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RESEARCH ARTICLE

On stable high order difference schemes for hyperbolic problems with the Neumann boundary conditions

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ABSTRACT

In this paper, third and fourth order of accuracy stable difference schemes for approximately solving multipoint nonlocal boundary value problems for hyperbolic equations with the Neumann boundary conditions are considered. Stability estimates for the solutions of these difference schemes are presented. Finite difference method is used to obtain numerical solutions. Numerical results of errors and CPU times are presented and are analyzed.

1. Introduction

Many mathematical models of natural and applied sciences phenomena such as fluid mechanics, hydrodynamics, electromagnetics and various areas of physics are based on hyperbolic partial differential equations. Modeling some of these phenomena, imposing nonlocal conditions may be more accurate than classical conditions. Nonlocal boundary condition is a relation between the values of unknown function on the boundary and inside of the given domain. Over the last decades, boundary value problems with nonlocal boundary conditions have become a rapidly growing area of research. Such types of boundary conditions are encountered in applications including thermoelasticity [1], climate control systems [2] and financial mathematics [3]. Boundary value problems for parabolic, elliptic and equations of mixed types are actively studied by many scientists for decades (see [4]- [27]). Stability has been an important research area in the development of numerical methods. Particulary, in this work stability analysis is performed by suitable unconditionally stable difference schemes with an unbounded operator. Some results of this paper, without proof, are presented in [27].

In the present paper, third and fourth order of accuracy stable difference schemes for approximately solving the multipoint nonlocal boundary value problem (NBVP)

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} = f(t,x), \\ x = (x_1, \dots, x_m) \in \Omega, \ 0 < t < 1, \\ u(0,x) = \sum\limits_{j=1}^{n} \alpha_j u(\lambda_j, x) + \varphi(x), x \in \overline{\Omega}, \\ u_t(0,x) = \sum\limits_{j=1}^{n} \beta_j u_t(\lambda_j, x) + \psi(x), x \in \overline{\Omega} \end{cases} \tag{1}$$

for the multidimensional hyperbolic equation with the Neumann boundary condition

$$\frac{\partial u(t,x)}{\partial \vec{n}}|_{x\in S} = 0, x \in S$$

or mixed conditions

$$u(t,x)|_{x\in S_1} = 0, \ \frac{\partial u(t,x)}{\partial \vec{n}}|_{x\in S_2} = 0,$$

$$x \in S, S = S_1 \cup S_2$$

are considered.

Here

$$\Omega = \{x = (x_1, \cdots, x_m) : 0 < x_j < 1, 1 \leq j \leq m\}$$

is the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m$, with boundary S, $\overline{\Omega} = \Omega \cup S$ and $a_r(x)$ $(a_r(x) \geq a > 0, x \in \Omega)$, $\varphi(x), \psi(x)$ $\left(x \in \overline{\Omega}\right)$, $f(t,x)$ $(t \in (0,1), x \in \Omega)$ are given smooth functions.

2. Stability Estimates for High Order Difference Schemes

In the present section the third and the fourth order absolutely stable difference schemes and stability estimates for the solutions of these difference schemes are presented. These difference schemes are obtained in [18]. The discretization of problem (1) with Neumann condition or mixed conditions is carried out in two steps. In the first step, the grid sets are defined as

$$\widetilde{\Omega}_h = \{x = x_r = (h_1 r_1, \ldots, h_m r_m),$$

$$r = (r_1, \cdots, r_m), 0 \leq r_j \leq N_j,$$

$$h_j N_j = 1, j = 1, \cdots, m\},$$

$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S,$$

and difference operator A_h^x is given by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m} \left(a_r(x) u_{\overline{x}_r}^h\right)_{x_r, j_r} \qquad (2)$$

acting in the space of grid functions $u^h(x)$ for all $x \in S_h$. Note that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$ with the domain $D\left(A_h^x\right) = \left\{u\left(x\right) \in W_{2h}^2\left(\widetilde{\Omega}_h\right), \frac{\partial u}{\partial \overrightarrow{n}} = 0 \text{ on } S_h\right\}$.

The spaces $L_{2h} = L_2(\widetilde{\Omega}_h)$, $W_{2h}^1 = W_{2h}^1\left(\widetilde{\Omega}_h\right)$ and $W_{2h}^2 = W_{2h}^2\left(\widetilde{\Omega}_h\right)$ of the grid functions

$$\varphi^h(x) = \{\varphi(h_1 r_1, \ldots, h_m r_m)\}$$

are defined on $\widetilde{\Omega}_h$, equipped with norms

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \ldots h_m\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^1} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{m} \left|\left(\varphi^h\right)_{\overline{x}_r, j_r}\right|^2 h_1 \ldots h_m\right)^{1/2},$$

and

$$\left\|\varphi^h\right\|_{W_{2h}^2} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{m} \left|\left(\varphi^h\right)_{\overline{x}_r}\right|^2 h_1 \ldots h_m\right)^{1/2} + \left(\sum_{x \in \overline{\Omega}_h} \sum_{r=1}^{m} \left|\left(\varphi^h\right)_{x_r \overline{x}_r, j_r}\right|^2 h_1 \ldots h_m\right)^{1/2},$$

respectively.

Using difference operator A_h^x the following NBVP

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), \\ 0 < t < 1, \ x \in \Omega_h, \\ v^h(0,x) = \sum\limits_{j=1}^{n} \alpha_j v^h\left(\lambda_j, x\right) + \varphi^h(x), x \in \widetilde{\Omega}_h, \\ \frac{dv^h(0,x)}{dt} = \sum\limits_{j=1}^{n} \beta_j v_t^h(\lambda_j, x) + \psi^h(x), x \in \widetilde{\Omega}_h \end{cases} \qquad (3)$$

is obtained.

In the next step problem (3) is replaced by the third order of accuracy difference scheme

$$\begin{cases} \tau^{-2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + \frac{2}{3} A_h^x u_k^h(x) \\ +\frac{1}{6} A_h^x \left(u_{k+1}^h(x) + u_{k-1}^h(x)\right) + \frac{1}{12}\tau^2 \left(A_h^x\right)^2 u_{k+1}^h(x) \\ = f_k^h(x), f_k^h(x) = \frac{2}{3} f^h(t_k, x) + \frac{1}{6}\left(f^h(t_{k+1}, x)\right) \\ +f^h(t_{k-1}, x)) - \frac{1}{12}\tau^2 \left(-Af^h(t_{k+1}, x) + f_{tt}^h(t_{k+1}, x)\right), \\ t_k = k\tau, 1 \leq k \leq N-1, N\tau = 1, x \in \Omega_h, \\ u_0^h(x) = \sum\limits_{j=1}^{n} \alpha_j \left\{u_{[\lambda_j/\tau]}^h(x)\right. \\ +\tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)(\lambda_j - [\lambda_j/\tau]\tau) \\ +\frac{3}{2}\left(f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)\right)(\lambda_j - [\lambda_j/\tau]\tau)^2 \\ +\frac{7}{6}\left(f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x \left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\right) \\ \left.\times(\lambda_j - [\lambda_j/\tau]\tau)^3\right\} + \varphi^h(x), x \in \Omega_h, \\ \left(I + \tau^2 \left(A_h^x\right)^4\right)\tau^{-1}\left(u_1^h(x) - u_0^h(x)\right) \\ = \sum\limits_{j=1}^{n} \beta_j \left\{\tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\right. \\ + \left(f_{[\lambda_j/\tau]} - A_h^x u_{[\lambda_j/\tau]}^h(x)\right)(\lambda_j - [\lambda_j/\tau]\tau) \\ +\frac{1}{2!}\left(f'_{[\lambda_j/\tau]} - \tau^{-1} A_h^x \left(u_{[\lambda_j/\tau]}^h(x) - u_{[\lambda_j/\tau]-1}^h(x)\right)\right) \\ \times(\lambda_j - [\lambda_j/\tau]\tau)^2 + \frac{1}{3!}\left(f''_{[\lambda_j/\tau]} - A_h^x f_{[\lambda_j/\tau]}\right. \\ \left. + \left(A_h^x\right)^2 u_{[\lambda_j/\tau]}^h(x)\right)\left.(\lambda_j - [\lambda_j/\tau]\tau)^3\right\} + \psi^h(x), \\ x \in \Omega_h, f_{1,1}^h\left(x\right) = \frac{1}{2} f^h\left(0, x\right) + \frac{\tau}{6} f_t^h\left(0, x\right). \end{cases} \qquad (4)$$

Theorem 1. *Let τ and $|h|$ be sufficiently small numbers. Then, the solution of difference scheme (4) satisfies the following stability estimates:*

$$\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^1}$$

$$\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\varphi^h\right\|_{W_{2h}^1}\right.$$

$$\left.+\tau\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{1,1}^h\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}$$

$$+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}\right.$$

$$+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi^h\right\|_{W_{2h}^1}$$

$$\left.+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|\varphi^h\right\|_{W_{2h}^3}+\tau\left\|f_{1,1}^h\right\|_{W_{2h}^1}\right]$$

where M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x), f_{1,1}^h$ and $f_k^h, 1\le k<N$.

This theorem is proved in [25] under the following assumption

$$\sum_{k=1}^{n}|\alpha_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.$$

$$\left.+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.$$

$$\left.+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3\right\}$$

$$+\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.$$

$$\left.+\frac{7}{12}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4+\frac{7}{36}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\right\}<1. \tag{5}$$

In the third step replacing problem (3) by the fourth order of accuracy difference scheme problem

$$\left\{\begin{array}{l}
\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{5}{6}A_h^x u_k^h(x)\\
+\frac{1}{12}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)-\frac{1}{72}\tau^2\left(A_h^x\right)^2u_k^h(x)+\frac{\tau^2}{144}\left(A_h^x\right)^2\\
\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x), f_k^h(x)=\frac{5}{6}f^h(t_k,x)\\
+\frac{1}{12}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)+\frac{\tau^2}{72}\left(-A_h^xf^h\left(t_k,x\right)+f_{tt}^h\left(t_k,x\right)\right)\\
-\frac{\tau^2}{144}\left(-A_h^x\left(f^h\left(t_{k+1},x\right)+f^h\left(t_{k-1},x\right)\right)\right.\\
\left.+f_{tt}^h\left(t_{k+1},x\right)+f_{tt}^h\left(t_{k-1},x\right)\right), x\in\Omega_h,\\
t_k=k\tau, 1\le k\le N-1, N\tau=1,\\
u_0^h(x)=\left(I-\frac{i\tau}{2}\left(A_h^x\right)^{1/2}+\frac{\tau^2}{12}\left(A_h^x\right)^3\right)^{-1}\\
\sum\limits_{k=1}^{n}\alpha_k\Big\{\left(\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)-\frac{7\tau^2}{6}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\right)\\
\times\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)+\Big(1-\frac{3\tau^2}{2}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2\\
+\frac{\tau^4}{24}\left(A_h^x\right)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\Big)u_{[\lambda_j/\tau]}^h+\frac{3\tau^2}{2}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2f_{[\lambda_j/\tau]}\\
+\frac{7\tau^3}{6}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3f'_{[\lambda_j/\tau]}+\frac{\tau^4}{24}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4f''_{[\lambda_j/\tau]}\\
-\frac{\tau^4}{24}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4f_{[\lambda_j/\tau]}\Big\}+\varphi^h(x), x\in\widetilde{\Omega}_h,\\
\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)\\
=\left(I-\frac{\tau^2}{12}A_h^x\right)\left(I+\frac{i\tau}{2}\left(A_h^x\right)^{1/2}+\frac{\tau^2}{12}\left(A_h^x\right)^3\right)^{-1}\\
\times\sum\limits_{k=1}^{n}\beta_k\Big\{\Big(\frac{1}{\tau}-\frac{\tau}{2}A_h^x\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2\\
+\frac{\tau^3}{24}\left(A_h^x\right)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\Big)\left(u_{[\lambda_j/\tau]}-u_{[\lambda_j/\tau]-1}\right)\\
+\left(-A_h^x\tau\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)+\frac{\tau^3}{6}\left(A_h^x\right)^2\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\right)u_{[\lambda_j/\tau]}\\
+\tau f_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)+\frac{\tau^2}{2}f'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^2\\
+\frac{\tau^3}{6}f''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3+\frac{\tau^4}{24}f'''_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\\
-\frac{\tau^3}{6}A_h^xf_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^3\\
-\frac{\tau^4}{24}A_h^xf'_{[\lambda_j/\tau]}\left(\frac{\lambda_k}{\tau}-[\lambda_j/\tau]\right)^4\Big\}+\psi^h(x), x\in\widetilde{\Omega}_h.
\end{array}\right. \tag{6}$$

is obtained.

Theorem 2. *Let τ and h be sufficiently small numbers. Then, solution of difference scheme (6) obeys the following stability estimates:*

$$\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^1}+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{L_{2h}}$$

$$\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\varphi^h\right\|_{W_{2h}^1}+\tau\left\|f_{2,2}^h\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}$$

$$+\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-u_{k-1}^h}{2\tau}\right\|_{W_{2h}^1}$$

$$+\max_{1\le k\le N}\left\|\frac{u_k^h+u_{k-1}^h}{2}\right\|_{W_{2h}^2}\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}\right.$$

$$+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}$$

$$\left.+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{2,2}^h\right\|_{W_{2h}^1}\right].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$, $f_{2,2}^h$ and $f_k^h, 1 \le k < N$.

This theorem is proved in [25] under the following assumption

$$
\begin{aligned}
&\left\{\sum_{k=1}^{n}|\alpha_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{3}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.\right.\\
&\qquad\left.+\frac{7}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}\\
&+\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.\\
&\qquad\left.+\frac{1}{6}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^3+\frac{1}{24}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4\right\}\\
&\qquad+\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|\left\{1+\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^2\right.\\
&\qquad\quad+\frac{1}{2}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^4+\frac{1}{9}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^6\\
&\qquad\qquad\left.\left.+\frac{1}{576}\left|\frac{\lambda_k}{\tau}-\left[\frac{\lambda_k}{\tau}\right]\right|^8\right\}<1. \qquad (7)
\end{aligned}
$$

3. Numerical Analysis

In the present section some examples are presented to verify theoretical statements. Finite difference method is used and symbolic computations are carried out by Matlab. Three problems for one dimensional hyperbolic equations with the Neumann boundary conditions and mixed type boundary conditions are considered. Results of numerical experiments are presented in tables and are analyzed.
The grid set $[0,1]_\tau\times\ [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h with

$$[0,1]_\tau \times [0,\pi]_h = \{(t_k,x_n) : t_k = k\tau, 0 \le k \le N,$$

$$N\tau = 1, x_n = nh,\ \ 0 \le n \le M, Mh = \pi\}$$

is considered.

Example 1. *Let us consider problem*

$$
\left\{
\begin{array}{l}
\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = e^{-t}(\sin^2 x - 2\cos 2x),\\
0<t<1, 0<x<\pi,\\
u(0,x) = \frac{1}{10}u(1,x) + \frac{1}{10}u(\frac{1}{2},x)\\
+(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\
u_t(0,x) = \frac{1}{10}u_t(1,x) + \frac{1}{10}u_t(\frac{1}{2},x)\\
+(-1+\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\
u_x(t,0) = u_x(t,\pi) = 0
\end{array}
\right.
\tag{8}
$$

for one-dimensional hyperbolic equation with constant coefficients.

The exact solution of this problem is

$$u\left(t,x\right) = e^{-t}\sin^2 x.$$

In approximately solving problem (8), third and fourth order of accuracy difference schemes (4) and (6) are used respectively.

In the first step, applying simple formulas

$$\tfrac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2} - u^{''}\left(x_n\right) = O\left(h^2\right), \quad (9)$$

$$\tfrac{35u(0)-104u(0+\tau)+114u(0+2\tau)-56u(0+3\tau)+11u(0+4\tau)}{12\tau^2}$$

$$- u^{''}\left(0\right) = O\left(\tau^3\right), \qquad (10)$$

$$\tfrac{-5u(0)+18u(h)-24u(2h)+14u(3h)-3u(4h)}{2\tau^3}$$

$$- u^{'''}\left(0\right) = O\left(\tau^4\right), \qquad (11)$$

and using difference scheme (4), the second order of accuracy in t third order of accuracy in x difference scheme

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{2}{3}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) \\ -\frac{1}{6}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} + \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\ +\frac{\tau^2}{12}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right) = \varphi_n^k, \\ \varphi_n^k = \left\{\frac{2}{3}e^{-t_k} + \frac{1}{6}(e^{-t_{k+1}} + e^{-t_{k-1}}) \right. \\ \left. -\frac{\tau^2}{12}e^{-t_{k+1}}\right\} \sin^2 x_n - 2\left\{\frac{2}{3}e^{-t_k} \right. \\ \left. +\frac{1}{6}(e^{-t_{k+1}} + e^{-t_{k-1}}) + \frac{\tau^2}{3}e^{-t_{k+1}}\right\} \cos 2x_n, \\ t_k = k\tau,\ 1 \le k \le N-1, N\tau = 1, \\ x_n = nh, 2 \le n \le M-2, Mh = \pi, \\ u_n^0 - \frac{1}{8}u_n^{(N/2)} - \frac{1}{8}u_n^N \\ = (1 - \frac{1}{10}e^{-1} - \frac{1}{10}e^{-\frac{1}{2}}) \sin^2 x_n, 0 \le n \le M, \\ (u_n^1 - u_n^0) \\ -\frac{\tau^2}{12}\left(\frac{(u_{n+1}^1-u_{n+1}^0)-2(u_n^1-u_n^0)+(u_{n-1}^1-u_{n-1}^0)}{h^2}\right) \\ +\frac{\tau^4}{144}\left[\frac{(u_{n+2}^1-u_{n+2}^0)-4(u_{n+1}^1-u_{n+1}^0)+6(u_n^1-u_n^0)}{h^4} \right. \\ \left. +\frac{-4(u_{n-1}^1-u_{n-1}^0)+(u_{n-2}^1-u_{n-2}^0)}{h^4}\right] = \varphi_n^N, \\ \varphi_n^N = (-\tau + \frac{\tau^2}{2} - \frac{\tau^3}{6} + \frac{\tau^4}{6}) \sin^2 x \\ +(\frac{\tau^3}{6} + \frac{\tau^4}{12} + \frac{35}{36}\tau^5 - \frac{5}{18}\tau^6 - \frac{5}{54}\tau^7) \cos 2x \\ +(\frac{1}{10}e^{-1} + \frac{1}{10}e^{-\frac{1}{2}}), \quad 2 \le n \le M-2 \end{cases} \tag{12}$$

for the approximate solution of problem (8) is obtained. By rearranging like terms of the problem, the following linear system

$$AU_{n+2}+BU_{n+1}+CU_n+DU_{n-1}+EU_{n-2} = R\varphi_n, \tag{13}$$

$$2 \le n \le M-2$$

with $(N+1)\times(N+1)$ matrix coefficients

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & x & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & x & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & x & 0 \\ -r & r & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ y & w & v & \dots & 0 & 0 & 0 \\ 0 & y & w & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & w & v & 0 \\ 0 & 0 & 0 & \dots & y & w & v \\ s & -s & 0 & \dots & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \dots 0 & \frac{-1}{8} & 0\dots & 0 & \frac{-1}{8} \\ l & n & m & 0 & \dots & 0 & 0 & 0 \\ 0 & l & n & m & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \ddots & n & m & 0 \\ 0 & 0 & 0 & \dots & \dots & l & n & m \\ -t & t & 0 & \dots & \dots & 0 & 0 & 0 \end{bmatrix},$$

$$D = B,\ E = A,$$

$$R = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

where the entries are

$$x = \frac{\tau^2}{12h^4}, v = -\frac{1}{6h^2} - \frac{\tau^2}{3h^4}, w = -\frac{2}{3h^2},$$

$$y = -\frac{1}{6h^2}, m = \frac{1}{\tau^2} + \frac{1}{3h^2} + \frac{\tau^2}{2h^4},$$

$$n = -\frac{2}{\tau^2} + \frac{4}{3h^2}, l = \frac{1}{\tau^2} + \frac{1}{3h^2},$$

$$r = \frac{\tau^4}{144h^4}, s = \frac{\tau^2}{12h^2} + \frac{\tau^4}{36h^4}$$

$$t = 1 + \frac{\tau^2}{6h^2} + \frac{\tau^4}{24h^4},$$

and $(N+1)\times 1$ column matrices

$$\varphi_n^k = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, 0 \le k \le N,$$

with

$$\varphi_n^0 = (1 - \frac{1}{10}e^{-1} - \frac{1}{10}e^{-\frac{1}{2}}) \sin^2(x_n), 0 \le n \le M,$$

$$\varphi_n^N = \{-\tau + \frac{\tau^2}{2} - \frac{\tau^3}{6} + \frac{\tau^4}{6}\} \sin^2(x_n)$$
$$+\left\{\frac{\tau^3}{6} + \frac{\tau^4}{12} + \frac{35}{36}\tau^5\right\}$$
$$\left. -\frac{5}{18}\tau^6 - \frac{5}{54}\tau^7\right\} \cos 2x_n$$
$$+(\frac{1}{10}e^{-1} + \frac{1}{10}e^{-\frac{1}{2}}) \sin^2(x_n)$$

$$\varphi_n^k = \left\{ \frac{2}{3}e^{-t_k} + \frac{1}{6}(e^{-t_{k+1}} + e^{-t_{k-1}}) - \frac{\tau^2}{12}e^{-t_{k+1}} \right\} \sin^2 x_n,$$

$$+2\left\{ \frac{2}{3}e^{-t_k} + \frac{1}{6}(e^{-t_{k+1}} + e^{-t_{k-1}}) + \frac{\tau^2}{3}e^{-t_{k+1}} \right\} \cos 2x_n,$$

$$1 \le k \le N-1,$$

$$U_s^k = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

$$0 \le k \le N, s = n-2, n-1, n, n+1, n+2$$

is obtained.

The modified Gauss elimination method is used and the following formula

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1},$$

$$n = M-2, ...2, 1, 0$$

is applied where α_j, β_j $(j = 1, \ldots, M)$ are $(N+1)\times(N+1)$ square matrices and γ_j are $(N+1)\times 1$ column matrices for the solution of difference scheme (12). From that one can obtain formulas $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$

$$\begin{cases} \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}A, \\ \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \\ \times(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} = +(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \\ \times(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \end{cases} \tag{14}$$

where $n = 2 : M-2$ and

$$\gamma_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} 4/5 & 0 & \ldots & 0 \\ 0 & 4/5 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 4/5 \end{bmatrix},$$

$$\beta_2 = \begin{bmatrix} -1/5 & 0 & \ldots & 0 \\ 0 & -1/5 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1/5 \end{bmatrix}.$$

In a similar manner the following formulas

$$U_M = -[P + Q(4I - \alpha_{M-1})^{-1}(\beta_{M-1} + 3I)]^{-1} \times \{R + Q(4I - \alpha_{M-1})^{-1}\gamma_{M-1}\} \tag{15}$$

$$U_{M-1} = -(P + Q)^{-1}R \tag{16}$$

$$U_{M-2} = (4I - \alpha_{M-2})^{-1} \times \{(5I + \beta_{M-2})U_{M-1} + \gamma_{M-2}\}, \tag{17}$$

where

$$P = \frac{1}{6h}(11I + 9\beta_{M-1} - 2\alpha_{M-2}\beta_{M-1}),$$

$$Q = \frac{1}{6h}\,(-18I + 9\alpha_{M-1} - 2(\alpha_{M-2}\alpha_{M-1} + \beta_{M-2}))$$

$$R = \frac{1}{6h}(9\gamma_{M-1} - 2\alpha_{M-2}\gamma_{M-1} - 2\gamma_{M-2})$$

are obtained. The system

$$U_0 = \alpha_1 U_1 + \beta_1 U_2 + \gamma_1 \tag{18}$$

where

$$\alpha_1 = \frac{-1}{h}T^{-1}, \beta_1 = 0, \gamma_1 = \frac{h}{2}T^{-1}\varphi_n^0$$

is used for the boundary condition $u_x(t, 0) = 0$ of third order of accuracy difference scheme. Here

$$T = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & 0 & \ldots & 0 \\ a & b & a & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & a & b & a & 0 & \ldots & \ldots & \vdots \\ \vdots & 0 & a & b & a & 0 & \ldots & \vdots \\ \vdots & \ldots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ldots & \ldots & 0 & a & b & a & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & a & b & a \\ 0 & \ldots & 0 & \lambda_5 & \lambda_4 & \lambda_3 & \lambda_2 & \lambda_1 \end{bmatrix} \tag{19}$$

with

$$
\begin{aligned}
\lambda_1 &= \left(-\frac{1}{h}-\frac{35h}{24\tau^2}+\frac{5h^2}{12\tau^3}\right), \\
\lambda_2 &= \left(\frac{104h}{24\tau^2}-\frac{18h^2}{12\tau^3}\right), \\
\lambda_3 &= \left(-\frac{114h}{24\tau^2}+\frac{24h^2}{12\tau^3}\right), \\
\lambda_4 &= \left(\frac{56h}{24\tau^2}-\frac{14h^2}{12\tau^3}\right), \\
\lambda_5 &= \left(-\frac{11h}{24\tau^2}+\frac{3h^2}{12\tau^3}\right), \\
a &= -\frac{h}{2\tau^2}\ , \quad b=\left(-\frac{1}{h}+\frac{h}{\tau^2}\right).
\end{aligned}
$$

In the next step difference scheme (6) and the formulas

$$\frac{-3u\left(1\right)+4u\left(1-h\right)-u\left(1-2h\right)}{2h}-u^{'}\left(1\right)=O\left(h^2\right),$$

$$\frac{1}{4\tau^3}\left(-17u\left(0\right)+71u\left(0+\tau\right)-118u\left(0+2\tau\right)\right.$$

$$\left.+98u\left(0+3\tau\right)-41u\left(0+4\tau\right)+7u(0+5\tau)\right)$$

$$-u^{'''}\left(0\right)=O\left(\tau^3\right),$$

$$\frac{u\left(0\right)-2u\left(0+\tau\right)+u\left(0+2\tau\right)}{\tau^2}-u^{''}\left(0\right)=O\left(\tau^3\right)$$

are used to obtain second order of accuracy in t and fourth order of accuracy in x difference scheme

$$
\left\{
\begin{array}{l}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{5}{6}\left(\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\right) \\
-\frac{1}{12}\left(\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}\right) \\
-\frac{\tau^2}{72}\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}\right) \\
+\frac{\tau^2}{144}\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}\right. \\
\left.+\frac{u_{n+2}^{k-1}-4u_{n+1}^{k-1}+6u_n^{k-1}-4u_{n-1}^{k-1}+u_{n-2}^{k-1}}{h^4}\right)=\varphi_n^k, \\
\varphi_n^k=\left\{(\frac{5}{6}+\frac{\tau^2}{72})e^{-t_k}\right. \\
\left.+(\frac{1}{12}-\frac{\tau^2}{144})(e^{-t_{k+1}}+e^{-t_{k-1}})\right\}\sin^2(x_n) \\
+\left\{(-\frac{5}{3}+\frac{\tau^2}{9})e^{-t_k}\right. \\
\left.-(\frac{1}{6}+\frac{\tau^2}{18})(e^{-t_{k+1}}+e^{-t_{k-1}})\right\}\cos 2x_n \\
t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\
x_n=nh, 1\le n\le M-1, Mh=\pi, \\
\varphi_n^0=(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2\left(x_n\right), \\
0\le n\le M, \left(u_n^1-u_n^0\right) \\
-\frac{\tau^2}{12}\left(\frac{\left(u_{n+1}^1-u_{n+1}^0\right)-2\left(u_n^1-u_n^0\right)+\left(u_{n-1}^1-u_{n-1}^0\right)}{h^2}\right) \\
+\frac{\tau^4}{144h^4}\left[(u_{n+2}^1-u_{n+2}^0)-4\left(u_{n+1}^1-u_{n+1}^0\right)\right. \\
+6\left(u_n^1-u_n^0\right)-4\left(u_{n-1}^1-u_{n-1}^0\right) \\
\left.+(u_{n-2}^1-u_{n-2}^0)\right]=\varphi_n^N, \\
\varphi_n^N=\left(-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{6}+\frac{\tau^4}{24}-\frac{\tau^5}{24}\right)\sin^2\left(x_n\right) \\
+\left(\frac{\tau^3}{6}-\frac{\tau^4}{12}-\frac{7}{36}\tau^5-\frac{15}{144}\tau^6\right. \\
\left.-\frac{25}{432}\tau^7-\frac{5}{432}\tau^8\right)\cos 2x_n \\
+\tau(\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2\left(x_n\right), \\
2\le n\le M-2, 0\le k\le N, \\
u_1^0-u_0^0=\frac{h}{2}\left(\frac{45u_0^0-154u_0^1+214u_0^2-156u_0^3+61u_0^4-10u_0^5}{12\tau^2}\right) \\
-\frac{h^2}{64\tau^3}\left(17u_0^0-71u_0^1+118u_0^2\right. \\
\left.-98u_0^3+41u_0^4-7u_0^5\right) \\
u_1^N-u_0^N=\frac{h}{212\tau^2}\left(45u_0^N-154u_0^{N-1}+214u_0^{N-2}\right. \\
\left.-156u_0^{N-3}+61u_0^{N-4}-10u_0^{N-5}\right) \\
-\frac{h^2}{64\tau^3}\left(17u_0^N-71u_0^{N-1}+118u_0^{N-2}\right. \\
\left.-98_0^{N-3}+41u_0^{N-4}-7u_0^{N-5}\right), u_1^k-u_0^k \\
=\frac{h}{2}\left(\frac{u_0^{k+1}-2u_0^k+u_0^{k-1}}{\tau^2}-\varphi_0^k\right),\quad 1\le k\le N-1
\end{array}
\right.
\tag{20}
$$

for the approximate solution of problem (8). By rearranging coefficients in the problem we have again the $(N+1)\times(N+1)$ linear system (13) with matrix coefficients

$$
A=\begin{bmatrix}
0 & 0 & 0 & \dots & 0 & 0 & 0 \\
x & y & x & \dots & 0 & 0 & 0 \\
0 & x & y & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & y & x & 0 \\
0 & 0 & 0 & \dots & x & y & x \\
-r & r & 0 & \dots & 0 & 0 & 0
\end{bmatrix},
$$

$$B=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0\\ v & w & v & \dots & 0 & 0 & 0\\ 0 & v & w & \ddots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ddots & w & v & 0\\ 0 & 0 & 0 & \dots & v & w & v\\ s & -s & 0 & \dots & 0 & 0 & 0 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & \dots 0 & \frac{-1}{8} & 0\dots & 0 & \frac{-1}{8}\\ m & n & m & 0 & \dots & 0 & 0 & 0\\ 0 & m & n & m & \dots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \ddots & n & m & 0\\ 0 & 0 & 0 & \dots & \dots & m & n & m\\ -t & t & 0 & \dots & \dots & 0 & 0 & 0 \end{bmatrix},$$

$$D=B,\ E=A,$$

$$R=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

and with entries

$$x=\frac{\tau^2}{144h^4}, y=-\frac{\tau^2}{72h^4}, v=-\frac{1}{12h^2}-\frac{\tau^2}{36h^4},$$

$$w=-\frac{5}{6h^2}+\frac{\tau^2}{18h^4},$$

$$m=\frac{1}{\tau^2}+\frac{1}{6h^2}+\frac{\tau^2}{24h^4},$$

$$n=-\frac{2}{\tau^2}+\frac{5}{3h^2}-\frac{\tau^2}{12h^4},$$

$$r=\frac{\tau^4}{144h^4}, s=\frac{\tau^4}{36h^4}+\frac{\tau^2}{12h^2},$$

$$t=1+\frac{\tau^2}{6h^2}+\frac{\tau^4}{24h^4}.$$

Here U_s^k and φ_n^k are defined as

$$U_s^k=\begin{bmatrix} u_s^0\\ u_s^1\\ \vdots\\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

$$0\le k\le N, s=n-2, n-1, n, n+1, n+2.$$

$$\varphi_n^k=\begin{bmatrix} \varphi_n^0\\ \varphi_n^1\\ \cdot\\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, 0\le k\le N,$$

$$\varphi_n^0=(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2(x_n), 0\le n\le M,$$

$$\varphi_n^N=\left(-\tau+\frac{\tau^2}{2}-\frac{\tau^3}{6}+\frac{\tau^4}{24}+\frac{\tau^5}{24}\right)\sin^2(x_n)$$

$$\varphi_n^k=\left\{(\frac{5}{6}+\frac{\tau^2}{72})e^{-t_k}\right.$$

$$\left.+(\frac{1}{12}-\frac{\tau^2}{144})(e^{-t_{k+1}}+e^{-t_{k-1}})\right\}\sin^2(x_n)$$

$$+\left\{(-\frac{5}{3}+\frac{\tau^2}{9})e^{-t_k}\right.$$

$$\left.-(\frac{1}{6}+\frac{\tau^2}{18})(e^{-t_{k+1}}+e^{-t_{k-1}})\right\}\cos 2x_n$$

$$+\left\{\frac{\tau^3}{6}-\frac{\tau^4}{12}-\frac{7}{36}\tau^5-\frac{15}{144}\tau^6\right.$$

$$\left.-\frac{25}{432}\tau^7-\frac{5}{432}\tau^8\right\}\cos 2x_n$$

$$+(\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2(x_n).$$

In exactly the same manner as Example 1 the linear system for the fourth order of accuracy difference scheme is solved with the following new formulas

$$U_M=-[P+Q(4I-\alpha_{M-1})^{-1}(\beta_{M-1}+3I)]^{-1}$$

$$\times\left[R+Q(4I-\alpha_{M-1})^{-1}\gamma_{M-1}\right], \quad (21)$$

$$U_{M-1}=[(\beta_{M-2}+5I)-(4I-\alpha_{M-2})\alpha_{M-1}]^{-1}$$

$$\times[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}] \quad (22)$$

$$U_{M-2}=(4I-\alpha_{M-2})^{-1} \quad (23)$$

$$\times\{(5I+\beta_{M-2})U_{M-1}+\gamma_{M-2}\}$$

where

$$P=\frac{1}{12h}[25I+36\beta_{M-1}-16\alpha_{M-2}\beta_{M-1}$$

$$+3(\alpha_{M-3}\alpha_{M-2}\beta_{M-1}+\beta_{M-3}\beta_{M-1})],$$

$$Q=\frac{1}{12h}\left[-48I+36\alpha_{M-1}\right.$$

$$-16(\alpha_{M-2}\alpha_{M-1}+\beta_{M-1})$$

$$\left.+3(\alpha_{M-3}\alpha_{M-2}\alpha_{M-1}+\alpha_{M-3}\beta_{M-2}+\alpha_{M-1}\beta_{M-3})\right],$$

$$R=\frac{1}{12h}\left[36\gamma_{M-1}-16(\alpha_{M-2}\gamma_{M-1}+\gamma_{M-2})\right.$$

$$+3\left(\alpha_{M-3}\alpha_{M-2}\gamma_{M-1}\right.$$

$$\left.\left.+\alpha_{M-3}\gamma_{M-2}+\beta_{M-3}\gamma_{M-1}+\gamma_{M-3}\right)\right].$$

For the boundary condition $u_x(t,0)=0$, the system (18) with the matrix

$$T=\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & 0 & \dots & 0\\ a & b & a & 0 & \dots & \dots & \dots & \dots & 0\\ 0 & a & b & a & 0 & \dots & \dots & \dots & 0\\ 0 & 0 & a & b & a & 0 & \dots & \dots & \vdots\\ \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \vdots\\ 0 & \dots & \dots & \dots & \dots & a & b & a & 0\\ 0 & \dots & 0 & \lambda_6 & \lambda_5 & \lambda_4 & \lambda_3 & \lambda_2 & \lambda_1 \end{bmatrix} \tag{24}$$

and the new entries

$$\lambda_1=\left(-\frac{1}{h}-\frac{45h}{24\tau^2}+\frac{17h^2}{24\tau^3}\right),$$

$$\lambda_2=\left(\frac{154h}{24\tau^2}-\frac{71h^2}{24\tau^3}\right),$$

$$\lambda_3=\left(-\frac{214h}{24\tau^2}+\frac{118h^2}{24\tau^3}\right),$$

$$\lambda_4=\left(\frac{156h}{24\tau^2}-\frac{98h^2}{12\tau^3}\right),$$

$$\lambda_5=\left(-\frac{61h}{24\tau^2}+\frac{41h^2}{24\tau^3}\right),\ \lambda_6=\left(\frac{10h}{24\tau^2}-\frac{7h^2}{24\tau^3}\right),$$

$$a=-\frac{h}{2\tau^2}\ ,\quad b=\left(-\frac{1}{h}+\frac{h}{\tau^2}\right)$$

is considered.

Example 2. *Consider*

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}=e^{-t}(\sin^2 x-2\cos 2x),\\ 0<t<1, 0<x<\pi,\\ u(0,x)=\frac{1}{10}u(1,x)+\frac{1}{10}u(\frac{1}{2},x)\\ +(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\ u_t(0,x)=\frac{1}{10}u_t(1,x)+\frac{1}{10}u_t(\frac{1}{2},x)\\ +(-1+\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2 x, 0\le x\le\pi,\\ u(t,0)=u_x(t,\pi)=0,\ 0\le t\le 1 \end{cases} \tag{25}$$

for one dimensional hyperbolic equation.

Note that this problem is similar to Example 1, with different mixed boundary conditions. Again exact solution of the problem is

$$u(t,x)=e^{-t}\sin^2 x.$$

In finding the approximate solution of problem (25), the method of first example is applied. Third and fourth orders of accuracy difference schemes (4), (6) are used. Approximating the boundary condition $u_x(t,\pi)=0$ the following formulas

$$U_M=-[P+Q(4I-\alpha_{M-1})^{-1}(\beta_{M-1}+3)]^{-1}$$

$$\times\{R+Q(4-\alpha_{M-1})^{-1}\gamma_{M-1}\}$$

$$U_{M-1}=-(P+Q)^{-1}R$$

$$U_{M-2}=(4I-\alpha_{M-2})^{-1}\{(5I+\beta_{M-2})U_{M-1}+\gamma_{M-2}\}$$

where

$$P=\frac{1}{6h}(11I+9\beta_{M-1}-2\alpha_{M-2}\beta_{M-1})U_M,$$

$$Q=\frac{1}{6h}\left[-18I+9\alpha_{M-1}\right.$$

$$\left.-2\left(\alpha_{M-2}\alpha_{M-1}+\beta_{M-2}\right)\right]U_{M-1}$$

$$R=\frac{1}{6h}(9\gamma_{M-1}-2\alpha_{M-2}\gamma_{M-1}-2\gamma_{M-2}),$$

for the third order of accuracy difference scheme and

$$U_M=-[P+Q(4I-\alpha_{M-1})^{-1}(\beta_{M-1}+3I)]^{-1}$$

$$\times\{R+Q(4I-\alpha_{M-1})^{-1}\gamma_{M-1}\},$$

$$U_{M-1}=[(\beta_{M-2}+5I)-(4I-\alpha_{M-2})\alpha_{M-1}]^{-1}$$

$$\times[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}]$$

$$U_{M-2}=(4I-\alpha_{M-2})^{-1}\left\{(5I+\beta_{M-2})U_{M-1}+\gamma_{M-2}\right\},$$

where

$$P=\frac{1}{12h}\left[25I+36\beta_{M-1}-16\alpha_{M-2}\beta_{M-1}\right.$$

$$\left.+3(\alpha_{M-3}\alpha_{M-2}\beta_{M-1}+\beta_{M-3}\beta_{M-1})\right],$$

$$Q=\frac{1}{12h}\left[-48I+36\alpha_{M-1}\right.$$

$$-16(\alpha_{M-2}\alpha_{M-1}+\beta_{M-1})+3\left(\alpha_{M-3}\alpha_{M-2}\alpha_{M-1}\right.$$

$$\left.\left.+\alpha_{M-3}\beta_{M-2}+\alpha_{M-1}\beta_{M-3}\right)\right],$$

$$R=\frac{1}{12h}\left[36\gamma_{M-1}-16(\alpha_{M-2}\gamma_{M-1}+\gamma_{M-2})\right.$$

$$+3((\alpha_{M-3}\alpha_{M-2}\gamma_{M-1}$$

$$\left.+\alpha_{M-3}\gamma_{M-2}+\beta_{M-3}\gamma_{M-1}+\gamma_{M-3})\right]$$

for the fourth order of accuracy difference scheme are used. For the boundary condition $u(t,0)=0$ the following initial matrices

$$\alpha_1=\begin{bmatrix}0&0&\dots&0\\0&0&\dots&0\\\vdots&\vdots&\dots&\vdots\\0&0&\dots&0\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_1=\begin{bmatrix}0&0&\dots&0\\0&0&\dots&0\\\vdots&\vdots&\dots&\vdots\\0&0&\dots&0\end{bmatrix}_{(N+1)\times(N+1)}$$

$$\gamma_1=\gamma_2=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}_{(N+1)\times 1},$$

$$\alpha_2=\begin{bmatrix}4/5&0&\dots&0\\0&4/5&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&4/5\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_2=\begin{bmatrix}-1/5&0&\dots&0\\0&-1/5&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&-1/5\end{bmatrix}_{(N+1)\times(N+1)}$$

are used in the formulae which were presented in (14).

Example 3. *Consider the NBVP with mixed condition*

$$\begin{cases}\frac{\partial^2u(t,x)}{\partial t^2}-\frac{\partial^2u(t,x)}{\partial x^2}=e^{-t}(\sin^2x-2\cos 2x),\\ 0<t<1,0<x<\pi,\\ u(0,x)=\frac{1}{10}u(1,x)+\frac{1}{10}u(\frac{1}{2},x)\\ +(1-\frac{1}{10}e^{-1}-\frac{1}{10}e^{-\frac{1}{2}})\sin^2x,0\le x\le\pi,\\ u_t(0,x)=\frac{1}{10}u_t(1,x)+\frac{1}{10}u_t(\frac{1}{2},x)\\ +(-1+\frac{1}{10}e^{-1}+\frac{1}{10}e^{-\frac{1}{2}})\sin^2x,0\le x\le\pi,\\ u_x(t,0)=u(t,\pi)=0,\ \ 0\le t\le 1\end{cases}\tag{26}$$

for one dimensional hyperbolic equation.

Note that this problem is similar to problem of Example1 with different boundary conditions. Exact solution of this problem is

$$u\left(t,x\right)=e^{-t}\sin^2x.$$

The approximate solution of problem (26) is obtained by a similar procedure as in the first example. Third and fourth order of accuracy difference schemes (4), (6) are used and the system

$$U_0=\alpha_1U_1+\beta_1U_2+\gamma_1$$

with

$$\alpha_1=\frac{-1}{h}T^{-1},\beta_1=0,\gamma_1=\frac{h}{2}T^{-1}\varphi_n^0$$

is considered. Matrices $T,\lambda_i,i=1,...,6;a,b$ are defined by (19) and (24) and are considered for the boundary condition $u_x(t,0)=0$. Approximating boundary condition $u(t,\pi)=0$, the following formulas

$$\begin{cases}U_{M-2}=\alpha_{M-1}U_{M-1}+\gamma_{M-1},\\ U_{M-3}=\alpha_{M-2}U_{M-2}+\beta_{M-2}U_{M-1}+\gamma_{M-2},\\ U_{M-3}=4U_{M-2}-5U_{M-1},\end{cases}$$

and

$$U_{M-1}=\left[(\beta_{M-2}+5I)-(4I-\alpha_{M-2})\alpha_{M-1}\right]^{-1}$$

$$\times[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}]$$

are used.

The errors for the approximations are computed by the formula

$$E_M^N=\max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u\left(t_k,x_n\right)-U_n^k\right|^2h\right)^{\frac{1}{2}}.$$

Table 1. Error analysis for the approximate solutions of (8).

	N=20, M=400	**N=30, M=900**	**N=40, M=1600**
Third order of accuracy difference scheme	0,000211	0,00006181	0,00002605
Fourth order of accuracy difference scheme	0,00009415	0,00001866	0,000005752

Table 2. CPU times for the approximate solutions of (8).

	N=20, M=400	**N=30, M=900**	**N=40, M=1600**
Third order of accuracy difference scheme	2.3078	13.5915	68.6596
Fourth order of accuracy difference scheme	2.3473	13.5283	67.7495

Table 3. Error analysis for the approximate solutions of (25).

	N=20, M=400	**N=30, M=900**	**N=40, M=1600**
Third order of accuracy difference scheme	0,0004817	0,0002047	0,0001138
Fourth order of accuracy difference scheme	0,00009781	0,00001979	0,00001954

Table 4. CPU times for the approximate solutions of (25).

	N=20, M=400	**N=30, M=900**	**N=40, M=1600**
Third order of accuracy difference scheme	2.3031	13.5165	68.2402
Fourth order of accuracy difference scheme	1.7361	13.5241	68.8427

Table 5. Error analysis for the approximate solutions of (26).

	N=20, M=400	**N=30, M=900**	**N=40, M=1600**
Third order of accuracy difference scheme	0,0037	0,0011	0,0004602
Fourth order of accuracy difference scheme	0,00009415	0,00001866	0,000005752

Table 6. CPU times for the approximate solutions of (26).

	N=20, M=400	**N=30, M=900**	**N=40, M=1600**
Third order of accuracy difference scheme	1.7006	13.4295	68.1728
Fourth order of accuracy difference scheme	1.7401	13.4628	68.0591

Here $u(t_k, x_n)$ represents exact solution and U_n^k represents numerical solution at (t_k, x_n). We denote the third order of accuracy difference scheme (4) as TO and the fourth order of accuracy difference scheme (6) as FO. Errors and the related CPU times are represented in Table 1,3,5 and Table 2,4,6 respectively, for different M and N values. The implementations are carried out by Matlab 7.9.0 software package and obtained by a PC System 64bit, Intel R Core TM i5 CPU, 3.20 GHz, 3.60Hz, 4000Mb of RAM.

The following conclusions can be noted from the tables above for the comparison of the numerical results presented in the tables.

- From Table 1 and Table 2, it can be noticed that approximately the same accuracy is achieved by TO with data error ,N=40, M=1600 and by FO with data error N=30, M=900 in different CPU times; 68.6596s and 13.5283s, respectively. This means the use of the difference scheme FO accelerates the computation with a ratio of more than 68.66/13.5≈5.08 times, that is, FO is considerably faster than TO.
- In Table 3 and Table 4, almost the same accuracy is achieved by TO with error ,N=40, M=1600 and by FO with error N=20, M=400 in different CPU

times; 68.6596s and 13.5283s, respectively, which means that the use of the difference scheme FO accelerates the computation with a ratio of more than 68.24/1.73≈39.44 times, which shows that FO is faster than TO.

- In Table 5 and Table 6, it is noted that approximately similar accuracy is achieved by TO with data error ,N=40, M=1600 and by FO with data error N=20, M=400 in different CPU times; 68.1728s and 1.7401s, respectively. This means that the use of the difference scheme FO accelerates the computation with a ratio of more than 68.17/1.74≈39.17 times, that is, FO is approximately faster than TO.
- It can be concluded from the tables that numerical results become approximately the same for larger N and M values for each difference scheme in the reliable range of the CPU times and this shows that the approximate solutions of problem (8), (25), (26) are accurate.
- In conclusion, the fourth order of accuracy difference scheme is more accurate than the third order of accuracy difference scheme when considering the CPU times and the error levels.

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