An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.8, No.2, pp.176-182 (2018)
http://doi.org/10.11121/ijocta.01.2018.00544

RESEARCH ARTICLE

# A conformable calculus of radial basis functions and its applications

Fuat Usta

*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*
*fuatusta@duzce.edu.tr*

ARTICLE INFO

Article History:
*Received 03 October 2017*
*Accepted 06 April 2018*
*Available 22 April 2018*

Keywords:
*Conformable fractional derivative*
*Radial basis functions*
*Kansa collocation technique*

AMS Classification 2010:
*65L60, 26A33*

ABSTRACT

In this paper we introduced the conformable derivatives and integrals of radial basis functions (RBF) to solve conformable fractional differential equations via RBF collocation method. For that, firstly, we found the conformable derivatives and integrals of power, Gaussian and multiquadric basis functions utilizing the rule of conformable fractional calculus. Then by using these derivatives and integrals we provide a numerical scheme to solve conformable fractional differential equations. Finally we presents some numerical results to confirmed our method.



## 1. Introduction

Recently, the question of how to take non-integer order of derivative or integration was phenomenon among the scientists. However together with the development of mathematics knowledge, this question was answered via Fractional Calculus which is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. Then In conjunction with the development of theoretical progress of fractional calculus, a number of mathematicians have started to applied the obtained results to real world problems consist of fractional derivatives and integrals [1, 2].

An significant point is that the fractional derivative at a point $x$ is a local property only when a is an integer; in non-integer cases we cannot say that the fractional derivative at $x$ of a function $f$ depends only on values of $f$ very near $x$, in the way that integer-power derivatives certainly do. Therefore it is expected that the theory involves some sort of boundary conditions, involving information on the function further out. To use a metaphor, the fractional derivative requires some peripheral vision. As far as the existence of such a theory is concerned, the foundations of the subject were laid by Liouville in a paper from 1832. The fractional derivative of a function to order a is often now defined by means of the Fourier or Mellin integral transforms. Various types of fractional derivatives were introduced: Riemann- Liouville, Caputo, Hadamard, Erdelyi-Kober, Grunwald-Letnikov, Marchaud and Riesz are just a few to name [3, 4].

Now, all these definitions satisfy the property that the fractional derivative is linear. This is the only property inherited from the first derivative by all of the definitions. However, all definitions do not provide some properties such as Product Rule (Leibniz Rule), Quotient Rule, Chain Rule, Rolls Theorem and Mean Value Theorem. In addition most of the fractional derivatives except Caputo-type derivatives, do not satisfy $D^{\alpha}(f)(1) = 0$ if $\alpha$ is not a natural number.

Recently, a new local, limit-based definition of a so-called conformable derivative has been formulated in [5, 6], with several follow-up papers [1, 2, 7–16]. This new idea was quickly generalized by Katugampola [17, 18]. This new definition forms the basis for this work and is referred to here as the Conformable derivative ($D_\alpha$ will henceforth be referring to the Conformable derivative). This definition has several practical properties which are summarized below.

Note that if $f$ is fully differentiable at $t$; then the derivative is $D_\alpha(f)(t) = t^{1-\alpha} f'(t)$. (Here, operators of a very similar form, $t^\alpha D_1$, have been applied in combinatorial theory [18]). Of course, for $t = 0$ this is not valid and it would be useful to deal with equations and solutions with singularities. Additionally it must be noted that conformable derivative is conformable at $\alpha = 1$, as

$$\lim_{\alpha \to 1} D^\alpha(f) = f',$$

but

$$\lim_{\alpha \to 0^+} D^\alpha(f) \neq f'.$$

On the other hand radial basis functions method is one of the more practical ways of solving fractional order of models. The most significant property of an RBF technique is that there is no need to generate any mesh so it called meshfree method. One only requires the pairwise distance between points for an RBF approximation. Therefore it can be easily applied to high dimensional problems since the computation of distance in any dimensions is straightforward. On the other hand in order to solve partial differential equations (PDEs) in [19, 20] Kansa proposed RBF collocation method which is meshfree and easy-to-handle in comparison with the other methods. Not only integer order PDEs [21] but also Kansa's approach has been used fractional order of PDEs [22].

In this paper we find the conformable derivatives and integrals of needed function of RBF interpolation such as powers, Gaussians and multiquadric. This derivatives play a significant role in the numerical solution of conformable differential equations by the help of RBF method.

The remainder of this work is organized as follows: In Section 2, the related definitions and theorems are summarised. In Section 3, the conformable derivative and integrals have been obtained for the radial basis functions which will use in the RBF computations. Numerical experiments are given in Section 4, while some conclusions and further directions of research are discussed in Section 5.

## 2. Preliminaries

### 2.1. Review of fractional derivatives and integrals

Here we review the Riemann-Liouville fractional derivatives and integrals introduced in [3, 4, 23].

**Definition 1.** *The left-sided Riemann-Liouville fractional derivative of order $\alpha$ of function $u(t)$ is described as*

$$^{\alpha}\mathcal{D}_a^t u(t) = \frac{1}{\Gamma(\tau-\alpha)} \int_a^t (t-\xi)^{\tau-\alpha-1} u(\xi) d\xi, t > a$$

*where $\tau = \lceil \alpha \rceil$.*

**Definition 2.** *The right-sided Riemann-Liouville fractional derivative of order $\alpha$ of function $u(t)$ is described as*

$$^{\alpha}\mathcal{D}_t^b u(t) = \frac{(-1)^\tau}{\Gamma(\tau-\alpha)} \int_t^b (\xi - t)^{\tau-\alpha-1} u(\xi) d\xi, t < b$$

*where $\tau = \lceil \alpha \rceil$.*

**Definition 3.** *The left-sided Riemann-Liouville fractional integral of order $\alpha$ of function $u(t)$ is described as*

$$^{\alpha}\mathcal{I}_a^t u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} u(\xi) d\xi, \quad t > a$$

**Definition 4.** *The right-sided Riemann-Liouville fractional integral of order $\alpha$ of function $u(t)$ is described as*

$$^{\alpha}\mathcal{I}_t^b u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\xi - t)^{\alpha-1} u(\xi) d\xi, \quad t < b$$

Then Khalil et.al. [6] have introduced the conformable fractional derivative and integrals by following definition.

**Definition 5.** *Let $u : [0, \infty) \to \mathbb{R}$. The conformable derivative of $u(t)$ of order $\alpha$ described by*

$$^{\alpha}\mathfrak{D} u(t) = \lim_{\eta \to 0} \frac{u(t + \eta t^{1-\alpha}) - u(t)}{\eta}$$

*where $\alpha \in (0, 1)$ and for all $t > 0$. In other words if $u(t)$ is differentiable, then*

$$^{\alpha}\mathfrak{D} u(t) = t^{1-\alpha} f'(t),$$

*where prime denotes the classical derivative operator.*

Similarly, one can define the conformable fractional integral operator.

**Definition 6.** *Let $u : [0, \infty) \to \mathbb{R}$. The left sided conformable integral of $u(t)$ of order $\alpha$ described by*

$$^{\alpha}\mathfrak{I}_a^t u(t) = \int_a^t t^{\alpha-1} u(t) dt, \qquad t > a$$

*where $\alpha \in (0, 1)$ and the integral is classical integral operator.*

**Definition 7.** *Let $u : [0, \infty) \to \mathbb{R}$. The right sided conformable integral of $u(t)$ of order $\alpha$ described by*

$$^{\alpha}\mathfrak{I}_t^b u(t) = \int_t^b (-t)^{\alpha-1} u(t) dt, \qquad t < b$$

*where $\alpha \in (0, 1)$ and the integral is classical integral operator.*

### 2.2. Radial basis function method

One of the properly approach to solving PDE is radial basis functions (RBFs). The main idea of the RBFs is to calculate distance to any fixed center points $x_i$ with the form $\varphi(\|x - x_i\|_2)$. Additionally RBF may also have scaling parameter called shape parameter $\varepsilon$. This can be done in the manner that $\varphi(r)$ is replaced by $\varphi(\varepsilon r)$. Generally shape parameter have been chosen arbitrarily because there are no exact consequence about how to choose best shape parameter. Some of the RBFs are listed in Table 1.

**Table 1.** Radial basis functions.

| RBFs | $\varphi(r)$ |
|---|---|
| Multiquadric ($MQ$) | $\sqrt{1+r^2}$ |
| Inverse Multiquadric ($IMQ$) | $\frac{1}{\sqrt{1+r^2}}$ |
| Inverse Quadratic ($IQ$) | $\frac{1}{1+r^2}$ |
| Gaussian ($GA$) | $e^{-r^2}$ |

The main advantageous of RBF technique is that it does not require any mesh hence it called meshfree method. Therefore the RBF interpolation can be represent as a linear combination of RBFs as follows:

$$s = \sum_{i=1}^{N} a_i \varphi(\|x - x_i\|_2)$$

where the $a_i$'s the coefficients which are usually calculated by collocation technique. Some of the greatest advantages of RBF interpolation method lies in its practicality in almost any dimension and their fast convergence to the approximated target function.

## 3. Conformable derivatives of RBFs in one dimension

In order to construct conformable derivatives and integrals we will make use of the fractional calculus. Namely the relationship between Riemann-Liouville fractional integral and conformable fractional integral can be given as follows:

**Definition 8.** *Let $\alpha \in (\epsilon, \epsilon+1]$, then the left sided relationship between Riemann-Liouville fractional integral and conformable fractional integral is*

$${}^{\alpha}\mathfrak{I}_a^t u(t) = ^{\epsilon+1} \mathcal{I}_a^t\{(t-a)^{\theta-1}u(t)\}$$

Here if $\alpha = \epsilon + 1$ then $\theta = 1$ since $\theta = \alpha - \epsilon$.

**Theorem 1.** *Let $\theta > -1$ and $t > a$*

$${}^{\alpha}\mathfrak{I}_a^t (t-a)^{\gamma} = \frac{\Gamma(\alpha-\epsilon+\gamma)}{\Gamma(\alpha+1+\gamma)}(t-a)^{\alpha+\gamma}$$

***Proof.***

$$\begin{aligned} {}^{\alpha}\mathfrak{I}_a^t (t-a)^{\gamma} &= ^{\epsilon+1} \mathcal{I}_a^t\{(t-a)^{\theta-1}(t-a)^{\gamma}\} \\ &= ^{\epsilon+1} \mathcal{I}_a^t (t-a)^{\gamma+\alpha-\epsilon-1} \\ &= \frac{1}{\Gamma(\epsilon+1)}\int_a^t (t-\xi)^{\epsilon} \\ &\times (\xi-a)^{\gamma+\alpha-\epsilon-1} d\xi \\ &= \frac{\Gamma(\gamma+\alpha-\epsilon)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha} \end{aligned}$$

□

**Theorem 2.** *Let $\theta > -1$ and $t > a$*

$${}^{\alpha}\mathfrak{I}_t^b (b-t)^{\gamma} = \frac{\Gamma(\alpha-\epsilon+\gamma)}{\Gamma(\alpha+1+\gamma)}(b-t)^{\alpha+\gamma}$$

***Proof.*** The proof is similar to Theorem 1. □

For instance if we take $a = 0$ and $b = 0$ for the above results, we obtain ${}^{\alpha}\mathfrak{I}_0^t (t)^{\gamma} = \frac{\Gamma(\alpha-\epsilon+\gamma)}{\Gamma(\alpha+1+\gamma)}(t)^{\alpha+\gamma}$ and ${}^{\alpha}\mathfrak{I}_t^0 (-t)^{\gamma} = \frac{\Gamma(\alpha-\epsilon+\gamma)}{\Gamma(\alpha+1+\gamma)}(-t)^{\alpha+\gamma}$ respectively. Now, similarly, we can get the conformable derivative of function $(t-a)^{\gamma}$. Namely, the derivative of $(t-a)^{\gamma}$ is

$${}^{\alpha}\mathfrak{D}(t-a)^{\gamma} = \gamma t^{1-\alpha}(t-a)^{\gamma-1}.$$

and again if we choose $a = 0$, we get

$${}^{\alpha}\mathfrak{D}(t)^{\gamma} = \gamma t^{\gamma-\alpha}.$$

Now, by using the above results, one can find the conformable derivatives and integration of radial basis functions. Additionally throughout this and next sections ${}^nC_k$ denotes the combination of $n$ and $k$ such that ${}^nC_k = \frac{n!}{(n-k)!k!}$.

### 3.1. For $\varphi(t) = t^m$ (power basis function)

**Theorem 3.** *For $a \neq 0$, $t > a$ and $m \in \mathbb{N}$*

$${}^{\alpha}\mathfrak{I}_a^t t^m = (t-a)^{\alpha} \sum_{k=0}^{m} {}^mC_k a^{m-k} \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k.$$

***Proof.*** In order to prove the above theorem we use the Taylor expansion of $t^m$ about the point $t = a$. Namely,

$$t^m = \sum_{k=0}^{m} {}^mC_k a^{m-k}(t-a)^k. \quad (1)$$

If we substitute the equation (1) into conformable integration definition, we have

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_a^t t^m &=^{\alpha} \mathfrak{I}_a^t \sum_{k=0}^{m} {}^mC_k a^{m-k}(t-a)^k \\
&= \sum_{k=0}^{m} {}^mC_k a^{m-k\alpha}\mathfrak{I}_a^t (t-a)^k \\
&= \sum_{k=0}^{m} {}^mC_k a^{m-k} \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^{\alpha+k} \\
&= (t-a)^{\alpha} \sum_{k=0}^{m} {}^mC_k a^{m-k} \\
&\times \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k.
\end{aligned}
$$

□

**Theorem 4.** *For $b \neq 0$, $b > t$ and $m \in \mathbb{N}$*

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_t^b t^m = (b-t)^{\alpha} \sum_{k=0}^{m}\sum_{k=0}^{m} {}^mC_k a^{m-k} \\
\times \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-b)^k.
\end{aligned}
$$

***Proof.*** The proof is similar to Theorem 3. □

**Theorem 5.** *For $a \neq 0$, $t > a$ and $m \in \mathbb{N}$*

$$
{}^{\alpha}\mathfrak{D}(t)^m = \frac{t^{1-\alpha}}{t-a} \sum_{k=0}^{m} {}^mC_k k a^{m-k}(t-a)^k. \quad (2)
$$

***Proof.*** In order to prove the above theorem we use the Taylor expansion of $t^m$ about the point $t = a$ again. In other words if we substitute the equation (1) into conformable integration definition, we have

$$
\begin{aligned}
{}^{\alpha}\mathfrak{D}t^m &=^{\alpha} \mathfrak{D} \sum_{k=0}^{m} {}^mC_k a^{m-k}(t-a)^k \\
&= \sum_{k=0}^{m} {}^mC_k a^{m-k\alpha}\mathfrak{D}(t-a)^k \\
&= \sum_{k=0}^{m} {}^mC_k a^{m-k} t^{1-\alpha} k(t-a)^{k-1} \\
&= \frac{t^{1-\alpha}}{t-a} \sum_{k=0}^{m} {}^mC_k k a^{m-k}(t-a)^k.
\end{aligned}
$$

□

### 3.2. For $\varphi(t) = e^{(-t^2/2)}$ (Gaussian basis function)

Now we can make use of the conformable derivatives and integration of power basis function, we are able to find out the Gaussian basis function derivatives and integrations.

**Theorem 6.** *For $a \neq 0$, $t > a$ and $m \in \mathbb{N}$*

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_a^t e^{-t^2/2} &= (t-a)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \sum_{k=0}^{2m} {}^{2m}C_k a^{2m-k} \\
&\times \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k.
\end{aligned}
$$

***Proof.*** In order to prove the above theorem we use the Taylor expansion of $e^{-t^2/2}$ about the point $t = 0$. Namely,

$$
e^{-t^2/2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!}(t)^{2m}. \quad (3)
$$

If we substitute the equation (3) into conformable integration definition, we have

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_a^t e^{(-t^2/2)} &=^{\alpha} \mathfrak{I}_a^t \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!}(t)^{2m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} {}^{\alpha}\mathfrak{I}_a^t t^{2m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \left[ (t-a)^{\alpha} \sum_{k=0}^{2m} \right. \\
&\left. \times {}^{2m}C_k a^{2m-k} \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k \right] \\
&= (t-a)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \sum_{k=0}^{2m} \\
&\times {}^{2m}C_k a^{2m-k} \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k.
\end{aligned}
$$

□

**Theorem 7.** *For $a \neq 0$, $b > t$ and $m \in \mathbb{N}$*

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_t^b e^{-t^2/2} = (b-t)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \sum_{k=0}^{2m} {}^{2m}C_k a^{2m-k} \\
\times \frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-b)^k.
\end{aligned}
$$

***Proof.*** The proof is similar to Theorem 7. □

**Theorem 8.** *For $a \neq 0$, $t > a$ and $m \in \mathbb{N}$*

$$
{}^{\alpha}\mathfrak{D}e^{-t^2/2} = t^{1-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m 2m}{2^m m!}(t)^{2m-1}.
$$

***Proof.*** Similarly by using the Taylor expansion of Gaussian function about $t = 0$ we can calculate the conformable derivative of it. That is,

$$
\begin{aligned}
{}^{\alpha}\mathfrak{D}e^{-t^2/2} &= {}^{\alpha}\mathfrak{D}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}(t)^{2m} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}{}^{\alpha}\mathfrak{D}t^{2m} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}t^{1-\alpha}2mt^{2m-1} \\
&= t^{1-\alpha}\sum_{m=0}^{\infty}\frac{(-1)^m 2m}{2^m m!}(t)^{2m-1}.
\end{aligned}
$$

□

### 3.3. For $\varphi(t) = \sqrt{1+t^2}$ (Multiquadric basis function)

Similarly one can compute the conformable derivatives and integrations.

**Theorem 9.** *For $a \neq 0$, $t > a$ and $m \in \mathbb{N}$*

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_a^t\sqrt{1+t^2} &= (t-a)^{\alpha}\sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m} \\
&\quad\times\sum_{k=0}^{2m}{}^{2m}C_k a^{2m-k}\frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k.
\end{aligned}
$$

***Proof.*** In order to prove the above theorem we use the Taylor expansion of $\sqrt{1+t^2}$ about the point $t = 0$. Namely,

$$
\sqrt{1+t^2} = \sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m}(t)^{2m}. \qquad (4)
$$

If we substitute the equation (4) into conformable integration definition, we have

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_a^t\sqrt{1+t^2} &= {}^{\alpha}\mathfrak{I}_a^t\sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m}(t)^{2m} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m}{}^{\alpha}\mathfrak{I}_a^t t^{2m} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m} \\
&\quad\times\Bigg[(t-a)^{\alpha}\sum_{k=0}^{2m}{}^{2m}C_k a^{2m-k} \\
&\quad\times\frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k\Bigg] \\
&= (t-a)^{\alpha}\sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m} \\
&\quad\times\sum_{k=0}^{2m}{}^{2m}C_k a^{2m-k}\frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(t-a)^k.
\end{aligned}
$$

□

**Theorem 10.** *For $a \neq 0$, $b > t$ and $m \in \mathbb{N}$*

$$
\begin{aligned}
{}^{\alpha}\mathfrak{I}_t^b\sqrt{1+t^2} &= (b-t)^{\alpha}\sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m} \\
&\quad\times\sum_{k=0}^{2m}{}^{2m}C_k a^{2m-k}\frac{\Gamma(\alpha-\epsilon+k)}{\Gamma(\alpha+1+k)}(b-t)^k.
\end{aligned}
$$

***Proof.*** The proof is similar to Theorem 9. □

**Theorem 11.** *For $a \neq 0$, $t > a$ and $m \in \mathbb{N}$*

$$
{}^{\alpha}\mathfrak{D}\sqrt{1+t^2} = t^{1-\alpha}\sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m 2m}{(1-2m)4^m}t^{2m-1}.
$$

***Proof.*** Similarly by using the Taylor expansion of multiquadric basis function about $t = 0$ we can calculate the conformable derivative of it. That is,

$$
\begin{aligned}
{}^{\alpha}\mathfrak{D}\sqrt{1+t^2} &= {}^{\alpha}\mathfrak{D}\sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m}(t)^{2m} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m}t^{2m} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m}{(1-2m)4^m}t^{1-\alpha}2mt^{2m-1} \\
&= t^{1-\alpha}\sum_{m=0}^{\infty}\frac{(-1)^{m\ 2m}C_m 2m}{(1-2m)4^m}t^{2m-1}.
\end{aligned}
$$

□

## 4. Numerical example

In this section we will give some results of numerical solution of conformable differential equations to validate our numerical scheme. For that we will use RBF interpolation method by the help of collocation technique. Consider the general form of following conformable differential equation:

$$
{}^{\alpha}\mathfrak{D}y(t) + p(t)y(t) = q(t), \qquad y_0(t) = y(t_0). \quad (5)
$$

Let $t_j$ be equally spaced grid points in the interval $0 \leq t_j \leq K$ such that $1 \leq j \leq L$, $t_1 = 0$ and $t_L = K$. Additionally, because collocation approach has been used we not only require an expression for the value of the function

$$y(t) = \sum_{k=1}^{L} a_k \psi(\|x - x_k\|) \quad (6)$$

but also for the conformal derivative given in (5). Thus, by conformal differentiating (6), we get

$${}^{\alpha}\mathfrak{D}y(t) = \sum_{k=1}^{L} a_k^{\alpha}\mathfrak{D}\psi(\|t - t_k\|)$$

where ${}^{\alpha}\mathfrak{D}$ denotes the conformable derivative the with respect to $t$. In order to compute conformable derivative of radial basis functions we take the advantage of formulas which are derived in the previous section. Then using the RBF collocation method, one can compute the unknown coefficients $a_k$'s by solving following matrix system:

$$\sum_{k=1}^{L} a_k^{\alpha}\mathfrak{D}\psi(\|x_j - x_k\|) + p(t)\sum_{k=1}^{L} a_k \psi(\|x_j - x_k\|) = q(t), \qquad j = 2, \ldots, L.$$

with boundary condition. In order to illustrate this scheme by numerically we take the following conformable differential equations:

(1)

$${}^{\alpha}\mathfrak{D}y(t) + y(t) = 0$$

$$y_0(t) = 1, \quad y_{exact}(t) = e^{-\frac{1}{\alpha}t^{\alpha}}$$

(2)

$${}^{\alpha}\mathfrak{D}y(t) + \alpha y(t) = 1 + t^{\alpha}$$

$$y_0(t) = 0, \quad y_{exact}(t) = \frac{t^{\alpha}}{\alpha}$$

(3)

$${}^{\alpha}\mathfrak{D}y(t) + y(t) = \sqrt{1 + \sin\left(\frac{2t^{\alpha}}{\alpha}\right)}$$

$$y_0(t) = 0, \quad y_{exact}(t) = \sin\left(\frac{t^{\alpha}}{\alpha}\right)$$

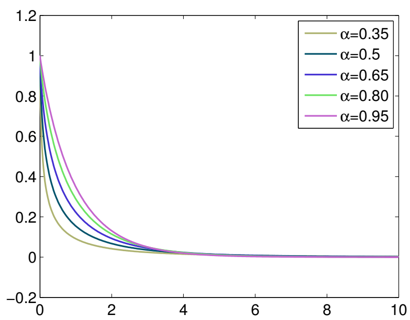


**Figure 1.** $y(t)$ versus $t$ using multiquadric basis function with $\varepsilon = 10^{-4}$ for $p(t) = 1$ and $q(t) = 0$ for different value of $\alpha$.

Here we use the multiquadric basis function with $\varepsilon = 10^{-4}$. In Figures 1, 2 and 3, we present the numerical solutions of given conformable differential equations with different $\alpha$ values. These results are in accord with the exact solutions of them.

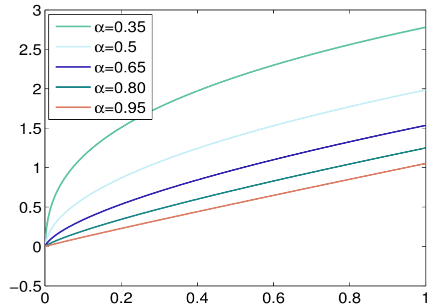


**Figure 2.** $y(t)$ versus $t$ using multiquadric basis function with $\varepsilon = 10^{-4}$ for $p(t) = \alpha$ and $q(t) = 1 + t^{\alpha}$ for different value of $\alpha$.

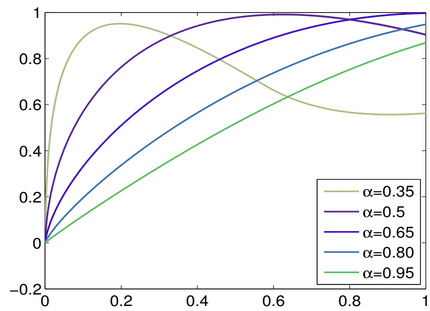


**Figure 3.** $y(t)$ versus $t$ using Multiquadric basis function with $\varepsilon = 10^{-4}$ for $p(t) = 1$ and $q(t) = \sqrt{1 + \sin\left(\frac{2t^{\alpha}}{\alpha}\right)}$ for different value of $\alpha$.

## 5. Conclusion

In this paper we gave the derivatives and integrals of three kinds of radial basis functions such as powers, Gaussians and multiquadric by using the conformable derivatives and integrals which are new type of fractional calculus. These findings allow to solve conformable differential equations by the RBF's. Then we gave three differential equations to show that this technique is applicable. These differential equations are solved by the help of RBF collocation method.

## References

[1] Hammad, M.A. and Khalil, R. (2014). Conformable fractional heat differential equations, *International*

*Journal of Pure and Applied Mathematics*, 94(2), 215-221.

[2] Hammad, M.A. and Khalil, R. (2014). Abel's formula and wronskian for conformable fractional differential equations, *International Journal of Differential Equations and Applications*, 13(3), 177-183.

[3] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). *Theory and applications of fractional differential equations*, North-Holland Mathematical Studies, Vol.204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York.

[4] Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993). Fractional Integrals and Derivatives: Theory and Applications, *Gordonand Breach, Yverdon et alibi.*

[5] Abdeljawad, T. (2015). On conformable fractional calculus, *Journal of Computational and Applied Mathematics*, 279, 57-66.

[6] Khalil, R., Al horani, M, Yousef, A. and Sababheh, M. (2014). A new definition of fractional derivative, *Journal of Computational Apllied Mathematics*, 264, 65-70.

[7] Anderson, D.R. and Ulness, D.J. (2016). Results for conformable differential equations, *preprint*.

[8] Atangana, A., Baleanu, D. and Alsaedi, A. (2015) New properties of conformable derivative, *Open Mathematics*, 13, 889-898.

[9] Iyiola, O.S. and Nwaeze, E.R. (2016). Some new results on the new conformable fractional calculus with application using D'Alambert approach, *Progress in Fractional Differentiation and Applications*, 2(2), 115-122.

[10] Zheng, A., Feng, Y. and Wang, W. (2015). The Hyers-Ulam stability of the conformable fractional differential equation. *Mathematica Aeterna*, 5(3), 485-492.

[11] Usta, F. and Sarkaya, M.Z. (2017). Some improvements of conformable fractional integral. *Inequalities International Journal of Analysis and Applications*, 14(2), 162-166.

[12] Sarkaya, M.Z., Yaldz, H. and Budak, H. (2017). On weighted Iyengar-type inequalities for conformable fractional integrals, *Mathematical Sciences*, 11(4), 327-331.

[13] Akkurt, A., Yldrm, M.E. and Yldrm, H. (2017). On Some Integral Inequalities for Conformable Fractional Integrals, *Asian Journal of Mathematics and Computer Research*, 15(3), 205-212.

[14] Usta, F. (2016). A mesh-free technique of numerical solution of newly defined conformable differential equations. *Konuralp Journal of Mathematics*, 4(2), 149-157.

[15] Usta, F. (2016). Fractional type Poisson equations by radial basis functions Kansa approach. *Journal of Inequalities and Special Functions*, 7(4), 143-149.

[16] Usta, F. (2017). Numerical solution of fractional elliptic PDE's by the collocation method. *Applications and Applied Mathematics: An International Journal (AAM)*, 12 (1), 470-478.

[17] Katugampola, U.N. (2014). A new fractional derivative with classical properties, *ArXiv:1410.6535v2*.

[18] Katugampola, U.N. (2015). Mellin transforms of generalized fractional integrals and derivatives, *Applied Mathematics and Computation*, 257, 566580.

[19] Kansa, E.J. (1990), Multiquadrics a scattered data approximation scheme with applications to computational filuid-dynamics. I. Surface approximations and partial derivative estimates, *Computers and Mathematics with Applications*, 19(8-9), 127-145.

[20] Kansa, E.J. (1990), Multiquadrics a scattered data approximation scheme with applications to computational filuid-dynamics. II. Solutions to parabolic, hyperbolic and elliptic partial differential equations, *Computers and Mathematics with Applications*, 19(8-9), 147-161.

[21] Zerroukat, M., Power, H. and Chen, C.S. (1998). A numerical method for heat transfer problems using collocation and radial basis functions. *International Journal for Numerical Methods in Engineering*, 42, 12631278.

[22] Chen, W., Ye, L. and Sun, H. (2010). Fractional diffusion equations by the Kansa method, *Computers and Mathematics with Applications*, 59, 16141620.

[23] Podlubny, I. (1999), *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their Solution and some of their applications*, Academic Press, San Diego, CA.

***Fuat Usta*** *received his BSc (Mathematical Engineering) degree from Istanbul Technical University, Turkey in 2009 and MSc (Mathematical Finance) from University of Birmingham, UK in 2011 and PhD (Applied Mathematics) from University of Leicester, UK in 2015. At present, he is working as a Assistant Professor in the Department of Mathematics at Düzce University (Turkey). He is interested in Approximation Theory, Multivariate approximation using Quasi Interpolation, Radial Basis Functions and Hierarchical/Wavelet Bases, High-Dimensional Approximation using Sparse Grids. Financial Mathematics, Integral Equations, Fractional Calculus, Partial Differantial Equations.*