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RESEARCH ARTICLE

Sinc-Galerkin method for solving hyperbolic partial differential equations

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ABSTRACT

In this work, we consider the hyperbolic equations to determine the approximate solutions via Sinc-Galerkin Method (SGM). Without any numerical integration, the partial differential equation transformed to an algebraic equation system. For the numerical calculations, *Maple* is used. Several numerical examples are investigated and the results determined from the method are compared with the exact solutions. The results are illustrated both in table and graphically.

1. Introduction

Most numerical methods are established on the basis of polynomials. One of these methods is Sinc methods. Frank Stenger firstly introduced the Sinc methods in his works [1, 22] to determine the solutions of some differential equations. The actual detailed analysis on Sinc functions were firstly made by Whittaker in the papers [20-21]. Lund has some works on two-point boundary-value problems [7, 9]. Lewis, Lund and Bowers investigated the parabolic and hyperbolic problems in [6, 12]. In [4] Bowers and Lund worked on singular Poisson and elliptic problems. Numerical solutions of the problems are found by means of SGM. Lund, Bowers and McArthur introduced the Symmetric SGM in [8]. A kind of Sinc methods which is called Sinc Domain Decomposition Method is illustrated in [10, 11, 14, and 15]. Moreover, iterative methods for symmetric Sinc-Galerkin systems are considered in [3, 16, and 17]. Some applications in the various areas of the science and engineering can be seen in [2, 5, 13, 18 and 19]. In the work of Morlet, Lybeck and Bowers in [15], a Volterra integro-differential equation is investigated via the Sinc-collocation method. In paper [1], Stenger made some applications with SGM for the approximate solutions of ODEs and some elliptic and parabolic PDEs. Koonprasert developed a fully SGM for some complex-valued PDEs with time-dependent boundary conditions in [5]. In the work of Stenger [23], some problems related to medical problems are taken and numerical results are found using Sinc methods. In [24], a new algorithm based on Sinc method is applied for the solution of a nonlinear set of PDEs. A new SGM is illustrated for the numerical solutions of convection diffusion equations on half-infinite intervals in [25]. The work in [26] Gamel, Behiry and Hashish dealt with the SGM for solving nonlinear ODEs with various boundary conditions. In [27], sinc-Galerkin method is also applied to a class of the second-order nonlinear ODEs. In the work [28], Zamani focused on some differential operators in one dimension and a Helmholtz eigenvalue problem in two dimensions. In [29], authors use sinc-Galerkin method to obtain approximate solution of fractional partial differential equations. In [30,32] Sinc methods is also applied for the solution of the second-order ODEs with homogeneous Dirichlet-Type boundary conditions. In [31], sinc-Galerkin method is used for solving fractional boundary value problems approximately. In this paper, we use sinc-Galerkin method to obtain approximate solution of a class of hyperbolic partial differential equations. The rest of this paper is organized as follows. In section 2, we give some definitions and theorems for sinc methods. In section 3, some test problems are given to compare the ability of present methods by using tables and graphics. Finally, in section 4, the paper is completed with a conclusion.

2. Sinc-Galerkin method

2.1. Sinc-Approximation formula for hyperbolic

We consider the following hyperbolic partial differential equation.

$$\begin{array}{l} u_{tt} - u_{xx} = F(x,t), \\ u(0,t) = u(1,t) = 0, 0 < x < 1, \\ u(x,0) = 0, t > 0, \\ u_t(x,0) = 0, t > 0. \end{array} \tag{1}$$

To determine the approximate solution of this equation, Sinc-Galerkin method is used. For the equation given above, the sinc-Galerkin method can be developed in both space and time direction as following:

In general, approximations can be constructed for infinite, semi-infinite, and infinite intervals and both spatial and time spaces will be introduced.

Let us define the function ϕ as

$$\phi(z) = \ln\left(\frac{z}{1-z}\right). \tag{2}$$

Here ϕ is a conformal mapping from D_E, the eye-shaped domain in the z plane, onto the infinite strip, D_S where

$$D_E = \left\{ z = x + iy : \left| \arg\left(\frac{z}{1-z}\right) \right| < d \le \frac{\pi}{2} \right\}. \tag{3}$$

This is shown in Figure 1.

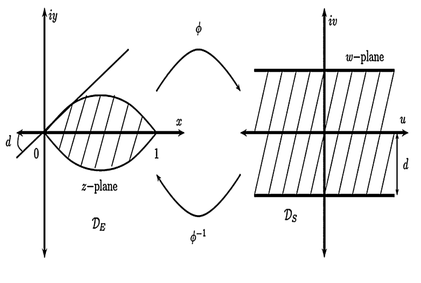

Figure 1. The connection between eye shape domain and the infinite strip [32].

A more general form of the sinc basis according to intervals can be given as following way

$$\begin{array}{c} S(m,h_x)\circ\phi(x) = Sinc\left(\frac{\phi(x) - mh_x}{h_x}\right), \\ m = -N_x, \ldots, N_x, \\ S(k,h_t)\circ\gamma(t) = Sinc\left(\frac{\gamma(t) - kh_t}{h_t}\right), \\ k = -N_t, \ldots, N_t, \end{array} \tag{4}$$

where

$$Sinc(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z} & , z \neq 0 \\ 1 & , z = 0 \end{cases} \tag{5}$$

and

$$Sinc(k,h)(z) = Sinc\left(\frac{z-kh}{h}\right) = \begin{cases} \frac{\sin\left(\pi\frac{z-kh}{h}\right)}{\pi\frac{z-kh}{h}}, & z \neq kh \\ 1, & z = kh \end{cases}, \tag{6}$$

and the conformal maps for both direction as follows

$$\begin{cases} \phi(x) = \ln\left(\frac{x}{l-x}\right) & , x \in (0,l) \\ \gamma(t) = \ln(t) & , t \in (0,\infty), \end{cases} \tag{7}$$

are used to define the basis functions on the intervals $(0,l)$ and $(0,\infty)$ respectively. $h_x, h_t > 0$ represents the mesh sizes in the space direction and the time direction respectively. The sinc nodes x_i and t_j are chosen so that $x_i = \phi^{-1}(ih_x)$,$t_j = \gamma^{-1}(jh_t)$. Here the function $x = \phi^{-1}(x) = \frac{e^x}{1+e^x}$ is an inverse mapping of $\phi = \phi(x)$.

We may define the range of ϕ^{-1}on the real line as

$$\Gamma_1 = \left\{\phi^{-1}(u) \in D_E : -\infty < u < \infty\right\}. \tag{8}$$

For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}} \tag{9}$$

where $0 < x_k < 1$, for all k.

The sinc basis functions in (4) do not have a derivative when x tends to 0 or 1. We modify the sinc basis functions as

$$\frac{S(m,h_x)\circ\phi(x)}{\phi'(x)}=\frac{Sinc\left(\frac{\phi(x)-mh_x}{h_x}\right)}{\phi'(x)} \tag{10}$$

where

$$\frac{1}{\phi'(x)}=x(1-x). \tag{11}$$

The modified sinc basis functions is shown in Figure 2.

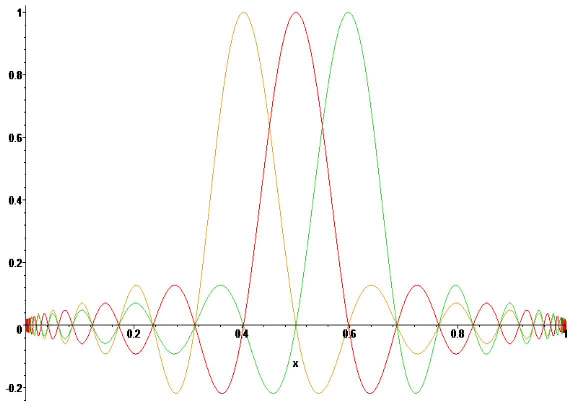

Figure 2. The modified sinc basis on (0, 1) [32].

For the transient space, we generate an approximation via defining the function

$$w=\gamma(r)=\ln(r). \tag{12}$$

Here, w is a conformal mapping from D_W, the wedge-shaped domain in the r-plane onto the infinite strip, D_S, where

$$D_W=\left\{r=t+is:|\arg(r)|<d<\frac{\pi}{2}\right\}. \tag{13}$$

For the SGM, the basis functions are determined from composite translated functions,

$$S(k,h_t)\circ\gamma(t)=Sinc\left(\frac{\gamma(t)-kh_t}{h_t}\right),\quad k=-N_t,.....,N_t. \tag{14}$$

The functions are given in Figure 3 for real values of t.

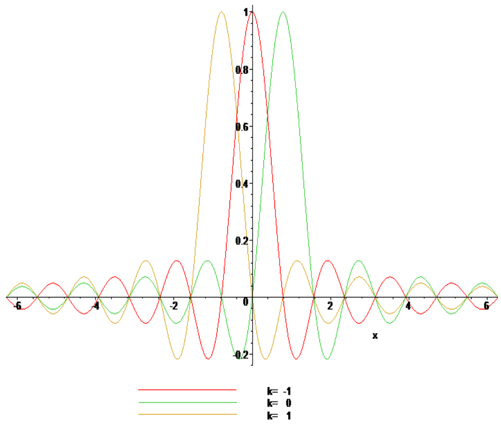

Figure 3. Adjacent members of $S(k,h)\circ\gamma(t)$ when $k=-1,0,1$ and $h=\frac{\pi}{8}$on $(0,\infty)$ [7].

In (14), $w=\gamma(r)$ and $\gamma^{-1}(w)=r=e^{w}$. We may define γ^{-1}on the real line as

$$\Gamma_2=\left\{\gamma^{-1}(u)\in D_w:-\infty<u<\infty\right\}. \tag{15}$$

For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$t_k=\gamma^{-1}(kh)=e^{kh}, \tag{16}$$

where $0<t_k<\infty$, for all k. A list of conformal mappings may be found in Table 1 below, [9].

Table 1. Conformal mappings and nodes for several subintervals of R.

(a,b)		$\phi(z)$	z_k
a	b	$\ln\left(\frac{z-a}{b-z}\right)$	$\frac{a+be^{kh}}{1+e^{kh}}$
0	1	$\ln\left(\frac{z}{1-z}\right)$	$\frac{e^{kh}}{1+e^{kh}}$
0	∞	$\ln(z)$	e^{kh}
0	∞	$\ln(\sinh(z))$	$\ln\left(e^{kh}+\sqrt{e^{2kh}+1}\right)$
$-\infty$	∞	z	kh
$-\infty$	∞	$\sinh^{-1}(z)$	kh

Definition 1. *Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy*

$$\int_{\psi(L+u)}|F(z)|\,dz\to 0,\quad \text{as } u=\mp\infty, \tag{17}$$

where

$$L=\left\{iy:|y|<d\le\frac{\pi}{2}\right\}, \tag{18}$$

and on the boundary of D_E satisfy

$$T(F) = \int_{\partial D_E} |F(z)dz| < \infty. \tag{19}$$

The proof of following theorems can be found in [1].

Theorem 1. *Let* Γ *be* $(0,1)$, $F \in B(D_E)$, *then for* $h > 0$ *sufficiently small*

$$\int_\Gamma F(z)dz - h\sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D} \frac{F(z)k(\phi,h)(z)}{\sin(\pi\phi(z)/h)}dz, \equiv I_F, \tag{20}$$

where

$$|k(\phi,h)|_{z\in\partial D} = \left| e^{\left[\frac{i\pi\phi(z)}{h} sgn(Im\phi(z))\right]}\right|_{z\in\partial D} = e^{\frac{-\pi d}{h}}. \tag{21}$$

For the SGM, the infinite quadrature rule must be truncated to a finite sum; the following theorem demonstrates the conditions under which exponential convergence results.

Theorem 2. *If there exist positive constants* α, β *and* C *such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C \begin{cases} e^{-\alpha|\phi(x)|}, x \in \psi((-\infty,\infty)) \\ e^{-\beta|\phi(x)|}, x \in \psi((0,\infty)), \end{cases} \tag{22}$$

then the error bound for the quadrature rule (20) *is*

$$\left|\int_\Gamma F(x)dx - h\sum_{j=-N}^{N} \frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha Nh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + |I_F|. \tag{23}$$

The infinite sum in (20) is truncated with the use of (21) to arrive at this inequality (23). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \tag{24}$$

$$N \equiv \left\|\frac{\alpha N}{\beta} + 1\right\|, \tag{25}$$

where $\|.\|$ is integer part of statement, then

$$\int_\Gamma F(x)dx = h\sum_{j=-N}^{N} \frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha d N)^{1/2}}\right). \tag{26}$$

Theorems 1 and 2 can be used to approximate the integrals that arise in the formulation of the discrete systems.

2.2. Discrete solutions scheme for hyperbolic PDEs

In ordinary differential equations

$$Lu = f, \tag{27}$$

on Γ_1, sinc solution is assumed as an approximate solution u_m in the form of series which $m = 2N+1$ terms

$$u_m(z) = \sum_{j=-N}^{N} c_j S(j,h) \circ \phi(z). \tag{28}$$

The coefficients $\{c_j\}_{j=-N}^{N}$ are determined by orthogonalizing the residual $Lu - f$ with respect to the sinc basis functions $\{S_j\}_{j=-N}^{N}$ where $S_j(z) = S(j,h) \circ \phi(z)$. An inner product for two continuous function such as f_1 and f_2 can be given by the following formula

$$\langle f_1, f_2\rangle = \int_\Gamma f_1 f_2 w dz, \tag{29}$$

where w is the weight function and chosen depending on boundary conditions. If we implement above inner product rule in orthogonalization this yields the discrete sinc-Galerkin system

$$\int_\Gamma (Lu_m - f)(z) S(k,h) \circ \phi(z) \cdot w(z)dz = 0, \quad -N \le k \le N. \tag{30}$$

Now, we are going to derive discrete sinc-Galerkin system for PDEs. Let we assume u_{m_z,m_t} is the approximate solution of equation (1). Then, the discrete system takes the following form

$$u_{m_z,m_t}(z,t) = \sum_{j=-N}^{N}\sum_{k=-N}^{N} c_{jk} S(j,h) \circ \phi(z) \cdot S(k,h) \circ \gamma(t). \tag{31}$$

The coefficients $\{c_{jk}\}_{j,k=-N}^{N}$ are determined by orthogonalizing the residual $Lu_{m_z,m_t} - f$ with respect to the sinc basis functions $\{S_j S_k\}_{j,k=-N}^{N}$ where $S_j S_k(z,t) = S(j,h) \circ \phi(z) S(k,h) \circ \gamma(t)$ for $-N \le j,k \le N$. In this case the inner product takes the following form

$$\langle f_1, f_2\rangle = \int_{\Gamma_t}\int_{\Gamma_z} f_1(z,t) f_2(z,t) w(z,t) dz dt. \tag{32}$$

The choice of the weight function $w(z,t)$ in the double integrand depends on the boundary conditions, the domain, and the partial differential equation. Therefore the discrete Galerkin system is

$$\int_{\Gamma_t}\int_{\Gamma_z}\left(Lu_{m_z m_t}-f\right)(z,t)\cdot S(j,h)\circ\phi(z)\cdot S(k,h)\circ\gamma(t)\cdot w(z,t)dzdt=0. \quad (33)$$

2.3. Matrix representation of the derivatives of sinc basis functions at nodal points

The sinc-Galerkin method actually requires the evaluated derivatives of sinc basis functions at the sinc nodes$z=z_j$. The rth derivative of $S_k(z)=S(k,h)\circ\phi(z)$with respect to$\phi$, evaluated at the nodal point $z_j$is denoted by

$$\frac{1}{h^r}\delta_{jk}^{(r)}=\frac{d^r}{d\phi^r}\left(S(k,h)\circ\phi(z)\right)\Big|_{z=z_j.} \quad (34)$$

Here, for each k and j can be stored in a matrix$I^{(r)}=\left[\delta_{jk}^{(r)}\right]$. For $r=0,1,2,...$

$$\begin{aligned}I^{(0)}=\delta_{jk}^{(0)}&=\left[S(j,h)\circ\phi(x)\right]\big|_{x=x_k}\\&=\begin{cases}1, k=j\\0, k\neq j,\end{cases}\end{aligned} \quad (35)$$

$$\begin{aligned}I^{(1)}=\delta_{jk}^{(1)}&=h\frac{d}{d\phi}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k}\\&=\begin{cases}0, k=j\\\frac{(-1)^{k-j}}{(k-j)}, k\neq j,\end{cases}\end{aligned} \quad (36)$$

$$\begin{aligned}I^{(2)}=\delta_{jk}^{(2)}&=h\frac{d^2}{d\phi^2}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k}\\&=\begin{cases}\frac{-\pi^2}{3}, k=j\\\frac{-2(-1)^{k-j}}{(k-j)^2}, k\neq j,\end{cases}\end{aligned} \quad (37)$$

where

$$I_m^{(0)}=\begin{bmatrix}1&0&0&...&0\\0&1&0&...&0\\0&0&1&...&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&...&1\end{bmatrix}=\left[\delta_{jk}^{(0)}\right], \quad (38)$$

$$\begin{aligned}I_m^{(1)}&=\begin{bmatrix}0&-1&\frac{1}{2}&...&\frac{1}{2N}\\1&0&-1&...&-\frac{1}{2N-1}\\-\frac{1}{2}&1&0&...&\frac{1}{2N-2}\\\vdots&\vdots&\vdots&\ddots&\vdots\\-\frac{1}{2N}&\frac{1}{2N-1}&\frac{1}{2N-2}&...&0\end{bmatrix}\\&=\left[\delta_{jk}^{(1)}\right],\end{aligned} \quad (39)$$

$$\begin{aligned}I_m^{(2)}&=\begin{bmatrix}-\frac{\pi^2}{3}&\frac{2}{1^2}&-\frac{2}{2^2}&...&-\frac{2}{(2N)^2}\\\frac{2}{1^2}&-\frac{\pi^2}{3}&\frac{2}{1^2}&...&\frac{2}{(2N-1)^2}\\-\frac{2}{2^2}&\frac{2}{1^2}&-\frac{\pi^2}{3}&...&-\frac{2}{(2N-2)^2}\\\vdots&\vdots&\vdots&\ddots&\vdots\\-\frac{2}{(2N)^2}&\frac{2}{(2N-1)^2}&-\frac{2}{(2N-2)^2}&...&-\frac{\pi^2}{3}\end{bmatrix}\\&=\left[\delta_{jk}^{(2)}\right].\end{aligned} \quad (40)$$

The chain rule has been used for the z-derivative of product sinc functions. For example, when $S_j(z)=S(j,h)\circ\phi(z)$;

$$\begin{aligned}\frac{d\left(S_j(z)w(z)\right)}{dz}&=\left(\frac{dS_j(z)}{d\phi(z)}\cdot\frac{d\phi(z)}{dz}\right)w(z)\\&\quad+S_j(z)\frac{dw(z)}{dz}\\&=\frac{dS_j(z)}{d\phi}\phi'(z)w(z)+S_j(z)w'(z),\end{aligned} \quad (41)$$

and

$$\begin{aligned}\frac{d^2\left(S_j(z)w(z)\right)}{dz^2}&=\frac{d}{dz}\left(\frac{dS_j(z)}{d\phi}\phi'(z)w(z)+S_j(z)w'(z)\right)\\&=\frac{d^2S_j(z)}{d\phi^2}\left(\phi'(z)\right)^2w(z)+\frac{dS_j(z)}{d\phi}\phi''(z)w(z)\\&\quad+2\cdot\frac{dS_j(z)}{d\phi}\phi'(z)w'(z)+S_j(z)w''(z).\end{aligned} \quad (42)$$

Now, we are going to develop discrete form for the equation (1). We choose for special case the parameters as follows for the spatial dimension:

$$\left.\begin{array}{l}\phi(z)=\ln\left(\frac{z}{1-z}\right),\\w_X(z)=\frac{1}{\phi'(z)},\\\frac{1}{\phi'(z)}=z(1-z),\end{array}\right\} \quad (43)$$

and for the temporal space as;

$$\left.\begin{array}{l}\gamma(t)=\ln(t),\\w_T(t)=\frac{1}{\gamma'(t)},\\\frac{1}{\gamma'(t)}=t.\end{array}\right\} \quad (44)$$

The discrete form of equation (1) can be given the following form

$$\begin{aligned}\langle Lu - F, S_k S_l\rangle &= \int_{\Gamma_t}\int_{\Gamma_z} (Lu - F)\, S(k,h)\circ\phi(z)\\ &\cdot w_X(x) S(l,s)\circ\gamma(t)\cdot w_T(t)\,dzdt\\ &= \int_{\Gamma_t}\int_{\Gamma_z} (u_{tt} - u_{xx} - F)\, S(k,h)\circ\phi(z)\\ &\cdot w_X(x) S(l,s)\circ\gamma(t)\cdot w_T(t)\,dzdt. \end{aligned} \tag{45}$$

We solve this by taking our approximating basis functions to be

$$\begin{cases} S_k(x) = w_X S(k,h)\circ\phi(x),\\ w_X = \frac{1}{\phi'(x)} = x(1-x),\\ \phi(x) = \ln(\frac{x}{1-x}),\\ S_l(t) = w_T S(l,s)\circ\gamma(t),\\ w_T = \frac{1}{\gamma'(t)} = t,\\ \gamma(t) = \ln(t). \end{cases} \tag{46}$$

If we apply sinc-quadrature rules on the definite integral given (45) by using (46) we can get a matrix system. For this purpose, let $A_m(u)$ be a diagonal matrix, whose diagonal elements are $u(x_{-N}), u(x_{-N+1}), ..., u(x_N)$ and non-diagonal elements are zero. Then (45) reproduces following matrixes accordingly:

$$C = \begin{pmatrix} A_1\\ A_2\\ A_3\\ A_4\\ A_5 \end{pmatrix} \tag{47}$$

where

$$\begin{aligned} A_1 &= c_{-N,-N}\ c_{-N,-N+1}\ c_{-N,-N+2}\ \ \dots\ c_{-N,N},\\ A_2 &= c_{-N+1,-N}\ c_{-N+1,-N+1}\ c_{-N+1,-N+2}\ \dots\ c_{-N+1,N},\\ A_3 &= c_{-N+2,-N}\ c_{-N+2,-N+1}\ c_{-N+2,-N+2}\ \dots\ c_{-N+2,N},\\ A_4 &= \vdots\quad \vdots\quad \vdots\quad \ddots\quad \vdots,\\ A_5 &= c_{N,-N}\ c_{N,-N+1}\ \ c_{N,-N+2}\ \dots\ c_{N,N}, \end{aligned}$$

and

$$\begin{cases} B = -2hI_m^{(0)}\left(A_m(w_X)\right) + I_m^{(1)}\left(A_m(w_X')\right) + \frac{I_m^{(2)}}{h},\\ G = A_m(w_T)\left[sI_m^{(0)} - I_m^1\right],\\ D = hA_m(\frac{w_X}{\phi'}),\\ E = sA_m(\frac{w_T}{\gamma'}). \end{cases} \tag{48}$$

Also, for the right side function F given equation (1) can be written as following matrix form;

$$F = \begin{pmatrix} B_1\\ B_2\\ B_3\\ B_4\\ B_5 \end{pmatrix} \tag{49}$$

where

$$\begin{aligned} B_1 &= F_{-N,-N}\ F_{-N,-N+1}\ F_{-N,-N+2}\ \ \dots\ F_{-N,N},\\ B_2 &= F_{-N+1,-N}\ F_{-N+1,-N+1}\ F_{-N+1,-N+2}\ \dots\ F_{-N+1,N},\\ B_3 &= F_{-N+2,-N}\ F_{-N+2,-N+1}\ F_{-N+2,-N+2}\ \dots\ F_{-N+2,N},\\ B_4 &= \vdots\quad \vdots\quad \vdots\quad \ddots\quad \vdots,\\ B_5 &= F_{N,-N}\ F_{N,-N+1}\ \ F_{N,-N+2}\ \dots\ F_{N,N}, \end{aligned}$$

Therefore, we arrive at a matrix system for equation (1) as follows:

$$D^{-1}BC + CGE^{-1} = F \tag{50}$$

Finally, by using Maple Computer Algebra Software, the matrix system (50) can be solved by using LU or QR decomposition method and can be found unknown coefficients. After calculation of C we get approximate solution as follows:

$$u_{x,t} = \sum_{j=-N}^{N}\sum_{k=-N}^{N} c_{jk} S(j,h)\circ\phi(x)\cdot S(k,h)\circ\gamma(t). \tag{51}$$

3. Numerical Examples

In this section, the presented method will be tested on two different problems.

Example 1. *The following hyperbolic equation given*

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = f(x,t), \tag{52}$$

where

$$f(x,t) = \frac{e^{-t}\left(A + t^2\left(12 - 11x - 8x^2 + 4x^3\right)\right)}{4\sqrt{t}\sqrt{1-x}},$$

and $A = 3(-1+x)^2 x - 12t(-1+x)^2 x.$

The exact solution of equation (52) given as follows

$$u(x,t) = t^{3/2}e^{-t}x(1-x)^{3/2}. \tag{53}$$

For the equation (52), we choose sinc components here in below:

$$h = s = \frac{0.75}{\sqrt{N}}, x_k = \frac{e^{kh}}{1+e^{kh}}, t_l = e^{sl}, \phi(x) = \ln\left(\frac{x}{1-x}\right), \gamma(t) = \ln(t), w_X = \frac{1}{\phi'(x)}, w_T = \frac{1}{\gamma'(t)}. \tag{54}$$

In the light of the above parameters, the numerical results obtained by SGM for equation (52) are indicated in Table 2 and Table 3. Moreover, the graphs of the exact and the approximate solutions for different values are showed in Figure 4, 5 and Figure 6.

Table 2. Numerical results for $N = 5$.

t	x	Exact Sol.	Num. Sol.	Error
0.03	0.3	0.000885	0.002565	0.001679
	0.6	0.000765	0.001809	0.001044
	0.9	0.000143	0.001596	0.001453
0.06	0.3	0.002431	0.014390	0.011958
	0.6	0.002100	0.014650	0.012549
	0.9	0.000393	0.005685	0.005291
0.09	0.3	0.004335	0.018336	0.014000
	0.6	0.003745	0.017395	0.013649
	0.9	0.000702	0.007839	0.007137

Table 3. Numerical results for $N = 20$.

t	x	Exact Sol.	Num. Sol.	Error
0.03	0.3	0.000885	0.001020	0.000134
	0.6	0.000765	0.000474	0.000291
	0.9	0.000143	−0.000060	0.000203
0.06	0.3	0.002431	0.002705	0.000273
	0.6	0.002100	0.001608	0.000492
	0.9	0.000393	0.000033	0.000360
0.09	0.3	0.004335	0.004734	0.000399
	0.6	0.003745	0.003049	0.000695
	0.9	0.000702	0.000186	0.000515

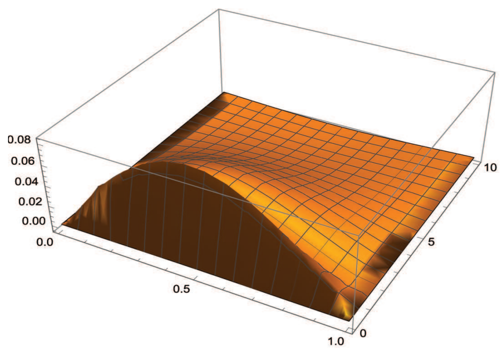

Figure 4. Numerical Simulation of equation (52) according to $N = 5$.

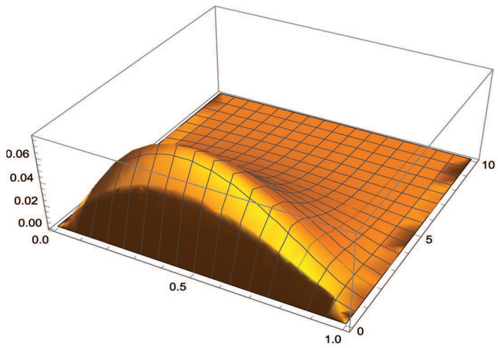

Figure 5. Numerical Simulation of equation (52) according to $N = 20$.

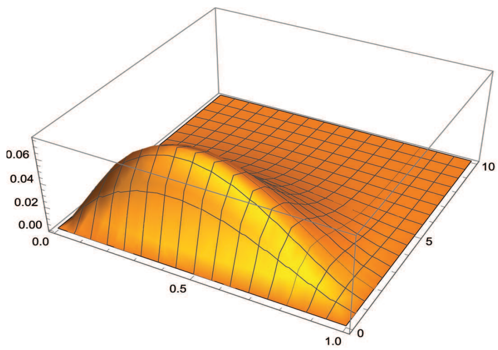

Figure 6. Graph of exact solution of equation (52).

Example 2. *The following hyperbolic equation given*

$$\frac{\partial^2}{\partial t^2} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = f(x,t), \tag{55}$$

where

$$f(x,t) = e^{-t}\left(B + t^2\left(2 + x - x^2\right)\right),$$

and $B = -2(-1+x)x + 4t(-1+x)x$.

The exact solution of equation (55) given as follows

$$u(x,t) = e^{-t}t^2(1-x)x. \tag{56}$$

For the equation (55), we choose sinc components here in below:

$$h = s = \frac{0.75}{\sqrt{N}}, x_k = \frac{e^{kh}}{1+e^{kh}}, t_l = e^{sl},$$
$$\phi(x) = \ln\left(\frac{x}{1-x}\right), \gamma(t) = \ln(t), w_X = \frac{1}{\phi'(x)},$$
$$w_T = \frac{1}{\gamma'(t)}. \tag{57}$$

According to the above parameters, the numerical solutions which are obtained by using the sinc-Galerkin method for equation (55) are presented in Table 4 and Table 5 for different values. Also the graphs of exact and approximate solutions for different values are presented in Figure 7, 8 and Figure 9.

Table 4. Numerical results for $N = 5$.

t	x	Exact Sol.	Num. Sol.	Error
0.03	0.3	0.000183	0.000078	0.000104
	0.6	0.000209	−0.0023475	0.002556
	0.9	0.000078	0.001439	0.001360
0.06	0.3	0.000711	0.003170	0.002458
	0.6	0.000813	0.000211	0.000601
	0.9	0.000305	0.003259	0.002954
0.09	0.3	0.001554	0.003741	0.002187
	0.6	0.001776	−0.002408	0.004185
	0.9	0.000666	0.005409	0.004743

Table 5. Numerical results for $N = 20$.

t	x	Exact Sol.	Num. Sol.	Error
0.03	0.3	0.000183	0.000180	2.86×10^{-6}
	0.6	0.000209	0.000203	5.75×10^{-6}
	0.9	0.000078	0.000077	1.52×10^{-6}
0.06	0.3	0.000711	0.000711	5.46×10^{-7}
	0.6	0.000813	0.000811	2.43×10^{-6}
	0.9	0.000305	0.000304	4.24×10^{-7}
0.09	0.3	0.001554	0.001554	5.03×10^{-7}
	0.6	0.001776	0.001777	1.12×10^{-6}
	0.9	0.000666	0.000666	2.11×10^{-7}

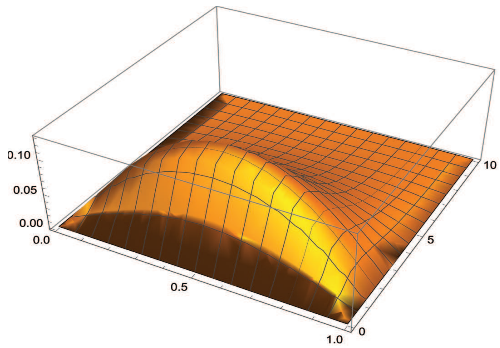

Figure 7. Numerical Simulation of equation (55) according to $N = 5$.

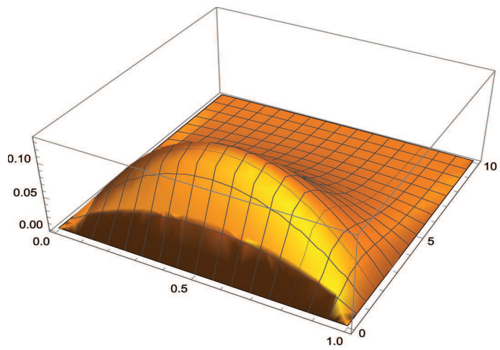

Figure 8. Numerical Simulation of equation (55) according to $N = 20$.

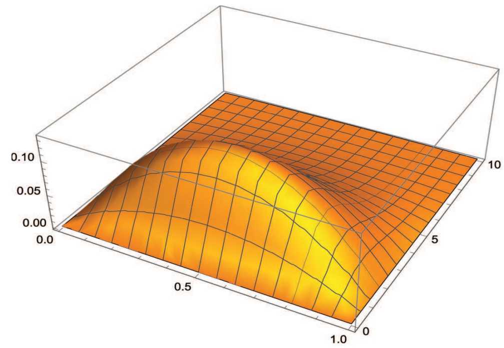

Figure 9. Graph of exact solution of equation (55).

4. Conclusion

The SGM is operated to determine the approximate solutions of second order PDEs. According to the obtained results in numerical examples, sinc-Galerkin method seems to be an efficient method in the sense that selection parameters and changing boundary conditions and also giving different problems to the algorithms. The accuracy of the solutions can be developed by increasing the number of grid points N. In this work, we improve a powerful algorithm for the solution with SGM via Maple. Various PDEs are solved in means of our technique in less than 20 seconds.

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