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RESEARCH ARTICLE

Hermite collocation method for fractional order differential equations

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ABSTRACT

This paper focuses on the approximate solutions of the higher order fractional differential equations with multi terms by the help of Hermite Collocation method (HCM). This new method is an adaptation of Taylor's collocation method in terms of truncated Hermite Series. With this method, the differential equation is transformed into an algebraic equation and the unknowns of the equation are the coefficients of the Hermite series solution of the problem. This method appears as an useful tool for solving fractional differential equations with variable coefficients. To show the pertinent feature of the proposed method, we test the accuracy of the method with some illustrative examples and check the error bounds for numerical calculations.

1. Introduction

In many branches of science, mathematical models of physical processes require differential equations and nowadays, it is verified that some of these models can be better defined by fractional order equations due to the material and hereditary properties. Consequently, too many applications of the fractional order differential equations exist (see [1–5]). Unfortunately, model equations are usually in complex nature and involves nonlinear terms, therefore, analytical solutions can not easily be obtained. As a result, we still need more powerful numerical or approximate methods. Nowadays, many researchers are studying on numerical or approximate solutions of the fractional order equations and some new techniques have been introduced or adopted with the existent ones for ordinary case. For instance, finite difference [6–10], fractional linear multistep methods [11–13], Adomian decomposition [14–16], variational iteration method [16–18], differential transform or Taylor collocation method [19, 20] and spectral method [21–24] can be cited here. For some classes of fractional differential equations, Kumar and Agarwal mentioned about polynomial approximation methods and detailed information can be found in [25–27]. There are also some other studies which worth to cite here [28–30]. These are some valuable studies on fractional partial differential equations. In the recent years, many works have also been published on solving fractional differential equations but most of them have been concerned with a single term and the order is less than one. However, here, we adopt the Hermite Collocation method (HCM) for obtaining solutions to higher order multi-term fractional differential equations with variable coefficients. This technique evaluates an analytical solution in the form of a truncated Hermite series with unknown coefficients. In many physical problems, orthogonal functions or polynomials are used as a basis for obtaining solutions to the problems. On the other hand, the orthogonal Hermite polynomials are extensively

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used in some problems of hydrodynamics and meteorology [31]. The present method uses Hermite polynomials smilar to the Taylor collocation method and so is called Hermite Collocation Method which was first developed for higher-order linear Fredholm integro differential equation [32]. Using this method has advantages on some partical types of physical processes as we mentioned above.

The second section of this study involves preliminary definitions and related theorems of the fractional calculus. In section 3, we recall the fundamental properties of the Hermite series and making adaptation of the method to the fractional order equation. Section 4 deals with the error bounds for the calculations and the section 5 involves some illustrative examples. Finally, we conclude the research with some highlights.

2. Preliminary information and notations

We start with the definition of Caputo derivative which was first introduced by Caputo ([33]). This derivative is preferred by many researches to make it easier to incorporate the initial and boundary conditions to the problem. Therefore, all the derivatives will be defined as Caputo derivatives throughout this study.

Definition 1. *[1] Let $f \in AC^n[a,b]$ then, the Caputo fractional derivative of a function f of order $\alpha > 0$ is defined by*

$$({}^C D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} (\frac{d}{dt})^n f(t)dt, \tag{1}$$

where Γ is the gamma function and $n-1 < \alpha < n, n \in N$.

Additionally, we can state the Caputo fractional derivative of a power function as follows. Let $\alpha \geq 0$, and $f(x) = (x-c)^\beta$ for some $\beta \geq 0$. Letting c is any number, then

$$ {}^C D_a^\alpha f(x) = \begin{cases} 0, & if\ \beta \in \{0,1,2,...,n-1\} \text{ and } \beta < n, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-c)^{\beta-\alpha}, & \beta \in N \text{ and } \beta \geq n \text{ or}, \\ & \beta \notin N \text{ and } \beta. > n-1 \end{cases} \tag{2}$$

Some properties of the Caputo derivative can be given as follows:

Lemma 1. *[1] Let $\alpha > 0$ and let $y \in L^\infty(a,b)$ or $C([a,b])$. Then, $({}^C D_a^\alpha I^\alpha y)(x) = y(x)$, where I^α defines the integral operator.*

Lemma 2. *[1] Let $\alpha > 0$ and $n = [\alpha]+1$ where $[\alpha]$ is the integer part of α. If $y \in AC^n([a,b])$ or $y \in C^n([a,b])$, then*

$(I^{\alpha C} D_a^\alpha y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k.$

Theorem 1. *[34] For every $\alpha, \beta \in \mathbb{R}_+$ the following relation holds,*

$$ {}^C D_a^{\alpha C} D_a^\beta f(x) = {}^C D_a^{\alpha+\beta} f(x).$$

3. Hermite-collocation method for fractional order differential equations

This section deals with the establishment of the theory of HCM for solving following multi-term fractional differential equations with variable coefficients,

$$\sum_{k=0}^{m} P_k(x)^C D_a^{k\alpha} y(x) = g(x), \tag{3}$$

where $a \leq x \leq b, n-1 < m\alpha < n$ $(0 < \alpha < 1)$, $n > 1, n\epsilon \mathbf{N}$, $P_k(x)$ and $g(x)$ continuous on $a \leq x \leq b$. Initial conditions are:

$$ {}^C D^j y(a) = \lambda_j, \ j = 0,1,2,...,m\alpha-1. \tag{4}$$

In Eqs.(3)-(4), D_a^α or, for convenience D^α defines the Caputo derivative of order α and ,$m\alpha-1$ is an integer number. We approximate the solution of the form as the following truncated Hermite series,

$$y(x) = \sum_{k=0}^{N} a_k H_k(x^\alpha), \tag{5}$$

where a_k are unknown Hermite coefficients and $N \in \mathbf{N}^+$ which satisfies $N \geq m\alpha$. To obtain the solution of Eq.(3) of the form Eq.(5), we first define the collocation points as $x_i = a + (\frac{b-a}{N})i$ ($i = 0,1,2,...N$, and $x_0 = a, x_N = b$). Now, letting that $H(x^\alpha) = [H_0(x^\alpha)\ H_1(x^\alpha)\ H_2(x^\alpha)...H_N(x^\alpha)]$, $A = [a_0\ a_1\ a_2...a_N]^T$ then, Eq.(5) is written in matrix form as follow:

$$[y(x)] = H(x^\alpha)A. \tag{6}$$

Eventually, at collocation points, Eq.(6) is shown by $[y(x_i)] = H(x_i^\alpha)A$.

3.1. Fractional hermite collocation method

The Hermite polynomials of degree n are generated by the very well known formula,

$H_n(x) = \sum_{n=0}^{[|\frac{n}{2}|]} (-1)^m \frac{n!}{(n-2m)!m!} x^{n-2m}$ $-\infty < x < \infty$, $n = 0, 1, 2..., N$.

Now we can define them in matrix notation (see [32])as below. If N is an odd number, then the matrix notation of Hermite polynomials is written as

$$\underbrace{\begin{bmatrix} H_0(x^\alpha) \\ H_1(x^\alpha) \\ \vdots \\ H_{N-1}(x^\alpha) \\ H_N(x^\alpha) \end{bmatrix}}_{H^T(x^\alpha)} = \underbrace{\begin{bmatrix} 2^0 & 0 & \dots & 0 & 0 \\ 0 & 2^1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{\left(\frac{N-5}{2}\right)} \frac{2^0}{0!} \frac{(N-1)!}{\left(\frac{N-1}{2}\right)!} & 0 & \dots & 2^{N-1} & 0 \\ 0 & (-1)^{\left(\frac{N-1}{2}\right)} \frac{2^1}{1!} \frac{N!}{\left(\frac{N-1}{2}\right)!} & \dots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 \\ x^\alpha \\ \vdots \\ x^{\alpha(N-1)} \\ x^{\alpha N} \end{bmatrix}}_{X^T(x^\alpha)}, \tag{7}$$

if N is even then it follows,

$$\underbrace{\begin{bmatrix} H_0(x^\alpha) \\ H_1(x^\alpha) \\ \vdots \\ H_{N-1}(x^\alpha) \\ H_N(x^\alpha) \end{bmatrix}}_{H^T(x^\alpha)} = \underbrace{\begin{bmatrix} 2^0 & 0 & \dots & 0 & 0 \\ 0 & 2^1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^{\left(\frac{N-2}{2}\right)} \frac{2^1}{1!} \frac{(N-1)!}{\left(\frac{N-2}{2}\right)!} & \dots & 2^{N-1} & 0 \\ (-1)^{\left(\frac{N-4}{2}\right)} \frac{2^0}{0!} \frac{N!}{\left(\frac{N}{2}\right)!} & 0 & \dots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 \\ x^\alpha \\ \vdots \\ x^{\alpha(N-1)} \\ x^{\alpha N} \end{bmatrix}}_{X^T(x^\alpha)}. \tag{8}$$

Consequently, we can write the above matrices shortly,

$$H^T(x^\alpha) = FX^T(x^\alpha),$$

or

$$H(x^\alpha) = X(x^\alpha)F^T. \tag{9}$$

More generally, if we show that

$$X(x^\alpha) = \left[(x-c)^0(x-c)^{1\alpha} \cdots (x-c)^{(N-1)\alpha}(x-c)^{N\alpha}\right],$$

then, the substitution of Eq.(9) into Eq.(6) yields,

$$y(x) = X(x^\alpha)F^T A. \tag{10}$$

3.2. Caputo derivatives of operational matrix

Now, we need to determine any $k\alpha$ th order Caputo fractional derivatives of Eq.(10) by the following procedure,

$$^C D^{k\alpha} y(x) = ^C D^{k\alpha} X(x^\alpha) F^T A. \tag{11}$$

$$^C D^{k\alpha} X(x^\alpha) = [^C D^{k\alpha}(x-c)^0 \ ^C D^{k\alpha}(x-c)^{1\alpha} \cdots ^C D^{k\alpha}(x-c)^{(N-1)\alpha} \ ^C D^{k\alpha}(x-c)^{N\alpha}] \tag{12}$$

$$\underbrace{\begin{bmatrix} ^C D^\alpha(x-c)^0 \\ ^C D^\alpha(x-c)^{1\alpha} \\ ^C D^\alpha(x-c)^{2\alpha} \\ \vdots \\ ^C D^\alpha(x-c)^{(N-1)\alpha} \\ ^C D^\alpha(x-c)^{N\alpha} \end{bmatrix}}_{^C D^{k\alpha} X^T(x^\alpha)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \Gamma(\alpha+1) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} (x-c)^0 \\ (x-c)^{1\alpha} \\ (x-c)^{2\alpha} \\ \vdots \\ (x-c)^{(N-1)\alpha} \\ (x-c)^{N\alpha} \end{bmatrix}}_{X^T(x^\alpha)}$$

Consequently, one can write that ${}^{C}D^{\alpha}X(x^{\alpha}) = X(x^{\alpha})B^{T}$ and the following theorem holds.

Theorem 2. *Let $X(x^{\alpha})$ be the Hermite polynomial vector, for any $\alpha > 0$, then we have,*

$${}^{C}D^{k\alpha}y(x) = X(x^{\alpha})(B^{T})^{k}F^{T}A,$$

Proof. By the help of Theorem 1, the successive α th order Caputo fractional derivatives of $X(x^{\alpha})$ become,

$$\begin{aligned}
{}^{C}D^{\alpha C}D^{\alpha}X(x^{\alpha}) &= \underbrace{{}^{C}D^{\alpha}X(x^{\alpha})}_{X(x^{\alpha})B^{T}}B^{T},\\
{}^{C}D^{2\alpha}X(x^{\alpha}) &= X(x^{\alpha})(B^{T})^{2}, \qquad (13)\\
&\vdots,\\
{}^{C}D^{k\alpha}X(x^{\alpha}) &= X(x^{\alpha})(B^{T})^{k}.
\end{aligned}$$

Hence, substitution of Eq.(13) into Eq.(11) gives,

$${}^{C}D^{k\alpha}y(x) = X(x^{\alpha})(B^{T})^{k}F^{T}A. \qquad (14)$$

Eq.(14), is also shown by the following formula at collocation points $x = x_i$ as,

$${}^{C}D^{k\alpha}y(x_i) = X(x_i^{\alpha})(B^{T})^{k}F^{T}A. \qquad (15)$$

Now, let recall the differential equation redefined at collocation points as same as below,

$$\sum_{k=0}^{m} P_k(x_i)^{C}D^{k\alpha}y(x_i) = g(x_i), \quad i = 0, 1, 2, \ldots, N, \qquad (16)$$

therefore, Eq.(16) is written in the following matrix form:

$$\underbrace{\begin{bmatrix} P_k(x_0) & 0 & \cdots & 0\\ 0 & P_k(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & P_k(x_N)\end{bmatrix}}_{P_k} \times \underbrace{\begin{bmatrix} {}^{C}D^{k\alpha}y(x_0)\\ {}^{C}D^{k\alpha}y(x_1)\\ \vdots\\ {}^{C}D^{k\alpha}y(x_N)\end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{bmatrix}}_{G}.$$

In the compact form, Eq.(16) can be given as,

$$\sum_{k=0}^{m} P_k Y^{k\alpha} = G. \qquad (17)$$

On the other hand, we have by Eq.(15),

$$\underbrace{\begin{bmatrix} {}^{C}D^{k\alpha}y(x_0)\\ {}^{C}D^{k\alpha}y(x_1)\\ \vdots\\ {}^{C}D^{k\alpha}y(x_N)\end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} X(x_0^{\alpha})\\ X(x_1^{\alpha})\\ \vdots\\ X(x_N^{\alpha})\end{bmatrix}}_{X^{\alpha}} \big[(B^{T})^{k}F^{T}A\big],$$

where the matrix X^{α} is equivalent to

$$X^{\alpha} = \underbrace{\begin{bmatrix} X(x_0^{\alpha})\\ X(x_1^{\alpha})\\ \vdots\\ X(x_N^{\alpha})\end{bmatrix}}_{X^{\alpha}} = \begin{bmatrix} 1 & (x_0-c)^{1\alpha} & \cdots & (x_0-c)^{(N-1)\alpha} & (x_0-c)^{N\alpha}\\ 1 & (x_1-c)^{1\alpha} & \cdots & (x_1-c)^{(N-1)\alpha} & (x_1-c)^{N\alpha}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & (x_N-c)^{1\alpha} & \cdots & (x_N-c)^{(N-1)\alpha} & (x_N-c)^{N\alpha}\end{bmatrix}.$$

Hence, we can rewrite Eq.(15) as follows,

$$Y^{k\alpha} = X^{\alpha}(B^{T})^{k}F^{T}A. \qquad (18)$$

Finally, the substitution Eq.(18) into Eq. (17) gives the fundamental matrix equation such as

$$\sum_{k=0}^{m} P_k X^{\alpha}(B^{T})^{k}F^{T}A = G. \qquad (19)$$

Moreover, denoting

$$W = \sum_{k=0}^{m} P_k(x)X^{\alpha}(B^{T})^{k}F^{T},$$

where $W = [w_{ij}]$ ($i, j = 0, 1, 2, \ldots N$), then, Eq. (19) is shown by,

$$W.A = G. \qquad (20)$$

Now, Eq. (20) generates an algebraic system which consists of (N+1) rows and (N+1) columns. Then, the augmented matrix of the system is written by,

$$[W;G]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1)\\ \vdots & \vdots & \vdots & \vdots & ; & \vdots\\ w_{(N-1)0} & w_{(N-1)1} & \cdots & w_{(N-1)N} & ; & g(x_{N-1})\\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(x_N)\end{bmatrix} \tag{21}$$

This method can be modified to handle the initial conditions defined at particular point a. Therefore, we recall the initial Eq. (4),

$$^{C}D^{j}y(a)=\lambda_j, j=0,1,2,\ldots m\alpha-1.$$

Hence, substitution of these conditions into Eq.(15) yields,

$$X^{\alpha}(a)(B^T)^jF^TA=\lambda_j. \tag{22}$$

Therefore, defining U_j as,

$$U_j=X^{\alpha}(a)(B^T)^jF^T=\begin{bmatrix} u_{j0} & u_{j1} & u_{j2} & \cdots & u_{jN}\end{bmatrix}$$

then, Eq.(22) can be shown by

$$U_jA=\lambda_j, \tag{23}$$

and corresponding augmented matrix is written of the form,

$$[U_j;\lambda_j], j=0,1,2,\ldots,m\alpha-1,$$

and denoted by

$$[U_j;\lambda_j]=\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0\\ u_{10} & u_{01} & \cdots & u_{1N} & ; & \lambda_1\\ \vdots & \vdots & \vdots & \vdots & ; & \vdots\\ u_{(m\alpha-1)0} & u_{(m\alpha-1)1} & \cdots & u_{(m\alpha-1)N} & ; & \lambda_{m\alpha-1}\end{bmatrix}. \tag{24}$$

Now, if the $m\alpha$th row of the the augmented matrix Eq.(21)of the system is replaced by the augmented matrix of initial conditions Eq.(24), then one can write the following matrix form,

$$\left[\widetilde{W};\widetilde{G}\right]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0)\\ w_{10} & w_{01} & \cdots & w_{1N} & ; & g(x_1)\\ \vdots & \vdots & \vdots & \vdots & ; & \vdots\\ w_{(N-1-m\alpha)0} & w_{(N-1-m\alpha)1} & \cdots & w_{(N-1-m\alpha)N} & ; & g(x_{N-1-m\alpha})\\ w_{(N-m\alpha)0} & w_{(N-m\alpha)1} & \cdots & w_{(N-m\alpha)N} & ; & g(x_{N-m\alpha})\\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0\\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1\\ \vdots & \vdots & \vdots & \vdots & ; & \vdots\\ u_{(m\alpha-1)0} & u_{(m\alpha-1)1} & \cdots & u_{(m\alpha-1)N} & ; & \lambda_{m\alpha-1}\end{bmatrix}$$

Hence, the system of algebraic equations are shown by the following notation,

$$\widetilde{W}A=\widetilde{G}. \tag{25}$$

□

Remark 1. *Now, let us consider the system* $\widetilde{W}A=\widetilde{G}$.

If $rank\widetilde{W}=rank\,[\widetilde{W},\widetilde{G}]=N+1$, (i.e $\det(\widetilde{W})\neq 0$) *then we can write*

$$A=(\widetilde{W})^{-1}\widetilde{G}. \tag{26}$$

Consequently, the Hermite coefficients a_k $(k=0,1,2,...,N)$ can be uniquely determined by Eq.(25). As a result, the truncated Hermite series is written as follows,

$$y(x)=\sum_{k=0}^{N}a_kH_k(x^{\alpha}). \tag{27}$$

4. Error bounds

Eq.(27) is the approximate solution to Eq.(3) with the initial conditions, Eq.(4). Therefore, substitution the truncated Hermite series into the problem, we obtain the residuals;

$$\left|\sum_{k=0}^{m}P_k(x_i)^{C}D^{k\alpha}y(x_i)-g(x_i)\right|$$

at $x=x_i$ $(-\infty<a\leq x\leq b<\infty)$, $i=0,1,2,\ldots,N$. Then, we call the error function as $E(x_i)$ and this function should be less than ϵ, which is a positive number and can arbitrarily be chosen as $10^{-k_i\alpha}$. As a result, error function becomes, $E(x_i) \le 10^{-k_i\alpha}$ where $k_i > 0$ is any constant. If the $\max(10^{-k_i\alpha}) = 10^{-k\alpha}$ is desired accuracy then, the truncation limit N is incerased untill $E(x_i)$ approaches zero. Besides, the global error function is defined as follows,

$$E_N(x) = \sum_{k=0}^{m} P_k(x)^C D^{k\alpha} y(x) - g(x).$$

Consequently, the global error, $E_N(x) \to 0$ when N is sufficiently large.

5. Illustrative examples

The method which was mentioned so far has been used to solve multi term fractional order differential equations. To show the accuracy of the method, the following examples have been solved. All the numerical calculations have been performed by using Matlab v7.5. and the results have been given by Figure 1 for different values of α. The comparisons between exact and Hermite polynomial solution approximation have been made and shown by Table 1.

Example 1. *First we consider Bagley-Torvik equation [35];*

$$D^2 y(x) + D^{3/2} y(x) + y(x) = x + 1, \qquad (28)$$

where $\alpha = 1/2, m = 4$ *and* $g(x) = x + 1$. *To find HCM solution of the problem here, for conveniance, we choose* $N = 2$. *Because the analytical solution of the problem for* $\alpha = 1$ *is easily obtainable. Therefore, we only concentrate on the different values of* α. *Hence,the approximate solution can be written by the following truncated Hermite series:*

$$y(x) = \sum_{n=0}^{2} a_n H_n(x^\alpha). \qquad (29)$$

The coefficients of the differential equation are $P_0(x) = P_3(x) = P_4(x) = 1, P_1(x) = P_2(x) = 0$ *Since* $N = 2$ *then, collocation points are taken as* $\{x_0 = 0, x_1 = 1/2, x_2 = 1\}$. *From the fundamental matrix equation Eq.(19),one can write that*

$$\{P_0 X + P_3 X (B^T)^3 + P_4 X (B^T)^4\} F^T A = G.$$

After evaluating the matrices B and F and substituting them into the above equation, then, the augmented matrix is obtained as follows,

$$W.A = G \Rightarrow [W; G] = \begin{bmatrix} 1 & 0 & -2 & ; & 1 \\ 1 & 1.4 & 0 & ; & 1.5 \\ 1 & 2 & 2 & ; & 2 \end{bmatrix}$$

Since $\det(W) \neq 0$, *finding the solution of the system defines the coefficients of the truncated series as*

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \\ 0.25 \end{bmatrix}$$

Finally, substituting these coefficients into Eq.(29),

$$y(x) = a_0 H_0(x^\alpha) + a_1 H_1(x^\alpha) + a_2 H_2(x^\alpha)$$

then, we obtain Hermite polynomial solution of the problem as

$$y(x) = \frac{3}{2} + \frac{1}{4}(4x^\alpha - 2).$$

From here, if we substitute $\alpha = 1$ *in* $y(x)$ *then, we obtain* $y(x) = x + 1$. *This is the exact solution of the problem for integer order case.*

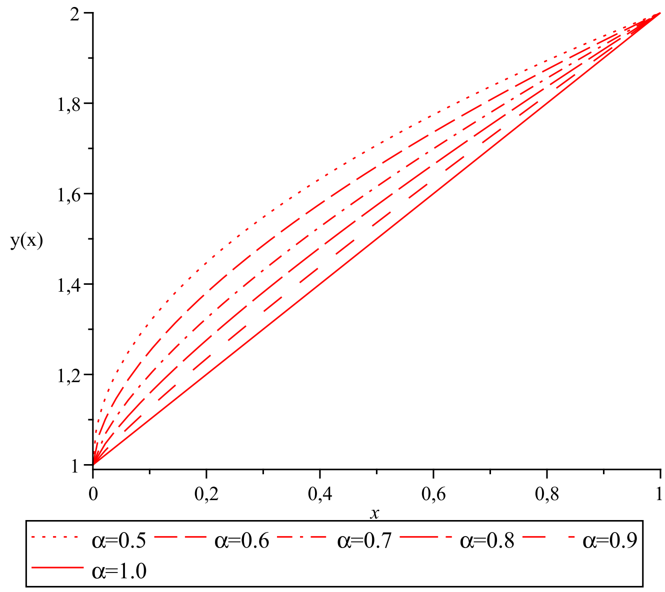

Figure 1. The solution curves of Example 1 for different values of α. The α values change from 0.5 to 1.

Example 2. *Next we consider an initial value problem which is studied in [36] where* $0 \le x \le 1$ *and* $\alpha \in (0, 1), \beta \le 0$. *Therefore, we can write the equation as:*

$$D^{\alpha}y(x)=\beta y(x)+g(x). \tag{30}$$

We assume that $\beta=-1$ *and* $g(x)=x^2+\frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}$ *defined on [0,1] and the initial condition is* $y(0)=0$. *If we apply the HCM for* $\alpha=\frac{1}{2}$ *and* $N=4$, *it is obvious that* m *is 2 and* $P_0(x)=P_1(x)=1, P_2(x)=0$. *Therefore, we can write the following fundamental matrix of Eq.(30) as,*

$$\left\{P_0X+P_1XB^T+P_2X(B^T)^2\right\}F^TA=G.$$

Therefore, we can easily establish the system $WA=G$ *and constitute* $[W;G]$ *matrix as,*

$$[W;G]=\begin{bmatrix} 1.0000 & 1.7725 & -2.0000 & -10.6347 & 12.0000 & ; & 0.0000\\ 1.0000 & 2.7725 & 1.2568 & -12.9760 & -23.0721 & ; & 0.2506\\ 1.0000 & 3.1867 & 3.1915 & -10.9742 & -37.7877 & ; & 0.7818\\ 1.0000 & 3.5045 & 4.9088 & -7.8548 & -46.2706 & ; & 1.5397\\ 1.0000 & 3.7725 & 6.5135 & -4.0000 & -50.0901 & ; & 0.0000 \end{bmatrix}.$$

On the other hand, the augmented matrix, which corresponds to initial condition, is obtained by substitution the row,

$$y(0)=X(\mathbf{0})F^TA=\lambda_0=0,$$

or

$$y(0)=\begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}F^TA=0.$$

Finally, we find that $[U_0;\lambda_0]=\begin{bmatrix}1 & 0 & -2 & 0 & 12 & ; & 0\end{bmatrix}$. Therefore, the augmented matrix of the system, $\widetilde{W}A=\widetilde{G}$, is obtained from Eq.(25) as follows,

$$\left[\widetilde{W};\widetilde{G}\right]=\begin{bmatrix} 1.0000 & 1.7725 & -2.0000 & -10.6347 & 12.0000 & ; & 0.0000\\ 1.0000 & 2.7725 & 1.2568 & -12.9760 & -23.0721 & ; & 0.2506\\ 1.0000 & 3.1867 & 3.1915 & -10.9742 & -37.7877 & ; & 0.7818\\ 1.0000 & 3.5045 & 4.9088 & -7.8548 & -46.2706 & ; & 1.5397\\ 1.0000 & 0.0000 & -2.0000 & 0.0000 & 12.0000 & ; & 0.0000 \end{bmatrix}$$

Consequently, solution of the above system gives the approximate solution of the problem as,

$$y(x)=0,986076x10^{-31}x^{1/2}+0,104468x10^{-13}x^{3/2}+x^2.$$

We note here that our solution is very close to the exact solution, $y(x)=x^2$, since the first two terms vanishes (see Figure 2).

As a result of all these, the HCM solution is very good approximation to the problem even for small N. Table 1 lists both the exact solution and error function at particular x, corresponding to the Example.

Table 1. The exact solution of Example 2 and error function $E(x_i)$ for HCM solution.

x values	Exact Solution	$E(x_i)$ For HCM solution, $N=4$
0.0	0.0000	9.0206e-17
0.2	4.0000e-2	9.0206e-17
0.4	1.6000e-1	2.7756e-16
0.6	3.6000e-1	4.9960e-16
0.8	6.4000e-1	7.7716e-16
1.0	1.0000	1.1102e-15

In Figure 2, both the analytical and the HCM solutions (for $N=2$ and $N=4$) are given. It is clear that the series (for $N=4$) and the analytical solution are identical.

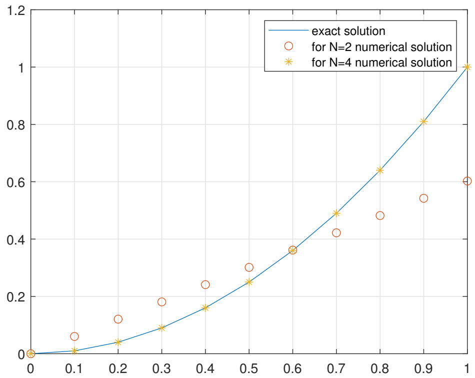

Figure 2. Comparison of Analytical solution and HCM solution for Example 2.

6. Conclusion

The objective of this study is to apply the HCM method for solving higher order multi-term fractional order differential equations. The motivation of this work is that obtaining considerable simplifications in the solutions of the multi-term fractional order differential equations by using HCM, since the analytical solutions of such equations cannot easily be obtained. By using any symbolic toolbox of Matlab programme, the Hermite polynomial coefficients of the solution can be obtained easily. Illustrated examples determine the reliability of the algorithm and give chance to apply the method for wider classes of equations. As a further work, the method will be considered for solving nonlinear fractional order differential equations.

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