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RESEARCH ARTICLE

Some new integral inequalities for Lipschitzian functions

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ABSTRACT

This paper is about obtaining some new type of integral inequalities for functions from the Lipschitz class. For this, some new integral inequalities related to the differences between the two different types of integral averages for Lipschitzian functions are obtained. Moreover, applications for some special means as arithmetic, geometric, logarithmic, p-logarithmic, harmonic, identric are given.

1. Introduction

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences.

Definition 1. A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

The research of beautiful inequalities which have symmetry is very interesting and important to Analysis and PDE. A well-known example is the famous Hermite-Hadamard inequality which was first published in [1].

If $f: I \to \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (1)$$

This double inequality is known as Hermite-Hadamard integral inequality for convex functions in the literature. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if mapping f is concave.

Definition 2. [2] Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f: I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then the function f is said to be harmonically concave.

Definition 3. [2] Let $f: I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$, If $f \in L[a,b]$ then the following inequalities hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2} \quad (2)$$

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Definition 4. (Beta function) The beta function denoted by $\beta(m,n)$ is defined as

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx.$$

Definition 5. (Hypergeometric function) [3] The hypergeometric function denoted by ${}_2F_1\ (a,b;c;z)$ is defined by the integral equality

$${}_2F_1\ (a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,$$
$$c>b>0, |z|<1.$$

Definition 6. (M-Lipschitz Condition) [4] $f: I\to\mathbb{R}$ is said to satisfy the Lipschitz condition if there is a constant $M>0$ such that

$$|f(x)-f(y)|\le M|x-y| \quad \forall x,y\in I.$$

Theorem 1. [4] If $f: I\to\mathbb{R}$ is convex, then f satisfies a Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Consequently, f is absolutely continuous on $[a,b]$ and continuous on I°.

In [5], the inequalities related to left-hand side and right-hand side of the inequality (1) for Lipschitzian mappings as follow:

Theorem 2. [5] Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be an M-Lipschitzian mapping on I and $a,b\in I$ with $a<b$. Then we have the inequalities

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{M}{4}(b-a)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{M}{3}(b-a).$$

Corollary 1. Let $f:\mathrm{I}\subseteq\mathbb{R}\to\mathbb{R}$ be a convex and differentiable function on interval I and $a,b\in I$ with $a<b$ and $M=sup_{t\in[a,b]}|f'(t)|<\infty$. Then we have the inequalities

$$0\le\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\le\frac{M}{4}(b-a)$$

and

$$0\le\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\le\frac{M}{3}(b-a)$$

See [5-8] and references therein for more information about the Hadamard-type inequalities for the Lipschitzian functions.

2. Main results

In this section, we obtain some new inequalities related to integral means given in the inequalities (1) and (2) for Lipschitzian mappings.

Theorem 3. Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be an M-Lipschitzian mapping on interval I and $a,b\in I$ with $a<b$. Then following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \le\frac{M(b-a)^2}{6b}\ {}_2F_1\left(1,2;4;1-\frac{a}{b}\right)$$

Proof: Since f is M-Lipschitzian function on interval I, for $\forall x,y\in[a,b]$

$$|f(x)-f(y)|\le M|x-y|.$$

Here, for arbitrary $t\in[0,1]$, if we take

$$x=tb+(1-t)a, \qquad y=\frac{ab}{ta+(1-t)b}$$

then

$$\left|f[(tb+(1-t)a]-f\left(\frac{ab}{ta+(1-t)b}\right)\right|$$
$$\le M\left|(tb+(1-t)a-\frac{ab}{ta+(1-t)b}\right|$$
$$=M\left|\frac{t^2ab+t(1-t)(b^2+a^2)+(1-t)^2ab-ab}{ta+(1-t)b}\right|$$
$$=\frac{Mt(1-t)(b-a)^2}{ta+(1-t)b}.$$

Consequently, we get the following inequality:

$$\left|f(tb+(1-t)a-f\left(\frac{ab}{ta+(1-t)b}\right)\right| \le\frac{Mt(1-t)(b-a)^2}{b-t(b-a)}$$

If we take integral the last inequality on $t\in[0,1]$ and use property of modulus, we have

$$\left|\int_0^1 f[tb+(1-t)a]dt-\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\right|$$
$$\le\int_0^1\left|f(tb+(1-t)a-f\left(\frac{ab}{ta+(1-t)b}\right)\right|dt$$
$$\le M(b-a)^2\int_0^1\frac{t(1-t)}{b\left[1-t\left(1-\frac{a}{b}\right)\right]}dt$$

$$= \frac{M(b-a)^2}{b} \ {}_2F_1\left(1,2;4;1-\frac{a}{b}\right)\beta(2,2)$$

If we make the change of variables $u = tb + (1-t)a$ and $u = \frac{ab}{ta+(1-t)b}$ in the integrals on the left side of the last inequality respectively, we have the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \le \frac{M(b-a)^2}{6b} \ {}_2F_1\left(1,2;4;1-\frac{a}{b}\right)$$

This completes the proof of theorem.

Proposition 1. Let $p \in (1,\infty)\backslash\{2\}$ and $a,b \in \mathbb{R}$ with $0 < a < b$. Then

$$\left|L_p^p - G^2 L_{p-2}^{p-2}\right| \le \frac{pb^{p-2}(b-a)^2}{6} \ {}_2F_1\left(1,2;4;1-\frac{a}{b}\right),$$

where $G = G(a,b)$ and $L_p = L_p(a,b)$ are geometric and p-logarithmic means respectively.

Proof: If the $f(x) = x^p$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 3, the inequality

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| = \left|\frac{1}{b-a}\int_a^b u^p du - \frac{ab}{b-a}\int_a^b \frac{u^p}{u^2}du\right|$$

is obtained. If the integral is calculated,

$$\left|\frac{1}{b-a}\int_a^b u^p du - \frac{ab}{b-a}\int_a^b \frac{u^p}{u^2}du\right| = \left|\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)} - \frac{ab(b^{p-1}-a^{p-1})}{(b-a)(p-1)}\right| = \left|L_p^p - G^2 L_{p-2}^{p-2}\right|.$$

From Corollary 1, if $M = sup_{t\in[a,b]}|f'(t)| < \infty$ is taken for the right side of the inequality, then $M = pb^{p-1}$. So, we get

$$\left|L_p^p - abL_{p-2}^{p-2}\right| \le \frac{pb^{p-2}(b-a)^2}{6} \ {}_2F_1\left(1,2;4;1-\frac{a}{b}\right).$$

Proposition 2. Let $p \in (1,\infty)\backslash\{2\}$ and $a,b \in \mathbb{R}$ with $0 < a < b$. Then

$$\left|L^{-1} - \frac{A}{G^2}\right| \le \frac{(b-a)^2}{6ba^2} \ {}_2F_1\left(1,2;4;1-\frac{a}{b}\right),$$

where $G = G(a,b)$, $A = A(a,b)$ and $L = L(a,b)$ are geometric, arithmetic and logarithmic means respectively.

Proof: If the $f(x) = \frac{1}{x}$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 3, we have the following equality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| = \left|\frac{1}{b-a}\int_a^b u^{-1}du - \frac{ab}{b-a}\int_a^b u^{-3}du\right| = \left|L^{-1} - \frac{A}{G^2}\right|.$$

Using the Corollary 1, if $M = sup_{t\in[a,b]}|f'(t)| < \infty$ is taken for the right side of the inequality, then $M = \frac{1}{a^2}$. So, we get

$$\left|L^{-1} - \frac{A}{G^2}\right| \le \frac{(b-a)^2}{6ba^2} \ {}_2F_1\left(1,2;4;1-\frac{a}{b}\right).$$

Theorem 4. Let $f: \mathrm{I} \subset \mathbb{R} \to \mathbb{R}$ be an M-Lipschitzian function on interval I and $a,b,x,y \in I$ with $a \le x < y$ and $a < b$. Then following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \le \frac{M}{2}[|b-y| + x - a].$$

Proof: Since f is an M-Lipschitzian function on interval I, for $\forall v,w \in I$

$$|f(v) - f(w)| \le M|v-w|.$$

Here, for arbitrary $t \in [0,1]$, if we take

$$v = tb + (1-t)a, \qquad w = ty + (1-t)x,$$

then

$$|f[tb + (1-t)a] - f[ty + (1-t)x]| \le M|t(b-y) + (1-t)(a-x)| \le M[t|b-y| + (1-t)|a-x|].$$

If we take integral the last inequality on $t \in [0,1]$ and use the property of modulus, we have

$$\left|\int_0^1 f[tb+(1-t)a]dt - \int_0^1 f[ty+(1-t)x]dt\right|$$
$$\leq \left|\int_0^1 (f[tb+(1-t)a] - f[ty+(1-t)x])dt\right|$$
$$\leq \int_0^1 |f[tb+(1-t)a] - f[ty+(1-t)x]|dt$$
$$\leq M\int_0^1 [t|b-y| + (1-t)(x-a)]dt.$$

If we make the change of variables $u = tb + (1-t)a$ and $u = ty + (1-t)x$ in the integrals on the left side of the last inequality respectively, we have the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \leq \frac{M}{2}[|b-y| + x - a].$$

This completes the proof of theorem.

Proposition 3. Let $p > 1$ and $a, b, x, y \in \mathbb{R}$ with $0 < a \leq x < y$ and $a < b$. Then

$$\left|L_p(a,b) - L_p(x,y)\right| \leq \frac{pb^{p-1}}{2}[|b-y| + x - a],$$

where $L_p = L_p(a,b)$ is p-logarithmic mean.

Proof: If the $f(x) = x^p$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 4, we have the following equality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right|$$
$$= \left|\frac{1}{b-a}\int_a^b u^p du - \frac{1}{y-x}\int_x^y u^p du\right|$$
$$= \left|\frac{b^{p+1} - a^{p+1}}{(b-a)(p+1)} - \frac{y^{p+1} - x^{p+1}}{(y-x)(p+1)}\right|$$
$$= \left|L_p(a,b) - L_p(x,y)\right|$$
$$\leq \frac{pb^{p-1}}{2}[|b-y| + x - a],$$

where $M = pb^{p-1}$.

Proposition 4. Let $a, b, x, y \in \mathbb{R}$ with $0 < a \leq x < y$ and $a < b$. Then

$$|L^{-1}(a,b) - L^{-1}(x,y)| \leq \frac{1}{2a^2}[|b-y| + x - a],$$

where $L = L(a,b)$ is logarithmic mean.

Proof: If the $f(x) = \frac{1}{x}$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 4, we have the following equality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right|$$
$$= \left|\frac{1}{b-a}\int_a^b \frac{1}{x}dx - \frac{1}{y-x}\int_x^y \frac{1}{x}dx\right|$$
$$= \left|\frac{1}{b-a}ln\frac{b}{a} - \frac{1}{y-x}ln\frac{y}{x}\right|$$
$$= |L^{-1}(a,b) - L^{-1}(x,y)|.$$

From Corollary 1, since $M = \frac{1}{a^2}$, the following inequality

$$|L^{-1}(a,b) - L^{-1}(x,y)| \leq \frac{1}{2a^2}[|b-y| + x - a]$$

is obtained.

Proposition 5. Let $a, b, x, y \in \mathbb{R}$ with $0 < a \leq x < y$ and $a < b$. Then

$$\left|\frac{e^b - e^a}{b-a} - \frac{e^y - e^x}{y-x}\right| \leq \frac{e^b}{2}[|b-y| + x - a].$$

Proof: If the $f(x) = e^x$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 4, we have the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right|$$
$$= \left|\frac{1}{b-a}\int_a^b e^u du - \frac{1}{y-x}\int_x^y e^u du\right|$$
$$= \left|\frac{e^b - e^a}{b-a} - \frac{e^y - e^x}{y-x}\right|$$
$$\leq \frac{e^b}{2}[|b-y| + x - a].$$

Proposition 6. Let $a, b, x, y \in \mathbb{R}$ with $0 < a \leq x < y$ and $a < b$. Then

$$|lnI(x,y) - lnI(a,b)| \leq \frac{1}{2a}[|b-y| + x - a],$$

where $I = I(a,b)$ is identric mean.

Proof: If the $f(x) = -lnx$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 4, we have the following equality:

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{1}{y-x}\int_x^y f(u)du\right|$$
$$=\left|\frac{1}{b-a}\int_a^b -lnudu-\frac{1}{y-x}\int_x^y -lnudu\right|$$

$$=\left|1-\frac{lnb^b-lna^a}{b-a}-\left(1-\frac{lny^y-lnx^x}{y-x}\right)\right|$$
$$=\left|-\left[ln\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\right]+\left[ln\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}\right]\right|$$
$$=|lnI(x,y)-lnI(a,b)|.$$

From Corollary 1, since $M=\frac{1}{a}$, the following inequality

$$|lnI(x,y)-lnI(a,b)|\le\frac{1}{2a}[|b-y|+x-a].$$

Theorem 5. Let $f\colon \mathrm{I}\subset(0,\infty)\to\mathbb{R}$ be an M-Lipschitzian function on interval I and $a,b,x,y\in I$ with $a\le x<y$ and $a<b$. Then following inequality holds:

$$\left|\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du-\frac{xy}{y-x}\int_x^y\frac{f(u)}{u^2}du\right|$$
$$\le\frac{M}{b-a}\{ax|b-y|[bL^{-1}(ay,bx)-L^{-1}(x,y)]$$
$$+by|a-x|[L^{-1}(x,y)-aL^{-1}(ay,bx)]\}$$

Proof: Since f is an M-Lipschitzian function on interval I, for $\forall v,w\in I$

$$|f(v)-f(w)|\le M|v-w|.$$

Here, for arbitrary $t\in[0,1]$, if we take

$$v=\frac{ab}{ta+(1-t)b},\qquad w=\frac{xy}{tx+(1-t)y}$$

then

$$\left|f\left(\frac{ab}{ta+(1-t)b}\right)-f\left(\frac{xy}{tx+(1-t)y}\right)\right|$$
$$\le M\left|\frac{ab}{ta+(1-t)b}-\frac{xy}{tx+(1-t)y}\right|$$
$$\le M\frac{tax|b-y|+(1-t)by|a-x|}{[ta+(1-t)b][tx+(1-t)y]}.$$

If we take integral the last inequality on $t\in[0,1]$ and use the property of modulus and changing variable, we have

$$\left|\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du-\frac{xy}{y-x}\int_x^y\frac{f(u)}{u^2}du\right|$$
$$\le M\left\{ax|b-y|\int_0^1\frac{t}{[ta+(1-t)b][tx+(1-t)y]}dt\right.$$
$$\left.+by|a-x|\int_0^1\frac{1-t}{[ta+(1-t)b][tx+(1-t)y]}dt\right\}$$
$$=M\left\{ax|b-y|\int_0^1\frac{tdt}{[b+t(a-b)][y+t(x-y)]}\right.$$
$$\left.+by|a-x|\int_0^1\frac{t}{[a+t(b-a)][x+t(y-x)]}dt\right\}$$
$$\le\left\{\frac{ax|b-y|}{(b-a)(y-x)}\int_0^1\frac{tdt}{\left[t+\frac{b}{a-b}\right]\left[t+\frac{y}{x-y}\right]}\right.$$
$$\left.+\frac{by|a-x|}{(b-a)(y-x)}\int_0^1\frac{tdt}{\left[t+\frac{a}{b-a}\right]\left[t+\frac{x}{y-x}\right]}\right\}\qquad(2)$$

If the integrals in (2) are calculated, we get

$$\int_0^1\frac{t}{\left[t+\frac{b}{a-b}\right]\left[t+\frac{y}{x-y}\right]}dt$$
$$=\frac{1}{(b-a)(y-x)}\left[ln\frac{x}{y}+\frac{b(y-x)}{ay-bx}ln\frac{ay}{bx}\right]$$
$$=\frac{1}{b-a}[bL^{-1}(ay,bx)-L^{-1}(x,y)],\qquad(3)$$

and

$$\int_0^1\frac{t}{\left[t+\frac{a}{b-a}\right]\left[t+\frac{x}{y-x}\right]}dt$$
$$=\frac{1}{b-a}[L^{-1}(x,y)-aL^{-1}(ay,bx)].\qquad(4)$$

By substituting (3) and (4) in (2), desired result can be obtained.

This completes the proof of theorem.

Proposition 7. Let $p\in(1,\infty)\backslash\{2\}$ and $a,b\in\mathbb{R}$ with $0<a<b$. Then

$$\left|G^2(a,b)L_{p-2}^{p-2}(a,b)-G^2(x,y)L_{p-2}^{p-2}(x,y)\right|$$
$$\le\frac{pb^{p-1}}{b-a}\{ax|b-y|[bL^{-1}(ay,bx)-L^{-1}(x,y)]$$

$+by|a-x|[L^{-1}(x,y)-aL^{-1}(ay,bx)]\}$,

where $G=G(a,b)$, $L_p=L_p(a,b)$ and $L=L(a,b)$are geometric, p-logarithmic and logarithmic means respectively.

Proof: If the$f(x)=x^p$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 5, we have the following equality:

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du-\frac{xy}{y-x}\int_x^y \frac{f(u)}{u^2}du\right|$$
$$=\left|\frac{ab}{b-a}\int_a^b \frac{u^p}{u^2}dx-\frac{xy}{y-x}\int_x^y \frac{u^p}{u^2}dx\right|$$
$$=\left|\frac{ab(b^{p-1}-a^{p-1})}{(b-a)(p-1)}-\frac{xy(x^{p-1}-y^{p-1})}{(y-x)(p-1)}\right|$$

From Corollary 1, since $M=pb^{p-1}$, the following inequality

$$\left|\frac{ab(b^{p-1}-a^{p-1})}{(b-a)(p-1)}-\frac{xy(x^{p-1}-y^{p-1})}{(y-x)(p-1)}\right|$$
$$=\left|G^2(a,b)L_{p-2}^{p-2}(a,b)-G^2(x,y)L_{p-2}^{p-2}(x,y)\right|$$
$$\le\frac{pb^{p-1}}{b-a}\{ax|b-y|[bL^{-1}(ay,bx)-L^{-1}(x,y)]$$
$$+by|a-x|[L^{-1}(x,y)-aL^{-1}(ay,bx)]\}$$

Proposition 8. Let $p\ge 1$ and $a,b\in\mathbb{R}$ with $0<a<b$. Then

$$|H^{-1}(a,b)-H^{-1}(x,y)|$$
$$\le\frac{1}{a^2(b-a)}\{ax|b-y|[bL^{-1}(ay,bx)-L^{-1}(x,y)]$$
$$+by|a-x|[L^{-1}(x,y)-aL^{-1}(ay,bx)]\},$$

where $H=H(a,b)$ and $L=L(a,b)$ are harmonic and logarithmic means respectively.

Proof: If the $f(x)=\frac{1}{x}$ convex mapping defined on interval $[a,b]$ is applied to the left side of the inequality in Theorem 5, we have the following equality:

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du-\frac{xy}{y-x}\int_x^y \frac{f(u)}{u^2}du\right|$$
$$=\left|\frac{ab}{b-a}\int_a^b \frac{\frac{1}{x}}{x^2}dx-\frac{xy}{y-x}\int_x^y \frac{\frac{1}{x}}{x^2}dx\right|$$

So, we get

$$|H^{-1}(a,b)-H^{-1}(x,y)|$$
$$\le\frac{1}{a^2(b-a)}\{ax|b-y|[bL^{-1}(ay,bx)-L^{-1}(x,y)]$$
$$+by|a-x|[L^{-1}(x,y)-aL^{-1}(ay,bx)]\}.$$

3. Conclusions

In this paper, some new type integral inequalities related to the differences between the two different types of integral averages for Lipschitzian functions are obtained. The significance of the obtained inequalities is that: some approaches of the same type averages to each other at different points are given in here first time. Similar studies can also be obtained for fractional integrals.

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