RESEARCH ARTICLE

The structure of one weight linear and cyclic codes over $\mathbb{Z}_2^r \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^s$

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ARTICLE INFO	ABSTRACT
	Inspired by the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, linear codes over $\mathbb{Z}_2^r \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^s$ have been introduced by Aydogdu et al. more recently. Although these family of codes are similar to each other, linear codes over $\mathbb{Z}_2^r \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^s$ have some advantages compared to $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. A code is called constant weight (one weight) if all the nonzero codewords have the same weight. It is well known that constant weight or one weight codes have many important applications. In this paper, we study the structure of one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear and cyclic codes. We classify one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes and also give some illustrative examples. (cc) BY

1. Introduction

In algebraic coding theory, the most important class of codes is the family of linear codes. A linear code of length n is a subspace $\mathcal{C}$ of a vector space F_q^n where F_q is a finite field of size q. When $q = 2$ then we have linear codes over F_2 which are called binary codes. Binary linear codes have very special and important place all among the finite field codes because of their easy implementations and applications. Beginning with a remarkable paper by Hammons et al. [1], interest of codes over variety of rings have been increased. Such studies motivate the researchers to work on different rings even over other structural algebras such as groups or modules. A $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$ is called a quaternary linear code. The structure of binary linear codes and quaternary linear codes have been studied in details for the last two decades. The reader can see some of them in [2–4]. In 2010, Borges et al. introduced a new class of error correcting codes over the ring $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ called additive codes that generalizes the class of binary linear codes and the class of quaternary linear codes in [5]. A $\mathbb{Z}_2\mathbb{Z}_4$-additive code

$\mathcal{C}$ is defined to be a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ where $\alpha + 2\beta = n$. If $\beta = 0$ then $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are just binary linear codes, and if $\alpha = 0$, then $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are the quaternary linear codes over $\mathbb{Z}_4$. $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have been generalized to $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes in 2013 by Aydogdu and Siap in [6], and recently this generalization has been extended to $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes, for a prime p, by the same authors in [7]. Later, cyclic codes over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ have been introduced in [8] in 2014 and more recently, in [9], one weight codes over such a mixed alphabet have been studied. A code $\mathcal{C}$ is said to be one weight code if all the nonzero codewords in $\mathcal{C}$ have the same Hamming weight where the Hamming weight of any string is the number of symbols that are different from the zero symbol of the alphabet used. In [10], Carlet determined one weight linear codes over $\mathbb{Z}_4$ and in [11], Wood studied linear one weight codes over $\mathbb{Z}_m$. Constant weight codes are very useful in a variety of applications such as data storage, fault-tolerant circuit design and computing, pattern generation for circuit testing, identification coding, and optical overlay networks [12].

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Moreover, the reader can find the other applications of constant weight codes; determining the zero error decision feedback capacity of discrete memoryless channels in [13], multiple access communications and spherical codes for modulation in [14, 15], DNA codes in [16, 17], powerline communications and frequency hopping in [18].

Another important ring of four elements other than the ring $\mathbb{Z}_4$, is the ring $\mathbb{Z}_2 + u\mathbb{Z}_2 = R = \{0, 1, u, 1 + u\}$ where $u^2 = 0$. For some of the works done in this direction we refer the reader to [19–21]. It has been shown that linear and cyclic codes over this ring have advantages compared to the ring $\mathbb{Z}_4$. For an example; the finite field $GF(2)$ is a subring of the ring R. So factorization over $GF(2)$ is still valid over the ring R. The Gray image of any linear code over R is always a binary linear code which is not always the case for $\mathbb{Z}_4$.

In this work, we are interested in studying one weight codes over $\mathbb{Z}_2^r \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^s = \mathbb{Z}_2^r \times R^s$. This family of codes are special subsets of $\mathbb{Z}_2^r \times R^s$ which their all nonzero codewords have the same weight. Since the structure of one weight binary linear codes were well classified by Bonisoli [22], we conclude some results that coincides with the results in [22] for $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes, and we classify cyclic codes over $\mathbb{Z}_2^r \times R^s$ and also we give some one weight linear and cyclic code examples. Furthermore, we look at the Gray (binary) images of one weight cyclic codes over $\mathbb{Z}_2^r \times R^s$ and we determine their parameters.

2. Preliminaries

Let $R = \mathbb{Z}_2 + u\mathbb{Z}_2 = \{0, 1, u, 1 + u\}$ be the four-element ring with $u^2 = 0$. It is easily seen that the ring $\mathbb{Z}_2$ is a subring of the ring R. Then let us define the set

$$\mathbb{Z}_2\mathbb{Z}_2[u] = \{(a, b) \mid a \in \mathbb{Z}_2 \text{ and } b \in R\}.$$

But we have a problem here, because the set $\mathbb{Z}_2\mathbb{Z}_2[u]$ is not well-defined with respect to the usual multiplication by $u \in R$. So, we must define a new method of multiplication on $\mathbb{Z}_2\mathbb{Z}_2[u]$ to make this set as an R-module. Now define the mapping

$$\begin{aligned} \eta &: & R \to \mathbb{Z}_2 \\ \eta(p + uq) &= & p. \end{aligned}$$

which means; $\eta(0) = 0$, $\eta(1) = 1$, $\eta(u) = 0$ and $\eta(1 + u) = 1$. It can be easily shown that η is a ring homomorphism. Furthermore, for any

element $e \in R$, we can also define a scalar multiplication on $\mathbb{Z}_2\mathbb{Z}_2[u]$ as follows.

$$e(a, b) = (\eta(e)a, eb).$$

This multiplication can be extended to $\mathbb{Z}_2^r \times R^s$ for $e \in R$ and $v = (a_0, a_1, ..., a_{r-1}, b_0, b_1, ..., b_{s-1}) \in \mathbb{Z}_2^r \times R^s$ as,

$$ev = (\eta(e)a_0, \eta(e)a_1, ..., \eta(e)a_{r-1}, eb_0, eb_1, ..., eb_{s-1}).$$

Lemma 1. $\mathbb{Z}_2^r \times R^s$ *is an $R-$module under the multiplication defined above.*

Definition 1. *A non-empty subset $\mathcal{C}$ of $\mathbb{Z}_2^r \times R^s$ is called a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code if it is an R-submodule of $\mathbb{Z}_2^r \times R^s$.*

Now, take any element $a \in R$, then there exist unique $p, q \in \mathbb{Z}_2$ such that $a = p + uq$. Also note that the ring R is isomorphic to $\mathbb{Z}_2^2$ as an additive group. Therefore, any $\mathbb{Z}_2\mathbb{Z}_2[u]-$linear code $\mathcal{C}$ is isomorphic to an abelian group of the form $\mathbb{Z}_2^{k_0+k_2} \times \mathbb{Z}_2^{2k_1}$, where k_0, k_2 and k_1 are nonnegative integers. Now define the following sets.

$$\mathcal{C}_s^F = \langle \{(a, b) \in \mathbb{Z}_2^r \times R^s \mid b \text{ free over } R^s\} \rangle$$

where if $\langle b \rangle = R^s$ then b is called free over R^s.

$$\begin{aligned} \mathcal{C}_0 &= \langle \{(a, ub) \in \mathbb{Z}_2^r \times R^s \mid a \neq 0\} \rangle \subseteq \mathcal{C} \backslash \mathcal{C}_s^F \\ \mathcal{C}_1 &= \langle \{(a, ub) \in \mathbb{Z}_2^r \times R^s \mid a = 0\} \rangle \subseteq \mathcal{C} \backslash \mathcal{C}_s^F. \end{aligned}$$

Therefore, denote the dimension of $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_s^F$ as k_0, k_2 and k_1 respectively. Under these parameters, we say that such a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$ is of type $(r, s; k_0, k_1, k_2)$. $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes can be considered as binary codes under a special Gray map. For $(x, y) \in \mathbb{Z}_2^r \times R^s$, where $(x, y) = (x_0, x_1, \ldots, x_{r-1}, y_0, y_1, \ldots, y_{s-1})$ and $y_i = p_i + uq_i$ the Gray map is defined as follows.

$$\Phi : \mathbb{Z}_2^r \times R^s \to \mathbb{Z}_2^n$$
$$\Phi(x_0, \ldots x_{r-1}, p_0 + uq_0, \ldots p_{s-1} + uq_{s-1})$$
$$= (x_0, \ldots x_{r-1}, q_0, \ldots, q_{s-1}, p_0 + q_0, \ldots, p_{s-1} + q_{s-1}),$$
$$(1)$$

where $n = r + 2s$.

The Hamming distance between two strings x and y of the same length over a finite alphabet Σ denoted by $d(x, y)$ is defined as the number of positions at which these two strings differ. The Hamming weight of a string x over an alphabet Σ is

defined as the number of its nonzero symbols in the string. More formally, the Hamming weight of a string is $wt(x) = |\{i : x_i \neq 0\}|$. Also note that $wt(x - y) = d(x, y)$.

The minimum distance of a linear code $\mathcal{C}$, denoted by $d(\mathcal{C})$ is defined by

$$d(\mathcal{C}) = min\{d(c_1, c_2) : c_1, \ c_2 \in \mathcal{C}, c_1 \neq c_2\}.$$

The Lee distance for the codes over R is the Lee weight of their differences where the Lee weights of the elements of R are defined as $wt_L(0) = 0$, $wt_L(1) = 1$, $wt_L(u) = 2$ and $wt_L(1 + u) = 1$. The Gray map defined above is a distance preserving map which transforms the Lee distance in $\mathbb{Z}_2^r \times R^s$ to the Hamming distance in $\mathbb{Z}_2^n$. Furthermore, for any $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$, we have that $\Phi(\mathcal{C})$ is a binary linear code as well. This property is not valid for the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. And also, we define

$$wt(v) = wt_H(v_1) + wt_L(v_2),$$

where $v = (v_1, v_2)$, $wt_H(v_1)$ is the Hamming of weight of v_1 and $wt_L(v_2)$ is the Lee weight of v_2. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(r, s; k_0, k_1, k_2)$ then the binary image $C = \Phi(\mathcal{C})$ is a binary linear code of length $n = r + 2s$ and size 2^n. It is also called a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code. Now, let $v = (a_0, \ldots, a_{r-1}, b_0, \ldots, b_{s-1})$, $w = (d_0, \ldots, d_{r-1}, e_0, \ldots, e_{s-1}) \in \mathbb{Z}_2^r \times R^s$ be any two elements. Then we can define the inner product as

$$\langle v, w \rangle = \left(u \sum_{i=0}^{r-1} a_i d_i + \sum_{j=0}^{s-1} b_j e_j \right) \in \mathbb{Z}_2 + u\mathbb{Z}_2.$$

According to this inner product, the dual linear code $\mathcal{C}^\perp$ of a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$ is also defined in a usual way,

$$\mathcal{C}^\perp = \{w \in \mathbb{Z}_2^r \times R^s | \ \langle v, w \rangle = 0 \text{ for all } v \in \mathcal{C}\}.$$

Hence, if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code, then $\mathcal{C}^\perp$ is also a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code.

The standard forms of generator and parity-check matrices of a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$ are given as follows.

Theorem 1. [23] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(r, s; k_0, k_1, k_2)$. Then the standard forms of the generator and the parity-check matrices of $\mathcal{C}$ are:

$$G = \left[\begin{array}{cc|ccc} I_{k_0} & A_1 & 0 & 0 & uT \\ 0 & S & I_{k_1} & A & B_1 + uB_2 \\ 0 & 0 & 0 & uI_{k_2} & uD \end{array} \right]$$

$$H = \left[\begin{array}{cc|ccc} -A_1^t & I_{r-k_0} & -uS^t & 0 & 0 \\ -T^t & 0 & -(B_1 + uB_2)^t + D^t A^t & -D^t & I_{s-k_1-k_2} \\ 0 & 0 & -uA^t & uI_{k_2} & 0 \end{array} \right]$$

where A, A_1, B_1, B_2, D, S and T are matrices over $\mathbb{Z}_2$.

Therefore, we can conclude the following corollary.

Corollary 1. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(r, s; k_0, k_1, k_2)$ then $\mathcal{C}^\perp$ is of type $(r, s; r - k_0, s - k_1 - k_2, k_2)$.

The weight enumerator of any $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$ of type $(r, s; k_0, k_1, k_2)$ is defined as

$$W_\mathcal{C}(x, y) = \sum_{c \in \mathcal{C}} x^{n-wt(c)} y^{wt(c)}$$

where, $n = r + 2s$. Moreover, the MacWilliams relations for codes over $\mathbb{Z}_2\mathbb{Z}_2[u]$ can be given as follows.

Theorem 2. [23] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]-$linear code. The relation between the weight enumerators of $\mathcal{C}$ and its dual is

$$W_{\mathcal{C}^\perp}(x, y) = \frac{1}{|\mathcal{C}|} W_\mathcal{C}(x + y, x - y).$$

We have given some information about the general concept of codes over $\mathbb{Z}_2^r \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^s$. To make reader understanding the paper easily we give the following example.

Example 1. Let $\mathcal{C}$ be a linear code over $\mathbb{Z}_2^3 \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^4$ with the following generator matrix.

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 0 & u & u & u \\ 0 & 1 & 1 & 1 & 1 + u & u & 0 \\ 0 & 1 & 0 & u & u & u & 0 \end{array} \right].$$

We will find the standard form of the generator matrix of $\mathcal{C}$ and then using this standard form, we find the generator matrix of the linear dual code $\mathcal{C}^\perp$ and also we determine the types of both $\mathcal{C}$ and its dual.

Now, applying elementary row operations to above generator matrix, we have the standard form as follows.

$$G = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & u & 0 & u \\ 0 & 1 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 1 & 1 & 1 + u & 0 & 0 \end{array} \right].$$

Since, G is in the standard form we can write this matrix as

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & u & 0 & u \\ 0 & 1 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 1 & 1 & 1+u & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_{k_0} & A_1 & 0 & uT \\ 0 & S & I_{k_1} & B_1 + uB_2 \end{bmatrix}.$$

Hence, with the help of Theorem 1 the parity-check matrix of $\mathcal{C}$ is

$$H = \begin{bmatrix} 0 & 0 & 1 & u & 0 & 0 & 0 \\ 1 & 0 & 0 & 1+u & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

- *$\mathcal{C}$ is of type $(3, 4; 2, 1, 0)$ and has $2^2 4^1 = 16$ codewords.*
- *$\mathcal{C}^\perp$ is of type $(3, 4; 1, 3, 0)$ and has $2^1 4^3 = 128$ codewords.*
- *$\mathcal{C} = \{(0,0,0,|0,0,0,0), (1,0,0,|0,u,0,u), (0,1,0,|0,0,u,0), (0,0,1,|1,\bar{u},0,0), (0,0,0,|u,u,0,0), (0,0,1,|\bar{u},1,0,0), (1,1,0,|0,u,u,u), (1,0,1,|1,1,0,u), (0,1,1,|1,\bar{u},u,0), (1,1,1,|1,1,u,u), (1,0,0,|u,0,0,u), (0,1,0,|u,u,u,0), (1,1,0,|u,0,u,u), (1,0,1,|\bar{u},\bar{u},0,u), (0,1,1,|\bar{u},1,u,0), (1,1,1,|\bar{u},\bar{u},u,u)\}$, where $\bar{u} = 1+u$.*
- *$W_{\mathcal{C}}(x, y) = x^{11} + 3x^8 y^3 + x^7 y^4 + 2x^6 y^5 + 4x^5 y^6 + x^4 y^7 + 2x^3 y^8 + 2x^2 y^9$.*
- *$W_{\mathcal{C}^\perp}(x, y) = \frac{1}{|\mathcal{C}|} W_{\mathcal{C}}(x+y, x-y) = x^{11} + 6x^9 y^2 + 8x^8 y^3 + 16x^7 y^4 + 32x^6 y^5 + 26x^5 y^6 + 24x^4 y^7 + 15x^3 y^8$.*
- *The Gray image $\Phi(\mathcal{C})$ of $\mathcal{C}$ is a $[11, 4, 3]$ binary linear code.*
- *$\Phi(\mathcal{C}^\perp)$ is a $[11, 7, 2]$ binary linear code.*

3. The Structure of One Weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear Codes

In this part of the paper, we study the structure of one weight codes over $\mathbb{Z}_2^r \times R^s$. Since the binary(Gray) images of $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes are always linear, our results about the one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes will coincide with the results of the paper [22]. So, in this section of the paper we will prepare for Section 4 and also we give some fundamental definitions and illustrative examples of one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes.

Definition 2. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code. $\mathcal{C}$ is called a one (constant) weight code if all of its nonzero codewords have the same weight. Furthermore, if such weight is m then $\mathcal{C}$ is called a code with weight m.*

Definition 3. *Let c_1, c_2, e_1, e_2 be any four distinct codewords of a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$. If the distance between c_1 and e_1 is equal to the distance between c_2 and e_2, that is, $d(c_1, e_1) = d(c_2, e_2)$, then $\mathcal{C}$ is said to be equidistant.*

Theorem 3. *[22] Let C be a $[n, k]$ linear code over $\mathbb{F}_q$ with all nonzero codewords of the same weight. Assume that C is nonzero and no column of a generator matrix is identically zero. Then C is equivalent to the λ-fold replication of a simplex (i.e., dual of the Hamming) code.*

Corollary 2. *Let $\mathcal{C}$ be an equidistant $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code with distance m. Then $\mathcal{C}$ is a one weight code with weight m. Moreover, the binary image $\Phi(\mathcal{C})$ of $\mathcal{C}$ is also a one weight code with weight m.*

Example 2. *It is worth to note that the dual of a one weight code is not necessarily a one weight code. Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(2, 2; 0, 1, 0)$ with $\mathcal{C} = \langle(1, 1|1 + u, 1 + u)\rangle$. Then $\mathcal{C} = \{(0,0|0,0), (1,1|1 + u, 1 + u), (1,1|1,1), (0,0|u,u)\}$ and $\mathcal{C}$ is a one weight code with weight $m = 4$. On the other hand, the dual code $\mathcal{C}^\perp$ is generated by $\langle(1, 0|u, 0), (0, 1|u, 0), (0, 0|1, 1)\rangle$ and of type $(2, 2; 2, 1, 0)$. But $d(\mathcal{C}^\perp) = 2$ and $\mathcal{C}^\perp$ is not a one weight code.*

Remark 1. *The dual code for length greater than 3 is never a one weight code.*

Example 3. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code with the standard form of the generator matrix $\begin{bmatrix} 1 & 0 & 1 & 0 & u \\ 0 & 1 & 1 & 1 & 1+u \end{bmatrix}$, then $\mathcal{C}$ is of type $(3, 2; 1, 1, 0)$ and one weight code with weight 4. Furthermore, $\Phi(\mathcal{C})$ is a binary linear code with parameters $[7, 3, 4]$. Here, note that the binary image of $\mathcal{C}$ is the binary simplex code of length 7, which is the dual of the $[7, 4, 3]$ Hamming code.*

Now, we give a theorem which gives a construction of one weight codes over $\mathbb{Z}_2^r \times R^s$.

Corollary 3. *Let $\mathcal{C}$ be a one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(r, s; k_0, k_1, k_2)$ and weight m. Then, a one weight code of type $(\gamma r, \gamma s; k_0, k_1, k_2)$ with weight γm exists, where γ is a positive integer.*

Definition 4. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code. Let $\mathcal{C}_r$ (respectively $\mathcal{C}_s$) be the punctured code of $\mathcal{C}$ by*

deleting the coordinates outside r (respectively s). If $C = C_r \times C_s$ then C is called separable.

Corollary 4. *There do not exist separable one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes.*

Proof. Since $\Phi(C_r \times C_s) = \Phi(C_r) \times \Phi(C_s)$, the proof is obvious. □

Corollary 5. *If C is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(r, s; k_0, k_1, k_2)$ with no all zero columns in the generator matrix of C. Then the sum of the weights of all codewords of C is equal to $\frac{|C|}{2}(r+2s)$.*

Proof. From [22], since the sums of the weights of a binary linear code $[n, k]$ is $n2^{k-1}$, the sum of the all codewords of C is

$$\sum_{c \in C} wt(c) = r\frac{|C|}{2} + s|C| = \frac{|C|}{2}(r + 2s).$$

□

Corollary 6. *Let C be a one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(r, s; k_0, k_1, k_2)$ and weight $\mathtt{m}$. If there is no zero columns in the generator matrix of C, then;*

    i) $\mathtt{m} = \alpha\ 2^{(k_0+2k_1+k_2)-1}$ *where α is a positive integer satisfying $(r + 2s) = \alpha\left(2^{k_0+2k_1+k_2} - 1\right)$. In addition, if $\mathtt{m}$ is an odd integer, then r is also odd and $C = \langle(\underbrace{1 \cdots 1}_{r\ times} | \underbrace{u \cdots u}_{s\ times})\rangle$.*

    ii) $d(C^\perp) \geq 2$. *Also, $d(C^\perp) \geq 3$ if and only if $\alpha = 1$.*

    iii) *for $\alpha = 1$, if $|C| \geq 4$ then $d(C^\perp) = 3$.*

We have known from the above corollary that if C is a one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(r, s; k_0, k_1, k_2)$ and weight $\mathtt{m}$ then there is a positive integer α such that $\mathtt{m} = \alpha\ 2^{(k_0+2k_1+k_2)-1}$, so the minimum distance for a one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code must be even. In the following, we characterize the structure of $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes.

Theorem 4. *Let C be a one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code over $\mathbb{Z}_2^r \times R^s$ with generator matrix G and weight $\mathtt{m}$.*

    i) *If $v = (a|b)$ is an any row of G, where $a = (a_0, \ldots, a_{r-1}) \in \mathbb{Z}_2^r$ and $b = (b_0, \ldots, b_{s-1}) \in R^s$, then the number of units(1 or $1 + u$) in b is either zero or $\frac{\mathtt{m}}{2}$.*

    ii) *If $v = (a|b)$ and $w = (c|d)$ are two distinct rows of G, where b and d are free over R^s, then the coordinate positions where b has units (1 or $1 + u$) are the same that the coordinate positions where d has units.*

    iii) *If $v = (a|b)$ and $w = (c|d)$ are two distinct rows of G, where b and d are free over R^s, then $|\{j : b_j = d_j = 1\ or\ 1 + u\}| = |\{j : b_j = 1, d_j = 1+u\ or\ b_j = 1+u, d_j = 1\}| = \frac{\mathtt{m}}{4}$.*

Proof.   i) The weight of $v = (a|b)$ is $wt(v) = wt_H(a) + wt_L(b) = \mathtt{m}$. Since C is linear $uv = (0|ub)$ is also in C then, if $ub = 0$ then b does not contain units. If $ub \neq 0$, then $wt(v) = \mathtt{m} = 0 + wt_L(ub)$ and therefore, $wt_L(ub) = 2|\{j : b_j = 1\ or\ 1 + u\}| = \mathtt{m}$. Hence, the number of units in b is $\frac{\mathtt{m}}{2}$.

ii) Multiplying v and w by u we have, $uv = (0|ub)$ and $uw = (0|ud)$. If v and w have units in the same coordinate positions then we get $uv + uw = 0$. So, assume that they have some units in different coordinates. Since C is a one weight code with weight $\mathtt{m}$, if $uv + uw \neq 0$ then the number of coordinates where b and d have units in different places must be $\frac{\mathtt{m}}{2}$. To obtain this, the number of coordinates where $\{b_j = 1 = d_j\}$ and $\{b_j = 1 + u = d_j\}$ has to be $\frac{\mathtt{m}}{2}$, and in all other coordinates where $\{b_j = 1\ or\ 1 + u\}$ we need $\{d_j = 0\ or\ u\}$, and also in all other coordinates where $\{b_j = 0\ or\ u\}$ we need $\{d_j = 1\ or\ 1+u\}$. Hence, consider the vector $v + (1 + u)w$. This vector has the same weight as $v + w$ in the first r coordinates but for the last s coordinates, it has u's in the coordinates where $\{b_j = 1 = d_j\}$ and $\{b_j = 1 + u = d_j\}$, so its weight is greater than $\mathtt{m}$. This contradiction gives the result.

iii) Let $x = v + w$ and $y = v + (1+u)w$ be two vectors in C. The binary parts of these two vectors are the same, and for the coordinates over R^s we know from ii) that v and w have units in the same coordinate positions, and for the all other coordinates in R^s, the values of x and y are the same. Therefore, the sum of the weights of the units in v must be same in x and y. So, they also have the same number of coordinates with u. But this is only possible if $|\{j : b_j = d_j = 1\ or\ 1 + u\}| = |\{j : b_j = 1, d_j = 1 + u\ or\ b_j = 1 + u, d_j = 1\}|$. We also know from i) that the number of units in v is $\frac{\mathtt{m}}{2}$, so we have the result.

□

Theorem 5. *Let C be a one weight code of type $(r, s; k_0, k_1, k_2)$. Then $k_1 \leq 1$ and C has the following standard form of the generator matrices. If $k_1 = 0$ then*

$$G = \begin{bmatrix} I_{k_0} & A_1 & 0 & uT \\ 0 & 0 & uI_{k_2} & uD \end{bmatrix}.$$

If $k_1 = 1$ then

$$G = \begin{bmatrix} I_{k_0} & A_1 & 0 & 0 & uT \\ 0 & s & 1 & a & b_1 + ub_2 \\ 0 & 0 & 0 & uI_{k_2} & uD \end{bmatrix}$$

where s, a, b_1, b_2 are vectors over $\mathbb{Z}_2$.

Proof. From Theorem 4 i), we know that any two distinct free vectors have their units in the same coordinate positions. So, if we add the first free row of the generator matrix to the other rows, we have only one free row in the generator matrix. Hence, $k_1 \leq 1$ and considering this and using the standard form of the generator matrix for a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$ given in Theorem 1, we have the result. $\qquad\square$

4. One Weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic Codes

In this section, we study the structure of one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes. At the beginning, we give some fundamental definitions and theorems about $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes. This information about $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes was given in [24], with details.

Definition 5. *An R-submodule $\mathcal{C}$ of $\mathbb{Z}_2^r \times R^s$ is called a $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code if for any codeword $v = (a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}) \in \mathcal{C}$, its cyclic shift*

$$T(v) = (a_{r-1}, a_0, \ldots, a_{r-2}, b_{s-1}, b_0, \ldots, b_{s-2})$$

is also in $\mathcal{C}$.

Any codeword $c = (a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}) \in \mathbb{Z}_2^r \times R^s$ can be identified with a module element such that

$$\begin{aligned} c(x) &= (a_0 + a_1 x + \ldots + a_{r-1} x^{r-1}, b_0 + b_1 x \\ &\quad + \ldots + b_{s-1} x^{s-1}) \\ &= (a(x), b(x)) \end{aligned}$$

in $R_{r,s} = \mathbb{Z}_2[x]/(x^r - 1) \times R[x]/(x^s - 1)$. This identification gives a one-to-one correspondence between elements in $\mathbb{Z}_2^r \times R^s$ and elements in $R_{r,s}$.

Theorem 6. *[24] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code in $R_{r,s}$. Then we can identify $\mathcal{C}$ uniquely as $\mathcal{C} = \langle (f(x), 0), (l(x), g(x) + ua(x)) \rangle$, where $f(x) | (x^r - 1) \pmod 2$, and $a(x) | g(x) | (x^s - 1) \pmod 2$, and $l(x)$ is a binary polynomial satisfying $\deg(l(x)) < \deg(f(x))$,*

$f(x) | \left(\dfrac{x^s - 1}{a(x)} \right) l(x) \pmod 2$ *and* $f(x) \neq$

$\left(\dfrac{x^s - 1}{a(x)} \right) l(x) \pmod 2$.

Considering the theorem above, the type of $\mathcal{C} = \langle (f(x), 0), (l(x), g(x) + ua(x)) \rangle$ can be written in terms of the degrees of the polynomials $f(x)$, $a(x)$ and $g(x)$. Let $t_1 = \deg f(x)$, $t_2 = \deg g(x)$ and $t_3 = \deg a(x)$. Then $\mathcal{C}$ is of type ([24])

$$(r, s; r - t_4, s - t_2, t_2 + t_4 - t_1 - t_3)$$

where $d_1(x) = \gcd\left(f(x), \dfrac{x^s - 1}{g(x)} l(x) \right)$ and $t_4 = \deg d_1(x)$.

Corollary 7. *If $\mathcal{C}$ is a one weight cyclic code generated by $(l(x), g(x) + ua(x)) \in R_{r,s}$ with weight m then $m = 2s$.*

Proof. We know from Theorem 5 that if $\mathcal{C}$ is a one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code then k_1, which generates the free part of the code, is less than or equal to 1. So, in the case where $\mathcal{C}$ is cyclic, it means that $s - t_2 \leq 1$, where $t_2 = \deg g(x)$. Therefore we have $\deg g(x) = s - 1$ and the polynomial $g(x) + ua(x)$ generates the vector with all unit entries and length s. If we multiply the whole vector (length= $r + s$) by u, then we have a vector with all entries 0 in the first r coordinates and all coordinates u in the last s coordinates. So the weight of this vector is $2s$. Hence the weight of $\mathcal{C}$ must be $2s$. $\qquad\square$

Theorem 7. *[24] Let $\mathcal{C} = \langle (f(x), 0), (l(x), g(x) + ua(x)) \rangle$ be a cyclic code in $R_{r,s}$ where $f(x)$, $l(x)$, $g(x)$ and $a(x)$ are as in Theorem 6 and $f(x) h_f(x) = x^r - 1$, $g(x) h_g(x) = x^s - 1$, $g(x) = a(x) b(x)$.*
Let

$$S_1 = \bigcup_{i=0}^{deg(h_f)-1} \left\{ x^i * (f(x), 0) \right\},$$

$$S_2 = \bigcup_{i=0}^{deg(h_g)-1} \left\{ x^i * (l(x), g(x) + ua(x)) \right\}$$

and

$$S_3 = \bigcup_{i=0}^{deg(b)-1} \left\{ x^i * (h_g(x) l(x), u h_g(x) a(x)) \right\}.$$

Then $S = S_1 \cup S_2 \cup S_3$ forms a minimal spanning set for $\mathcal{C}$ as an R-module.

Let $\mathcal{C} = \langle (f(x), 0), (l(x), g(x) + ua(x)) \rangle$ be a one weight cyclic code in $R_{r,s}$. Consider the codewords $(v, 0) \in \langle (f(x), 0) \rangle$ and $(w_1, w_2) \in \langle (l(x), g(x) + ua(x)) \rangle$. Since $\mathcal{C}$ is a one weight code, $wt(v, 0) = wt(w_1, w_2)$. Further, since $\mathcal{C}$ is an R-submodule, $u(w_1, w_2) = (0, uw_2) \in \mathcal{C}$ and $wt(v, 0) = wt(0, uw_2)$. Moreover, $(v, uw_2) \in \mathcal{C}$ because of the linearity of $\mathcal{C}$. But it is clear that $wt(v, uw_2) \neq wt(v, 0)$ and $wt(v, uw_2) \neq wt(0, uw_2)$. Hence, $\langle (f(x), 0) \rangle$ can not generate a one weight code.

Now, let us suppose that $\mathcal{C} = \langle (l(x), g(x) + ua(x)) \rangle$ is a one weight cyclic code in $R_{r,s}$. We know from Corollary 7 that $\deg g(x) = s - 1$, $\mathtt{m} = 2s$ and $g(x)$ generates a vector of length s with all unit entries. Therefore, $l(x)$ also must generate a vector over $\mathbb{Z}_2$ with weight s. Hence, to generate such a cyclic one weight code we have two different cases; $r = s$ and $r > s$.

If $r = s$ then, to generate a vector with weight s, the degree of $l(x)$ must be $s - 1$. So, $(l(x), g(x) + ua(x))$ generates the codeword $(\underbrace{1 \cdots 1}_{\text{length } s} \mid \underbrace{unit \cdots unit}_{\text{length } s})$.

Further, if we multiply $(l(x), g(x) + ua(x))$ by $h_g(x)$ we get $(h_g(x)l(x), uh_g(x)a(x))$ and it generates codewords of order 2. Since $r = s$ and the degrees of the polynomials $l(x)$ and $g(x)$ are $s - 1$ we have $h_g = x + 1$ and $h_g(x)l(x) = 0$. Hence, $uh_g(x)a(x)$ must generate a vector with weight $2s$, i.e, $h_g(x)a(x)$ must generate a vector of length s with all unit entries. This means that

$$h_g(x)a(x) = \frac{x^s - 1}{(x + 1)}$$
$$\Rightarrow (x + 1)a(x) = \frac{x^s - 1}{(x + 1)}$$
$$a(x) = \frac{x^s - 1}{(x + 1)^2}.$$

Hence we get $a(x) = \frac{x^s - 1}{(x+1)^2}$. But, since we always assume that s is an odd integer, $a(x)$ is not a factor of $(x^s - 1)$ and this contradicts with the assumption $a(x) | (x^s - 1)$. So, we can not allow $ua(x)h_g(x)$ to generate a vector, i.e, we must always choose $a(x) = g(x)$ to obtain $ua(x)h_g(x) = 0$. So in the case where $\mathcal{C}$ is a one weight cyclic code generated by $(l(x), g(x) + ua(x))$ in $R_{r=s,s}$, we only have $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code of type $(s, s; 0, 1, 0)$ with weight $\mathtt{m} = 2s$.

In the second case we have $r > s$. We know that $\mathcal{C}$ is a one weight cyclic code with weight $\mathtt{m} = 2s$ and $g(x) = \frac{x^s - 1}{x+1}$ generates a vector with exactly s nonzero and all unit entries. Let $v = (v_1, v_2)$

be a codeword of $\mathcal{C}$ such that $v_1 = \langle l(x) \rangle$ and $v_2 = \langle g(x) + ua(x) \rangle$. We can write v as

$$(\underbrace{a_0 a_1 \cdots a_{k-1} a_k}_{s \text{ nonzero entries}} \mid \underbrace{unit \cdots unit}_{\text{length } s})$$

where $a_i \in \mathbb{Z}_2, k \in \mathbb{Z}$. Since $\mathcal{C}$ is an R-submodule we can multiply v by u, then we have

$$(\underbrace{00 \cdots 0}_{\text{length } r} \mid \underbrace{u \cdots u}_{\text{length } s}).$$

Let $w = (w_1, w_2)$ be another codeword of $\mathcal{C}$ generated by $(h_g(x)l(x), ua(x)h_g(x))$. Since $\mathcal{C}$ is a one weight code of weight $2s$, we can write $w = (\underbrace{b_0 b_1 b_2 \cdots b_{t-1} b_t}_{2s - 2p \text{ nonzero entries}} \mid \underbrace{u0uu0 \cdots uu0u}_{p \text{ nonzero entries}})$, $b_i \in \mathbb{Z}_2, t \in \mathbb{Z}$. Since $w + uv$ must be a codeword in $\mathcal{C}$, we have

$$w + uv = (\underbrace{b_0 b_1 b_2 \cdots b_{t-1} b_t}_{2s - 2p \text{ nonzero entries}} \mid \underbrace{0u00u \cdots 00u0}_{s - p \text{ nonzero entries}}).$$

Therefore, $wt(w + uv) = 2s - 2p + 2s - 2p = 4s - 4p$ and since $\mathcal{C}$ is a one weight code with $\mathtt{m} = 2s$,

$$4s - 4p = 2s \implies 2s = 4p \implies s = 2p.$$

But this contradicts with our assumption, that is, s is an odd integer. Consequently, for $r > s$ and $g(x) \neq 0$ there is no one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code. Under the light of all this discussion, we can give the following proved theorem.

Theorem 8. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code in $R_{r=s,s}$ generated by $(l(x), g(x) + ua(x))$ with $\deg l(x) = \deg a(x) = \deg g(x) = s - 1$. Then $\mathcal{C}$ is a one weight cyclic code of type $(r, s; 0, 1, 0)$ with weight $\mathtt{m} = 2s$. Furthermore, there do not exist any other one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code with $g(x) \neq 0$.*

Example 4. *Let $\mathcal{C} = \langle (l(x), g(x) + ua(x)) \rangle$ be a cyclic code in $R_{7,7}$ with $l(x) = g(x) = a(x) = (1 + x + x^3)(1 + x^2 + x^3) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. Hence, $\mathcal{C}$ is a one weight code with weight $\mathtt{m} = 14$ and the following generator matrix,*

$$(\ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \mid 1+u \ \ 1+u \ \ 1+u \ \ 1+u \ \ 1+u \ \ 1+u \ \ 1+u \ ).$$

Furthermore, the dual cyclic code $\mathcal{C}^{\perp}$ has the following generator matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1+u \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1+u & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1+u & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1+u & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1+u & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1+u & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & u & u & u & u & u & u & u \end{pmatrix}.$$

It is obvious from this matrix that $\mathcal{C}^\perp$ is not a one weight code. However, it is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code of type $(7, 7; 7, 6, 0)$ and its image under the Gray map is a binary cyclic code with the parameters $[21, 19, 2]$.

5. Examples of One Weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic Codes

In this part of the paper, we give some examples of one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes. Furthermore, we look at their binary images under the Gray map that we defined in (1). Actually, according to the results of [22], any binary linear one (constant) weight code with no zero column is equivalent to a λ-fold replication of a simplex code. Hence, the examples of one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes that will be given in this section are all λ-fold replication of simplex code S_k. Therefore, any such code has length $n = \lambda 2^k - 1$, dimension k and weight (or minimum distance) $d = \lambda 2^{k-1}$. It is also well-known that a binary simplex code is cyclic in the usual sense.

If the minimum distance of any code $\mathcal{C}$ get the possible maximum value according to its length and dimension, then $\mathcal{C}$ is called optimal (distance-optimal) or good parameter code. For an example, the binary image of a dual code in Example 4 has the parameters $[21, 19, 2]$ which are optimal. Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code with minimum distance $d = 2t + 1$, then we say $\mathcal{C}$ is a t-error correcting code. Since, the Gray map preserves the distances, $\Phi(\mathcal{C})$ is also a t-error correcting code of length $r + 2s$ over $\mathbb{Z}_2$. Since, $|\Phi(\mathcal{C})| = |\mathcal{C}|$, we can write a sphere packing bound for a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code $\mathcal{C}$. With the help of usual sphere packing bound in $\mathbb{Z}_2$,

$$|\Phi(\mathcal{C})| \sum_{j=0}^{t} \binom{r + 2s}{j} \le |2^{r+2s}|,$$

we have

$$|\mathcal{C}| \sum_{j=0}^{t} \binom{r + 2s}{j} \le |2^{r+2s}| = |\mathbb{Z}_2^r \times R^s|.$$

If $\mathcal{C}$ attains the sphere packing bound above then it is called a *perfect code*. Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code of type $(3, 2; 2, 1, 0)$ with standard form of the generator matrix

$$G = \left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & u \\ 0 & 1 & 0 & 0 & u \\ 0 & 0 & 1 & 1 & 1+u \end{array} \right).$$

It is easy to check that $\mathcal{C}$ attains the sphere packing bound, so $\mathcal{C}$ is a perfect code. Moreover, the dual code $\mathcal{C}^\perp$ of $\mathcal{C}$ is generated by the matrix

$$H = \left(\begin{array}{ccc|cc} 1 & 0 & 1 & u & 0 \\ 1 & 1 & 0 & 1+u & 1 \end{array} \right) \qquad (2)$$

and $\mathcal{C}^\perp$ is a one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear code with weight $\mathtt{m} = 4$.

Plotkin bound for a code over F_q^n with the minimum distance d is given by,

1. If $d = \left(1 - \frac{1}{q}\right)n$, then $|\mathcal{C}| \le 2qn$.

2. If $d > \left(1 - \frac{1}{q}\right)n$, then $|\mathcal{C}| \le \frac{qd}{qd-(q-1)n}$.

If $\mathcal{C} \subseteq F_q^n$ attains the Plotkin bound then $\mathcal{C}$ is also an equidistant code [25]. Since any one weight binary linear code is a λ-fold replication of a simplex code and have the parameters $[\lambda(2^k - 1), k, \lambda(2^{k-1})]$, a binary image of any one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code always meet the Plotkin bound.

Finally, we will give the following examples of one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes. We also determine the parameters of the binary images of these one weight cyclic codes. Further we list some of them in Table 1.

Example 5. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic code in $R_{15,15}$ generated by $(l(x), g(x) + ua(x))$ where*

$$\begin{aligned} l(x) &= 1 + x^3 + x^4 + x^6 + x^8 + x^9 + x^{10} + x^{11}, \\ g(x) &= x^{15} - 1, \\ a(x) &= 1 + x^3 + x^4 + x^6 + x^8 + x^9 + x^{10} + x^{11}. \end{aligned}$$

Then $\mathcal{C}$ is a one weight code with weight $\mathtt{m} = 24$ and following generator matrix

$$\left(\begin{array}{ccccccccccccccc|ccccccccccccccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & u & 0 & 0 & u & u & 0 & u & 0 & u & u & u & u & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & u & 0 & 0 & u & u & 0 & u & 0 & u & u & u & u & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & u & 0 & 0 & u & u & 0 & u & 0 & u & u & u & u & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & u & 0 & 0 & u & u & 0 & u & 0 & u & u & u & u \end{array} \right).$$

Furthermore, the binary image $\Phi(\mathcal{C})$ *of* $\mathcal{C}$ *is a* $[45, 4, 24]$ *code, which is a binary optimal code* [26]. *Also, it is important to note that* $\Phi(C)$ *is a* 3-*fold replication of the simplex code* S_4 *of length* 15.

Example 6. *The* $\mathbb{Z}_2\mathbb{Z}_2[u]$-*cyclic code* $\mathcal{C} = \langle(l(x), g(x) + ua(x))\rangle$ *in* $R_{9,9}$ *is a one weight code with* $m = 18$, *where* $l(x) = g(x) = a(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$. $\mathcal{C}$ *has the generator matrix of the form,*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & | & \bar{u} & \bar{u} & \bar{u} & \bar{u} & \bar{u} & \bar{u} & \bar{u} & \bar{u} & \bar{u} \end{pmatrix}$$

where $\bar{u} = 1 + u$. *The Gray image of* $\mathcal{C}$ *is a* 9-*fold replication of the simplex code* S_2 *of length* 3 *with the optimal parameters* $[27, 2, 18]$.

Example 7. *Let* $\mathcal{C} = \langle(l(x), g(x) + ua(x))\rangle$, $l(x) = a(x) = 1 + x + x^2 + x^4$, $g(x) = x^7 - 1$, *be a cyclic code in* $R_{7,7}$. *Then the generator matrix of* $\mathcal{C}$ *is*

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & | & u & u & u & 0 & u & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & | & 0 & u & u & u & 0 & u & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & | & 0 & 0 & u & u & u & 0 & u \end{pmatrix}.$$

$\mathcal{C}$ *is a one weight code with* $m = 12$ *and* $\Phi(\mathcal{C})$ *is a* 3-*fold replication of the simplex code* S_3 *of length* 7 *with the parameters* $[21, 3, 12]$.

6. Conclusion

In this paper, we study the one weight linear and cyclic codes over $\mathbb{Z}_2^r \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^s$ where $u^2 = 0$. We also classify one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes and present some illustrative examples. We further list some binary linear codes with their parameters which are derived from the Gray images of one weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes.

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Table 1. Some Examples of One Weight $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic Codes

Generators	$\mathbb{Z}_2\mathbb{Z}_2[u]-$type	Binary Image
$l(x) = 1 + x + x^2 + x^4,\ g(x) = x^{21} - 1,\ a(x) = 1 + x + x^2 + x^4 + x^7 + x^8 + x^9 + x^{11} + x^{14} + x^{15} + x^{16} + x^{18}$	$[7, 21; 0; 0, 3]$	$[49, 3, 28]$
$l(x) = a(x) = 1 + x^2 + x^4 + x^5 + x^6 + x^8 + x^9 + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{20} + x^{21} + x^{23} + x^{26},\ g(x) = x^{31} - 1$	$[31, 31; 0; 0, 5]$	$[93, 5, 48]$
$l(x) = 1 + x + x^3 + x^4 + x^6 + x^7 + x^9 + x^{10} + x^{12} + x^{13} + x^{15} + x^{16} + x^{18} + x^{19} + x^{21} + x^{22} + x^{24} + x^{25},\ g(x) = x^{15} - 1,\ a(x) = 1 + x + x^3 + x^4 + x^6 + x^7 + x^9 + x^{10} + x^{12} + x^{13}$	$[27, 15; 0; 0, 2]$	$[57, 2, 38]$
$l(x) = 1 + x^2 + x^3 + x^4 + x^7 + x^9 + x^{10} + x^{11} + x^{14} + x^{16} + x^{17} + x^{18} + x^{21} + x^{23} + x^{24} + x^{25} + x^{28} + x^{30} + x^{31} + x^{32},\ g(x) = x^{21} - 1,\ a(x) = 1 + x^2 + x^3 + x^4 + x^7 + x^9 + x^{10} + x^{11} + x^{14} + x^{16} + x^{17} + x^{18}$	$[35, 21; 0, 0, 3]$	$[77, 3, 44]$

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