_1 + \alpha_5 + \alpha_7 + 3\mu$,

$$c_2 = (D\alpha_4 - S\alpha_1)(\alpha_7 + 2\mu) + (M\alpha_1 + \alpha_7)(D\alpha_4 + \alpha_5) + 2\mu(M\alpha_1 + \alpha_5) + 2\mu(\alpha_7 + \mu) + \mu^2$$

$$c_3 = (\alpha_7 + \mu)(\mu(D\alpha_4 - S\alpha_1) + MD\alpha_1\alpha_4 + \alpha_5\mu) + M\alpha_1\mu(\alpha_5 + \alpha_7) + \mu^2(M\alpha_1 + \alpha_7) + \mu^3.$$

$$c_1 c_2 - c_3 = (D\alpha_4 - S\alpha_1)$$

$$((\alpha_7 + \mu)(D\alpha_4 + 2M\alpha_1 + 2\alpha_5 + \alpha_7 + 4\mu)$$

$$+ M\alpha_1(D\alpha_4 + \alpha_5)) + (\alpha_7 + 2\mu)(M\alpha_1(3\alpha_5 + M\alpha_1)$$

$$+ (\alpha_7 + 4\mu)(M\alpha_1 + \alpha_5) + 2\mu(\alpha_7 + 2\mu) + \alpha_5^2)$$

$$+ M^2\alpha_1^2(D\alpha_4 + \alpha_5) + S^2\alpha_1^2(\alpha_7 + \mu)$$

$$+ M\alpha_1\alpha_5(\alpha_5 + D\alpha_4) + 2DM\alpha_1\alpha_4\mu$$

$c_1, c_2 > 0$ and $c_1 c_2 - c_3 > 0$ if $D\alpha_4 > S\alpha_1$.

With the help of Routh-Hurwitz criteria it is clear that all the roots of the characteristic polynomial have negative real part when $D\alpha_4 > S\alpha_1$, hence

$E_1 = (X_{(1)}^1, 0)$ is globally asymptotically stable under this condition.

Now,

$$G_1(X_{(1)}, Z_{(1)}) = \left(\frac{\alpha_2 S_1}{1 + \alpha_3 D} + \alpha_4 M_1 - \alpha_6 - (\mu + \mu_D) \right) D - \left(\frac{\alpha_2 D}{1 + \alpha_3 D} (S_1 - S) + \alpha_4 D(M_1 - M) \right) = \left(B_5 - \frac{\alpha_2 \alpha_3 D S_1}{1 + \alpha_3 D} \right) D - \hat{G}_1(X_{(1)}, Z_{(1)})$$

Here, $\hat{G}_1(X, Z) \geq 0$ hence, the conditions (H3) and (H4) stated above are satisfied. \square

4.2.3. Global stability of malaria-free equilibrium point (E_2)

The model system can be written as:

$$\begin{aligned} \frac{dX_{(2)}}{dt} &= F_2(X_{(2)}, Z_{(2)}) \\ \frac{dZ_{(2)}}{dt} &= G_2(X_{(2)}, Z_{(2)}), \quad G_2(X_{(2)}, 0) = 0 \end{aligned} \tag{6}$$

Here $X_{(2)} = X_{(2)}(X_1^2, X_2^2, X_3^2) \in \mathbb{R}^3$ represents the number of uninfected individuals and $Z_{(2)} = Z_2(Y_1^2) \in \mathbb{R}$ denotes the number of infected individuals. According to this notation the Dengue free equilibrium point is denoted by $E_2 = (X_{(2)}^1, 0)$, where

$$X_{(2)}^1 = (S_2, D_2, R_2).$$

The following two conditions will ensure global stability of the malaria-free equilibrium point:

[H5] $\frac{dX_{(2)}}{dt} = F_2(X_{(2)}, 0)$, $E_2 = (X_{(2)}^1, 0)$ is globally asymptotically stable.

[H6] $G_2(X_{(2)}, Z_{(2)}) = B_5M - \hat{G}_2(X_{(2)}, Z_{(2)})$, where

$$\hat{G}_2(X_{(2)}, Z_{(2)}) \geq 0 \text{ for } (X_{(2)}, Z_{(2)}) \in \Lambda.$$

Here $B_5 = D_M G_2(X_{(2)}^1, 0)$ is a M-matrix.

Lemma 3. *The fixed point $E_2 = (X_{(2)}^1, 0)$ is a globally asymptotically stable equilibrium of the system, provided $R_0 < 1$ and assumptions [H5] and [H6] are satisfied.*

Theorem 7. *For $R_0 < 1$, the disease-free equilibrium point is globally asymptotically stable if*

$$\frac{\alpha_2 S}{(1 + \alpha_3 D)^2} < \alpha_6 + \mu + \mu_D.$$

Proof. we begin by showing [H5] as:

$$F_2(X_{(2)}, 0) = \begin{bmatrix} B + \alpha_7 R - \frac{\alpha_2 SD}{1 + \alpha_3 D} - \mu S \\ \frac{\alpha_2 SD}{1 + \alpha_3 D} - \alpha_6 D - (\mu + \mu_D) D \\ \alpha_6 D - \alpha_7 R - \mu R \end{bmatrix} \text{ and}$$

$Ch_{(2)}(\lambda) = (\lambda^3 + d_1 \lambda^2 + d_2 \lambda + d_3) = 0$ is the characteristic polynomial of its Jacobian matrix.

Where, $d_1 = n_7 + n_8 + n_9$,

$$d_2 = (n_9 + \mu)(n_7 + n_8) + n_7(\mu_D + \alpha_6) - \mu(\mu_D + \alpha_6 + \mu)$$

$$\text{and } d_3 = (n_7 - \mu)(\mu_D n_9 + n_9 \mu + \alpha_6 \mu) + n_8 n_9 \mu$$

$$\text{where, } n_7 = \frac{\alpha_2 D}{1 + \alpha_3 D} + \mu,$$

$$n_8 = \frac{-\alpha_2 S}{(1 + \alpha_3 D)^2} + \alpha_6 + \mu + \mu_D \text{ and } n_9 = \alpha_7 + \mu.$$

$$d_1 d_2 - d_3 = (n_7 - \mu) \left[(n_7 + n_8)(\mu_D + \mu + \alpha_6) + \alpha_6(n_9 + \mu) + n_7 n_8(n_9 + \mu) + n_8 n_9(n_7 + n_8) + n_7 n_9(n_7 + n_9) + n_8(n_8 \mu + n_9^2) \right]$$

Clearly, $n_7 > \mu$ hence $d_3 > 0$ and $d_1 d_2 - d_3 > 0$. $d_1 > 0$

$$\text{if } \frac{\alpha_2 S}{(1 + \alpha_3 D)^2} < \alpha_6 + \mu + \mu_D.$$

With the help of Routh-Hurwitz criteria it is clear that all the roots of the characteristic polynomial have negative real part, hence $E_2 = (X_{(2)}^1, 0)$ is globally asymptotically stable.

Now,

$$G_2(X_{(2)}, Z_{(2)}) = (\alpha_1 S_2 - \alpha_4 D_2 - \alpha_5 - \mu) M - (\alpha_1(S_2 - S) + \alpha_4(D_2 - D)) M = B_5 M - \hat{G}_2(X_{(2)}, Z_{(2)})$$

Here, $\hat{G}_2(X, Z) \geq 0$ hence, the conditions (H5) and (H6) stated above are satisfied. \square

4.2.4. Global stability of endemic equilibrium point (E^*)

We analyze global stability of an endemic equilibrium point through a geometric approach described in [16], [27] and [15]. To use this method let we modify our system (1) as follow:

$$\begin{aligned} \frac{dS}{dt} &= B + \frac{\alpha_7(\alpha_6 D + \alpha_5 M)}{(\alpha_7 + \mu)} - \alpha_1 S M - \frac{\alpha_2 S D}{1 + \alpha_3 D} - \mu S \\ \frac{dM}{dt} &= \alpha_1 S M - \alpha_4 M D - \alpha_5 M - \mu M \end{aligned} \tag{7}$$

$$\frac{dD}{dt} = \frac{\alpha_2 S D}{1 + \alpha_3 D} + \alpha_4 M D - \alpha_6 D - (\mu + \mu_D) D$$

Let $K \subset \mathbb{R}^3$ be a simply connected open set and $f \in C^1(K)$. Further suppose that $\varphi(t)$ be a solution to the following system,

$$x' = f(x)$$

(7) Suppose $P(x)$ be a matrix valued function on K and let $Q = P_f P^{-1} + P M^{[2]} P^{-1}$.

Here, the matrix P_f is:

$$(P_{ij}(x))_f = \left(\frac{\partial P_{ij}(x)}{\partial x} \right) \cdot f(x) = \nabla P_{ij} \cdot f(x).$$

Jacobian matrix of an arbitrary point is $M = [c_{ij}]$:

$$\text{Where, } c_{11} = -\alpha_1 M - \frac{\alpha_2 D}{1 + \alpha_3 D} - \mu, \quad c_{12} = \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \alpha_1 S$$

$$c_{13} = \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} - \frac{\alpha_2 S}{(1 + \alpha_3 D)^2}, \quad c_{21} = \alpha_1 M, \quad c_{23} = -\alpha_4 M,$$

$$c_{22} = \alpha_1 S - \alpha_4 D - \alpha_5 - \mu, \quad c_{31} = \frac{\alpha_2 D}{1 + \alpha_3 D}, \quad c_{32} = \alpha_4,$$

$$c_{33} = \frac{\alpha_2 S}{(1 + \alpha_3 D)^2} + \alpha_4 M - \alpha_6 - (\mu + \mu_D).$$

The second additive compound matrix obtain from the Jacobian matrix M is $M^{[2]}$,

$$M^{[2]} = \begin{bmatrix} n_{10} & -\alpha_4 M & \frac{\alpha_2 S}{(1 + \alpha_3 D)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \\ \alpha_4 & n_{11} & \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \alpha_1 S \\ -\frac{\alpha_2 D}{1 + \alpha_3 D} & \alpha_1 M & n_{12} \end{bmatrix}$$

$$\text{Where } n_{10} = \alpha_1 S - \alpha_4 D - \alpha_5 - 2\mu - \alpha_1 M - \frac{\alpha_2 D}{1 + \alpha_3 D},$$

$$n_{11} = \frac{\alpha_2 S}{(1 + \alpha_3 D)^2} + \alpha_4 M - \alpha_6 - 2\mu - \mu_D - \alpha_1 M - \frac{\alpha_2 D}{1 + \alpha_3 D}$$

, and

$$n_{12} = \alpha_1 S - \alpha_4 D - \alpha_5 + \frac{\alpha_2 S}{(1 + \alpha_3 D)^2} + \alpha_4 M - \alpha_6 - 2\mu - \mu_D$$

Next, consider the following system:

$$\frac{dz}{dt} = \Phi(\varphi(t))z \tag{8}$$

And if (8) is stable then also the second compound

$$\text{equation } \frac{d\bar{z}}{dt} = M^{[2]}(\varphi(t))\bar{z} \text{ is stable, moreover } \varphi$$

belong to a set in which $|P^{-1}|$ is bounded. A set \bar{K} is absorbing with respect to (7) if solution exist for all $t \geq 0$ and $x(t, K_1) \subset \bar{K}$ for all t , where K_1 is any bounded subset of K .

To prove global stability through this approach we use techniques developed in [19].

Let, $P = \frac{1}{D} I_3$, where I_3 is an identity matrix of order

$$3. \text{ Hence, } P_f P^{-1} = -\frac{D'}{D} I_3.$$

Next, $Q = P_f P^{-1} + PM^{[2]}P^{-1}$

$$= \begin{bmatrix} n_{10} - \frac{D'}{D} & -\alpha_4 M & \frac{\alpha_2 S}{(1 + D\alpha_3)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \\ \alpha_4 & n_{11} - \frac{D'}{D} & \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \alpha_1 S \\ -\frac{\alpha_2 D}{1 + \alpha_3 D} & \alpha_1 M & n_{12} - \frac{D'}{D} \end{bmatrix}$$

Where,

$$n_{10} - \frac{D'}{D} = \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4)M - \frac{\alpha_2(S + D)}{1 + \alpha_3 D}$$

$$n_{11} - \frac{D'}{D} = \frac{\alpha_3 \alpha_2 S D}{(1 + D\alpha_3)^2} - \alpha_1 S - \frac{\alpha_2 D}{1 + \alpha_3 D} - \mu \text{ and}$$

$$n_{12} - \frac{D'}{D} = \frac{-\alpha_3 \alpha_2 S D}{(1 + D\alpha_3)^2} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu.$$

Let we define the following norm function as described in [2] for some $z = (z_1, z_2, z_3)$.

$$\|z\| = \begin{cases} \max\{|z_1| + |z_3|, |z_2| + |z_3|\} & \text{if } z_2 z_3 \geq 0 \\ \max\{|z_1| + |z_3|, |z_2|\} & \text{if } z_2 z_3 < 0 \end{cases}$$

Now we explore the existence of some $\chi > 0$, so that $D_+ \|z\| \leq -\chi \|z\|$. In this situation we have to analyses all eight possible cases.

Case 1 If $0 < z_1, z_2, z_3$ and $|z_1| + |z_3| > |z_2| + |z_3|$ then

$$\|z\| = |z_1| + |z_3| \text{ and } D_+ \|z\| = D_+ (|z_1| + |z_3|) = z'_1 + z'_3$$

$$\begin{aligned} D_+ \|z\| &= [\alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4)M \\ &\quad - \frac{\alpha_2(S + D)}{1 + \alpha_3 D} - \alpha_6 - \mu_D] z_1 + [-\alpha_4 M] z_2 \\ &\quad + \left[\frac{\alpha_2 S}{(D\alpha_3 + 1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \right] z_3 + \left[-\frac{\alpha_2 D}{D\alpha_3 + 1} \right] z_1 \\ &\quad + [\alpha_1 M] z_2 \\ &\quad + \left[\frac{-\alpha_2 \alpha_3 S D}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right] z_3 \\ &\leq \left[\alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4)M - \frac{\alpha_2(S + D)}{1 + \alpha_3 D} \right. \\ &\quad \left. - \alpha_6 - \mu_D \right] |z_1| + [\alpha_1 M - \alpha_4 M] |z_2| \\ &\quad + \left[\frac{\alpha_2 S(1 - \alpha_3 D)}{(D\alpha_3 + 1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} + \alpha_1 S - \alpha_4 D \right. \\ &\quad \left. - \alpha_5 - \mu \right] |z_3| \end{aligned}$$

Since $|z_1| > |z_2|$, we get

$$\begin{aligned} &\leq \left[\alpha_1 S - \alpha_4 D - \alpha_5 - \mu - \alpha_4 M - \frac{\alpha_2(S + D)}{1 + \alpha_3 D} \right. \\ &\quad \left. - \alpha_6 - \mu_D \right] |z_1| + \left[\frac{\alpha_2 S(1 - \alpha_3 D)}{(D\alpha_3 + 1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \right. \\ &\quad \left. + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right] |z_3| \\ D_+ \|z\| &\leq \max \left\{ \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - \alpha_4 M - \frac{\alpha_2(S + D)}{1 + \alpha_3 D} \right. \\ &\quad \left. - \alpha_6 - \mu_D, \frac{\alpha_2 S(1 - \alpha_3 D)}{(D\alpha_3 + 1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} + \alpha_1 S - \alpha_4 D \right. \\ &\quad \left. - \alpha_5 - \mu \right\} \|z\| \end{aligned}$$

Case 2 If $0 < z_1, z_2, z_3$ and $|z_1| + |z_3| < |z_2| + |z_3|$ then

$$\begin{aligned} \|z\| &= |z_2| + |z_3| \text{ and } D_+ \|z\| = z'_2 + z'_3 \\ D_+ \|z\| &= [\alpha_4] z_1 + \left[-\alpha_1 M - \frac{\alpha_2 \alpha_3 S D}{1 + \alpha_3 D} - \frac{\alpha_2 D}{D\alpha_3 + 1} - \mu \right] z_2 \\ &\quad + \left[\frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \alpha_1 S \right] z_3 + \left[-\frac{\alpha_2 D}{D\alpha_3 + 1} \right] z_1 + [\alpha_1 M] z_2 \\ &\quad + \left[\frac{-\alpha_2 \alpha_3 S D}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right] z_3 \\ &\leq \left[\alpha_4 - \frac{\alpha_2 D}{D\alpha_3 + 1} \right] |z_1| + \left[-\frac{\alpha_2 \alpha_3 S D}{1 + \alpha_3 D} - \frac{\alpha_2 D}{D\alpha_3 + 1} - \mu \right] |z_2| \\ &\quad + \left[\frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \frac{\alpha_2 \alpha_3 S D}{(D\alpha_3 + 1)^2} - \alpha_4 D - \alpha_5 - \mu \right] |z_3| \end{aligned}$$

Since $|z_1| < |z_2|$, we get

$$D_+ \|z\| \leq \left[\alpha_4 - \frac{\alpha_2 D}{D\alpha_3 + 1} - \frac{\alpha_2 \alpha_3 SD}{1 + \alpha_3 D} - \mu \right] |z_2| + \left[\frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \frac{\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} - \alpha_4 D - \alpha_5 - \mu \right] |z_3|$$

$$D_+ \|z\| \leq \max \left\{ \alpha_4 - \frac{\alpha_2 D}{D\alpha_3 + 1} - \frac{\alpha_2 \alpha_3 SD}{1 + \alpha_3 D} - \mu, \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \frac{\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} - \alpha_4 D - \alpha_5 - \mu \right\} \|z\|$$

Case 3 If $z_1 < 0 < z_2, z_3$ and $|z_1| + |z_3| > |z_2| + |z_3|$ then

$$\|z\| = |z_1| + |z_3| \text{ and } D_+ \|z\| = -z'_1 + z'_3$$

$$D_+ \|z\| = [-\alpha_1 S + \alpha_4 D + \alpha_5 + \mu + (\alpha_1 + \alpha_4) M + \frac{\alpha_2(S+D)}{1 + \alpha_3 D} + \alpha_6 + \mu_D] z_1 + [\alpha_4 M] z_2 + \left[-\frac{\alpha_2 S}{(D\alpha_3 + 1)^2} + \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \right] z_3 + \left[-\frac{\alpha_2 D}{D\alpha_3 + 1} \right] z_1 + [\alpha_1 M] z_2 + \left[\frac{-\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D \right] z_3 + \left[-\alpha_5 - \mu \right] z_3$$

$$\leq [-\alpha_1 S + \alpha_4 D + \alpha_5 + \mu + (\alpha_1 + \alpha_4) M + \frac{\alpha_2 S}{1 + \alpha_3 D} + \alpha_6 + \mu_D] |z_1| + [\alpha_1 M + \alpha_4 M] |z_2| + \left[\frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} - \frac{\alpha_2 S(1 + \alpha_3 D)}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D \right] |z_3| + \left[-\alpha_5 - \mu \right] |z_3|$$

Since $|z_1| > |z_2|$, we get

$$\leq \left[-\alpha_1 S + \alpha_4 D + \alpha_5 + \mu + (2\alpha_1 + \alpha_4) M + \alpha_4 M + \frac{\alpha_2 S}{1 + \alpha_3 D} + \alpha_6 + \mu_D \right] |z_1| + \left[\frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} - \frac{\alpha_2 S(1 + \alpha_3 D)}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right] |z_3|$$

$$D_+ \|z\| \leq \max \|z\| \left\{ -\alpha_1 S + \alpha_4 D + \alpha_5 + \mu + (2\alpha_1 + \alpha_4) M + \alpha_4 M + \frac{\alpha_2 S}{1 + \alpha_3 D} + \alpha_6 + \mu_D, \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} - \frac{\alpha_2 S(1 + \alpha_3 D)}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right\} \|z\|$$

Case 4 If $z_1 < 0 < z_2, z_3$ and $|z_1| + |z_3| < |z_2| + |z_3|$ then

$$\|z\| = |z_2| + |z_3| \text{ and } D_+ \|z\| = z'_2 + z'_3$$

$$D_+ \|z\| = [\alpha_4] z_1 + \left[-\alpha_1 M - \frac{\alpha_2 \alpha_3 SD}{1 + \alpha_3 D} - \frac{\alpha_2 D}{D\alpha_3 + 1} - \mu \right] z_2$$

$$+ \left[\frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \alpha_1 S \right] z_3 + \left[-\frac{\alpha_2 D}{D\alpha_3 + 1} \right] z_1 + [\alpha_1 M] z_2 + \left[\frac{-\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right] z_3$$

$$\leq \left[\alpha_4 - \frac{\alpha_2 D}{D\alpha_3 + 1} \right] |z_1| + \left[-\frac{\alpha_2 \alpha_3 SD}{1 + \alpha_3 D} - \frac{\alpha_2 D}{D\alpha_3 + 1} - \mu \right] |z_2| + \left[\frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \frac{\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} - \alpha_4 D - \alpha_5 - \mu \right] |z_3|$$

Since $|z_1| < |z_2|$, we get

$$D_+ \|z\| \leq \left[\alpha_4 - \frac{\alpha_2 D}{D\alpha_3 + 1} - \frac{\alpha_2 \alpha_3 SD}{1 + \alpha_3 D} - \mu \right] |z_2| + \left[\frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \frac{\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} - \alpha_4 D - \alpha_5 - \mu \right] |z_3|$$

$$D_+ \|z\| \leq \max \left\{ \alpha_4 - \frac{\alpha_2 D}{D\alpha_3 + 1} - \frac{\alpha_2 \alpha_3 SD}{1 + \alpha_3 D} - \mu, \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \frac{\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} - \alpha_4 D - \alpha_5 - \mu \right\} \|z\|$$

Case 5 If $z_2 < 0 < z_1, z_3$ and $|z_1| + |z_3| > |z_2|$,

then $\|z\| = |z_1| + |z_3|$ and $D_+ \|z\| = z'_1 + z'_3$

$$D_+ \|z\| = \left[\alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4) M - \frac{\alpha_2(S+D)}{1 + \alpha_3 D} - \alpha_6 - \mu_D \right] z_1 + [-\alpha_4 M] z_2 + \left[\frac{\alpha_2 S}{(D\alpha_3 + 1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \right] z_3 + \left[-\frac{\alpha_2 D}{D\alpha_3 + 1} \right] z_1 + [\alpha_1 M] z_2 + \left[\frac{-\alpha_2 \alpha_3 SD}{(D\alpha_3 + 1)^2} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right] z_3$$

$$\leq \left[\alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4) M - \frac{\alpha_2(S+D)}{1 + \alpha_3 D} - \alpha_6 - \mu_D \right] |z_1| + [\alpha_1 M - \alpha_4 M] |z_2| + \left[\frac{\alpha_2 S(1 - \alpha_3 D)}{(D\alpha_3 + 1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} + \alpha_1 S - \alpha_4 D - \alpha_5 - \mu \right] |z_3|$$

Since $|z_1| + |z_3| > |z_2|$, we get

$$\begin{aligned} &\leq \left[\begin{array}{l} \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - \alpha_4 M - \frac{\alpha_2(S+D)}{1+\alpha_3 D} \\ -\alpha_6 - \mu_D \end{array} \right] |z_1| \\ &+ \left[\begin{array}{l} \frac{\alpha_2 S(1-\alpha_3 D)}{(D\alpha_3+1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} + \alpha_1(S+M) \\ -\alpha_4(D+M) - \alpha_5 - \mu \end{array} \right] |z_3| \\ D_+ \|z\| &\leq \max \left\{ \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - \alpha_4 M - \frac{\alpha_2(S+D)}{1+\alpha_3 D} \right. \\ &\quad \left. - \alpha_6 - \mu_D, \frac{\alpha_2 S(1-\alpha_3 D)}{(D\alpha_3+1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} + \alpha_1(S+M) \right. \\ &\quad \left. - \alpha_4(D+M) - \alpha_5 - \mu \right\} \|z\| \end{aligned}$$

Case 6 If $z_2 < 0 < z_1, z_3$ and $|z_1| + |z_3| < |z_2|$

then $\|z\| = |z_2|$ and

$$\begin{aligned} D_+ \|z\| &= -z'_2 = [-\alpha_4] z_1 \\ &+ \left[\alpha_1 M + \frac{\alpha_2 \alpha_3 S D}{1+\alpha_3 D} + \frac{\alpha_2 D}{D\alpha_3+1} + \mu \right] z_2 \\ &+ \left[\alpha_1 S - \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} \right] z_3 \end{aligned}$$

Since $|z_1| + |z_3| < |z_2|$, we get

$$D_+ \|z\| \leq \left[\begin{array}{l} \frac{\alpha_2 D}{D\alpha_3+1} + \frac{\alpha_2 \alpha_3 S D}{1+\alpha_3 D} - \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} \\ +\alpha_1(M+S) - \alpha_4 + \mu \end{array} \right] \|z\|$$

Case 7 If $z_3 < 0 < z_1, z_2$ and $|z_1| + |z_3| > |z_2|$

then $\|z\| = |z_1| + |z_3|$ and

$$\begin{aligned} D_+ \|z\| &= z'_1 - z'_3 = \left[\begin{array}{l} \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4) M \\ -\frac{\alpha_2(S+D)}{1+\alpha_3 D} - \alpha_6 - \mu_D \end{array} \right] z_1 \\ &+ [-\alpha_4 M] z_2 + \left[\frac{\alpha_2 S}{(D\alpha_3+1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \right] z_3 \\ &+ \left[\frac{\alpha_2 D}{D\alpha_3+1} \right] z_1 + [-\alpha_1 M] z_2 \\ &+ \left[\frac{\alpha_2 \alpha_3 S D}{(D\alpha_3+1)^2} - \alpha_1 S + \alpha_4 D + \alpha_5 + \mu \right] z_3 \\ &\leq \left[\begin{array}{l} \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4) M \\ -\frac{\alpha_2 S}{1+\alpha_3 D} - \alpha_6 - \mu_D \end{array} \right] |z_1| \\ &+ [-\alpha_1 M - \alpha_4 M] |z_2| \\ &+ \left[\begin{array}{l} \frac{\alpha_2 S(1+\alpha_3 D)}{(D\alpha_3+1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} - \alpha_1 S + \alpha_4 D \\ +\alpha_5 + \mu \end{array} \right] |z_3| \end{aligned}$$

Since $|z_1| + |z_3| > |z_2|$, we get

$$\begin{aligned} &\leq \left[\begin{array}{l} \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4) M - \frac{\alpha_2 S}{1+\alpha_3 D} \\ -\alpha_6 - \mu_D \end{array} \right] |z_1| \\ &+ \left[\begin{array}{l} \frac{\alpha_2 S(1+\alpha_3 D)}{(D\alpha_3+1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} - \alpha_1(S+M) \\ -\alpha_4(M-D) + \alpha_5 + \mu \end{array} \right] |z_3| \\ D_+ \|z\| &\leq \max \left\{ \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - (\alpha_1 + \alpha_4) M \right. \\ &\quad \left. - \frac{\alpha_2 S}{1+\alpha_3 D} - \alpha_6 - \mu_D, \frac{\alpha_2 S(1+\alpha_3 D)}{(D\alpha_3+1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} \right. \\ &\quad \left. - \alpha_1(S+M) - \alpha_4(M-D) + \alpha_5 + \mu \right\} \|z\| \end{aligned}$$

Case 8 If $z_3 < 0 < z_1, z_2$ and $|z_1| + |z_3| < |z_2|$

then $\|z\| = |z_2|$ and

$$\begin{aligned} D_+ \|z\| &= z'_2 = [\alpha_4] z_1 \\ &+ \left[-\alpha_1 M - \frac{\alpha_2 \alpha_3 S D}{1+\alpha_3 D} - \frac{\alpha_2 D}{D\alpha_3+1} - \mu \right] z_2 \\ &+ \left[\frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \alpha_1 S \right] z_3 \end{aligned}$$

Since $|z_1| + |z_3| < |z_2|$, we get

$$D_+ \|z\| \leq \left[\begin{array}{l} \frac{\alpha_2 \alpha_3 S D}{1+\alpha_3 D} - \frac{\alpha_2 D}{D\alpha_3+1} + \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} \\ -\alpha_1(M+S) + \alpha_4 - \mu \end{array} \right] \|z\|$$

Combining all the eight cases, we got four independent inequality which are used in following theorem that proves the global stability of endemic equilibrium point.

Theorem 8. For $R_0 > 1$, the endemic equilibrium point E^* is globally asymptotically stable if the following inequality holds

$$\max(\chi_1, \chi_2, \chi_3, \chi_4) < -\chi$$

Where,

$$\begin{aligned} \chi_1 &= \max \left\{ \begin{array}{l} \alpha_4 - \frac{\alpha_2 D}{D\alpha_3+1} - \frac{\alpha_2 \alpha_3 S D}{1+\alpha_3 D} - \mu, \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} \\ -\frac{\alpha_2 \alpha_3 S D}{(D\alpha_3+1)^2} - \alpha_4 D - \alpha_5 - \mu \end{array} \right\} \\ \chi_2 &= \max \left\{ \begin{array}{l} \alpha_1 S - \alpha_4 D - \alpha_5 - \mu - \alpha_4 M - \frac{\alpha_2(S+D)}{1+\alpha_3 D} \\ -\alpha_6 - \mu_D, \frac{\alpha_2 S(1-\alpha_3 D)}{(D\alpha_3+1)^2} - \frac{\alpha_7 \alpha_6}{\alpha_7 + \mu} + \alpha_1(S+M) \\ -\alpha_4(D+M) - \alpha_5 - \mu \end{array} \right\} \\ \chi_3 &= \frac{\alpha_2 D}{D\alpha_3+1} + \frac{\alpha_2 \alpha_3 S D}{1+\alpha_3 D} - \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} + \alpha_1(M+S) - \alpha_4 + \mu \\ \chi_4 &= \frac{\alpha_2 \alpha_3 S D}{1+\alpha_3 D} - \frac{\alpha_2 D}{D\alpha_3+1} + \frac{\alpha_7 \alpha_5}{\alpha_7 + \mu} - \alpha_1(M+S) + \alpha_4 - \mu \end{aligned}$$

And χ is a positive number. \square

5. Optimal control

Mosquitoes are the most prolific killers of humans in the animal kingdom. One of the most ancient and deadly diseases that mosquitoes transmit are malaria and dengue. It has been hypothesized due to influences on immune responses that infection with malaria can alter to the course of infection of the dengue. An effective way to protect the people from dengue who are already affected by malaria is to control vector. Also medication plays a major role to control spread of vector borne diseases.

In present dynamical model, two bounded Lebesgue integrable controls are introduced say u_1 and u_2 . u_1 control is to minimize concurrent infection cases by vector control and u_2 is a treatment control which helps to improve recovery rate. After applying control system (1) will take form as follow:

$$\begin{aligned} \frac{dS}{dt} &= B + \alpha_7 R - \alpha_1 S M - \frac{\alpha_2 S D}{1 + \alpha_3 D} - \mu S \\ \frac{dM}{dt} &= \alpha_1 S M - \alpha_4 M D - u_1 M - \alpha_5 M - \mu M \\ \frac{dD}{dt} &= \frac{\alpha_2 S D}{1 + \alpha_3 D} + \alpha_4 M D + u_1 M - (\alpha_6 + u_2) D - (\mu + \mu_D) D \\ \frac{dR}{dt} &= (\alpha_6 + u_2) D + \alpha_5 M - \alpha_7 R - \mu R \end{aligned}$$

The objective function $J(u_i, \Omega)$ for the mathematical model along with the optimal control is given by:

$$J = \int_0^T (A_1 S^2 + A_2 M^2 + A_3 D^2 + A_4 R^2 + w_1 u_1^2 + w_2 u_2^2) dt$$

Here, Ω denotes set of all compartmental variables. A_i are small positive constants to keep a balance in the size of the respective compartments. w_1 and w_2 are positive weight parameter which is associated with the control u_1 and u_2 . The objective of our work is to maximize the total number of recovered individual by optimizing control variables u_1 and u_2 .

As, the weight parameters w_1 and w_2 are constant of the control rates applied as vector control and treatment control, from which the optimal control condition is normalized. Now, we will calculate the values of control variables from $t = 0$ to $t = T$ such that

$$J(u_1(t), u_2(t)) = \text{optimum} \{ J(u_i, \Omega) / (u_1, u_2) \in \phi \}$$

Where ϕ is a smooth function on the interval $[0,1]$. The optimal controls denoted by u_1^* and u_2^* are founded by accumulating all the integrands of equation (4) using the lower bounds and upper bounds respectively with the results of Fleming and Rishel [11].

To optimize controls using the Pontryagin’s principle we construct a Lagrangian function consisting of state equations and adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ as follows:

$$\begin{aligned} L(\Omega, A) &= A_1 S^2 + A_2 M^2 + A_3 D^2 + A_4 R^2 + w_1 u_1^2 + w_2 u_2^2 \\ &+ \lambda_1 \left(B + \alpha_7 R - \alpha_1 S M - \frac{\alpha_2 S D}{1 + \alpha_3 D} - \mu S \right) \\ &+ \lambda_2 \left(\alpha_1 S M - \alpha_4 M D - u_1 M - \alpha_5 M - \mu M \right) \\ &+ \lambda_3 \left(\frac{\alpha_2 S D}{1 + \alpha_3 D} + \alpha_4 M D + u_1 M - (\alpha_6 + u_2) D \right. \\ &\quad \left. - (\mu + \mu_D) D \right) + \lambda_4 \left((\alpha_6 + u_2) D + \alpha_5 M \right. \\ &\quad \left. - \alpha_7 R - \mu R \right) \end{aligned}$$

The partially differentiation of the Lagrangian function with respect to each compartmental variable gives the adjoint equation variables $A_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ corresponding to the system:

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial L}{\partial S} = -2A_1 S + (\lambda_1 - \lambda_2)\alpha_1 M \\ &\quad + (\lambda_1 - \lambda_3) \frac{\alpha_2 D}{1 + \alpha_3 D} + \mu \lambda_1 \\ \dot{\lambda}_2 &= -\frac{\partial L}{\partial M} \\ &= -2A_2 M + (\lambda_1 - \lambda_2)\alpha_1 S + (\lambda_2 - \lambda_3)(\alpha_4 D + u_1) \\ &\quad + (\lambda_2 - \lambda_4)\alpha_5 + \lambda_2 \mu - \frac{\lambda_3 \alpha_1 S}{1 + \alpha_3 D} \\ \dot{\lambda}_3 &= -\frac{\partial L}{\partial D} = -2A_3 D + (\lambda_1 - \lambda_3) \frac{\alpha_2 S}{(1 + \alpha_3 D)^2} \\ &\quad + (\lambda_2 - \lambda_3)\alpha_4 M + (\lambda_3 - \lambda_4)(\alpha_6 + u_2) \\ &\quad + \lambda_3(\mu + \mu_D) \\ \dot{\lambda}_4 &= -\frac{\partial L}{\partial R} = -2A_4 R + (\lambda_4 - \lambda_1)\alpha_7 + \lambda_4 \mu \end{aligned}$$

The necessary conditions for Lagrangian function L to be optimal are, $\dot{u}_1 = 0$ and $\dot{u}_2 = 0$. Hence we get,

$$u_1 = \frac{1}{2w_1} (\lambda_2 - \lambda_3) M, \quad u_2 = \frac{1}{2w_2} (\lambda_3 - \lambda_4) D$$

Formulated required optimal controls are:

$$\begin{aligned} u_1^* &= \max \left\{ a_1, \min \left(b_1, \frac{(\lambda_2 - \lambda_3) M}{2w_1} \right) \right\} \\ u_2^* &= \max \left\{ a_2, \min \left(b_2, \frac{(\lambda_3 - \lambda_4) D}{2w_2} \right) \right\} \end{aligned}$$

Thus, analytical results for optimized controls have been visualised in simulation part.

6. Numerical simulation

Bifurcation analysis helps to demonstrate the qualitative information about the equilibrium point. Figure 2 shows backward bifurcation diagram where $R_C = 0.12$ is a critical point from which system's stability switches from unstable to stable state. If $R_C < R_0$, then for the point of R_0 backward bifurcation exists, moreover equilibrium coexist when $R_C < R_0 < 1$ [14], [29].

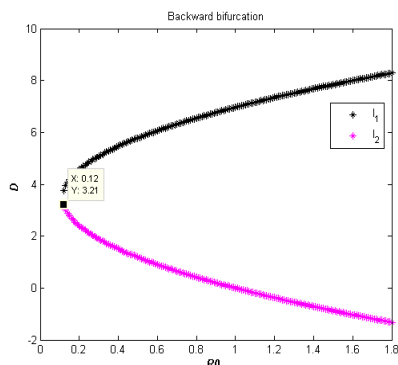


Figure 2. Bifurcation diagram for dengue infected individuals with R_0

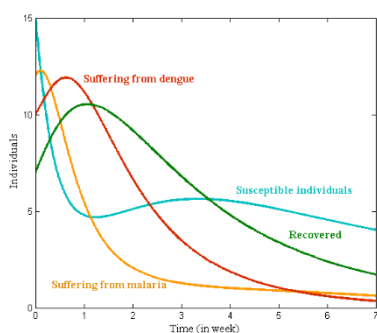


Figure 3. Time series of solution of malaria-dengue model

Figure 3 shows the flow of malaria-dengue model with time. It is observed that human immunity is more sensitive towards dengue compare to malaria infection moreover compare to dengue, recovery rate of malaria is higher. Hence we can say that medication is more effective on malaria infected compare to dengue infected. Compare to dengue, spread of malaria is easy to control by improving medication. Under proper medication both the diseases can be controlled in 7-8 weeks.

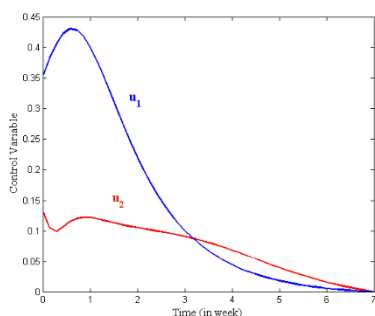


Figure 4. Variation in control variables with time

Figure 4 shows change in both the control variables needs to be done to stabilize the model. It is observed that initially 35% and 13%, u_1 and u_2 controls are needed to be applied respectively.

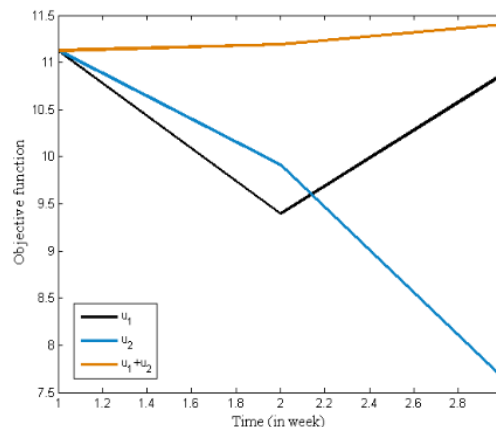


Figure 5. Change in objective function with time

Figure 5 gives change in objective function under influence of both the controls combine and individually which gives combine and individual effect of both the controls on malaria-dengue model. It is clearly visible that combine effect of controls gives more fruitful effect on the model compare to an individual effect.

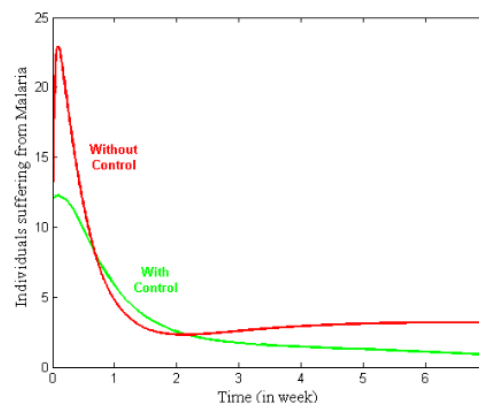
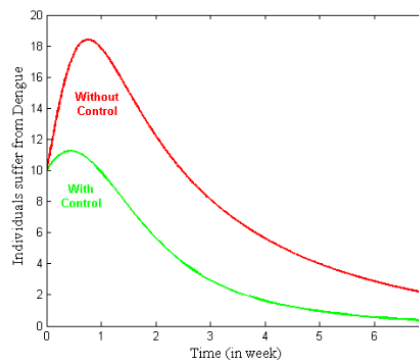
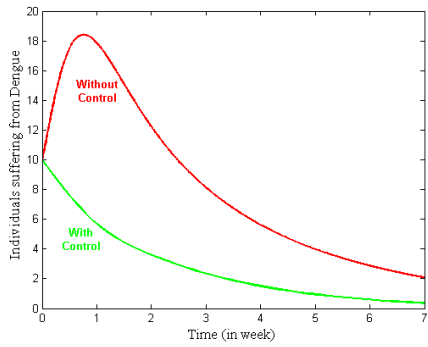


Figure 6. Impact of u_1 control on class of malaria infected individuals

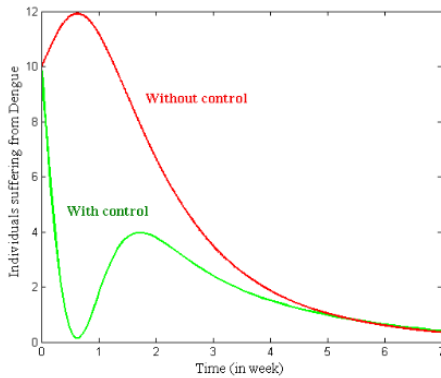
The simulation in figure 6 interprets that that chances to get infected by malaria decreases by 50% after applying control u_1 .



(a) Impact of u_1 control



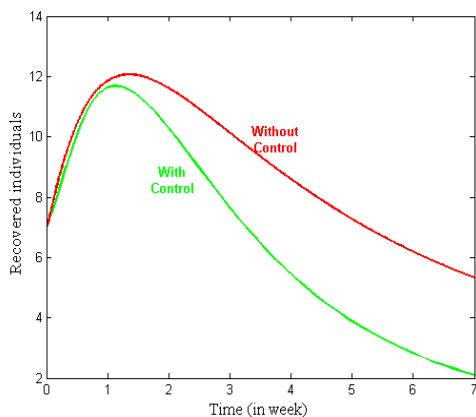
(b) Impact of u_2 control



(c) Impact of both the controls

Figure 7. Impact of controls on class of dengue infected individuals

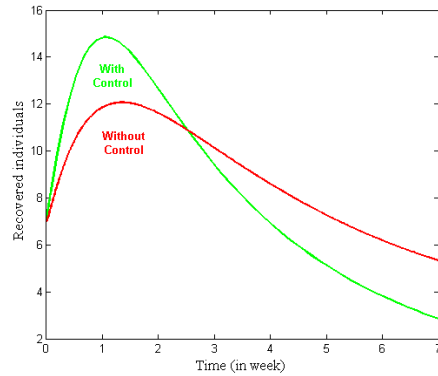
From figure 7(a) and 7(b), it is clear that for class of dengue infected individuals u_2 control is more effective compare to u_1 control moreover it is visualised in figure 7(c) that combine effect of both the control is even more effective which shoews only medication is not enough to minimize dengue infaction case, different acts which minimize concurrent infaction case also have a significant effect.



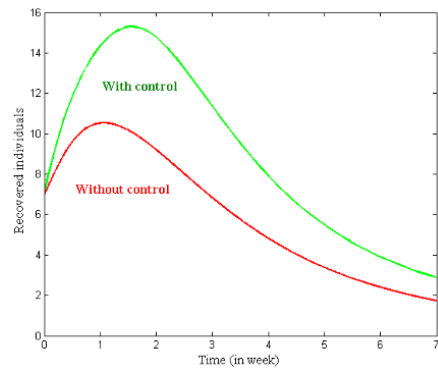
(b) Impact of u_1 control

Figure 8 shows separate and combine effect of both the controls on class of recovered individuals. From figure 8(c) we can observe better improvement in recovered class after applying both the controls at the same time. About 61% improvement is observed in recovery rate

white applying both the control together.



(b) Impact of u_2 control



(c) Impact of both the controls

Figure 8. Impact of controls on class of recovered individuals

Also figure 8(a) and 8(b) deplicate that compare to u_1 control, u_2 control gives better result which suggest that madication plays a major contribution to control the concurrent infection. Hence better medication facility and availability is good approach to control outbreak of malaria-dengue infections in endemic areas.

7. Conclusion

The fight against most deadly mosquito-borne diseases malaria and dengue is a challenge to the world. In the present study, the system of dynamical model for two different mosquito borne diseases is studied through the use of mathematical modeling. Moreover, Optimal control theory is also applied on the model to visualise the effect of controles on it. The model have four equilibrium points for four different possible cases including disease free society, case when only one individual infection is present and the case when both the diseases are present concurrently in society. It is proved that all four equilibrium points are local and globally asymptotically stable under some parametric conditions. The formula of basic reproduction number (R_0) used to calculate threshold value of the model. In this article, the basic reproduction number is formulated for malaria and dengue combinely, hence it




is unaffected by parameter α_4 . Threshold value increases as value of parameters α_1 and α_2 is increases, and it decreases as α_3 and α_6 increases. Which simply means threshold value can be controlled by improving recovery rates of both the diseases. Bifurcation analysis indicates that minimum rate of diseases spread is 12%. Threshold value signifies that there is 14.9% chance to get infected by malaria and dengue concurrently. In numerical simulation we have observed the effect of optimal controls individually as well as concurrently and more stability is observed when we apply both the controls at same time. Also it is analysed that 61% improvement in recovery rate is observed under the effect of both optimal controls, which suggest that vector control by using insecticide, treated mosquito nets and indoor residual spraying and medication to improve recovery are the main way to prevent and reduce malaria and dengue transmission.

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