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RESEARCH ARTICLE

New extensions of Chebyshev type inequalities using generalized Katugampola integrals via Pólya-Szegö inequality

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ABSTRACT

A number of Chebyshev type inequalities involving various fractional integral operators have, recently, been presented. In this work, motivated essentially by the earlier works and their applications in diverse research subjects, we establish some new Pólya-Szegö inequalities involving generalized Katugampola fractional integral operator and use them to prove some new fractional Chebyshev type inequalities which are extensions of the results in the paper: [On Pólya-Szegö and Chebyshev type inequalities involving the Riemann-Liouville fractional integral operators, J. Math. Inequal, 10(2) (2016)].

1. Introduction and preliminaries

This article is based on the well known Chebyshev functional [1]:

$$\begin{aligned} &T(f,g)\\ =\ &\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\\ &-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \end{aligned}$$

where f and g are two integrable functions which are synchronous on $[a,b]$, i.e.

$$(f(x)-f(y))(g(x)-g(y))\geq 0$$

for any $x,y\in[a,b]$, then the Chebyshev inequality states that $T(f,g)\geq 0$.
For some recent counterparts, generalizations of Chebyshev inequality, the reader is refer to [2–6].

We also need to introduce the Pólya and Szegö inequality [7]:

$$\frac{\int_a^b f^2(x)dx\int_a^b g^2(x)dx}{(\int_a^b f(x)g(x)dx)^2}\leq\frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2.$$

Using the above Pólya-Szegö inequality, Dragomir and Diamond [8] established the following Grüss type inequality:

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}_+$ be two integrable functions so that*

$$0<m\leq f(x)\leq M<\infty$$

and

$$0<n\leq g(x)\leq N<\infty$$

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for a.e. $x \in [a, b]$.
Then, we have

$$\begin{aligned} &|T(f, g; a, b)| \\ &\leq \frac{1}{4}\frac{(M-m)(N-n)}{\sqrt{mnMN}} \\ &\times \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx. \end{aligned} \tag{1}$$

The constant $\frac{1}{4}$ is best possible in (1) in the sense it can not be replaced by a smaller constant.

For our purpose, we recall some other preliminaries: We note that the beta function $B(\alpha, \beta)$ is defined by (see, e.g. [9, Section 1.1])

$$\begin{aligned} &B(\alpha,\, \beta) \\ &= \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha) > 0; \ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \end{cases} \end{aligned} \tag{2}$$

where Γ is the familiar Gamma function. Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers and non-positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$.

Definition 1. *(see, e.g., [10], [11]) Let $[a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals(left-sided) of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of a real function $f \in L(a, b)$, is defined:*

$$\begin{aligned} &\left(J_{a+}^{\alpha} f\right)(x) \\ &:= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt \quad (x > a). \end{aligned} \tag{3}$$

Definition 2. *(see, e.g., [10], [11]) Let (a, b) $(0 \leq a < b \leq \infty)$ be a finite or infinite interval on the half-axis $\mathbb{R}^+$. The Hadamard fractional integrals(left-sided) of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of a real function $f \in L(a, b)$ are defined by*

$$\begin{aligned} &\left(H_{a+}^{\alpha} f\right)(x) \\ &:= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}\,dt \quad (a < x < b) \end{aligned} \tag{4}$$

Definition 3. *(see, e.g., [10], [11]) Let (a, b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval on the half-axis $\mathbb{R}^+$. Also let $Re(\alpha) > 0$, $\sigma > 0$ and $\eta \in \mathbb{C}$. The Erdelyi-Kober fractional integrals (left-sided) of order $\alpha \in \mathbb{C}$ of a real function $f \in L(a, b)$ are defined by*

$$\begin{aligned} &\left(I_{a+,\sigma,\eta}^{\alpha} f\right)(x) \\ &= \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x \frac{t^{\sigma(\eta+1)-1}}{(x^{\alpha}-t^{\alpha})^{1-\alpha}} f(t)\,dt \\ &(0 \leq a < x < b \leq \infty). \end{aligned} \tag{5}$$

Definition 4. *[12] Let $[a, b] \subset \mathbb{R}$ be a finite interval. The Katugampola fractional integrals (left-sided) of order $\alpha \in \mathbb{C}$, $\rho > 0$ $Re(\alpha > 0)$ of a real function $f \in X_c^p(a, b)$ are defined by*

$$\begin{aligned} &\left({}^{\rho}I_{a+}^{\alpha} f\right)(x) \\ &:= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}} f(t)\,dt. \quad (x > a) \end{aligned} \tag{6}$$

Definition 5. *(see, e.g., [10], [11]) Let a continuous function by parts in $\mathbb{R} = (-\infty, \infty)$. The Liouville fractional integrals (left-sided) of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ of a real function f, are defined by*

$$\begin{aligned} &\left(I_{+}^{\alpha} f\right)(x) \\ &:= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt \quad (x \in \mathbb{R}). \end{aligned} \tag{7}$$

Here, the space $X_c^p(a, b)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ consists of those complex-valued Lebesgue measurable functions φ on (a, b) for which $\|\varphi\|_{X_c^p} < \infty$, with

$$\|\varphi\|_{X_c^p} = \left(\int_a^b |x^c\varphi(x)|^p \frac{dx}{x}\right)^{1/p} \quad (1 \leq p < \infty)$$

and

$$\|\varphi\|_{X_c^p} = \texttt{esssup}_{x\in(a,b)}[x^c|\varphi(x)|].$$

In particular, when $c = 1/p$ $(1 \leq p < \infty)$, the space $X_c^p(a, b)$ coincides with the classical $L^p(a, b)$ space.
Let $0 \leq a < x < b \leq \infty$. Also, let $\varphi \in X_c^p(a, b)$, $\alpha, \rho \in \mathbb{R}^+$, and β, η, $\kappa \in \mathbb{R}$. Then, the fractional integrals (left-sided and right-sided) of a function φ are defined, respectively, by (see [13])

$$\begin{aligned} &\left({}^{\rho}I_{a+,\eta,\kappa}^{\alpha,\beta}\varphi\right)(x) \\ &:= \frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\varphi(\tau)\,d\tau \end{aligned} \tag{8}$$

and

$$\left({}^{\rho}I^{\alpha,\beta}_{b-,\eta,\kappa}\varphi\right)(x) := \frac{\rho^{1-\beta}x^{\rho\eta}}{\Gamma(\alpha)}\int_x^b \frac{\tau^{\kappa+\rho-1}}{(\tau^{\rho}-x^{\rho})^{1-\alpha}}\varphi(\tau)\,d\tau. \tag{9}$$

Remark 1. The fractional integral (8) contains five well-known fractional integrals as its particular cases (see also [13–15]):

(i) Setting $\kappa = 0$, $\eta = 0$ and $\rho = 1$ in (8), the integral operator (8) reduces to the Riemann-Liouville fractional integral (3) (see also [10, p. 69]).
(ii) Setting $\kappa = 0$, $\eta = 0$, $a = -\infty$ and $\rho = 1$ in (8), the integral operator (8) reduces to the Liouville fractional integral (7) (see also [10, p.79]).
(iii) Setting $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, and taking the limit $\rho \to 0^+$ with L'Hôspital's rule in (8), the integral operator (8) reduces to the Hadamard fractional integral (4) (see also [10, p. 110]).
(iv) Setting $\beta = 0$ and $\kappa = -\rho(\alpha+\eta)$ in (8), the integral operator (8) reduces to the Erdélyi-Kober fractional integral (5) (see also [10, p. 105]).
(v) Setting $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$ in (8), the integral operator (8) reduces to the Katugampola fractional integral (6) (see also [12]).

The principle aim of the present paper is to establish new Pólya-Szegö inequalities and other of Chebyshev type by using generalized Katugampola fractional integration theory.

2. Main Results

In this section, we establish some new Chebyshev type inequalities involving the Katugampola fractional integration approach. Thanks to (2), we obtain (see [15, Eq. (3.1)])

$$\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}d\tau = \frac{x^{\kappa+\rho(\eta+\alpha)}\,\Gamma(\eta+1)}{\rho^{\beta}\,\Gamma(\alpha+\eta+1)} = \Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta) \tag{10}$$

$$\left(\alpha,\, x \in \mathbb{R}^+;\ \beta,\, \rho,\, \eta,\, \kappa \in \mathbb{R}\right).$$

We also let

$$\left({}^{\rho}I^{\alpha,\beta}_{0+,\eta,\kappa}\varphi\right)(x) := \left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\varphi\right)(x).$$

Lemma 1. *Let $\beta, \kappa \in \mathbb{R}$, x, α, $\rho \in \mathbb{R}^+$, and $\eta \in \mathbb{R}^+_0$. Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions υ_1, υ_2, ω_1 and ω_2, such that:*

$$\begin{aligned} &0 < \upsilon_1(\tau) \le f(\tau) \le \upsilon_2(\tau)\\ &0 < \omega_1(\tau) \le g(\tau) \le \omega_2(\tau)\\ &(\tau \in [0,x], x > 0)\end{aligned} \tag{11}$$

Then the following inequality holds:

$$\frac{{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{\omega_1\omega_2 f^2\right\}(x)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{\upsilon_1\upsilon_2 g^2\right\}(x)}{\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{(\upsilon_1\omega_1+\upsilon_2\omega_2)fg\right\}(x)\right)^2} \le \frac{1}{4}. \tag{12}$$

Proof. From (11), for $\tau \in [0,x]$, $x > 0$, we can write

$$\left(\frac{\upsilon_2(\tau)}{\omega_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \ge 0 \tag{13}$$

and

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{\upsilon_1(\tau)}{\omega_2(\tau)}\right) \ge 0. \tag{14}$$

Multiplying (13) and (14), we get

$$\left(\frac{\upsilon_2(\tau)}{\omega_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{\upsilon_1(\tau)}{\omega_2(\tau)}\right) \ge 0.$$

From the above inequality, we can write

$$\begin{aligned}&(\upsilon_1(\tau)\omega_1(\tau) + \upsilon_2(\tau)\omega_2(\tau))\,f(\tau)g(\tau)\\ &\ge \omega_1(\tau)\omega_2(\tau)f^2(\tau) + \upsilon_1(\tau)\upsilon_2(\tau)g^2(\tau).\end{aligned} \tag{15}$$

Multiplying both sides of (15) by

$$\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}$$

and integrating the resulting inequality with respect to τ over $(0,x)$, we get

$$\begin{aligned}&{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{(\upsilon_1\omega_1+\upsilon_2\omega_2)fg\right\}(x)\\ &\ge {}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{\omega_1\omega_2 f^2\right\}(x) + {}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{\upsilon_1\upsilon_2 g^2\right\}(x)\end{aligned}$$

Applying the AM-GM inequality, i.e. $(a+b \geq 2\sqrt{ab},\quad a,b \in \mathbb{R}^+)$, we have

$$
\begin{aligned}
&{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{(\upsilon_1\omega_1+\upsilon_2\omega_2)fg\right\}(x) \\
&\geq 2\sqrt{{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\omega_1\omega_2 f^2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1\upsilon_2 g^2\right\}(x)}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\omega_1\omega_2 f^2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1\upsilon_2 g^2\right\}(x) \\
&\leq \frac{1}{4}\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{(\upsilon_1\omega_1+\upsilon_2\omega_2)fg\right\}(x)\right)^2.
\end{aligned}
$$

So, we get the desired result. □

Corollary 1. *If $\upsilon_1 = m$, $\upsilon_2 = M$, $\omega_1 = n$ and $\omega_2 = N$, then we have*

$$
\frac{\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f^2\right)(x)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g^2\right)(x)}{\left(\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg\right)(x)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2
$$

Remark 2. *Setting $\kappa = 0$, $\eta = 0$ and $\rho = 1$ in Lemma 1, yields the inequality in [16, Lemma 3.1].*

Lemma 2. *Let $\beta, \kappa \in \mathbb{R}$, x, α, θ, $\rho \in \mathbb{R}^+$, and $\eta \in \mathbb{R}_0^+$. Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions υ_1, υ_2, ω_1 and ω_2 satisfying condition (11). Then the following inequality holds:*

$$
\begin{aligned}
&{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1\upsilon_2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\omega_1\omega_2\right\}(x) \\
&\times{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{f^2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{g^2\right\}(x) \\
&\leq \frac{1}{4}\Big({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1 f\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\omega_1 g\right\}(x) \qquad (16) \\
&+{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_2 f\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\upsilon_2 g\right\}(x)\Big)^2.
\end{aligned}
$$

Proof. From (11), we get

$$
\left(\frac{\upsilon_2(\tau)}{\omega_1(\xi)}-\frac{f(\tau)}{g(\xi)}\right) \geq 0
$$

and

$$
\left(\frac{f(\tau)}{g(\xi)}-\frac{\upsilon_1(\tau)}{\omega_2(\xi)}\right) \geq 0
$$

which lead to

$$
\begin{aligned}
&\left(\frac{\upsilon_1(\tau)}{\omega_2(\xi)}+\frac{\upsilon_2(\tau)}{\omega_1(\xi)}\right)\frac{f(\tau)}{g(\xi)} \\
&\geq \frac{f^2(\tau)}{g^2(\xi)}+\frac{\upsilon_1(\tau)\upsilon_2(\tau)}{\omega_1(\xi)\omega_2(\xi)}. \qquad (17)
\end{aligned}
$$

Multiplying both sides of (17) by $\omega_1(\xi)\omega_2(\xi)g^2(\xi)$, we get

$$
\begin{aligned}
&\upsilon_1(\tau)f(\tau)\omega_1(\xi)g(\xi)+\upsilon_2(\tau)f(\tau)\omega_2(\xi)g(\xi) \\
&\geq \omega_1(\xi)\omega_2(\xi)f^2(\tau)+\upsilon_1(\tau)\upsilon_2(\tau)g^2(\xi). \qquad (18)
\end{aligned}
$$

Multiplying both sides of (18) by

$$
\frac{\rho^{2(1-\beta)}x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}
$$

and integrating the resulting inequality with respect to τ and ξ over $(0,x)^2$, we get

$$
\begin{aligned}
&{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1 f\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\omega_1 g\right\}(x) \\
&+{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_2 f\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\upsilon_2 g\right\}(x) \\
&\geq {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{f^2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\omega_1\omega_2\right\}(x) \\
&+{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1\upsilon_2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{g^2\right\}(x).
\end{aligned}
$$

Applying the AM-GM inequality, we have

$$
\begin{aligned}
&{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1 f\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\omega_1 g\right\}(x) \\
&+{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_2 f\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\upsilon_2 g\right\}(x) \\
&\geq 2\sqrt{{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{f^2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{\omega_1\omega_2\right\}(x)} \\
&\times\sqrt{{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\left\{\upsilon_1\upsilon_2\right\}(x)\,{}^{\rho}I_{\eta,\kappa}^{\theta,\beta}\left\{g^2\right\}(x)}.
\end{aligned}
$$

So, we get the desired inequality of (16). □

Corollary 2. *If $\upsilon_1 = m$, $\upsilon_2 = M$, $\omega_1 = n$ and $\omega_2 = N$, then we have*

$$
\begin{aligned}
&\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta) \\
&\times\frac{\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f^2\right)(x)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g^2\right)(x)}{\left(\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^{\rho}I_{\eta,\kappa}^{\theta,\beta}g\right)(x)\right)^2} \\
&\leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2
\end{aligned}
$$

Remark 3. *Setting $\kappa = 0$, $\eta = 0$ and $\rho = 1$ in Lemma 2 yields the inequality in [16, Lemma 3.3].*

Lemma 3. *Suppose that all assumptions of Lemma 2 are satisfied. Then, we have:*

$$
\begin{aligned}
&{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{f^2\right\}(x)\,{}^{\rho}I^{\theta,\beta}_{\eta,\kappa}\left\{g^2\right\}(x) \qquad (19)\\
&\leq {}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{\frac{\upsilon_2 fg}{\omega_1}\right\}(x)\,{}^{\rho}I^{\theta,\beta}_{\eta,\kappa}\left\{\frac{\omega_2 fg}{\upsilon_1}\right\}(x).
\end{aligned}
$$

Proof. Using the condition (11), we get

$$
\begin{aligned}
&\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} f^2(\tau)d\tau\\
&\leq \frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\upsilon_2(\tau)}{\omega_1(\tau)} f(\tau)g(\tau)d\tau
\end{aligned}
$$

which leads to

$$
{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{f^2\right\}(x) \leq {}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{\frac{\upsilon_2 fg}{\omega_1}\right\}(x). \qquad (20)
$$

Similarly, we have

$$
\begin{aligned}
&\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\theta)}\int_0^x \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}} g^2(\xi)d\xi\\
&\leq \frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\theta)}\int_0^x \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}\frac{\omega_2(\xi)}{\upsilon_1(\xi)} f(\xi)g(\xi)d\xi,
\end{aligned}
$$

which implies

$$
{}^{\rho}I^{\theta,\beta}_{\eta,\kappa}\left\{g^2\right\}(x) \leq {}^{\rho}I^{\theta,\beta}_{\eta,\kappa}\left\{\frac{\omega_2 fg}{\upsilon_1}\right\}(x). \qquad (21)
$$

Multiplying (20) and (21), we get the inequality of (19). □

Corollary 3. *If $\upsilon_1 = m$, $\upsilon_2 = M$, $\omega_1 = n$ and $\omega_2 = N$, then we have*

$$
\frac{\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f^2\right)(x)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g^2\right)(x)}{\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg\right)(x)\left({}^{\rho}I^{\theta,\beta}_{\eta,\kappa}fg\right)(x)} \leq \frac{MN}{mn}.
$$

Remark 4. *Setting $\kappa = 0$, $\eta = 0$ and $\rho = 1$ in Lemma 3 yields the inequality in [16, Lemma 3.4].*

Theorem 2. *Let $\beta,\kappa \in \mathbb{R}$, x, α, θ, $\rho \in \mathbb{R}^+$, and $\eta \in \mathbb{R}^+_0$. Let f and g be two positive integrable functions on $[0,\infty)$. Assume also that there exist four positive integrable functions υ_1, υ_2, ω_1 and ω_2 satisfying the condition (11). Then the following inequality holds:*

$$
\begin{aligned}
&\Big|\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\left({}^{\rho}I^{\theta,\beta}_{\eta,\kappa}fg\right)(x)\\
&+\Lambda^{\rho,\beta}_{x,\kappa}(\theta,\eta)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg\right)(x)\\
&-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f\right)(x)\left({}^{\rho}I^{\theta,\beta}_{\eta,\kappa}g\right)(x) \qquad (22)\\
&-\left({}^{\rho}I^{\theta,\beta}_{\eta,\kappa}f\right)(x)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g\right)(x)\Big|\\
\leq\ &|G_1(f,\upsilon_1,\upsilon_2)(x)+G_2(f,\upsilon_1,\upsilon_2)(x)|^{1/2}\\
&\times|G_2(g,\omega_1,\omega_2)(x)+G_2(g,\omega_1,\omega_2)(x)|^{1/2}
\end{aligned}
$$

where

$$
\begin{aligned}
&G_1(f,\upsilon_1,\upsilon_2)(x)\\
&=\frac{\Lambda^{\rho,\beta}_{x,\kappa}(\theta,\eta)}{4}\\
&\times\frac{\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{(\upsilon_1+\upsilon_2)f\right\}(x)\right)^2}{{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}\left\{\upsilon_1\upsilon_2\right\}(x)}\\
&-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f\right)(x)\left({}^{\rho}I^{\theta,\beta}_{\eta,\kappa}f\right)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
&G_2(f,\omega_1,\omega_2)(x)\\
&=\frac{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)}{4}\\
&\times\frac{\left({}^{\rho}I^{\theta,\beta}_{\eta,\kappa}\left\{(\omega_1+\omega_2)f\right\}(x)\right)^2}{{}^{\rho}I^{\theta,\beta}_{\eta,\kappa}\left\{\omega_1\omega_2\right\}(x)}\\
&-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f\right)(x)\left({}^{\rho}I^{\theta,\beta}_{\eta,\kappa}f\right)(x).
\end{aligned}
$$

Proof. Let f and g be two positive integrable functions on $[0,\infty)$. For $\tau,\xi \in (0,x)$ with $x > 0$, we define $H(\tau,\xi)$ as

$$
H(\tau,\xi) = (f(\tau)-f(\xi))\,(g(\tau)-g(\xi)),
$$

equivalently,

$$
\begin{aligned}
&H(\tau,\xi) \qquad (23)\\
&= f(\tau)g(\tau)+f(\xi)g(\xi)-f(\tau)g(\xi)-f(\xi)g(\tau).
\end{aligned}
$$

Multiplying both sides of (23) by

$$
\frac{\rho^{2(1-\beta)}x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}
$$

and double integrating the resulting inequality with respect to τ and ξ over $(0,x)^2$, we get

$$\begin{aligned}
&\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\\
&\times\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}H(\tau,\xi)d\tau d\xi\\
&=\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}fg\right)(x)\\
&+\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}fg\right)(x)\\
&-\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}g\right)(x)\\
&-\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f\right)(x)\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}g\right)(x).
\end{aligned}\tag{24}$$

Applying the Cauchy-Schwarz inequality for double integrals, we can write

$$\begin{aligned}
&\Bigg|\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\\
&\times\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}H(\tau,\xi)d\tau d\xi\Bigg|\\
&\leq\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\\
&\times\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}f^2(\tau)d\tau d\xi+\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\\
&\times\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}f^2(\xi)d\tau d\xi\\
&-2\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\\
&\times\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}f(\tau)f(\xi)d\tau d\xi\Bigg]^{1/2}\\
&\times\Bigg[\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\\
&\times\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}g^2(\tau)d\tau d\xi+\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\\
&\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{\theta}}g^2(\xi)d\tau d\xi\\
&-2\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\\
&\times\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}g(\tau)g(\xi)d\tau d\xi\Bigg]^{1/2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Bigg|\frac{\rho^2(1-\beta)x^{2\kappa}}{\Gamma(\alpha)\Gamma(\theta)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\\
&\times\frac{\xi^{\rho(\eta+1)-1}}{(x^\rho-\xi^\rho)^{1-\theta}}H(\tau,\xi)d\tau d\xi\Bigg|\\
&\leq\Bigg[\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f^2\right)(x)\\
&+\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f^2\right)(x)\\
&-2\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f\right)(x)\Bigg]^{1/2}\\
&\times\Bigg[\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}g^2\right)(x)\\
&+\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}g^2\right)(x)\\
&-2\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}g\right)(x)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}g\right)(x)\Bigg]^{1/2}.
\end{aligned}\tag{25}$$

Applying Lemma 1 with $\omega_1(\tau)=\omega_2(\tau)=g(\tau)=1$, we get

$$\begin{aligned}
&\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f^2\right)(x)\\
&\leq\frac{\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)}{4}\frac{\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}\{(\upsilon_1+\upsilon_2)f\}(x)\right)^2}{{}^\rho I_{\eta,\kappa}^{\alpha,\beta}\{\upsilon_1\upsilon_2\}(x)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f^2\right)(x)\\
&-\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f\right)(x)\\
&\leq\frac{\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)}{4}\frac{\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}\{(\upsilon_1+\upsilon_2)f\}(x)\right)^2}{{}^\rho I_{\eta,\kappa}^{\alpha,\beta}\{\upsilon_1\upsilon_2\}(x)}\\
&-\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f\right)(x)\\
&=G_1(f,\upsilon_1,\upsilon_2)(x).
\end{aligned}\tag{26}$$

and

$$\begin{aligned}
&\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f^2\right)(x)\\
&-\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f\right)(x)\\
&\leq\frac{\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)}{4}\frac{\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}\{(\omega_1+\omega_2)f\}(x)\right)^2}{{}^\rho I_{\eta,\kappa}^{\theta,\beta}\{\omega_1\omega_2\}(x)}\\
&-\left({}^\rho I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^\rho I_{\eta,\kappa}^{\theta,\beta}f\right)(x)\\
&=G_2(f,\omega_1,\omega_2)(x).
\end{aligned}\tag{27}$$

Similarly, applying Lemma 1 with $\upsilon_1(\tau)=\upsilon_2(\tau)=f(\tau)=1$, we have

$$\Lambda_{x,\kappa}^{\rho,\beta}(\theta,\eta)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g^{2}\right)(x) - \left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g\right)(x)\left({}^{\rho}I_{\eta,\kappa}^{\theta,\beta}g\right)(x) \leq G_{1}(g,\omega_{1},\omega_{2})(x) \tag{28}$$

and

$$\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\left({}^{\rho}I_{\eta,\kappa}^{\theta,\beta}g^{2}\right)(x) - \left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g\right)(x)\left({}^{\rho}I_{\eta,\kappa}^{\theta,\beta}g\right)(x) \leq G_{2}(g,\omega_{1},\omega_{2})(x). \tag{29}$$

Using (26)-(29), we conclude the desired result. □

Remark 5. *Setting $\kappa=0$, $\eta=0$ and $\rho=1$ in Theorem 2, yields the inequality in [16, Theorem 3.6].*

Theorem 3. *Assume that all conditions of Theorem (2) are fulfilled. Then, we have:*

$$\left|\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg\right)(x) - \left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g\right)(x)\right| \leq \left|G(f,\upsilon_{1},\upsilon_{2})(x)G(g,\omega_{1},\omega_{2})(x)\right|^{1/2} \tag{30}$$

where

$$G(u,v,w)(x) = \frac{\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)}{4} \times \frac{\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\{(v+w)u\}(x)\right)^{2}}{{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\{vw\}(x)} - \left(\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u\right)(x)\right)^{2}.$$

Proof. Setting $\alpha=\theta$ in (22), we obtain (30). □

Corollary 4. *If $\upsilon_1=m$, $\upsilon_2=M$, $\omega_1=n$ and $\omega_2=N$, then we have*

$$G(f,m,M)(x) = \frac{(M-m)^{2}}{4Mm}\left(\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\right)^{2},$$

$$G(g,n,N)(x) = \frac{(N-n)^{2}}{4Nn}\left(\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g\right)(x)\right)^{2}.$$

Remark 6. We consider some particular cases of the result in Theorem 3.

(i) Setting $\kappa=0$, $\eta=0$ and $\rho=1$ in the result in Theorem 3 yields the inequality in [16, Theorem 3.7],

(ii) Setting $\beta=0$ and $\kappa=-\rho(\alpha+\eta)$ in the result in inequality 30 yields to

$$\left|\frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}\left(I_{0+,\rho,\eta}^{\alpha}fg\right)(x) - \left(I_{0+,\rho,\eta}^{\alpha}f\right)(x)\left(I_{0+,\rho,\eta}^{\alpha}g\right)(x)\right| \leq \left|G(f,\upsilon_{1},\upsilon_{2})(x)G(g,\omega_{1},\omega_{2})(x)\right|^{1/2}$$

where

$$G(u,v,w)(x) = \frac{\Gamma(\eta+1)}{4\Gamma(\alpha+\eta+1)} \times \frac{\left(I_{0+,\rho,\eta}^{\alpha}\{(v+w)u\}(x)\right)^{2}}{I_{0+,\rho,\eta}^{\alpha}\{vw\}(x)} - \left(\left(I_{0+,\rho,\eta}^{\alpha}u\right)(x)\right)^{2}$$

(iii) Setting $\beta=\alpha$, $\kappa=0$ and $\eta=0$ in the result in Theorem 3, under the corresponding reduced assumption, we obtain

$$\left|\frac{x^{\rho\alpha}}{\Gamma(\alpha+1)}\left({}^{\rho}I_{0+}^{\alpha}fg\right)(x) - \left({}^{\rho}I_{0+}^{\alpha}f\right)(x)\left({}^{\rho}I_{0+}^{\alpha}g\right)(x)\right| \leq \left|G(f,\upsilon_{1},\upsilon_{2})(x)G(g,\omega_{1},\omega_{2})(x)\right|^{1/2}$$

where

$$G(u,v,w)(x) = \frac{x^{\rho\alpha}}{4\Gamma(\alpha+1)} \times \frac{\left({}^{\rho}I_{0+}^{\alpha}\{(v+w)u\}(x)\right)^{2}}{{}^{\rho}I_{0+}^{\alpha}\{vw\}(x)} - \left(\left({}^{\rho}I_{0+}^{\alpha}u\right)(x)\right)^{2}.$$

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