RESEARCH ARTICLE

An application of the MEFM to the modified Boussinesq equation

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ABSTRACT

In this paper, some travelling wave solutions of the Modified Boussinesq (MBQ) equation are obtained by using the modified expansion function method (MEFM). When the obtained solutions are commented, trigonometric functions including hyperbolic features are obtained. The 2D and 3D graphics of the solutions have been investigated by selecting appropriate parameters. All the obtained solutions provide the MBQ equation. In this work, all mathematical calculations are done with Wolfram Mathematica software.

1. Introduction

The solution of nonlinear partial differential equations has a measure in real life. For this reason, many methods have been developed and applied to solve these equations. Some of these, respectively the trial equation method [1], the new function methods [2-6], the extended trial equation method [7], Kudryashov method [8], the sine-Gordon expansion method [9-10] and so on. In this study, we apply the modified expansion function method (MEFM) [11-13] to solve a nonlinear MBQ equation and find new interactions among travelling wave solutions. Boussinesq–type equations of higher order in dispersion as well as in nonlinearity are reproduced for wave–current interaction over an unbalanced bottom. There are various methods in the literature to obtain the solution of the equation. Some of those; tanh method, the modified decomposition method and bilinearization method

In Section 2, Information about the modified expansion function method will be given.

In Section 3 the modified expansion function method is applied to the MBQ equation and the new exact wave solution to this problem is obtained. The 2D and 3D graphics of the solutions were drawn by using the Mathematica software program.

The modified Boussinesq equation can be defined as follows [14-16],

$$u_{tt} - u_{xxtt} - u_{xx} + \frac{a}{2}\left(u^2\right)_{xx} = 0. \tag{1}$$

2. Modified Expansion Function method

In this part, we will be given information about MEFM. Consider the following nonlinear partial differential equation (NPDE):

$$P\left(u, u^2, u_x, u_t, u_{xx}, u_{tt}, \left(u^2\right)_{xx}, u_{xxt}, u_{xxtt}\right) = 0, \tag{2}$$

where $u = u(x,t)$ is unknown function, P is a polynomial in $u(x,t)$ and its derivatives.

The general form of the nonlinear partial differential equation (2) is given above. By applying wave conversion to NPDE expression (3), the general form of the following nonlinear ordinary differential equation (4) is obtained.

Step 1: Consider the following travelling wave transformation:

$$u(x,t) = u(\xi), \quad \xi = \nu\left(x - ct\right). \tag{3}$$

Substituting Eq. (3) into Eq. (2), gives the following nonlinear ordinary differential equation (NODE);

$$N\left(u, u^2, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \ldots\right) = 0. \tag{4}$$

Step 2: We assume the following solution;

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$$u(\xi) = \frac{\sum\limits_{i=0}^{m} A_i [e^{-\vartheta(\xi)}]^i}{\sum\limits_{j=0}^{n} B_j [e^{-\vartheta(\xi)}]^j} =$$

$$= \frac{A_0 + A_1 e^{-\vartheta} + \ldots + A_m e^{-m\vartheta}}{B_0 + B_1 e^{-\vartheta} + \ldots + B_n e^{-n\vartheta}},$$

(5)

where $A_i, B_j, (0 \le i \le m, 0 \le j \le n)$.

m, n are positive integers that can be obtained by using the balancing principle.

$$\vartheta'(\eta) = e^{-\vartheta(\eta)} + k e^{\vartheta(\eta)} + \lambda.$$

(6)

Eq.(6) has the following families of solutions [17]:

Family 1: When, $k \ne 0, \lambda^2 - 4k > 0$,

$$\vartheta(\eta) = ln(\frac{-\sqrt{\lambda^2 - 4k}}{2k}$$

$$tanh(\frac{\sqrt{\lambda^2 - 4k}}{2}(\eta + EE)) - \frac{\lambda}{2k}).$$

(7)

Family 2: When, $k \ne 0, \lambda^2 - 4k < 0$,

$$\vartheta(\eta) = ln(\frac{\sqrt{-\lambda^2 + 4k}}{2k}$$

$$tan(\frac{\sqrt{-\lambda^2 + 4k}}{2}(\eta + EE)) - \frac{\lambda}{2k}).$$

(8)

Family 3: When, $k = 0, \lambda \ne 0, \lambda^2 - 4k > 0$,

$$\vartheta(\eta) = -ln(\frac{\lambda}{e^{\lambda(\eta+EE)} - 1}).$$

(9)

Family 4: When, $k \ne 0, \lambda \ne 0, \lambda^2 - 4k = 0$,

$$\vartheta(\eta) = ln(-\frac{2\lambda(\eta + EE) + 4}{\lambda^2(\eta + EE)}).$$

(10)

Family 5: When, $k = 0, \lambda = 0, \lambda^2 - 4k = 0$,

$$\vartheta(\eta) = ln(\eta + EE),$$

(11)

Where, EE is a integral constant.

Step 3: By substituting Eq. (5) and its derivatives into Eq. (4), we get algebraic equation system. This system was solved by using the Mathematica software program and then the solutions of the MBQ equation were obtained.

3. Application

In this section, the modified expansion function method will be used to obtain solutions of the MBQ equation. Consider the following travelling wave transformation:

$$u(x,t) = u(\xi), \quad \xi = v(x - ct).$$

(12)

the following nonlinear ordinary differential equation is obtained,

$$au^2 + 2(c^2 - 1)u - 2c^2 v^2 u'' = 0.$$

(13)

If the balancing procedure is applied to equation (13), we get $n = m + 2$ equality.

Choosing $m = 1$, we get $n = 3$. Eq. (5) for m and n values is obtained as follows;

$$u(\xi) = \frac{A_0 + A_1 e^{-\vartheta} + A_2 e^{-2\vartheta} + A_3 e^{-3\vartheta}}{B_0 + B_1 e^{-\vartheta}}.$$

(14)

If Eq. (14) is regulated according to the necessary term in equation (13), then the following system of algebraic equations is obtained which consists of the coefficients of $e^{-\vartheta(\xi)}$.

Some suitable coefficients obtained by using the Mathematica package program are given below.

Case-1:

$$A_0 = \frac{12\mu v^2 B_0}{a - a(\lambda^2 - 4\mu)v^2},$$

$$A_1 = -\frac{12v^2(\lambda B_0 + \mu B_1)}{a(-1 + (\lambda^2 - 4\mu)v^2)},$$

$$A_2 = -\frac{12v^2(B_0 + \lambda B_1)}{a(-1 + (\lambda^2 - 4\mu)v^2)},$$

$$A_3 = \frac{12v^2 B_1}{a - a(\lambda^2 - 4\mu)v^2},$$

$$c = \frac{1}{\sqrt{1 - v^2(\lambda^2 - 4\mu)}}.$$

Substituting these coefficients into Eq. (14), the following solutions:

Family 1: When, $k \ne 0, \lambda^2 - 4k > 0$, solution of equation (1),

$$u_1(x,t) = \frac{(12(\lambda^2 - 4\mu)\mu v^2)}{(a(-1 + (\lambda^2 - 4\mu)v^2)(\lambda Cosh[\psi] + \sqrt{\lambda^2 - 4\mu}Sinh[\psi])^2)},$$

(15)

where,

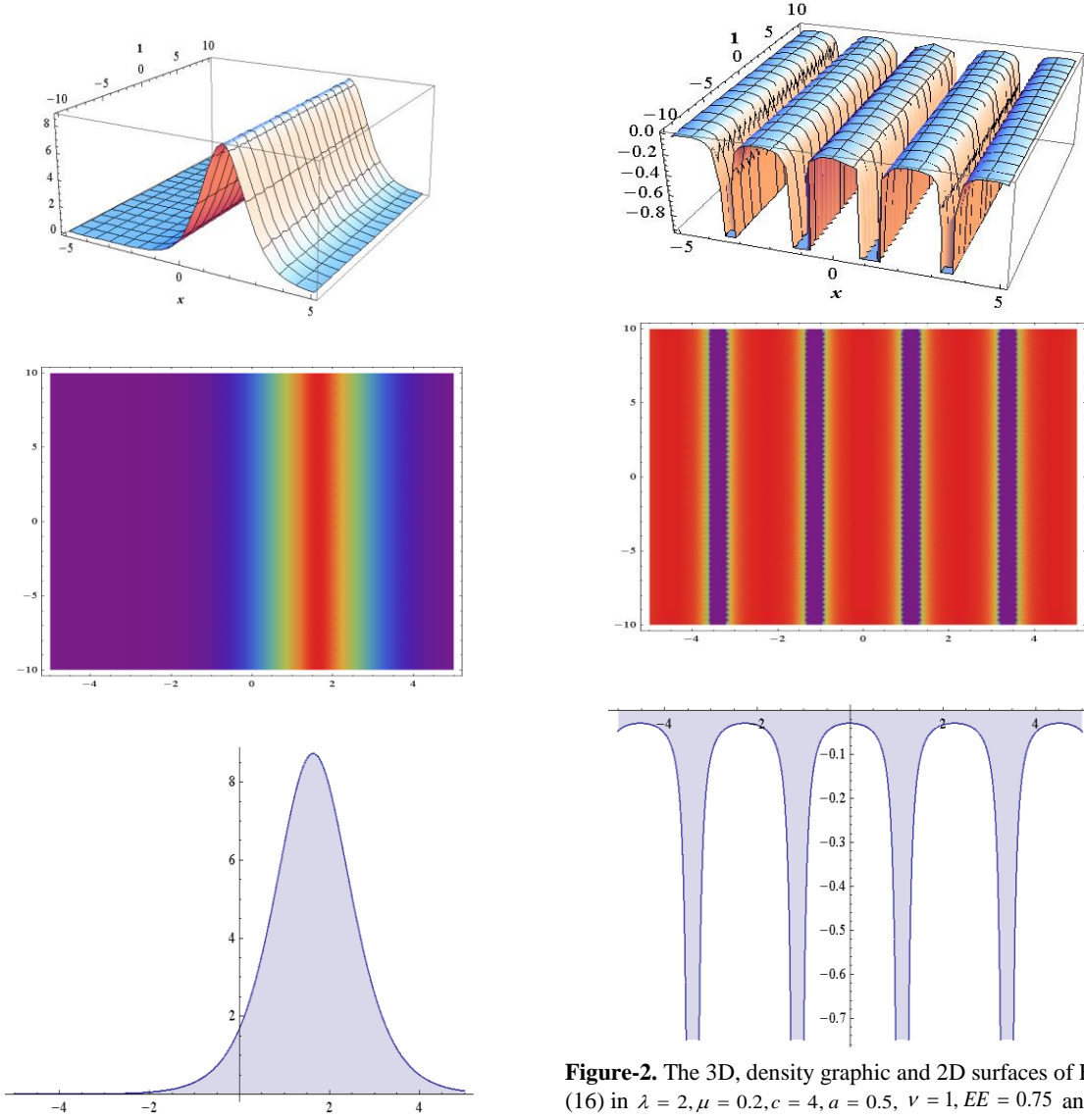$$(\psi = [\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(EE + \xi)v]).$$

Figure-1. The 3D, density graphic and 2D surfaces of Eq. (15) in $\lambda = 2, \mu = 0.2, c = 4, a = 0.5, \ \nu = 1, EE = 0.75$ and $t = 1$

Figure-2. The 3D, density graphic and 2D surfaces of Eq. (16) in $\lambda = 2, \mu = 0.2, c = 4, a = 0.5, \ \nu = 1, EE = 0.75$ and $t = 1$.

Family 2: When, $k \neq 0$, $\lambda^2 - 4k < 0$, we get,

$$u_2 (x,t) = \frac{\left(12 \left(\lambda^2 - 4\mu \right) \mu \nu^2 \right)}{\left(a \left(-1 + \left(\lambda^2 - 4\mu \right) \nu^2 \right) \left(\lambda Cos[\upsilon] - \sqrt{-\lambda^2 + 4\mu} Sin[\upsilon] \right)^2 \right)}, \tag{16}$$

where,

$$\left(\upsilon = \left[\frac{1}{2} \sqrt{-\lambda^2 + 4\mu} \left(EE + \xi \right) \nu \right] \right).$$

Family 3: $k = 0, \lambda \neq 0$, $\lambda^2 - 4k > 0$,

$$u_3 (x,t) = \left(-\frac{3\lambda^2 \nu^2 Csch \left[\frac{1}{2} \lambda \left(EE + \left(\xi \right) \nu \right) \right]^2}{-a + a\lambda^2 \nu^2} \right), \tag{17}$$
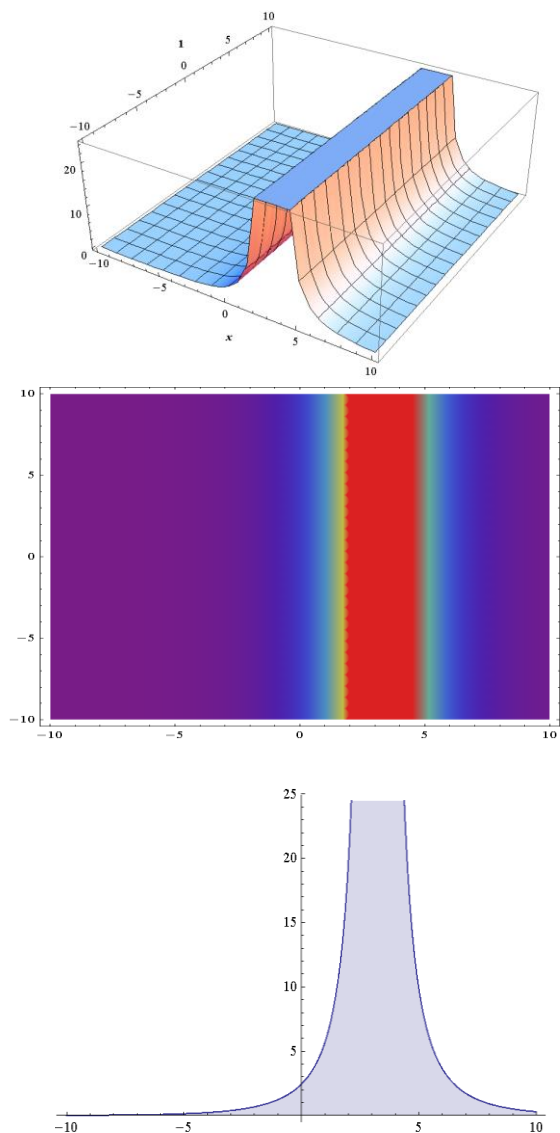
Figure-3. The 3D, density graphic and 2D surfaces of Eq. (17) in $\lambda = 0.5, \mu = 0, c = 4, a = 0.5, \nu = 1, EE = 0.75$ and $t = 1$.
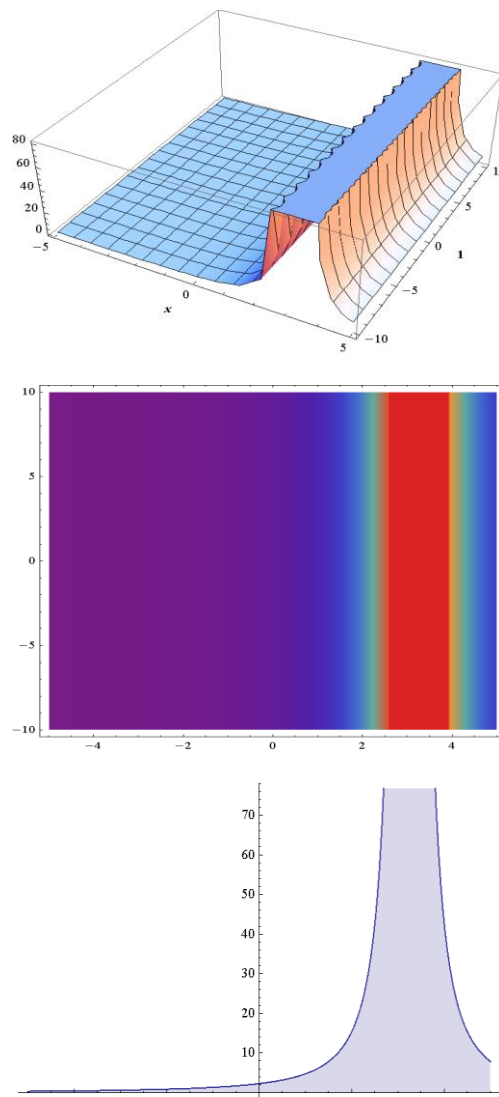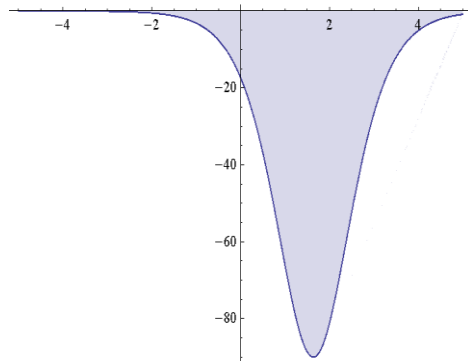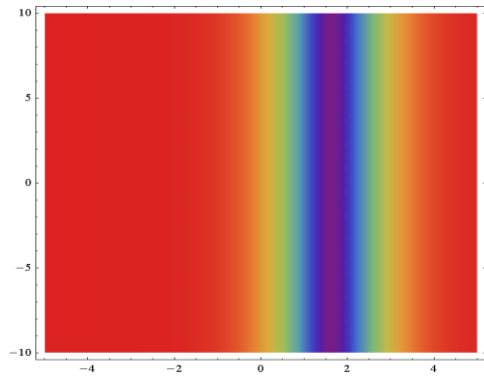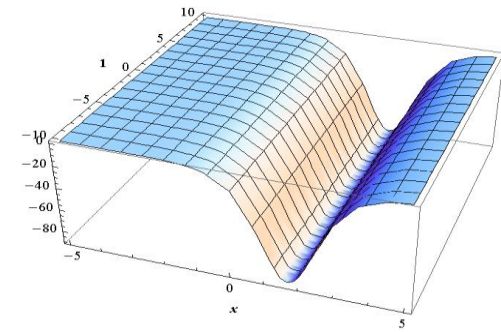

Figure-4. The 3D, density graphic and 2D surfaces of Eq. (18) in $\lambda = 0, \mu = 0, c = 4$ $a = 0.5, \nu = 1, EE = 0.75$ and $t = 1$.

According to Family-4, the solution does not exist.

Family 5: When $k = 0$, $\lambda = 0$ and $\lambda^2 - 4k = 0$,

$$u_5(x,t) = \left(\frac{12\nu^2}{a\left(EE + (\xi)\nu\right)^2} \right),\qquad(18)$$

Case-2:

$$A_0 = \frac{\left(\left(-1+c^2\right)\left(-\sqrt{c^4\left(\lambda^2-4\mu\right)^2} + c^2\left(\lambda^2+8\mu\right)\right)B_0\right)}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$A_1 = \frac{\left(\left(-1+c^2\right)\left(12c^2\lambda B_0 + \left(-\sqrt{c^4\left(\lambda^2-4\mu\right)^2} + c^2\left(\lambda^2+8\mu\right)\right)B_1\right)\right)}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$A_2 = \frac{\left(12c^2\left(-1+c^2\right)\left(B_0+\lambda B_1\right)\right)}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$A_3 = \frac{\left(12c^2\left(-1+c^2\right)B_1\right)}{a\sqrt{c^4\left(\lambda^2-4\mu\right)^2}},$$

$$\nu = -\frac{\sqrt{-1+c^2}}{\left(\sqrt{c^4\left(\lambda^2-4\mu\right)^2}\right)^{\frac{1}{4}}}.$$

Substituting these coefficients into Eq. (14), the following solutions:

Family 1: When, $k \neq 0$, $\lambda^2 - 4k > 0$, we get

$$u_1(x,t) = \frac{\left(-1+c^2\right) Sech[\psi]^2 \left(-2\left(5c^2(\chi) + \sqrt{c^4(\chi)^2}\right)\mu\right)}{\left(a\sqrt{c^4(\chi)^2}\left(\lambda + \sqrt{\chi}Tanh[\psi]\right)^2\right)},$$

(19)

where,

$$\left(\chi = \lambda^2 - 4\mu, \ \psi = \sqrt{\lambda^2 - 4\mu}\,Tanh\left[\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(EE + \xi\right)\right]\right),$$

Family 2: When, $k \neq 0$, $\lambda^2 - 4k < 0$,

$$u_2(x,t) = \frac{-\left(-1+c^2\right)\left(\sqrt{c^4\chi^2}\left(\lambda - \sqrt{-\chi}Tan[\tau]\right)^2\right) + c^2\chi\left(-\lambda^2 + 12\mu + 2\lambda\sqrt{-\chi}Tan[\tau]\right)}{\left(\left(\lambda^2 + 8\mu\right)Tan[\tau]^2\right)},$$

(20)

where,

$$\left(\tau = \frac{1}{2}\sqrt{-\chi}\left(EE + \xi\right)v\right).$$

Figure-5. The 3D, density graphic and 2D surfaces of Eq. (19) in $\lambda = 0.2, \mu = 2, c = 4$ $a = 0.5, v = 1, EE = 0.75$ and $t = 1$.
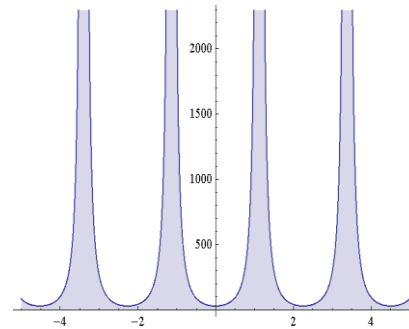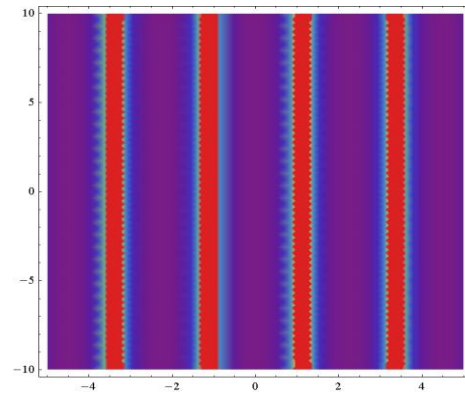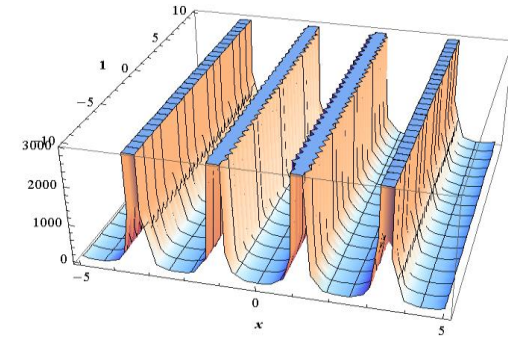

Figure-6. The 3D, density graphic and 2D surfaces of Eq. (20) in $\lambda = 0.5, \mu = 2, c = 4$ $a = 0.5, v = 1, EE = 0.75$ and $t = 1$.

Family 3: $k = 0, \lambda \neq 0$, $\lambda^2 - 4k > 0,$

$$u_3\left(x,t\right) = \frac{\left(-1+c^2\right)\left(-\sqrt{c^4\lambda^4}+c^2\lambda^4\left(1+3Csch\left[\frac{1}{2}\lambda\left(EE+\xi\right)v\right]^2\right)\right)}{a\sqrt{c^4\lambda^4}},$$
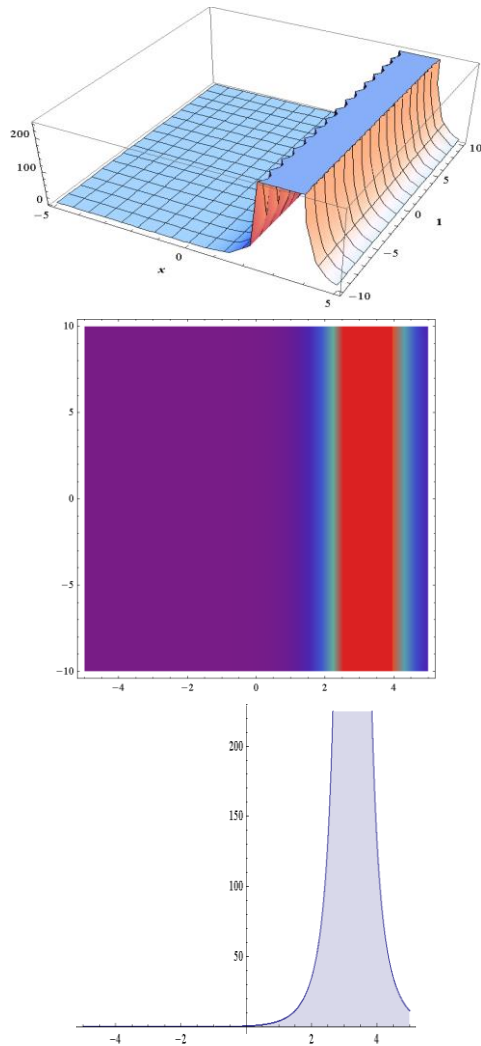
(21)

Figure-7. The 3D, density graphic and 2D surfaces of Eq. (21) in $\lambda = 0, \mu = 2, c = 4$ $a = 0.5, v = 1, EE = 0.75$ and $t = 1$.

Family-4 and Family-5, the solution does not exist.

4. Conclusion

In this study, we obtained some travelling wave solutions of Boussinesq equation by using modified expansion function method. The results show that the modified expansion function method is a suitable mathematical method for solving nonlinear partial differential equations. The resulting solutions were checked with the Mathematica software. These solutions have been obtained by MEFM for the first time in the literature.

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