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RESEARCH ARTICLE

Some Hermite-Hadamard type inequalities for (P, m)-function and quasi m-convex functions

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ABSTRACT

In this paper, we introduce a new class of functions called as (P, m)-function and quasi-m-convex function. Some inequalities of Hadamard's type for these functions are given. Some special cases are discussed. Results represent significant refinement and improvement of the previous results. We should especially mention that the definition of (P, m)-function and quasi-m-convexity are given for the first time in the literature and moreover, the results obtained in special cases coincide with the well-known results in the literature.

1. Preliminaries

Inequalities present an attractive and active field of research. In recent years, various inequalities for convex functions and their variant forms are being developed using innovative techniques. For some inequalities, generalizations and applications concerning convexity see [1, 2]. Recently, in the literature there are so many papers about P-function, quasi-convex and m-convex functions. Many papers have been written by a number of mathematicians concerning inequalities for P-function, quasi-convex functions and m-convex functions see for instance the recent papers [3–8] and the references within these papers.

Definition 1. *A function $\omega : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$\omega(t\lambda + (1-t)\mu) \leq t\omega(x) + (1-t)\omega(y)$$

is valid for all $\lambda, \mu \in I$ and $t \in [0, 1]$. If this inequality reverses, then ω is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature. Denote by $C(I)$ the set of the convex functions on the interval I.

Definition 2. *Let $\omega : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $\lambda, \mu \in I$ with $\lambda < \mu$. The following inequality*

$$\omega\left(\frac{\lambda+\mu}{2}\right) \leq \frac{1}{\mu-\lambda}\int_\lambda^\mu \omega(x)\,dx \leq \frac{\omega(\lambda)+\omega(\mu)}{2} \quad (1)$$

holds.

The inequality (1) is known as Hermite-Hadamard (H-H) integral inequality for convex functions in the literature.

Some refinements of the H-H inequality on convex functions have been extensively studied by researchers (*e.g.*, [1,9]) and the researchers obtained a new refinement of the H-H inequality for convex functions.

Definition 3. *A function $\omega : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex if the inequality*

$$\omega\left(t\lambda + (1-t)\mu\right) \leq \max\left\{\omega\left(\lambda\right), \omega(\mu)\right\}$$

holds for all $\lambda, \mu \in I$ and $t \in [0,1]$. Denote by $QC(I)$ the set of the quasi-convex functions on the interval I.

Definition 4. *A nonnegative function $\omega : I \subseteq \mathbb{R} \to \mathbb{R}$ is called P-function if the inequality*

$$\omega\left(t\lambda + (1-t)\,\mu\right) \leq \omega\left(\lambda\right) + \omega\left(\mu\right)$$

holds for all $\lambda, \mu \in I$ and $t \in (0,1)$.

We will denote by $P(I)$ the set of P-function on the interval I. Note that $P(I)$ contain all nonnegative quasi-convex and convex functions.

In [10], Dragomir et al. proved the following inequality of Hadamard type for class of P-functions.

Theorem 1. *Let $\omega \in P(I)$, $\lambda, \mu \in I$ with $\lambda < \mu$ and $\omega \in L[\lambda, \mu]$. Then*

$$\omega\left(\frac{\lambda+\mu}{2}\right) \leq \frac{2}{\mu-\lambda}\int_{\lambda}^{\mu} \omega(x)\,dx \leq 2\left[\omega\left(\lambda\right) + \omega\left(\mu\right)\right]. \tag{2}$$

Definition 5. *[11] The function $\omega : [0,\tau] \to \mathbb{R}$, $\tau > 0$, is said to be an m-convex function, where $m \in [0,1]$; if we have*

$$\omega\left(t\lambda + m\left(1-t\right)\mu\right) \leq t\omega\left(\lambda\right) + m\left(1-t\right)\omega\left(\mu\right)$$

for all $\lambda, \mu \in [0,\tau]$ and $t \in [0,1]$. We say that f is an m-concave function if $(-\omega)$ is m-convex. Denote by $K_m(\tau)$ the set of the m-convex functions on $[0,\tau]$ for which $\omega(0) \leq 0$.

Obviously, this definition recaptures the concept of standard convex functions on $[0,\tau]$ for $m = 1$; and the concept star-shaped functions for $m = 0$.

2. Some new definitions and their properties

In this section, we will define the (P, m) and quasi-m-convex function supply several properties of this kind of functions.

Definition 6. *A function $\omega : [0,\tau] \to \mathbb{R}$ is called quasi-m-convex if the inequality*

$$\omega\left(t\lambda + m\left(1-t\right)\mu\right) \leq \max\left\{\omega\left(\lambda\right), m\omega(\mu)\right\}$$

holds for all $\lambda, \mu \in [0,\tau]$, $m \in [0,1]$ and $t \in [0,1]$. We will denote by $Q_mC(\tau)$ the set of quasi-m-convex function on the interval $[0,\tau]$.

It is clear that quasi-convexity obtained in quasi-m-convexity for $m = 1$.

Definition 7. *A nonnegative function $\omega : [0,\tau] \to \mathbb{R}$ is called (P, m)-function if the inequality*

$$\omega\left(t\lambda + m\left(1-t\right)\mu\right) \leq \omega\left(\lambda\right) + m\omega\left(\mu\right)$$

holds for all $\lambda, \mu \in [0,\tau]$, $m \in [0,1]$ and $t \in (0,1)$. We will denote by $P_m(\tau)$ the set of (P, m)-function on the interval $[0,\tau]$.

It is clear that P-function obtained in (P, m)-function for $m = 1$. Note also that $P_m(\tau)$ contain all nonnegative *m-convex* and *quasi-m-convex* functions. Since

$$\begin{aligned} \omega\left(t\lambda + m\left(1-t\right)\mu\right) &\leq t\omega\left(\lambda\right) + m\left(1-t\right)\omega\left(\mu\right) \\ &\leq \omega\left(\lambda\right) + m\omega\left(\mu\right), \end{aligned}$$

$$\begin{aligned} \omega\left(t\lambda + m\left(1-t\right)\mu\right) &\leq \max\left\{\omega\left(\lambda\right), m\omega\left(\mu\right)\right\} \\ &\leq \omega\left(\lambda\right) + m\omega\left(\mu\right). \end{aligned}$$

Theorem 2. *Let $m \in [0,1]$ and $\omega : [0,\tau] \to \mathbb{R}$. If ω is a quasi-m-convex function, then, for $c \in \mathbb{R}$ $(c \geq 0)$, $c\omega$ is a quasi-m-convex function.*

Proof. For $c \in \mathbb{R}$ $(c \geq 0)$,

$$\begin{aligned} &\left(c\omega\right)\left(t\lambda + m\left(1-t\right)\mu\right) \\ \leq\ & c.\max\left\{\omega\left(\lambda\right), m\omega\left(\mu\right)\right\} \\ =\ & \max\left\{\left(c\omega\right)\left(\lambda\right), m\left(c\omega\right)\left(\mu\right)\right\}. \end{aligned}$$

□

Remark 1. *If ω and φ are quasi-m-convex functions, then it is not necessary that the function $\omega + \varphi$ is a quasi-m-convex function.*

Example 1. *Let $\omega, \varphi : [0,\tau] \to \mathbb{R}$, $\omega(u) = u$, $\varphi(u) = 1$. Then ω and φ are quasi-m-convex functions. Now, if we choose $\lambda, \mu \in [0,\tau]$, $m \in [0,1]$ as numbers which satisfy the conditions $m\mu \geq \lambda$ and $m(\mu+1) \leq \lambda + 1$. Then, $(\omega + \varphi)(u) = u + 1$. Moreover, $\omega + \varphi$ is not quasi-m-convex function. Indeed, we can write following equality: for all $t \in [0,1]$,*

$$\begin{aligned}&(\omega+\varphi)(t\lambda+m(1-t)\mu)\\=\;&t\lambda+m(1-t)\mu+1\\=\;&t(\lambda+1)+(1-t)(m\mu+1).\end{aligned}$$

Since $m\mu\geq\lambda$,

$$\begin{aligned}&(\omega+\varphi)(t\lambda+m(1-t)\mu)\\=\;&t(\lambda+1)+(1-t)(m\mu+1)\\\geq\;&t(\lambda+1)+(1-t)(\lambda+1)\\=\;&\lambda+1,\end{aligned}$$

and since $m(\mu+1)\leq\lambda+1$,

$$\begin{aligned}&(\omega+\varphi)(t\lambda+m(1-t)\mu)\\=\;&t(\lambda+1)+(1-t)(m\mu+1)\\\geq\;&tm(\mu+1)+(1-t)(m\mu+1).\end{aligned}$$

Since $m\leq 1$,

$$\begin{aligned}&(\omega+\varphi)(t\lambda+m(1-t)\mu)\\\geq\;&tm(\mu+1)+(1-t)(m\mu+m)\\=\;&tm(\mu+1)+m(1-t)(\mu+1)\\=\;&m(\mu+1).\end{aligned}$$

So,

$$\begin{aligned}&(\omega+\varphi)(t\lambda+m(1-t)\mu)\\\geq\;&\max\{\lambda+1,m(\mu+1)\}\\=\;&\max\{(\omega+\varphi)(\lambda),m(\omega+\varphi)(\mu)\}.\end{aligned}$$

Theorem 3. *Let* $m\in[0,1]$ *and* $\omega_\alpha:[0,\tau]\to\mathbb{R}$ *be an arbitrary family of quasi-m-convex functions and let* $\omega(x)=\sup_\alpha\omega_\alpha(x)$ *for all* $x\in[0,\tau]$. *If*

$$J=\{u\in[0,\tau]:\omega(u)<\infty\}$$

is nonempty, then J *is an interval and* ω *is a quasi-m-convex functions on* J.

Proof. Let $t\in[0,1]$ and $\lambda,\mu\in J$ be arbitrary. Then

$$\begin{aligned}&\omega(t\lambda+m(1-t)\mu)\\=\;&\sup_\alpha\omega_\alpha\,(t\lambda+m(1-t)\mu)\\\leq\;&\sup_\alpha[\max\{\omega_\alpha(\lambda),m\omega_\alpha(\mu)\}]\\\leq\;&\max\left\{\sup_\alpha\omega_\alpha(\lambda),m\sup_\alpha\omega_\alpha(\mu)\right\}\\\leq\;&\max\{\omega(\lambda),m\omega(\mu)\}<\infty\end{aligned}$$

This shows that J is an interval since it contains every point between any two of its points and ω *is a quasi-m-convex functions on* J. □

Theorem 4. *Let* $m\in[0,1]$ *and* $\omega:[0,\tau]\to\mathbb{R}$ *be a m-convex function. If* φ *is a quasi-m-convex functions and increasing on* $[0,\tau]$, *then the function* $\varphi\circ\omega$ *is a quasi-m-convex function.*

Proof. For $\lambda,\mu\in[0,\tau]$ and $t\in[0,1]$,

$$\begin{aligned}&(\varphi\circ\omega)(t\lambda+m(1-t)\mu)\\=\;&\varphi(\omega(t\lambda+m(1-t)\mu))\\\leq\;&\varphi(\omega(\lambda)+m(1-t)\omega(\mu))\\\leq\;&\max\{(\varphi\circ\omega)(\lambda),m(\varphi\circ\omega)(\mu)\}.\end{aligned}$$

□

Theorem 5. *Let* $m\in[0,1]$ *and* $\omega,\varphi:[0,\tau]\to\mathbb{R}$. *If* ω *is a quasi-m-convex and non-negative function,* φ *is a* (P,m)*-function. Then,* $\omega+\varphi$ *is a* (P,m)*-function.*

Proof. For $\lambda,\mu\in[0,\tau]$ and $t\in[0,1]$,

$$\begin{aligned}&(\omega+\varphi)(t\lambda+m(1-t)\mu)\\=\;&\omega(t\lambda+m(1-t)\mu)+\varphi(t\lambda+m(1-t)\mu)\\\leq\;&\max\{\omega(\lambda),m\omega(\mu)\}+\varphi(\lambda)+m\varphi(\mu)\\\leq\;&\omega(\lambda)+m\omega(\mu)+\varphi(\lambda)+m\varphi(\mu)\\=\;&(\omega+\varphi)(\lambda)+m(\omega+\varphi)(\mu).\end{aligned}$$

□

Theorem 6. *Let* $m\in[0,1]$ *and* $\omega,\varphi:[0,\tau]\to\mathbb{R}$. *If* ω *and* φ *are* (P,m)*-functions, then*

(1) $\omega+\varphi$ *is a* (P,m)*-function ,*
(2) *For* $c\in\mathbb{R}$ $(c\geq 0)$, $c\omega$ *is a* (P,m)*-function .*

Proof. **i)** For $\lambda,\mu\in[0,\tau]$ and $t\in[0,1]$,

$$\begin{aligned}&(\omega+\varphi)(t\lambda+m(1-t)\mu)\\=\;&\omega(t\lambda+m(1-t)\mu)+\varphi(t\lambda+m(1-t)\mu)\\\leq\;&\omega(\lambda)+m\omega(\mu)+\varphi(\lambda)+m\varphi(\mu)\\\leq\;&(\omega+\varphi)(\lambda)+m(\omega+\varphi)(\mu).\end{aligned}$$

ii) For $c\in\mathbb{R}$ $(c\geq 0)$,

$$\begin{aligned}(c\omega)(t\lambda+m(1-t)\mu) &\leq c[\omega(\lambda)+m\omega(\mu)]\\ &= (c\omega)(\lambda)+m(c\omega)(\mu).\end{aligned}$$

□

Theorem 7. *Let $m\in[0,1]$ and $\omega_\alpha:[0,\tau]\to\mathbb{R}$ be an arbitrary family of (P,m)-functions and let $\omega(x)=\sup_\alpha\omega_\alpha(x)$ for all $x\in[0,\tau]$. If*

$$J=\{u\in[0,\tau]:\ \omega(u)<\infty\}$$

is nonempty, then J is an interval and ω is a (P,m)-functions on J.

Proof. Let $t\in[0,1]$ and $\lambda,\mu\in J$ be arbitrary. Then

$$\begin{aligned}&\omega(t\lambda+m(1-t)\mu)\\ =\ &\sup_\alpha\omega_\alpha\ (t\lambda+m(1-t)\mu)\\ \leq\ &\sup_\alpha[\omega_\alpha(\lambda)+m\omega_\alpha(\mu)]\\ \leq\ &\sup_\alpha\omega_\alpha(\lambda)+m\sup_\alpha\omega_\alpha(\mu)\\ =\ &\omega(\lambda)+m\omega(\mu)<\infty.\end{aligned}$$

This shows simultaneously that J is an interval since it contains every point between any two of its points and ω *is a (P,m)-function on the interval J.* □

Theorem 8. *Let $m\in[0,1]$ and $\omega:[0,\tau]\to\mathbb{R}$ be an m-convex function. If the function φ is a (P,m)-function and increasing, then the function $\varphi o\omega$ is a (P,m)-function.*

Proof. For $\lambda,\mu\in I$ and $t\in[0,1]$,

$$\begin{aligned}&(\varphi\circ\omega)(t\lambda+m(1-t)\mu)\\ =\ &\varphi(\omega(t\lambda+m(1-t)\mu))\\ \leq\ &\varphi(t\omega(\lambda)+m(1-t)\omega(\mu))\\ \leq\ &\varphi(\omega(\lambda))+m\varphi(\omega(\mu))\\ =\ &(\varphi\circ\omega)(\lambda)+m(\varphi\circ\omega)(\mu).\end{aligned}$$

□

3. Hermite-Hadamard integral inequality for (P,m)-function and quasi-m-convex functions

The main purpose of this paper is to develop concepts of the (P,m)-function and quasi-m-convex functions and to obtain some inequalities of H-H type for these classes of functions.

Theorem 9. *Let $m\in[0,1]$ and $\omega:[0,\tau]\to\mathbb{R}$ be a (P,m)-function. If $0\leq\lambda<\mu<\tau$ and $\omega\in L[\lambda,\mu]$, then the following inequalities holds:*

$$\begin{aligned}&\frac{1}{m\mu-\lambda}\int_\lambda^{m\mu}\omega(x)dx\\ \leq\ &\min\{\omega(\lambda)+m\omega(\mu),\omega(\mu)+m\omega(\lambda)\}.\end{aligned}$$

Proof. By using (P,m)-function property of ω and changing variable as $u=t\lambda+m(1-t)\mu$

$$\begin{aligned}&\int_0^1\omega(t\lambda+m(1-t)\mu)\,dt\\ =\ &\frac{1}{m\mu-\lambda}\int_\lambda^{m\mu}\omega(u)du\\ \leq\ &\int_0^1[\omega(\lambda)+m\omega(\mu)]\,dt\\ =\ &\omega(\lambda)+m\omega(\mu)\end{aligned}$$

and similarly for $z=t\mu+m(1-t)\lambda$, then

$$\begin{aligned}&\int_0^1\omega(t\mu+m(1-t)\lambda)\,dt\\ =\ &\frac{1}{m\mu-\lambda}\int_\lambda^{m\mu}\omega(z)dz\\ \leq\ &\int_0^1[\omega(\mu)+m\omega(\lambda)]\,dt\\ =\ &\omega(\mu)+m\omega(\lambda).\end{aligned}$$

So, we have

$$\begin{aligned}&\frac{1}{m\mu-\lambda}\int_\lambda^{m\mu}\omega(x)dx\\ \leq\ &\min\{\omega(\lambda)+m\omega(\mu),\omega(\mu)+m\omega(\lambda)\}.\end{aligned}$$

□

Remark 2. *Under the conditions of Theorem 9, if $m=1$ then, the following inequality holds:*

$$\frac{1}{\mu-\lambda}\int_\lambda^{\mu}\omega(x)dx\leq\omega(\lambda)+\omega(\mu)$$

The above inequality is the right hand side of the inequality 2.

Theorem 10. *Let $m\in(0,1]$ and $\omega:[0,\tau]\to\mathbb{R}$be an (P,m)-function. If $0\leq\lambda<\mu<\tau$ and $\omega\in L[\lambda,\tau]$, then the following inequalities holds:*

$$\omega\left(\frac{\lambda+m\mu}{2}\right)\leq\frac{2}{m\mu-\lambda}\int_\lambda^{m\mu}\omega(x)dx.$$

Proof. By the (P,m)-function property of ω, we have

$$\begin{aligned} &\omega\left(\frac{\lambda+m\mu}{2}\right) \\ =\ &\omega\left(\frac{[t\lambda+m(1-t)\mu]+[(1-t)\lambda+mt\mu]}{2}\right) \\ =\ &\omega\left(\frac{1}{2}[t\lambda+m(1-t)\mu]+\frac{1}{2}[(1-t)\lambda+mt\mu]\right) \\ \leq\ &\omega\left(t\lambda+m(1-t)\mu\right)+\omega\left((1-t)\lambda+mt\mu\right). \end{aligned}$$

Now, if we take integral in the last inequality on $t\in[0,1]$ and choose $x=t\lambda+m(1-t)\mu$ and $y=(1-t)\lambda+mt\mu$, we deduce

$$\begin{aligned} \omega\left(\frac{\lambda+m\mu}{2}\right) \leq\ &\frac{1}{m\mu-\lambda}\int_{\lambda}^{m\mu}\omega(x)\,dx \\ &+\frac{1}{m\mu-\lambda}\int_{\lambda}^{m\mu}\omega(y)\,dy. \end{aligned}$$

□

Remark 3. *Under the conditions of Theorem 10, if $m=1$, then, the following inequality holds:*

$$\omega\left(\frac{\lambda+\mu}{2}\right)\leq\frac{2}{\mu-\lambda}\int_{\lambda}^{\mu}\omega\left(x\right)dx$$

This inequality is the left hand side of the inequality 2.

Theorem 11. *Let $m\in(0,1]$ and $\omega:[0,\tau]\to\mathbb{R}$ be a (P,m)-function. If $0\leq\lambda<\mu<\tau$ and $\omega\in L[\lambda,\mu]$, then the following inequalities holds:*

$$\begin{aligned} \omega\left(\frac{\lambda+\mu}{2}\right) \leq\ &\frac{1}{\mu-\lambda}\int_{\lambda}^{\mu}\left[\omega\left(x\right)+m\omega\left(\frac{x}{m}\right)\right]dx \\ \leq\ &\min\{I_1,I_2\}. \end{aligned} \tag{3}$$

where

$$I_1=\omega\left(\lambda\right)+m\omega\left(\frac{\mu}{m}\right)+m\omega\left(\frac{\lambda}{m}\right)+m^2\omega\left(\frac{\mu}{m^2}\right)$$

and

$$I_2=\omega\left(\mu\right)+m\omega\left(\frac{\lambda}{m}\right)+m\omega\left(\frac{\mu}{m}\right)+m^2\omega\left(\frac{\lambda}{m^2}\right).$$

Proof. Using the (P,m)-function property of ω, we have

$$\omega\left(\frac{x+y}{2}\right)\leq\omega\left(x\right)+m\omega\left(\frac{y}{m}\right)$$

for all $x,y\in[0,\tau]$. If we take $x=t\lambda+(1-t)\mu$, $y=(1-t)\lambda+t\mu$, we get

$$\begin{aligned} &\omega\left(\frac{\lambda+\mu}{2}\right) \\ \leq\ &\omega\left(t\lambda+(1-t)\mu\right)+m\omega\left((1-t)\frac{\lambda}{m}+t\frac{\mu}{m}\right) \end{aligned}$$

for all $t\in[0,1]$. Here, if we take integral over $t\in[0,1]$, we get

$$\begin{aligned} \omega\left(\frac{\lambda+\mu}{2}\right) \leq\ &\int_0^1\omega(t\lambda+(1-t)\mu)\,dt \\ &+m\int_0^1\omega\left((1-t)\frac{\lambda}{m}+t\frac{\mu}{m}\right)dt. \end{aligned} \tag{4}$$

Taking into account that

$$\int_0^1\omega(t\lambda+(1-t)\mu)\,dt=\frac{1}{\mu-\lambda}\int_{\lambda}^{\mu}\omega(x)dx,$$

and

$$\begin{aligned} &\int_0^1\omega\left((1-t)\frac{\lambda}{m}+t\frac{\mu}{m}\right)dt \\ =\ &\frac{m}{\mu-\lambda}\int_{\frac{\lambda}{m}}^{\frac{\mu}{m}}\omega(x)dx \\ =\ &\frac{1}{\mu-\lambda}\int_{\lambda}^{\mu}\omega\left(\frac{x}{m}\right)dx, \end{aligned}$$

we deduce from (4) the first part of (3). That is

$$\omega\left(\frac{\lambda+\mu}{2}\right)\leq\frac{1}{\mu-\lambda}\int_{\lambda}^{\mu}\left[\omega\left(x\right)+m\omega\left(\frac{x}{m}\right)\right]dx.$$

By the (P,m)-function property of ω we also have

$$\begin{aligned} &\omega\left(t\lambda+(1-t)\mu\right)+m\omega\left((1-t)\frac{\lambda}{m}+t\frac{\mu}{m}\right) \\ \leq\ &\omega\left(\lambda\right)+m\omega\left(\frac{\mu}{m}\right)+m\omega\left(\frac{\lambda}{m}\right)+m^2\omega\left(\frac{\mu}{m^2}\right). \end{aligned} \tag{5}$$

for all $t\in[0,1]$. Integrating the last equality (5) over t on $[0,1]$, we deduce

$$\begin{aligned} &\frac{1}{\mu-\lambda}\int_{\lambda}^{\mu}\left[\omega\left(x\right)+m\omega\left(\frac{x}{m}\right)\right]dx \\ \leq\ &\omega\left(\lambda\right)+m\omega\left(\frac{\mu}{m}\right)+m\omega\left(\frac{\lambda}{m}\right)+m^2\omega\left(\frac{\mu}{m^2}\right). \end{aligned} \tag{6}$$

By a similar argument, if we take

$$\begin{aligned}&\omega\left(t\mu+(1-t)\lambda\right)+m\omega\left(t\frac{\lambda}{m}+(1-t)\frac{\mu}{m}\right) \quad (7)\\ \leq\ &\omega\left(\mu\right)+m\omega\left(\frac{\mu}{m}\right)+m\omega\left(\frac{\lambda}{m}\right)+m^2\omega\left(\frac{\lambda}{m^2}\right),\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{\mu-\lambda}\int_{\lambda}^{\mu}\left[\omega\left(x\right)+m\omega\left(\frac{x}{m}\right)\right]dx \quad (8)\\ \leq\ &\omega\left(\mu\right)+\omega\left(\frac{\lambda}{m}\right)+\omega\left(\frac{\mu}{m}\right)+m^2\omega\left(\frac{\lambda}{m^2}\right).\end{aligned}$$

From (6) and (8), we obtain

$$\frac{1}{\mu-\lambda}\int_{\lambda}^{\mu}\left[\omega\left(x\right)+m\omega\left(\frac{x}{m}\right)\right]dx\leq\min\left\{I_1,I_2\right\}.$$

□

Remark 4. *For $m=1$, (3) exactly becomes the inequality 2 (the Hermite-Hadamard integral inequality for P-functions given in [10]).*

Theorem 12. *Let $m\in(0,1]$ and $\omega:[0,\tau]\to\mathbb{R}$ be a (P,m)-function. If $0\leq\lambda<\mu<\tau$ and $\omega\in L[\lambda,\mu]$, then the following inequalities holds:*

$$\begin{aligned}&\frac{1}{m\mu-\lambda}\int_{\lambda}^{m\mu}\omega(x)dx+\frac{1}{\mu-m\lambda}\int_{\lambda m}^{\mu}\omega(x)dx\\ \leq\ &(m+1)\left[\omega\left(\lambda\right)+\omega\left(\mu\right)\right] \quad (9)\end{aligned}$$

Proof. By the (P,m)-function property of ω we have that

$$\begin{aligned}\omega\left(t\lambda+m\left(1-t\right)\mu\right)&\leq\omega\left(\lambda\right)+m\omega\left(\mu\right),\\ \omega\left(t\mu+m(1-t)\lambda\right)&\leq\omega\left(\mu\right)+m\omega\left(\lambda\right)\end{aligned}$$

for all $t\in[0,1]$ and $\lambda,\mu\in[0,\tau]$. By adding the above inequalities we get

$$\begin{aligned}&\omega\left(t\lambda+m\left(1-t\right)\mu\right)+\omega\left(t\mu+m(1-t)\lambda\right)\\ \leq\ &(m+1)\left[\omega\left(\lambda\right)+\omega\left(\mu\right)\right].\end{aligned}$$

Integrating over $t\in[0,1]$, we obtain

$$\begin{aligned}&\int_0^1\omega(t\lambda+m\left(1-t\right)\mu)\,dt \quad (10)\\ &+\int_0^1\omega(t\mu+m(1-t)\lambda)\,dt\\ \leq\ &(m+1)\left[\omega\left(\lambda\right)+\omega\left(\mu\right)\right].\end{aligned}$$

As it is easy to see that

$$\begin{aligned}&\int_0^1\omega(t\lambda+m\left(1-t\right)\mu)\,dt\\ =\ &\frac{1}{m\mu-\lambda}\int_{\lambda}^{m\mu}\omega(x)dx\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\omega(t\mu+m(1-t)\lambda)\,dt\\ =\ &\frac{1}{\mu-m\lambda}\int_{m\lambda}^{\mu}\omega(x)dx,\end{aligned}$$

from (10) we deduce the desired result, namely, the inequality (9). □

Remark 5. *For $m=1$, (9) exactly becomes the right hand side of the inequality 2.*

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