RESEARCH ARTICLE

Novel solution methods for initial boundary value problems of fractional order with conformable differentiation

Mehmet Yavuz[*]

Department of Mathematics-Computer Sciences, Necmettin Erbakan University, Turkey
mehmetyavuz@konya.edu.tr

ARTICLE INFO	ABSTRACT
	In this work, we develop a formulation for the approximate-analytical solution of fractional partial differential equations (PDEs) by using conformable fractional derivative. Firstly, we redefine the conformable fractional Adomian decomposition method (CFADM) and conformable fractional modified homotopy perturbation method (CFMHPM). Then, we solve some initial boundary value problems (IBVP) by using the proposed methods, which can analytically solve the fractional partial differential equations (FPDE). In order to show the efficiencies of these methods, we have compared the numerical and exact solutions of the IBVP. Also, we have found out that the proposed models are very efficient and powerful techniques in finding approximate solutions for the IBVP of fractional order in the conformable sense.

1. Introduction

Fractional differential equations have an important role in modelling and describing certain problems such as diffusion processes, chemistry, engineering, economic, material sciences and other areas of application. Zhang [1] used a finite difference method for the fractional PDEs. Ibrahim [2] interpreted holomorphic solutions for nonlinear singular fractional differential equations. Odibat and Momani [3, 4] applied several different types of methods to fractional PDEs and compared the results they obtained.

On the other hand, several researchers [5-17] have applied the homotopy perturbation/analysis methods (HPM/HAM) and Adomian decomposition method (ADM) to solve different kinds of fractional ordinary differential equations (ODEs), fractional partial differential equations (ODEs), integral equations (IEs) and integro-differential equations (IDEs). Among them Javidi and Ahmad [18] proposed a numerical method which is based on the homotopy perturbation method and Laplace transform for fractional PDEs. In [19], LHPM which is a combination of the HPM and Laplace Transform (LT) has been employed for solving one-dimensional partial differential equations. Recently, [20-22] introduced a new fractional derivative called conformable derivative operator

(CDO) and by the help of this operator, the behaviors of many scientific problems have been solved and some solution methods have been developed. Many researchers [23-27] have studied on CDO in engineering, physical and applied mathematics problems. The aim of this study is to construct CADM and CMHPM by using conformable derivative. Many linear and nonlinear fractional PDEs can be solved with these methods. We have solved two fractional order PDEs with these mentioned methods and compared the numerical and approximate-analytical solutions in term of figures and tables. When looking at the results, it is obvious that these methods are very effective and accurate for solving fractional partial differential equations.

2. Some preliminaries

In this section, we give some basic concepts of conformable fractional derivative and its properties.

Definition 1. *Given a function* $f:[0,\infty)\to\mathbb{R}$. *Then the conformable derivative of* f *order* $\alpha\in(0,1]$ *is defined by [20]:*

$$CD_{*\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon} \qquad (1)$$

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[*]Corresponding author

for all $t > 0$.

Theorem 1. [20] *Let* $\alpha \in (0,1]$ *and* f, g *be* $\alpha-$ *differentiable at a point* $t > 0$. *Then;*

(i) $CD_{*\alpha}(af + bg) = aCD_{*\alpha}(f) + bCD_{*\alpha}(g)$ *for all* $a, b \in \square$,

(ii) $CD_{*\alpha}(t^k) = kt^{k-\alpha}$ *for all* $k \in \square$,

(iii) $CD_{*\alpha}(f(t)) = 0$ *for all constant functions* $f(t) = k$,

(iv) $CD_{*\alpha}(fg) = fCD_{*\alpha}(g) + gCD_{*\alpha}(f)$,

(v) $CD_{*\alpha}(f/g) = \dfrac{gCD_{*\alpha}(f) - fCD_{*\alpha}(g)}{g^2}$,

(vi) If $f(t)$ *is differentiable, then*
$$CD_{*\alpha}(f(t)) = t^{1-\alpha}\frac{d}{dt}f(t).$$

Definition 2. [20, 27] *Let* f *be an* $n-times$ *differentiable at* t. *Then the conformable derivative of* f *order* α *is defined as:*
$$CD_{*\alpha}(f(t)) = \lim_{\varepsilon \to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t + \varepsilon t^{(\lceil\alpha\rceil-\alpha)}\right) - f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}$$
for all $t > 0$, $\alpha \in (n, n+1]$. *Here* $\lceil\alpha\rceil$ *is the smallest integer greater than or equal to* α.

Lemma 1. [20, 27] *Let* f *be an* $n-times$ *differentiable at* t. *Then*
$$CD_{*\alpha}(f(t)) = t^{\lceil\alpha\rceil-\alpha}f^{\lceil\alpha\rceil}(t)$$
for all $t > 0$, $\alpha \in (n, n+1]$.

3. Conformable fractional adomian decomposition method

Consider the following nonlinear fractional partial differential equation:
$$L_{*\alpha}(u(x,t)) + R(u(x,t)) + N(u(x,t)) = v(x,t) \tag{2}$$

where $L_{*\alpha} = CD_{*\alpha}$ is a linear operator with conformable derivative of order α $(n < \alpha \leq n+1)$, R is the other part of the linear operator, N is a nonlinear operator and $v(x,t)$ is a non-homogeneous term. In Eq. (2), if we apply the linear operator to Lemma 1, we obtain the following equation [28]:
$$t^{\lceil\alpha\rceil-\alpha}\frac{\partial^{\lceil\alpha\rceil}u(x,t)}{\partial t^{\lceil\alpha\rceil}} + R(u(x,t)) + N(u(x,t)) = v(x,t).$$

Applying the inverse of linear operator $L_{*\alpha}^{-1} = \int_0^t\int_0^{\gamma_1}\cdots\int_n^{\gamma_{n-1}}\frac{1}{\gamma_n^{\lceil\alpha\rceil-\alpha}}(.)d\gamma_n d\gamma_{n-1}\cdots d\gamma_1$, to both sides of Eq. (2), we obtain

$$L_{*\alpha}^{-1}L_{*\alpha}(u(x,t)) + L_{*\alpha}^{-1}R(u(x,t)) + L_{*\alpha}^{-1}N(u(x,t)) = L_{*\alpha}^{-1}v(x,t). \tag{3}$$

The conformable ADM suggests the solution $u(x,t)$ be decomposed into the infinite series of components
$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t). \tag{4}$$

The nonlinear function in Eq. (2) is decomposed as follows:
$$N(u) = \sum_{n=0}^{\infty}A_n, \quad (u_0, u_1, \cdots, u_n), \tag{5}$$

where A_n is the so-called Adomian polynomials. These polynomials can be calculated for all forms of nonlinearity with respect to the algorithms developed by Adomian [29].

Substituting (4) and (5) into (3), we obtain
$$\sum_{n=0}^{\infty}u_n = u(x,0) + L_{*\alpha}^{-1}v - L_{*\alpha}^{-1}R\left(\sum_{n=0}^{\infty}u_n\right) - L_{*\alpha}^{-1}\left(\sum_{n=0}^{\infty}A_n\right). \tag{6}$$

By using Eq. (6), the iteration terms are obtained by the following way:
$$\begin{aligned}u_0 &= u(x,0) + L_{*\alpha}^{-1}v, \\ u_1 &= -L_{*\alpha}^{-1}Ru_0 - L_{*\alpha}^{-1}A_0, \\ &\vdots \\ u_{n+1} &= -L_{*\alpha}^{-1}Ru_n - L_{*\alpha}^{-1}A_n, \ n \geq 0.\end{aligned} \tag{7}$$

Then, the approximate-analytical solution of Eq. (2) is obtained by
$$\tilde{u}_k(x,t) = \sum_{n=0}^{k}u_n(x,t).$$

Finally, we obtain the exact solution of Eq. (2) as
$$u(x,t) = \lim_{k \to \infty}\tilde{u}_k(x,t).$$

4. Conformable fractional modified homotopy perturbation method

In this section, some basic solution steps and properties of modified homotopy perturbation method are given in the conformable sense (CMHPM) definition. We introduce a solution algorithm in an effective way for the nonlinear PDEs of fractional order. Firstly, we consider the following nonlinear fractional equation:
$$CD_{*t}^{\alpha}u(x,t) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + v(x,t), \tag{8}$$

where $t > 0$, L is a linear operator, N is a nonlinear operator, v is a known analytical function and CD_{*t}^{α}, $m-1 < \alpha \leq m$, is the Conformable fractional derivative of order α, subject to the initial conditions
$$u^k(x,0) = v_k(x), \ k = 0,1,\ldots,m-1.$$

According to the homotopy technique, we can construct the following homotopy:

$$\frac{\partial^m u}{\partial t^m} - L(u, u_x, u_{xx}) - v(x,t)$$
$$= p\left(\frac{\partial^m u}{\partial t^m} + N(u, u_x, u_{xx}) - CD_{*t}^\alpha u\right), \quad (9)$$

or evenly,

$$\frac{\partial^m u}{\partial t^m} - v(x,t)$$
$$= p\left(\frac{\partial^m u}{\partial t^m} + L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) - CD_{*t}^\alpha u\right), \quad (10)$$

where $p \in [0,1]$. Here, the homotopy parameter p always changes from zero to unity. In case $p = 0$, Eq. (9) becomes the linearized equation

$$\frac{\partial^m u}{\partial t^m} = L(u, u_x, u_{xx}) + v(x,t),$$

and Eq. (10) becomes the linearized equation

$$\frac{\partial^m u}{\partial t^m} = v(x,t).$$

If we take the homotopy parameter $p = 1$, Eq. (9) or Eq. (10) turns out to be the original differential equation of fractional order (8). As the basic assumption is that the solution of Eq. (10) can be written by using a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots.$$

At the end of the solution steps, we approximate the solution as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

5. Numerical examples

In this section of the study, we show the effectiveness and appropriateness of the CADM and CMHPM by applying them to two different problems.

Example 1. We consider the linear time-fractional initial boundary value problem [30]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \quad (11)$$

with the initial condition

$$u(x,0) = x \quad (12)$$

and the boundary conditions

$$u_x(x,0) = 1, \quad u(0,t) = 0. \quad (13)$$

Firstly, we will solve this problem by using the proposed conformable Adomian decomposition method of fractional order. Let $L_{*\alpha} = CD_{*\alpha} = \dfrac{\partial^\alpha}{\partial t^\alpha}$ be a linear operator, then if we apply the operator to Eq. (11) we have

$$CD_{*\alpha} u(x,t) = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u. \quad (14)$$

By using the Lemma 1, we can write the Eq. (14) as

$$t^{1-\alpha}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u. \quad (15)$$

Now, we apply the inverse of operator $L_{*\alpha}$ which is $L_{*\alpha}^{-1} = \displaystyle\int_0^t \frac{1}{\zeta^{1-\alpha}}(.)d\zeta$ to both sides of Eq. (15), we get

$$u(x,t) = u(x,0) + L_{*\alpha}^{-1}\left(\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u\right).$$

According to the iteration terms (7) and the initial condition (12), we can write the iterations and the decomposition series terms as:

$$u_0 = u(x,0) = x,$$
$$u_1 = L_{*\alpha}^{-1}\left(\frac{\partial^2 u_0}{\partial x^2} + x\frac{\partial u_0}{\partial x} + u_0\right) = 2x\frac{t^\alpha}{\alpha},$$
$$u_2 = L_{*\alpha}^{-1}\left(\frac{\partial^2 u_1}{\partial x^2} + x\frac{\partial u_1}{\partial x} + u_1\right) = 4x\frac{t^{2\alpha}}{2!\alpha^2},$$
$$u_3 = L_{*\alpha}^{-1}\left(\frac{\partial^2 u_2}{\partial x^2} + x\frac{\partial u_2}{\partial x} + u_2\right) = 8x\frac{t^{3\alpha}}{3!\alpha^3}, \quad (16)$$
$$\vdots$$
$$u_n = L_{*\alpha}^{-1}\left(\frac{1}{2}x^2\frac{\partial^2 u_{n-1}}{\partial x^2}\right) = 2^n x\frac{t^{n\alpha}}{n!\alpha^n}.$$

So, by using the decomposition series in Eq. (16), the approximate solution of Eq. (11) obtained by Adomian decomposition method in conformable sense is

$$\tilde{u}_k(x,t) = \sum_{n=0}^{k} u_n(x,t) = \sum_{n=0}^{k} 2^n x\frac{t^{n\alpha}}{n!\alpha^n}.$$

From the last equation we obtain the approximate analytical solution of the problem as

$$u(x,t) = \lim_{k\to\infty}\tilde{u}_k(x,t) = xe^{\frac{(2t)^\alpha}{\alpha}}.$$

Then the exact solution of the Eq. (11) subject to the initial condition (12) and the boundary conditions (13) for special case of $\alpha = 1$, is obtained as

$$u(x,t) = xe^{2t}.$$

Secondly, we solve the Eq. (11) by using the modified homotopy perturbation method in conformable sense. If we consider the initial condition (12) and according to the homotopy (9), we can obtain the following set of linear partial differential equations:

$$\frac{\partial u_0}{\partial t} = 0, \quad u_0(x,0) = x,$$
$$\frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{\partial^2 u_0}{\partial x^2} + x\frac{\partial u_0}{\partial x} + u_0 - CD_{*\alpha}u_0, \quad u_1(x,0) = 0,$$
$$\frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} + \frac{\partial^2 u_1}{\partial x^2} + x\frac{\partial u_1}{\partial x} + u_1 - CD_{*\alpha}u_1, \quad u_2(x,0) = 0,$$
$$\vdots$$
$$(17)$$

By solving the Eq. (17) according to u_0, u_1, u_2 and u_3, the first several components of the modified homotopy perturbation solution for Eq. (11) are

derived as follows:

$$u_0(x,t) = x,$$

$$u_1(x,t) = 2xt,$$

$$u_2(x,t) = x\left(2t + 2t^2 - \frac{2t^{2-\alpha}}{2-\alpha}\right),$$

$$u_3(x,t) = x\left(2t + 4t^2 + \frac{4t^3}{3} - \frac{4t^{2-\alpha}}{2-\alpha}\right)$$
$$- x\left(\frac{4t^{3-\alpha}}{(2-\alpha)(3-\alpha)} + \frac{4t^{3-\alpha}}{3-\alpha} - \frac{2t^{3-2\alpha}}{3-2\alpha}\right),$$

$$\vdots$$

and so on, in this way the rest of components of the homotopy can be obtained. Then the approximate solution of Eq. (11) is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$
$$= x\left(1 + 6t + 6t^2 + \frac{4t^3}{3} - \frac{6t^{2-\alpha}}{2-\alpha} - \frac{4t^{3-\alpha}}{(2-\alpha)(3-\alpha)}\right)$$
$$+ x\left(-\frac{4t^{3-\alpha}}{3-\alpha} + \frac{2t^{3-2\alpha}}{3-2\alpha} + \cdots\right)$$

Then the exact solution of the Eq. (11) subject to the initial condition (12) and the boundary conditions (13) for special case of $\alpha = 1$, is obtained with CMHPM as

$$u(x,t) = xe^{2t}.$$

The following Figure 1 shows CMHPM, CADM and exact solutions for various values of α. According to the Figure 1, it can be say that the numerical results found are very close to the exact solution results.
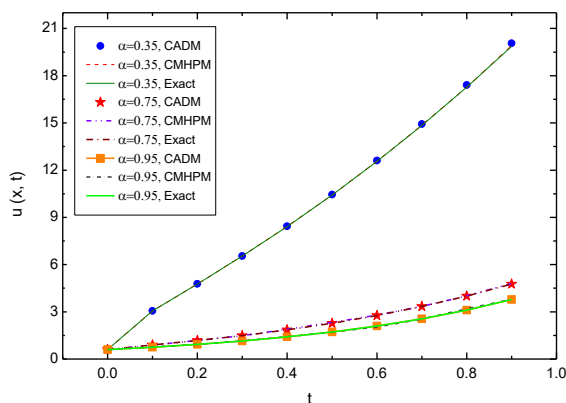

Figure 1. Comparison the numerical solutions and the exact solutions at $x = 0.6$ for various values of α.

Example 2. Now let us consider the following time-fractional diffusion equation [31]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \quad (18)$$

with the initial condition

$$u(x,0) = \sin x. \quad (19)$$

subject to the boundary conditions

$$u_x(x,0) = \cos x, \quad u(0,t) = 0. \quad (20)$$

Solve the problem by using CADM. Let us apply the linear operator to Eq. (18), then we obtain

$$CD_{*\alpha}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \quad (21)$$

Also, we can write the Eq. (21) as

$$t^{1-\alpha}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \quad (22)$$

Applying the inverse of operator $L_{*\alpha}$ to both sides of Eq. (22), we have

$$u(x,t) = u(x,0) + L_{*\alpha}^{-1}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right).$$

Using Eq. (7) and the initial condition (19), we can obtain the iterations in conformable sense as:

$$u_0 = u(x,0) = \sin x,$$

$$u_1 = L_{*\alpha}^{-1}\left(\frac{\partial^2 u_0}{\partial x^2}\right) = -\sin x \frac{t^\alpha}{\alpha},$$

$$u_2 = L_{*\alpha}^{-1}\left(\frac{\partial^2 u_1}{\partial x^2}\right) = \sin x \frac{t^{2\alpha}}{2!\alpha^2},$$

$$u_3 = L_{*\alpha}^{-1}\left(\frac{\partial^2 u_2}{\partial x^2}\right) = -\sin x \frac{t^{3\alpha}}{3!\alpha^3}, \quad (23)$$

$$\vdots$$

$$u_n = L_{*\alpha}^{-1}\left(\frac{\partial^2 u_{n-1}}{\partial x^2}\right) = \sin x (-1)^n \frac{t^{n\alpha}}{n!\alpha^n}.$$

Then, by using the obtained values in Eq. (23) the approximate solution of Eq. (18) is obtained as

$$\tilde{u}_k(x,t) = \sum_{n=0}^{k} u_n(x,t) = \sum_{n=0}^{k} \sin x (-1)^n \frac{t^{n\alpha}}{n!\alpha^n}.$$

Using the last equation we obtain the approximate analytical solution of the proposed problem

$$u(x,t) = \lim_{k \to \infty} \tilde{u}_k(x,t) = \sin x \, e^{-\frac{t^\alpha}{\alpha}}.$$

The exact solution of the Eq. (18) with the initial condition (19) for special case of $\alpha = 1$, is found as

$$u(x,t) = \sin x \, e^{-t}$$

which is the same solution with [31]. Now, let us consider the solution of problem (18) with CMHPM. In order to obtain the solution, we use the homotopy and following set of linear partial differential equations:

$$\frac{\partial u_0}{\partial t} = 0, \ u_0(x,0) = \sin x,$$

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{\partial^2 u_0}{\partial x^2} - CD_{*\alpha}u_0, \ u_1(x,0) = 0, \quad (24)$$

$$\frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} + \frac{\partial^2 u_1}{\partial x^2} - CD_{*\alpha}u_1, \ u_2(x,0) = 0,$$

$$\vdots$$

By solving Eq. (24) according to u_0, u_1 and u_2, the first three components of the modified homotopy perturbation solution for Eq. (18) are obtained as follows:

$$u_0(x,t) = \sin x,$$

$$u_1(x,t) = -t \sin x,$$

$$u_2(x,t) = \sin x\left(-t + \frac{t^2}{2} + \frac{t^{2-\alpha}}{2-\alpha}\right),$$

$$u_3(x,t) = \sin x\left(-t + t^2 - \frac{t^3}{6} + \frac{2t^{2-\alpha}}{2-\alpha}\right),$$

$$-\sin x\left(\frac{t^{3-\alpha}}{(2-\alpha)(3-\alpha)} + \frac{t^{3-\alpha}}{3-\alpha} + \frac{t^{3-2\alpha}}{3-2\alpha}\right)$$

$$\vdots$$

and so on, in this manner the rest of components of the homotopy can be obtained. The approximate solution of problem (18) is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= \sin x\left(1 - 3t + \frac{3t^2}{2} - \frac{t^3}{6} + \frac{3t^{2-\alpha}}{2-\alpha}\right)$$

$$-\sin x\left(\frac{t^{3-\alpha}}{(2-\alpha)(3-\alpha)} + \frac{t^{3-\alpha}}{3-\alpha} + \frac{t^{3-2\alpha}}{3-2\alpha}\right)$$

Then, for the special value of $\alpha = 1$, the exact solution of the Eq. (18) subject to the initial condition (19) is obtained with CMHPM as $u(x,t) = \sin x e^{-t}$ which is the same solution with obtained CADM one.

Table 1. Absolute errors $|\tilde{u}_k(x,t) - u(x,t)|$ obtained with CADM for Example 2.

x	α	t			
		0.1	**0.3**	**0.5**	**0.7**
0.1	$\alpha = 0.20$	6.08E-04	1.80E-03	3.45E-04	4.80E-02
	$\alpha = 0.45$	5.24E-03	1.80E-02	5.06E-02	3.56E-02
	$\alpha = 0.80$	6.49E-01	1.80E-02	1.28E-01	5.69E-01
0.4	$\alpha = 0.20$	6.82E-04	9.03E-04	3.33E-05	3.15E-04
	$\alpha = 0.45$	5.42E-04	6.07E-03	8.43E-03	8.62E-04
	$\alpha = 0.80$	3.96E-02	5.80E-03	1.45E-02	5.69E-03
0.7	$\alpha = 0.20$	5.39E-03	6.03E-04	1.75E-05	9.03E-05
	$\alpha = 0.45$	4.44E-02	5.24E-02	3.08E-02	5.56E-03
	$\alpha = 0.80$	8.43E-02	3.94E-01	7.80E-02	3.78E-02
1.0	$\alpha = 0.20$	4.32E-05	4.80E-03	3.35E-04	3.07E-03
	$\alpha = 0.45$	3.74E-04	6.92E-02	5.42E-02	9.10E-02
	$\alpha = 0.80$	6.20E-02	3.42E-01	5.06E-02	6.05E-02

According to Table 1, we can say about the solution of Eq. (18) that the absolute error values are very small for various values α, x and t.

In addition, in the following Figure 2 and Figure 3, the graphs of solution functions of Eq. (18) with respect to the CADM and the exact solution for $\alpha = 0.70$ are shown, respectively.
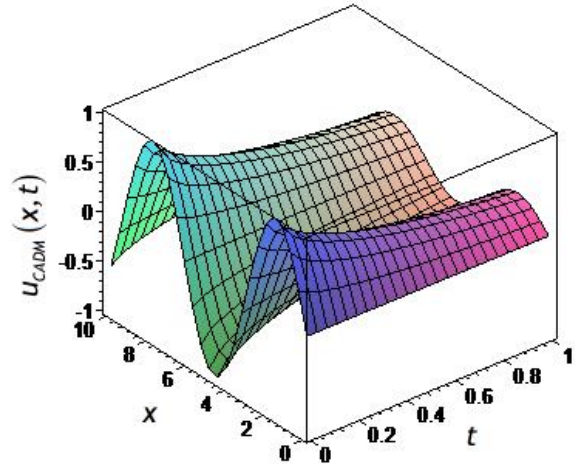

Figure 2. CADM solution with $\alpha = 0.70$ for Example 2.
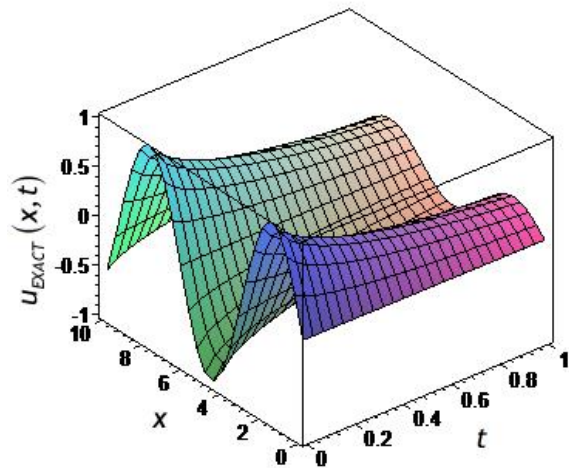

Figure 3. Exact solution with $\alpha = 0.70$ for Example 2.

In the following Figure 4 and Figure5, the sketches of solution functions of Eq. (18) with respect to the CMHPM and the exact solution for $\alpha = 0.30$ are shown, respectively.

According to the Figure 2, Figure 3, Figure 4 and Figure 5, we can say that the numerical results obtained from CADM and CMHPM are very close to the exact solution values.
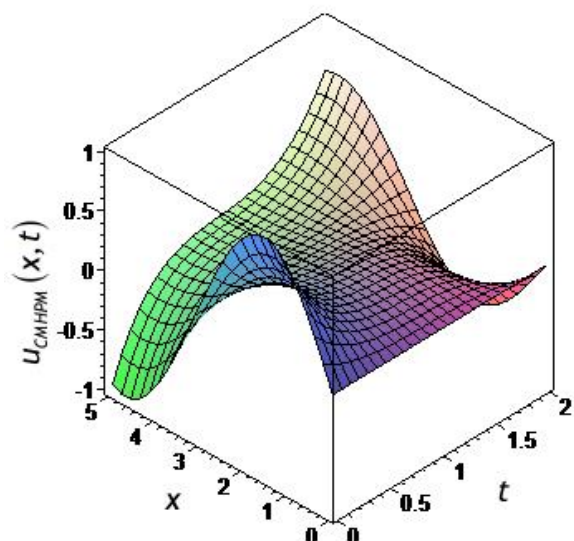
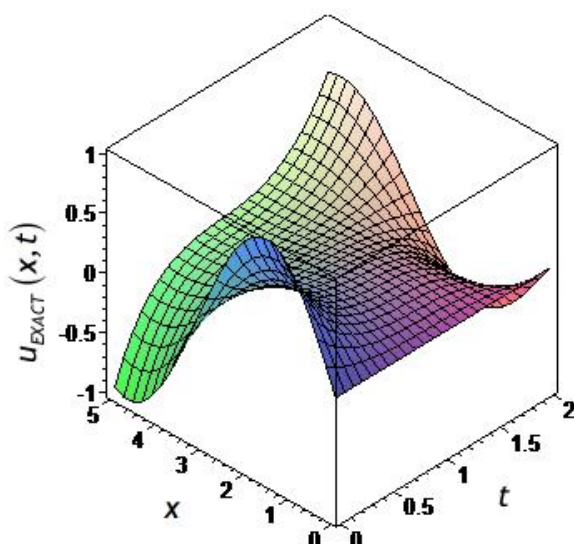**Figure 4.** CMHPM solution with $\alpha = 0.30$ for Example 2.

Figure 5. Exact solution with $\alpha = 0.30$ for Example 2.

6. Conclusion

We have found out approximate solutions with two numerical methods for time-fractional linear partial differential equations. These methods are based on conformable derivative (CD) which is extremely popular in the last years. In this study, firstly, by using the CD, we have redefined ADM and MHPM. Then we have demonstrated the efficiencies and accuracies of the proposed methods by applying them to two different problems. It is found that the approximate solutions generated by our methods are in complete agreement with the corresponding exact solutions. Besides, in view of their usability, our methods are applicable to many initial-boundary value problems and linear-nonlinear partial differential equations of fractional order.

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Mehmet Yavuz is a lecturer at the Department of Mathematics-Computer Sciences, Faculty of Science, Necmettin Erbakan University, Turkey. He received his B.Sc. (2009) degree from Department of Mathematics, Bulent Ecevit University, Turkey. He received his M.Sc. (2012) and Ph.D. (2016) degrees from Department of Mathematics, Balikesir University, Turkey. His research areas include fractional calculus, differential equations and financial mathematics.