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RESEARCH ARTICLE

Fractional trapezium type inequalities for twice differentiable preinvex functions and their applications

Artion Kashuri* and Rozana Liko

Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania
artionkashuri@gmail.com, rozanaliko86@gmail.com

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ABSTRACT

Trapezoidal inequalities for functions of divers natures are useful in numerical computations. The authors have proved an identity for a generalized integral operator via twice differentiable preinvex function. By applying the established identity, the generalized trapezoidal type integral inequalities have been discovered. It is pointed out that the results of this research provide integral inequalities for almost all fractional integrals discovered in recent past decades. Various special cases have been identified. Some applications of presented results to special means have been analyzed. The ideas and techniques of this paper may stimulate further research.

1. Introduction

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $a_1, a_2 \in I$ with $a_1 < a_2$. Then the following inequality holds:*

$$f\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1)+f(a_2)}{2}. \tag{1}$$

This inequality (1) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1]–[16], [19, 20, 22, 23].
The aim of this paper is to establish trapezoidal type generalized integral inequalities for preinvex functions. Interestingly, the special cases of presented results, are fractional integral inequalities. Therefore, it is important to summarize the study of fractional integrals. Let us recall some special functions and evoke some basic definitions as follows:

Definition 1. *[13] Let $f \in L[a_1, a_2]$. Then k–fractional integrals of order $\alpha, k > 0$ with $a_1 \geq 0$ are defined by*

$$I_{a_1^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{x}(x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a_1$$

and

$$I_{a_2^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{x}^{a_2}(t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad a_2 > x.$$

For $k = 1$, k–fractional integrals give Riemann–Liouville integrals. For $\alpha = k = 1$, k–fractional integrals give classical integrals.

Definition 2. *[21] A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S\times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.*

The invex set also termed as, an η–connected set.

*Corresponding Author

Definition 3. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y). \tag{2}$$

Also, let define a function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < +\infty, \tag{3}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\varphi(r)} \leq A \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{4}$$

$$\frac{\varphi(r)}{r^2} \leq B\frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{5}$$

$$\left|\frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2}\right| \leq C|r - s|\frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{6}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \geq 0$, then φ satisfies (3)–(6), see [18]. Therefore, the left–sided and right–sided generalized integral operators are defined as follows:

$$_{a_1^+}I_{\varphi}f(x) = \int_{a_1}^{x} \frac{\varphi(x - t)}{x - t} f(t)dt, \quad x > a_1,$$

$$_{a_2^-}I_{\varphi}f(x) = \int_{x}^{a_2} \frac{\varphi(t - x)}{t - x} f(t)dt, \quad x < a_2.$$

The most important feature of generalized integrals is that; they produce Riemann–Liouville fractional integrals, k–Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc.

Motivated by the above literatures, the main objective of this paper is to discover in section 2, an interesting identity in order to study some new bounds regarding general trapezoidal type integral inequalities. By using the established identity as an auxiliary result, some new estimates for trapezoidal type integral inequalities via generalized integrals are obtained. It is pointed out that some new fractional integral inequalities have been deduced from main results. In section 3, some applications to special means are given. In section 4, a briefly conclusion is provided as well. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

2. Main results

Throughout this study, let $P = [ma_1, a_2]$ with $a_1 < a_2$, $m \in (0, 1]$ be an invex subset with respect to $\eta : P \times P \longrightarrow \mathbb{R}$. Also, for brevity, we define

$$\Lambda_{m,n}^{(1)}(t) = \int_0^t \Delta_{m,n}^{(1)}(s)ds, \tag{7}$$

$$\Delta_{m,n}^{(1)}(s) = \int_0^s \frac{\varphi\left(\frac{\eta(x,ma_1)}{n+1}u\right)}{u} du < +\infty, \tag{8}$$

where $\eta(x, ma_1) > 0$ and

$$\Lambda_{m,n}^{(2)}(t) = \int_0^t \Delta_{m,n}^{(2)}(s)ds, \tag{9}$$

$$\Delta_{m,n}^{(2)}(s) = \int_0^s \frac{\varphi\left(\frac{\eta(a_2,mx)}{n+1}u\right)}{u} du < +\infty, \tag{10}$$

where $\eta(a_2, mx) > 0$.

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 1. *Let $f : P \longrightarrow \mathbb{R}$ be a twice differentiable mapping on (ma_1, a_2). If $f'' \in L(P)$, then the following identity for generalized fractional integrals hold:*

$$\begin{aligned}
&\frac{\eta(x, ma_1)\Lambda_{m,n}^{(1)}(1)}{(n+1)\Delta_{m,n}^{(1)}(1)} \times \frac{f'(ma_1) + f'(ma_1 + \eta(x, ma_1))}{2} \\
&- \frac{f(ma_1) + f(ma_1 + \eta(x, ma_1))}{2} - \frac{1}{2\Delta_{m,n}^{(1)}(1)} \\
&\times \left[{}_{(ma_1)^+}I_{\varphi}f\left(ma_1 + \frac{\eta(x, ma_1)}{n+1}\right) + {}_{(ma_1+\eta(x,ma_1))^-}I_{\varphi}f\left(ma_1 + \frac{n}{n+1}\eta(x, ma_1)\right)\right] \\
&+ \frac{\eta(a_2, mx)\Lambda_{m,n}^{(2)}(1)}{(n+1)\Delta_{m,n}^{(2)}(1)} \times \frac{f'(mx) + f'(mx + \eta(a_2, mx))}{2} \\
&- \frac{f(mx) + f(mx + \eta(a_2, mx))}{2} - \frac{1}{2\Delta_{m,n}^{(2)}(1)} \\
&\times \left[{}_{(mx)^+}I_{\varphi}f\left(mx + \frac{\eta(a_2, mx)}{n+1}\right) + {}_{(mx+\eta(a_2,mx))^-}I_{\varphi}f\left(mx + \frac{n}{n+1}\eta(a_2, mx)\right)\right] \\
&= \frac{\eta^2(x, ma_1)}{2(n+1)^2\Delta_{m,n}^{(1)}(1)} \\
&\times \int_0^1 \Lambda_{m,n}^{(1)}(t)\left[f''\left(ma_1 + \frac{(n+t)}{n+1}\eta(x, ma_1)\right) - f''\left(ma_1 + \frac{(1-t)}{n+1}\eta(x, ma_1)\right)\right]dt
\end{aligned} \tag{11}$$

$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta^{(2)}_{m,n}(1)}\times\int_0^1\Lambda^{(2)}_{m,n}(t)\left[f''\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)-f''\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)\right]dt.$$

We denote

$$I_{f,\Lambda^{(1)}_{m,n},\Lambda^{(2)}_{m,n},\Delta^{(1)}_{m,n},\Delta^{(2)}_{m,n}}(x,a_1,a_2)=\frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta^{(1)}_{m,n}(1)}\times\int_0^1\Lambda^{(1)}_{m,n}(t)\left[f''\left(ma_1+\frac{(n+t)}{n+1}\eta(x,ma_1)\right)-f''\left(ma_1+\frac{(1-t)}{n+1}\eta(x,ma_1)\right)\right]dt$$
$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta^{(2)}_{m,n}(1)}\times\int_0^1\Lambda^{(2)}_{m,n}(t)\left[f''\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)-f''\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)\right]dt. \tag{12}$$

Proof. Integrating by parts twice (12) and changing the variables of integration, we have

$$I_{f,\Lambda^{(1)}_{m,n},\Lambda^{(2)}_{m,n},\Delta^{(1)}_{m,n},\Delta^{(2)}_{m,n}}(x,a_1,a_2)=\frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta^{(1)}_{m,n}(1)}\times\left\{\frac{(n+1)\Lambda^{(1)}_{m,n}(t)f'\left(ma_1+\frac{(n+t)}{n+1}\eta(x,ma_1)\right)}{\eta(x,ma_1)}\Bigg|_0^1-\frac{(n+1)}{\eta(x,ma_1)}\times\int_0^1\Delta^{(1)}_{m,n}(t)f'\left(ma_1+\frac{(n+t)}{n+1}\eta(x,ma_1)\right)dt\right.$$
$$\left.+\frac{(n+1)\Lambda^{(1)}_{m,n}(t)f'\left(ma_1+\frac{(1-t)}{n+1}\eta(x,ma_1)\right)}{\eta(x,ma_1)}\Bigg|_0^1-\frac{(n+1)}{\eta(x,ma_1)}\times\int_0^1\Delta^{(1)}_{m,n}(t)f'\left(ma_1+\frac{(1-t)}{n+1}\eta(x,ma_1)\right)dt\right\}$$
$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta^{(2)}_{m,n}(1)}\times\left\{\frac{(n+1)\Lambda^{(2)}_{m,n}(t)f'\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)}{\eta(a_2,mx)}\Bigg|_0^1-\frac{(n+1)}{\eta(a_2,mx)}\times\int_0^1\Delta^{(2)}_{m,n}(t)f'\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)dt\right.$$
$$\left.+\frac{(n+1)\Lambda^{(2)}_{m,n}(t)f'\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)}{\eta(a_2,mx)}\Bigg|_0^1-\frac{(n+1)}{\eta(a_2,mx)}\times\int_0^1\Delta^{(2)}_{m,n}(t)f'\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)dt\right\}$$
$$=\frac{\eta^2(x,ma_1)}{2(n+1)^2\Lambda^{(1)}_{m,n}(1)}\times\left\{\frac{(n+1)\Lambda^{(1)}_{m,n}(1)f'(ma_1+\eta(x,ma_1))}{\eta(x,ma_1)}-\frac{(n+1)}{\eta(x,ma_1)}\times\left[\frac{(n+1)\Delta^{(1)}_{m,n}(1)f(ma_1+\eta(x,ma_1))}{\eta(x,ma_1)}-\frac{(n+1)}{\eta(x,ma_1)}\times{}_{(ma_1+\eta(x,ma_1))^-}I_\varphi f\left(ma_1+\frac{n}{n+1}\eta(x,ma_1)\right)\right]\right.$$
$$\left.+\frac{(n+1)\Lambda^{(1)}_{m,n}(1)f'(ma_1)}{\eta(x,ma_1)}-\frac{(n+1)}{\eta(x,ma_1)}\times\left[\frac{(n+1)\Delta^{(1)}_{m,n}(1)f(ma_1)}{\eta(x,ma_1)}-\frac{(n+1)}{\eta(x,ma_1)}\times{}_{(ma_1)^+}I_\varphi f\left(ma_1+\frac{\eta(x,ma_1)}{n+1}\right)\right]\right\}$$
$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Lambda^{(2)}_{m,n}(1)}\times\left\{\frac{(n+1)\Lambda^{(2)}_{m,n}(1)f'(mx+\eta(a_2,mx))}{\eta(a_2,mx)}-\frac{(n+1)}{\eta(a_2,mx)}\times\left[\frac{(n+1)\Delta^{(2)}_{m,n}(1)f(mx+\eta(a_2,mx))}{\eta(a_2,mx)}-\frac{(n+1)}{\eta(a_2,mx)}\times{}_{(mx+\eta(a_2,mx))^-}I_\varphi f\left(mx+\frac{n}{n+1}\eta(a_2,mx)\right)\right]\right.$$
$$+\frac{(n+1)\Lambda^{(2)}_{m,n}(1)f'(mx)}{\eta(a_2,mx)}-\frac{(n+1)}{\eta(a_2,mx)}$$

$$\times\left[\frac{(n+1)\Delta_{m,n}^{(2)}(1)f(mx)}{\eta(a_2,mx)}\right.$$
$$\left.-\frac{(n+1)}{\eta(a_2,mx)}\times {}_{(mx)^+}I_{\varphi}f\left(mx+\frac{\eta(a_2,mx)}{n+1}\right)\right\}$$
$$=\frac{\eta(x,ma_1)\Lambda_{m,n}^{(1)}(1)}{(n+1)\Delta_{m,n}^{(1)}(1)}$$
$$\times\frac{f'(ma_1)+f'(ma_1+\eta(x,ma_1))}{2}$$
$$-\frac{f(ma_1)+f(ma_1+\eta(x,ma_1))}{2}-\frac{1}{2\Delta_{m,n}^{(1)}(1)}$$
$$\times\left[{}_{(ma_1)^+}I_{\varphi}f\left(ma_1+\frac{\eta(x,ma_1)}{n+1}\right)\right.$$
$$\left.+{}_{(ma_1+\eta(x,ma_1))^-}I_{\varphi}f\left(ma_1+\frac{n}{n+1}\eta(x,ma_1)\right)\right]$$
$$+\frac{\eta(a_2,mx)\Lambda_{m,n}^{(2)}(1)}{(n+1)\Delta_{m,n}^{(2)}(1)}$$
$$\times\frac{f'(mx)+f'(mx+\eta(a_2,mx))}{2}$$
$$-\frac{f(mx)+f(mx+\eta(a_2,mx))}{2}-\frac{1}{2\Delta_{m,n}^{(2)}(1)}$$
$$\times\left[{}_{(mx)^+}I_{\varphi}f\left(mx+\frac{\eta(a_2,mx)}{n+1}\right)\right.$$
$$\left.+{}_{(mx+\eta(a_2,mx))^-}I_{\varphi}f\left(mx+\frac{n}{n+1}\eta(a_2,mx)\right)\right].$$

The proof of Lemma 1 is completed. □

Remark 1. *Taking* $m=1, n=0, x=\frac{a_1+a_2}{2}$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$ *and* $\varphi(t)=t$ *in Lemma 1, we get*

$$I_{f,\Lambda_{1,0}^{(1)},\Lambda_{1,0}^{(2)},\Delta_{1,0}^{(1)},\Delta_{1,0}^{(2)}}\left(\frac{a_1+a_2}{2},a_1,a_2\right)$$
$$=\left(\frac{a_2-a_1}{2}\right) \tag{13}$$
$$\times\left[\frac{f'(a_1)+2f'\left(\frac{a_1+a_2}{2}\right)+f'(a_2)}{2}\right]$$
$$-\left[\frac{f(a_1)+2f\left(\frac{a_1+a_2}{2}\right)+f(a_2)}{2}\right]$$
$$-\frac{2}{(a_2-a_1)}\int_{a_1}^{a_2}f(t)dt.$$

Theorem 2. *Let* $f:P\longrightarrow\mathbb{R}$ *be a twice differentiable mapping on* (ma_1,a_2). *If* $|f''|^q$ *is preinvex on* P *for* $q>1$ *and* $p^{-1}+q^{-1}=1$, *then the following inequality for generalized fractional integrals hold:*

$$\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|$$
$$\le\frac{\eta^2(x,ma_1)}{2(n+1)^2\sqrt[q]{2(n+1)}\Delta_{m,n}^{(1)}(1)} \tag{14}$$
$$\times\sqrt[p]{B_{\Lambda_{m,n}^{(1)}}(p)}$$
$$\times\left\{\sqrt[q]{|f''(ma_1)|^q+(2n+1)|f''(x)|^q}\right.$$
$$\left.+\sqrt[q]{(2n+1)|f''(ma_1)|^q+|f''(x)|^q}\right\}$$
$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\sqrt[q]{2(n+1)}\Delta_{m,n}^{(2)}(1)}\sqrt[p]{B_{\Lambda_{m,n}^{(2)}}(p)}$$
$$\times\left\{\sqrt[q]{|f''(mx)|^q+(2n+1)|f''(a_2)|^q}\right.$$
$$\left.+\sqrt[q]{(2n+1)|f''(mx)|^q+|f''(a_2)|^q}\right\},$$

where

$$B_{\Lambda_{m,n}^{(i)}}(p)=\int_0^1\left[\Lambda_{m,n}^{(i)}(t)\right]^p dt,\quad \forall\, i=1,2. \tag{15}$$

Proof. From Lemma 1, preinvexity of $|f''|^q$, Hölder's inequality and properties of the modulus, we have

$$\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|$$
$$\le\frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta_{m,n}^{(1)}(1)}$$
$$\times\left\{\int_0^1\Lambda_{m,n}^{(1)}(t)\left[\left|f''\left(ma_1+\frac{(n+t)}{n+1}\eta(x,ma_1)\right)\right|\right.\right.$$
$$\left.\left.+\left|f''\left(ma_1+\frac{(1-t)}{n+1}\eta(x,ma_1)\right)\right|\right]dt\right\}$$
$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta_{m,n}^{(2)}(1)}$$
$$\times\left\{\int_0^1\Lambda_{m,n}^{(2)}(t)\left[\left|f''\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)\right|\right.\right.$$
$$\left.\left.+\left|f''\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)\right|\right]dt\right\}$$
$$\le\frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta_{m,n}^{(1)}(1)}\left(\int_0^1\left[\Lambda_{m,n}^{(1)}(t)\right]^p dt\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left|f''\left(ma_1+\frac{(n+t)}{n+1}\eta(x,ma_1)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|f''\left(ma_1+\frac{(1-t)}{n+1}\eta(x,ma_1)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta_{m,n}^{(2)}(1)}\left(\int_0^1\left[\Lambda_{m,n}^{(2)}(t)\right]^p dt\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left|f''\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|f''\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta_{m,n}^{(1)}(1)}\sqrt[p]{B_{\Lambda_{m,n}^{(1)}}(p)}$$

$$\times\Bigg\{\Bigg[\int_0^1\Bigg[\left(1-\frac{n+t}{n+1}\right)\left|f''(ma_1)\right|^q + \frac{(n+t)}{n+1}\left|f''(x)\right|^q\Bigg]dt\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[\int_0^1\Bigg[\left(1-\frac{1-t}{n+1}\right)\left|f''(ma_1)\right|^q + \frac{(1-t)}{n+1}\left|f''(x)\right|^q\Bigg]dt\Bigg]^{\frac{1}{q}}\Bigg\}$$

$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta_{m,n}^{(2)}(1)}\sqrt[p]{B_{\Lambda_{m,n}^{(2)}}(p)}$$

$$\times\Bigg\{\Bigg[\int_0^1\Bigg[\left(1-\frac{1-t}{n+1}\right)\left|f''(mx)\right|^q + \frac{(1-t)}{n+1}\left|f''(a_2)\right|^q\Bigg]dt\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[\int_0^1\Bigg[\left(1-\frac{n+t}{n+1}\right)\left|f''(mx)\right|^q + \frac{(n+t)}{n+1}\left|f''(a_2)\right|^q\Bigg]dt\Bigg]^{\frac{1}{q}}\Bigg\}$$

$$=\frac{\eta^2(x,ma_1)}{2(n+1)^2\sqrt[q]{2(n+1)}\Delta_{m,n}^{(1)}(1)}\sqrt[p]{B_{\Lambda_{m,n}^{(1)}}(p)}$$

$$\times\Big\{\sqrt[q]{|f''(ma_1)|^q+(2n+1)|f''(x)|^q} + \sqrt[q]{(2n+1)|f''(ma_1)|^q+|f''(x)|^q}\Big\}$$

$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\sqrt[q]{2(n+1)}\Delta_{m,n}^{(2)}(1)}\sqrt[p]{B_{\Lambda_{m,n}^{(2)}}(p)}$$

$$\times\Big\{\sqrt[q]{|f''(mx)|^q+(2n+1)|f''(a_2)|^q} + \sqrt[q]{(2n+1)|f''(mx)|^q+|f''(a_2)|^q}\Big\}.$$

The proof of Theorem 2 is completed. □

We point out some special cases of Theorem 2.

Corollary 1. *Taking $p=q=2$ in Theorem 2, we have*

$$\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|$$

$$\leq \frac{\eta^2(x,ma_1)}{2(n+1)^2\sqrt{2(n+1)}\Delta_{m,n}^{(1)}(1)} \tag{16}$$

$$\times\sqrt{B_{\Lambda_{m,n}^{(1)}}(2)}$$

$$\times\Big\{\sqrt{|f''(ma_1)|^2+(2n+1)|f''(x)|^2} + \sqrt{(2n+1)|f''(ma_1)|^2+|f''(x)|^2}\Big\}$$

$$+\frac{\eta^2(a_2,mx)}{2(n+1)^2\sqrt{2(n+1)}\Delta_{m,n}^{(2)}(1)}\sqrt{B_{\Lambda_{m,n}^{(2)}}(2)}$$

$$\times\Big\{\sqrt{|f''(mx)|^2+(2n+1)|f''(a_2)|^2} + \sqrt{(2n+1)|f''(mx)|^2+|f''(a_2)|^2}\Big\}.$$

Corollary 2. *Taking $\varphi(t)=t$ in Theorem 2, we get*

$$\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|$$

$$\leq \frac{\eta^2(x,ma_1)}{4(n+1)^2\sqrt[q]{2(n+1)}\sqrt[p]{2p+1}} \tag{17}$$

$$\times\Big\{\sqrt[q]{|f''(ma_1)|^q+(2n+1)|f''(x)|^q} + \sqrt[q]{(2n+1)|f''(ma_1)|^q+|f''(x)|^q}\Big\}$$

$$+\frac{\eta^2(a_2,mx)}{4(n+1)^2\sqrt[q]{2(n+1)}\sqrt[p]{2p+1}}$$

$$\times\Big\{\sqrt[q]{|f''(mx)|^q+(2n+1)|f''(a_2)|^q} + \sqrt[q]{(2n+1)|f''(mx)|^q+|f''(a_2)|^q}\Big\}.$$

Corollary 3. *Taking $x=\frac{a_1+a_2}{2}$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$ and $\eta(a_2,mx)=a_2-mx$ in Corollary 2, we obtain*

$$\left|I_{f,\Lambda_{1,0}^{(1)},\Lambda_{1,0}^{(2)},\Delta_{1,0}^{(1)},\Delta_{1,0}^{(2)}}\left(\frac{a_1+a_2}{2},a_1,a_2\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{8\sqrt[q]{2}\sqrt[p]{2p+1}} \tag{18}$$

$$\times\left\{\sqrt[q]{|f''(a_1)|^q+\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q} + \sqrt[q]{\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+|f''(a_2)|^q}\right\}.$$

Corollary 4. *Taking $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2, we have*

$$\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|$$

$$\leq \frac{\eta^2(x,ma_1)}{4(n+1)^2\sqrt[q]{2(n+1)}\sqrt[p]{2p\alpha+1}} \tag{19}$$

$$\times\Big\{\sqrt[q]{|f''(ma_1)|^q+(2n+1)|f''(x)|^q} + \sqrt[q]{(2n+1)|f''(ma_1)|^q+|f''(x)|^q}\Big\}$$

$$+\frac{\eta^2(a_2,mx)}{4(n+1)^2\sqrt[q]{2(n+1)}\sqrt[p]{2p\alpha+1}}$$

$$\times\Big\{\sqrt[q]{|f''(mx)|^q+(2n+1)|f''(a_2)|^q}$$

$$+\sqrt[q]{(2n+1)|f''(mx)|^q+|f''(a_2)|^q}\Big\}.$$

Corollary 5. *Taking $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2, we get*

$$\begin{aligned}
&\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right| \\
&\leq \frac{\eta^2(x,ma_1)}{4(n+1)^2\sqrt[q]{2(n+1)}\sqrt[p]{\frac{2p\alpha}{k}+1}} \qquad (20)\\
&\times\Big\{\sqrt[q]{|f''(ma_1)|^q+(2n+1)|f''(x)|^q} \\
&+\sqrt[q]{(2n+1)|f''(ma_1)|^q+|f''(x)|^q}\Big\} \\
&+\frac{\eta^2(a_2,mx)}{4(n+1)^2\sqrt[q]{2(n+1)}\sqrt[p]{\frac{2p\alpha}{k}+1}} \\
&\times\Big\{\sqrt[q]{|f''(mx)|^q+(2n+1)|f''(a_2)|^q} \\
&+\sqrt[q]{(2n+1)|f''(mx)|^q+|f''(a_2)|^q}\Big\}.
\end{aligned}$$

Corollary 6. *Taking $\varphi(t)=t(a_2-t)^{\alpha-1}$ and $f(x)$ is symmetric to $x=\frac{ma_1+a_2}{2}$, in Theorem 2, we obtain*

$$\begin{aligned}
&\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}\left(\frac{ma_1+a_2}{2},a_1,a_2\right)\right| \\
&\leq \frac{\frac{\alpha\eta^2\left(\frac{ma_1+a_2}{2},ma_1\right)}{2(n+1)^2\sqrt[q]{2(n+1)}}\sqrt[p]{B^*_{\Lambda_{m,n}^{(1)}}(p)}}{\left[a_2^{\alpha}-\frac{(n+1)}{(\alpha+1)\eta\left(\frac{ma_1+a_2}{2},ma_1\right)}\left(a_2^{\alpha+1}-\left(a_2-\frac{\eta\left(\frac{ma_1+a_2}{2},ma_1\right)}{n+1}\right)^{\alpha+1}\right)\right]} \qquad (21)\\
&\times\left\{\sqrt[q]{|f''(ma_1)|^q+(2n+1)\left|f''\left(\frac{ma_1+a_2}{2}\right)\right|^q}\right. \\
&\left.+\sqrt[q]{(2n+1)|f''(ma_1)|^q+\left|f''\left(\frac{ma_1+a_2}{2}\right)\right|^q}\right\} \\
&+\frac{\frac{\alpha\eta^2\left(a_2,m\frac{(ma_1+a_2)}{2}\right)}{2(n+1)^2\sqrt[q]{2(n+1)}}\sqrt[p]{B^*_{\Lambda_{m,n}^{(2)}}(p)}}{\left[a_2^{\alpha}-\frac{(n+1)}{(\alpha+1)\eta\left(a_2,m\frac{(ma_1+a_2)}{2}\right)}\left(a_2^{\alpha+1}-\left(a_2-\frac{\eta\left(a_2,m\frac{(ma_1+a_2)}{2}\right)}{n+1}\right)^{\alpha+1}\right)\right]} \\
&\times\left\{\sqrt[q]{\left|f''\left(m\frac{(ma_1+a_2)}{2}\right)\right|^q+(2n+1)|f''(a_2)|^q}\right. \\
&\left.+\sqrt[q]{(2n+1)\left|f''\left(m\frac{(ma_1+a_2)}{2}\right)\right|^q+|f''(a_2)|^q}\right\},
\end{aligned}$$

where

$$\begin{aligned}
B^*_{\Lambda_{m,n}^{(1)}}(p)&=\frac{1}{\alpha} \qquad (22)\\
&\times\int_0^1\left[a_2^{\alpha}t-\frac{(n+1)}{(\alpha+1)\eta\left(\frac{ma_1+a_2}{2},ma_1\right)}\right. \\
&\times\left.\left(a_2^{\alpha+1}-\left(a_2-\frac{\eta\left(\frac{ma_1+a_2}{2},ma_1\right)t}{n+1}\right)^{\alpha+1}\right)\right]^p dt
\end{aligned}$$

and

$$\begin{aligned}
B^*_{\Lambda_{m,n}^{(2)}}(p)&=\frac{1}{\alpha} \qquad (23)\\
&\times\int_0^1\left[a_2^{\alpha}t-\frac{(n+1)}{(\alpha+1)\eta\left(a_2,m\frac{(ma_1+a_2)}{2}\right)}\right. \\
&\times\left.\left(a_2^{\alpha+1}-\left(a_2-\frac{\eta\left(a_2,m\frac{(ma_1+a_2)}{2}\right)t}{n+1}\right)^{\alpha+1}\right)\right]^p dt.
\end{aligned}$$

Corollary 7. *Taking $\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha\in(0,1)$, in Theorem 2, we have*

$$\begin{aligned}
&\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right| \\
&\leq\frac{(\alpha-1)\eta^2(x,ma_1)}{2(n+1)^2\sqrt[q]{2(n+1)}} \qquad (24)\\
&\times\frac{1}{\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(x,ma_1)}{n+1}\right]-1\right\}} \\
&\times\sqrt[p]{B^{\diamond}_{\Lambda_{m,n}^{(1)}}(p)} \\
&\times\Big\{\sqrt[q]{|f''(ma_1)|^q+(2n+1)|f''(x)|^q} \\
&+\sqrt[q]{(2n+1)|f''(ma_1)|^q+|f''(x)|^q}\Big\} \\
&+\frac{(\alpha-1)\eta^2(a_2,mx)}{2(n+1)^2\sqrt[q]{2(n+1)}\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(a_2,mx)}{n+1}\right]-1\right\}} \\
&\times\sqrt[p]{B^{\diamond}_{\Lambda_{m,n}^{(2)}}(p)} \\
&\times\Big\{\sqrt[q]{|f''(mx)|^q+(2n+1)|f''(a_2)|^q} \\
&+\sqrt[q]{(2n+1)|f''(mx)|^q+|f''(a_2)|^q}\Big\},
\end{aligned}$$

where

$$\begin{aligned}
B^{\diamond}_{\Lambda_{m,n}^{(1)}}(p)&=\frac{1}{(\alpha-1)^p} \qquad (25)\\
&\times\int_0^1\left[\frac{(n+1)\alpha}{(\alpha-1)\eta(x,ma_1)}\right. \\
&\times\left.\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(x,ma_1)}{(n+1)}t\right]-(t+1)\right\}\right]^p dt
\end{aligned}$$

and

$$\begin{aligned}
B^{\diamond}_{\Lambda_{m,n}^{(2)}}(p)&=\frac{1}{(\alpha-1)^p} \qquad (26)\\
&\times\int_0^1\left[\frac{(n+1)\alpha}{(\alpha-1)\eta(a_2,mx)}\right.
\end{aligned}$$

$$\times\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(a_2,mx)}{(n+1)}t\right]-(t+1)\right\}\Bigg]^p dt.$$

Theorem 3. *Let $f: P \longrightarrow \mathbb{R}$ be a twice differentiable mapping on (ma_1, a_2). If $|f''|^q$ is preinvex on P for $q \geq 1$, then the following inequality for generalized fractional integrals hold:*

$$\begin{aligned}&\left|I_{f,\Lambda^{(1)}_{m,n},\Lambda^{(2)}_{m,n},\Delta^{(1)}_{m,n},\Delta^{(2)}_{m,n}}(x,a_1,a_2)\right|\\
&\leq\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\frac{\eta^2(x,ma_1)}{2\Delta^{(1)}_{m,n}(1)}\\
&\times\left(B_{\Lambda^{(1)}_{m,n}}(1)\right)^{1-\frac{1}{q}}\\
&\times\left\{\sqrt[q]{C_{\Lambda^{(1)}_{m,n}}|f''(ma_1)|^q+D_{\Lambda^{(1)}_{m,n}}(n)|f''(x)|^q}\right.\\
&\left.+\sqrt[q]{D_{\Lambda^{(1)}_{m,n}}(n)|f''(ma_1)|^q+C_{\Lambda^{(1)}_{m,n}}|f''(x)|^q}\right\}\\
&+\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\frac{\eta^2(a_2,mx)}{2\Delta^{(2)}_{m,n}(1)}\left(B_{\Lambda^{(2)}_{m,n}}(1)\right)^{1-\frac{1}{q}}\\
&\times\left\{\sqrt[q]{C_{\Lambda^{(2)}_{m,n}}|f''(mx)|^q+D_{\Lambda^{(2)}_{m,n}}(n)|f''(a_2)|^q}\right.\\
&\left.+\sqrt[q]{D_{\Lambda^{(2)}_{m,n}}(n)|f''(mx)|^q+C_{\Lambda^{(2)}_{m,n}}|f''(a_2)|^q}\right\},\end{aligned}\tag{27}$$

where

$$C_{\Lambda^{(i)}_{m,n}}=\int_0^1(1-t)\Lambda^{(i)}_{m,n}(t)dt,\quad \forall\, i=1,2 \tag{28}$$

$$D_{\Lambda^{(i)}_{m,n}}(n)=\int_0^1(n+t)\Lambda^{(i)}_{m,n}(t)dt,\ \forall\, i=1,2 \tag{29}$$

and $B_{\Lambda^{(i)}_{m,n}}(1)$, $\forall\, i=1,2$, are defined as in Theorem 2, where $p=1$.

Proof. From Lemma 1, preinvexity of $|f''|^q$, power mean inequality and properties of the modulus, we have

$$\begin{aligned}&\left|I_{f,\Lambda^{(1)}_{m,n},\Lambda^{(2)}_{m,n},\Delta^{(1)}_{m,n},\Delta^{(2)}_{m,n}}(x,a_1,a_2)\right|\\
&\leq\frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta^{(1)}_{m,n}(1)}\\
&\times\left\{\int_0^1\Lambda^{(1)}_{m,n}(t)\left[\left|f''\left(ma_1+\frac{(n+t)}{n+1}\eta(x,ma_1)\right)\right|\right.\right.\\
&\left.\left.+\left|f''\left(ma_1+\frac{(1-t)}{n+1}\eta(x,ma_1)\right)\right|\right]dt\right\}\\
&+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta^{(2)}_{m,n}(1)}\\
&\times\left\{\int_0^1\Lambda^{(2)}_{m,n}(t)\left[\left|f''\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)\right|\right.\right.\\
&\left.\left.+\left|f''\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)\right|\right]dt\right\}\\
&\leq\frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta^{(1)}_{m,n}(1)}\left(\int_0^1\Lambda^{(1)}_{m,n}(t)dt\right)^{1-\frac{1}{q}}\\
&\times\left\{\left(\int_0^1\Lambda^{(1)}_{m,n}(t)\left|f''\left(ma_1+\frac{(n+t)}{n+1}\eta(x,ma_1)\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\Lambda^{(1)}_{m,n}(t)\left|f''\left(ma_1+\frac{(1-t)}{n+1}\eta(x,ma_1)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta^{(2)}_{m,n}(1)}\left(\int_0^1\Lambda^{(2)}_{m,n}(t)dt\right)^{1-\frac{1}{q}}\\
&\times\left\{\left(\int_0^1\Lambda^{(2)}_{m,n}(t)\left|f''\left(mx+\frac{(1-t)}{n+1}\eta(a_2,mx)\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\Lambda^{(2)}_{m,n}(t)\left|f''\left(mx+\frac{(n+t)}{n+1}\eta(a_2,mx)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{\eta^2(x,ma_1)}{2(n+1)^2\Delta^{(1)}_{m,n}(1)}\left(B_{\Lambda^{(1)}_{m,n}}(1)\right)^{1-\frac{1}{q}}\\
&\times\left\{\left[\int_0^1\Lambda^{(1)}_{m,n}(t)\right.\right.\\
&\times\left.\left[\left(1-\frac{n+t}{n+1}\right)|f''(ma_1)|^q+\frac{(n+t)}{n+1}|f''(x)|^q\right]dt\right]^{\frac{1}{q}}\\
&+\left[\int_0^1\Lambda^{(1)}_{m,n}(t)\right.\\
&\times\left.\left.\left[\left(1-\frac{1-t}{n+1}\right)|f''(ma_1)|^q+\frac{(1-t)}{n+1}|f''(x)|^q\right]dt\right]^{\frac{1}{q}}\right\}\\
&+\frac{\eta^2(a_2,mx)}{2(n+1)^2\Delta^{(2)}_{m,n}(1)}\left(B_{\Lambda^{(2)}_{m,n}}(1)\right)^{1-\frac{1}{q}}\\
&\times\left\{\left[\int_0^1\Lambda^{(2)}_{m,n}(t)\right.\right.\\
&\times\left.\left[\left(1-\frac{1-t}{n+1}\right)|f''(mx)|^q+\frac{(1-t)}{n+1}|f''(a_2)|^q\right]dt\right]^{\frac{1}{q}}\\
&+\left[\int_0^1\Lambda^{(2)}_{m,n}(t)\right.\\
&\times\left.\left.\left[\left(1-\frac{n+t}{n+1}\right)|f''(mx)|^q+\frac{(n+t)}{n+1}|f''(a_2)|^q\right]dt\right]^{\frac{1}{q}}\right\}\\
&=\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\frac{\eta^2(x,ma_1)}{2\Delta^{(1)}_{m,n}(1)}\left(B_{\Lambda^{(1)}_{m,n}}(1)\right)^{1-\frac{1}{q}}\\
&\times\left\{\sqrt[q]{C_{\Lambda^{(1)}_{m,n}}|f''(ma_1)|^q+D_{\Lambda^{(1)}_{m,n}}(n)|f''(x)|^q}\right.\end{aligned}$$

$$+\sqrt[q]{D_{\Lambda_{m,n}^{(1)}}(n)|f''(ma_1)|^q+C_{\Lambda_{m,n}^{(1)}}|f''(x)|^q}\Bigg\}$$

$$+\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\frac{\eta^2(a_2,mx)}{2\Delta_{m,n}^{(2)}(1)}\left(B_{\Lambda_{m,n}^{(2)}}(1)\right)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\sqrt[q]{C_{\Lambda_{m,n}^{(2)}}|f''(mx)|^q+D_{\Lambda_{m,n}^{(2)}}(n)|f''(a_2)|^q}$$

$$+\sqrt[q]{D_{\Lambda_{m,n}^{(2)}}(n)|f''(mx)|^q+C_{\Lambda_{m,n}^{(2)}}|f''(a_2)|^q}\Bigg\}.$$

The proof of Theorem 3 is completed. □

We point out some special cases of Theorem 3.

Corollary 8. *Taking $q=1$ in Theorem 3, we have*

$$\begin{aligned}&\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|\\&\le\frac{1}{(n+1)^3}\Bigg\{\frac{\eta^2(x,ma_1)}{2\Delta_{m,n}^{(1)}(1)}\\&\times\left(C_{\Lambda_{m,n}^{(1)}}+D_{\Lambda_{m,n}^{(1)}}(n)\right)\Big[|f''(ma_1)|+|f''(x)|\Big]\\&+\frac{\eta^2(a_2,mx)}{2\Delta_{m,n}^{(2)}(1)}\\&\times\left(C_{\Lambda_{m,n}^{(2)}}+D_{\Lambda_{m,n}^{(2)}}(n)\right)\Big[|f''(mx)|+|f''(a_2)|\Big]\Bigg\}.\end{aligned}\tag{30}$$

Corollary 9. *Taking $\varphi(t)=t$ in Theorem 3, we get*

$$\begin{aligned}&\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|\\&\le\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\frac{\eta^2(x,ma_1)}{12\sqrt[q]{4}}\\&\times\Big\{\sqrt[q]{|f''(ma_1)|^q+(4n+3)|f''(x)|^q}\\&+\sqrt[q]{(4n+3)|f''(ma_1)|^q+|f''(x)|^q}\Big\}\\&+\frac{\eta^2(a_2,mx)}{12\sqrt[q]{4}}\\&\times\Big\{\sqrt[q]{|f''(mx)|^q+(4n+3)|f''(a_2)|^q}\\&+\sqrt[q]{(4n+3)|f''(mx)|^q+|f''(a_2)|^q}\Big\}.\end{aligned}\tag{31}$$

Corollary 10. *Taking $x=\frac{a_1+a_2}{2}$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$ and $\eta(a_2,mx)=a_2-mx$ in Corollary 9, we obtain*

$$\begin{aligned}&\left|I_{f,\Lambda_{1,0}^{(1)},\Lambda_{1,0}^{(2)},\Delta_{1,0}^{(1)},\Delta_{1,0}^{(2)}}\left(\frac{a_1+a_2}{2},a_1,a_2\right)\right|\\&\le\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\frac{(a_2-a_1)^2}{48\sqrt[q]{4}}\\&\times\Bigg\{\sqrt[q]{|f''(a_1)|^q+3\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q}\\&+\sqrt[q]{3|f''(a_1)|^q+\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q}\\&+\sqrt[q]{\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+3|f''(a_2)|^q}\\&+\sqrt[q]{3\left|f''\left(\frac{a_1+a_2}{2}\right)\right|^q+|f''(a_2)|^q}\Bigg\}.\end{aligned}\tag{32}$$

Corollary 11. *Taking $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 3, we have*

$$\begin{aligned}&\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|\\&\le\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\\&\times\frac{\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\sqrt[q]{\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+4)}}\eta^2(x,ma_1)\\&\times\Bigg\{\sqrt[q]{|f''(ma_1)|^q+\big[n(\alpha+3)+(\alpha+2)\big]|f''(x)|^q}\\&+\sqrt[q]{\big[n(\alpha+3)+(\alpha+2)\big]|f''(ma_1)|^q+|f''(x)|^q}\Bigg\}\\&+\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\frac{\Gamma(\alpha+1)}{2\Gamma(\alpha+3)}\sqrt[q]{\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+4)}}\eta^2(a_2,mx)\\&\times\Bigg\{\sqrt[q]{|f''(mx)|^q+\big[n(\alpha+3)+(\alpha+2)\big]|f''(a_2)|^q}\\&+\sqrt[q]{\big[n(\alpha+3)+(\alpha+2)\big]|f''(mx)|^q+|f''(a_2)|^q}\Bigg\}.\end{aligned}\tag{33}$$

Corollary 12. *Taking $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 3, we get*

$$\begin{aligned}&\left|I_{f,\Lambda_{m,n}^{(1)},\Lambda_{m,n}^{(2)},\Delta_{m,n}^{(1)},\Delta_{m,n}^{(2)}}(x,a_1,a_2)\right|\\&\le\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\\&\times\frac{\Gamma_k(\alpha+k)}{2\Gamma_k(\alpha+k+2)}\sqrt[q]{\frac{\Gamma_k(\alpha+k+2)}{\Gamma_k(\alpha+k+3)}}\eta^2(x,ma_1)\\&\times\Bigg\{\sqrt[q]{|f''(ma_1)|^q+\left[n\left(\frac{\alpha}{k}+3\right)+\left(\frac{\alpha}{k}+2\right)\right]|f''(x)|^q}\\&+\sqrt[q]{\left[n\left(\frac{\alpha}{k}+3\right)+\left(\frac{\alpha}{k}+2\right)\right]|f''(ma_1)|^q+|f''(x)|^q}\Bigg\}\\&+\left(\frac{1}{n+1}\right)^{\frac{2q+1}{q}}\\&\times\frac{\Gamma_k(\alpha+k)}{2\Gamma_k(\alpha+k+2)}\sqrt[q]{\frac{\Gamma_k(\alpha+k+2)}{\Gamma_k(\alpha+k+3)}}\eta^2(a_2,mx)\end{aligned}\tag{34}$$

$$\times\Bigg\{\sqrt[q]{|f''(mx)|^q+\left[n\left(\frac{\alpha}{k}+3\right)+\left(\frac{\alpha}{k}+2\right)\right]|f''(a_2)|^q}+\sqrt[q]{\left[n\left(\frac{\alpha}{k}+3\right)+\left(\frac{\alpha}{k}+2\right)\right]|f''(mx)|^q+|f''(a_2)|^q}\Bigg\}.$$

Remark 2. *Applying our Theorems 2 and 3, for $n\in\mathbb{N}^*$ and appropriate choices of function $\varphi(t)=t$; $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$; $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi(t)=t(a_2-t)^{\alpha-1}$, where $f(x)$ is symmetric to $x=\frac{ma_1+a_2}{2}$ and $m\in(0,1]$ is a fixed number; $\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$, for $\alpha\in(0,1)$; such that $\eta(x,ma_1)=x-ma_1$ and $\eta(a_2,mx)=a_2-mx$, $\forall\, x\in P$, we can deduce some new general fractional integral inequalities. We omit their proofs and the details are left to the interested readers.*

3. Applications to special means

Consider the following special means for different real numbers α,β and $\alpha\beta\neq 0$, as follows:

(1) The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2},$$
(2) The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},$$
(3) The logarithmic mean:
$$L:=L(\alpha,\beta)=\frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},$$
(4) The generalized log–mean:
$$L_r:=L_r(\alpha,\beta)=\left[\frac{\beta^{r+1}-\alpha^{r+1}}{(r+1)(\beta-\alpha)}\right]^{\frac{1}{r}},$$
where $r\in\mathbb{Z}\setminus\{-1,0\}$.

It is well known that L_r is monotonic nondecreasing over $r\in\mathbb{Z}$ with $L_{-1}:=L$. In particular, we have the following inequality $H\leq L\leq A$. Now, using the theory results in section 2, we give some applications to special means for different real numbers.

Proposition 1. *Let $a_1,a_2\in\mathbb{R}\setminus\{0\}$, where $a_1<a_2$ and $x\in[a_1,a_2]$. Then for $r\in\{2,3,\dots\}$, where $q>1$ and $p^{-1}+q^{-1}=1$, the following inequality hold:*

$$\left|r\left(\frac{a_2-a_1}{2}\right)\left[A\left(a_1^{r-1},a_2^{r-1}\right)+A^{r-1}(a_1,a_2)\right]-\left[A\left(a_1^{r},a_2^{r}\right)+A^{r}(a_1,a_2)\right]-2L_r^r\left(a_1,a_2\right)\right|$$
$$\leq\frac{r(r-1)(a_2-a_1)^2}{8\sqrt[p]{2p+1}}\tag{35}$$
$$\times\left\{\sqrt[q]{A\left(|a_1|^{q(r-2)},\left|\frac{a_1+a_2}{2}\right|^{q(r-2)}\right)}+\sqrt[q]{A\left(\left|\frac{a_1+a_2}{2}\right|^{q(r-2)},|a_2|^{q(r-2)}\right)}\right\}.$$

Proof. Applying Theorem 2 for $x=\frac{a_1+a_2}{2}$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$, $f(x)=x^r$ and $\varphi(t)=t$, one can obtain the result immediately. □

Proposition 2. *Let $a_1,a_2\in\mathbb{R}\setminus\{0\}$, where $a_1<a_2$ and $x\in[a_1,a_2]$. Then, for $q>1$ and $p^{-1}+q^{-1}=1$, the following inequality hold:*

$$\left|\left(\frac{a_1-a_2}{2}\right)\left[\frac{1}{H\left(a_1^2,a_2^2\right)}+\frac{1}{A^2\left(a_1,a_2\right)}\right]-\left[\frac{1}{H\left(a_1,a_2\right)}+\frac{1}{A\left(a_1,a_2\right)}\right]-\frac{2}{L\left(a_1,a_2\right)}\right|$$
$$\leq\frac{(a_2-a_1)^2}{4\sqrt[p]{2p+1}}\tag{36}$$
$$\times\left\{\frac{1}{\sqrt[q]{H\left(|a_1|^{3q},\left|\frac{a_1+a_2}{2}\right|^{3q}\right)}}+\frac{1}{\sqrt[q]{H\left(\left|\frac{a_1+a_2}{2}\right|^{3q},|a_2|^{3q}\right)}}\right\}.$$

Proof. Applying Theorem 2 for $x=\frac{a_1+a_2}{2}$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$, $f(x)=\frac{1}{x}$ and $\varphi(t)=t$, one can obtain the result immediately. □

Proposition 3. *Let $a_1,a_2\in\mathbb{R}\setminus\{0\}$, where $a_1<a_2$ and $x\in[a_1,a_2]$. Then, for $r\in\{2,3,\dots\}$ and $q\geq 1$, the following inequality hold:*

$$\left|r\left(\frac{a_2-a_1}{2}\right)\left[A\left(a_1^{r-1},a_2^{r-1}\right)+A^{r-1}(a_1,a_2)\right]-\left[A\left(a_1^{r},a_2^{r}\right)+A^{r}(a_1,a_2)\right]-2L_r^r\left(a_1,a_2\right)\right|$$
$$\leq\frac{r(r-1)(a_2-a_1)^2}{48\sqrt[q]{2}}\tag{37}$$
$$\times\left\{\sqrt[q]{A\left(3|a_1|^{q(r-2)},\left|\frac{a_1+a_2}{2}\right|^{q(r-2)}\right)}+\sqrt[q]{A\left(3\left|\frac{a_1+a_2}{2}\right|^{q(r-2)},|a_1|^{q(r-2)}\right)}\right.$$

$$+\sqrt[q]{A\left(3|a_2|^{q(r-2)},\left|\frac{a_1+a_2}{2}\right|^{q(r-2)}\right)}$$
$$\left.+\sqrt[q]{A\left(3\left|\frac{a_1+a_2}{2}\right|^{q(r-2)},|a_2|^{q(r-2)}\right)}\right\}.$$

Proof. Applying Theorem 3 for $x=\frac{a_1+a_2}{2}$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$, $f(x)=x^r$ and $\varphi(t)=t$, one can obtain the result immediately. □

Proposition 4. *Let $a_1,a_2\in\mathbb{R}\setminus\{0\}$, where $a_1<a_2$ and $x\in[a_1,a_2]$. Then for $q\geq 1$, the following inequality hold:*

$$\left|\left(\frac{a_1-a_2}{2}\right)\left[\frac{1}{H\left(a_1^2,a_2^2\right)}+\frac{1}{A^2\left(a_1,a_2\right)}\right]\right.$$
$$\left.-\left[\frac{1}{H\left(a_1,a_2\right)}+\frac{1}{A\left(a_1,a_2\right)}\right]-\frac{2}{L\left(a_1,a_2\right)}\right|$$
$$\leq\sqrt[q]{\frac{3}{2}}\frac{(a_2-a_1)^2}{24}\tag{38}$$
$$\times\left\{\frac{1}{\sqrt[q]{H\left(3|a_1|^{3q},\left|\frac{a_1+a_2}{2}\right|^{3q}\right)}}\right.$$
$$+\frac{1}{\sqrt[q]{H\left(3\left|\frac{a_1+a_2}{2}\right|^{3q},|a_1|^{3q}\right)}}$$
$$+\frac{1}{\sqrt[q]{H\left(3|a_2|^{3q},\left|\frac{a_1+a_2}{2}\right|^{3q}\right)}}$$
$$\left.+\frac{1}{\sqrt[q]{H\left(3\left|\frac{a_1+a_2}{2}\right|^{3q},|a_2|^{3q}\right)}}\right\}.$$

Proof. Applying Theorem 3 for $x=\frac{a_1+a_2}{2}$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$, $f(x)=\dfrac{1}{x}$ and $\varphi(t)=t$, one can obtain the result immediately. □

Remark 3. *Applying our Theorems 2 and 3 for $x=\frac{a_1+a_2}{2}$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$ and appropriate choices of function $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\varphi(t)=t(a_2-t)^{\alpha-1}$, where $f(x)$ is symmetric to $x=\frac{a_1+a_2}{2}$, $\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$, for $\alpha\in(0,1)$, such that $|f''|^q$ to be preinvex, we can deduce some new general fractional integral inequalities using above special means. We omit their proofs and the details are left to the interested readers.*

4. Conclusion

It is expected that from the results obtained, and following the methodology applied, additional special functions may also be evaluated. Future works can be developed in the area of numerical analysis using the theorems and corollaries presented. The authors hope that the ideas and techniques of this paper will inspire interested readers working in this fascinating field.

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Artion Kashuri *received his PhD degree from University Ismail Qemali of Vlora in 2016 in the area of Analysis. His research areas are Mathematical Inequalities, Applied Mathematics, Fractional Calculus, Quantum Calculus, etc. He has vast experience of teaching such as Differential Equations, Numerical Analysis, Calculus, Real Analysis, Complex Analysis, Topology, etc. He has more than 100 published papers in international reputation indexed journals. His current position is Lecturer in University Ismail Qemali, Department of Mathematics.*
http://orcid.org/0000-0003-0115-3079

Rozana Liko *received her PhD degree from University Ismail Qemali of Vlora in 2018 in the area of Applied Mathematics. Her research areas are Mathematical Inequalities, Applied Mathematics, etc. She has vast experience of teaching such as Probability and Statistics, Calculus, Linear Algebra, Real Analysis, etc. She has more than 50 published papers in international reputation indexed journals. Her current position is Lecturer in University Ismail Qemali, Department of Mathematics.*
http://orcid.org/0000-0003-2439-8538