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RESEARCH ARTICLE

Boundary values for an eigenvalue problem with a singular potential

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ABSTRACT

In this paper we consider the inverse spectral problem on the interval [0,1]. This determines the three-dimensional Schrödinger equation with from singular symmetric potential. We show that the two spectrums uniquely identify the potential function $q(r)$ in a single Sturm-Liouville equation, and we obtain new evidence for the difference in the $q(r)-q(\tilde{r})$ of the Hochstadt theorem.

1. Introduction

Although the time-independent radial Schrödinger equation can not be solved for some simple spherical symmetric potentials, it is not possible to obtain a complete solution in complex situations. In these cases, it must resorted to some approach methods. For the calculation of stationary states and energy eigenvalues, these are perturbation theory, variational method and WKB approximation methods. Perturbation theory can be applied if case of Hamiltonian differs slightly from a fully soluble matter. The method of variation is suitable for calculating the ground state energy in the form of wave function.After that WKB method can be practically applied at the classical boundary.

Given a Schrödinger operator with a symmetrical potential, the separation of variables leads to the global Schrödinger operator

$$L=\frac{-d^2}{dr^2}+\frac{l(l+1)}{r^2}+q(r),\ l=0,1,2,...$$

Many results have been obtained considering the eigenvalues of this problem which have been of great interest in the past. The initial results are studied by Guillot and Raltson [1]. They examined the problem case of $l=1$ and $q\in L^2(0,1)$. Then, the results were extended by Carlson [2] and by Serier [3] for any $l\in N_0$. However, the assumption $q\in L^2(0,1)$ clearly excludes the physically interesting case of a Coulomb type singularity $q(r)=\frac{\gamma}{r}+...$ (γ is the Euler-Mascheroni constant). This corresponds to the work of Savchuk and Shkalikov [4], who considered $q\in W^{-1,2}(0,1)$ for $l=0$. It is extended by Albeverio, Hryniv and Mykytyuk and the case $q\in W^{-1,p}(0,1)$ $p\in[1,\infty)$, was first discussed for $l=0$ and then in [5] all $l\in N_0$ are expanded using the double commutation method. Panakhov and Sat [6], examined the inverse problem for the internal spectral data of the hydrogen atom equation [7]. The basic reference for this study is Zhornitskaya and Serov [8] who treat the general case $l\geq-\frac{1}{2}$ under the assumption $q\in L^1(0,1)$. Moreover, they show [8] that the Dirichlet eigenvalues satisfy

$$\lambda_n=\left(j_{l+\frac{1}{2},n}+\varepsilon_n\right)^2$$

where $j_{l+\frac{1}{2},n}=\pi(n+\frac{1}{2})+O(n^{-1})$ are the zeros of the Bessel function $J_{l+\frac{1}{2}}$ and the error satisfies $|\varepsilon_n|\leq\frac{\pi}{4}$ (this is claimed for all *n* but only proven for large n). We want to extend the results a bit further:

First we change $q(r) \in L^1(0,1)$ by $rq(r) \in L^1(0,1)$. We will show that the error satisfies the following equalit

$$\varepsilon_n = O\left(\int_0^1 \frac{y|g(y)|}{1+ny} dy \right).$$

In particular, we obtain let $\varepsilon_n = O(n^{-1}\log(n))$ for a Coulomb type singularity be $q(r) = \frac{\gamma}{r}$.

Based on this information, in addition to the Dirichlet spectrum, we will obtain, spectral data that uniquely specifies q and l, which again correspond to the case $rq(r) \in L^1(0,1)$ [8].

We can apply this method to some spherically symmetric potentials by obtaining a simple method for the general solution of time independent radials. We will examine this problem for the Schrödinger equation with spherical symmetric potential. We will now discuss the problem

$$\psi'' + (\lambda - q(r) - \frac{l(l+1)}{r^2})\psi = 0$$

$$\psi(0) = 0 \ , \ \psi'(1) + h\psi(1) = 0 \ , \quad l \geq -\frac{1}{2}$$

and show that we have identified the potential function of the two spectra only and that we are dealing with,new evidence for the difference in the $q(r) - q(\tilde{r})$ of the Hochstadt theorem.

2. Time independent radial Schrödinger equation for spherical symmetric potentials

The time-independent Schrödinger equation in three dimensions is given by

$$\Delta\Psi(\tilde{r}) + \frac{2m}{\hbar^2}[E - V(\tilde{r})]\Psi(\tilde{r}) = 0 \qquad (1)$$

where, m is the mass of particle and E, V are the total and potential energies, respectively. Spherical polar coordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ are appropriate for the symmetry of the problem. The Schrödinger Eq. (1), expressed as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]\Psi(r,\theta,\phi) + \frac{1}{r^2}\tilde{L}^2(\theta,\phi)\Psi(r,\theta,\phi)$$

$$+ \frac{2m}{\hbar^2}[E - V(r,\theta,\phi)]\Psi(r,\theta,\phi) = 0 \qquad (2)$$

in these coordinates, where $\tilde{L}^2(\theta,\phi) = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$. The potential energy of a particle which moves in a central, spherically symmetric field of force depends only upon the distance r between the particle and the centre of force. Thus, the potential energy should be $V(r,q,j) = V(r)$. Solution of the Eq. (2) can be found by the method of separation of variables. To apply this method, we use a solution of the form

$$\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \qquad (3)$$

where $R(r)$ is independent of the angles, and $Y(q,j)$ is independent of r. By substituting Eq. (3) into Eq. (2) and rearranging, we obtain,

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{2}{r}\frac{\partial R(r)}{\partial r} + \left\{\frac{2m}{\hbar^2}[E - V(r)] - \frac{C}{r^2}\right\}R(r) = 0 \qquad (4a)$$

$$\tilde{L}^2(\theta,\phi)Y(\theta,\phi) + CY(\theta,\phi) = 0 \qquad (4b)$$

where C is fixed. Eq. (4b) is independent of total energy E and potential energy $V(r)$. In this case, the angular dependence of wave functions is determined by the global symmetry property, and Eq. (4b) are valid for every spherical symmetric system, regardless of the specific form of the potential function. Eq. (4b) are known as spherical harmonic functions, $Y(\theta,\phi)$ where $C = l(l+1); l = 0,1,2,3,..$ are positive integers. Substituting $C = l(l+1)$ value and $\psi(r) = rR(r)$ to Eq. (4a), we obtain the radial wave equation as:

$$\frac{\partial^2}{\partial r^2} + \left\{\frac{2m}{\hbar^2}[E - U(r)]\right\}\psi(r) = 0 \qquad (5)$$

where $U(r) = V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$ is the effective potential energy.

The problem of radially symmetric potential determination in the three-dimensional Schrödinger equation given as follows

$$-\Delta\Psi + q(X)\Psi = \lambda\Psi, \quad \Psi(X) = 0 \,|X| = 1 \qquad (6)$$

in the unit ball R^3 with a central potential $q(x)$. By looking for solutions in the separated form

$$\Psi(r,\theta,\phi) = \frac{\psi(r)}{r}Y_l^m(\theta,\phi) \qquad (7)$$

in which $\Psi = (r,\theta,\phi)$ are spherical coordinates in R^3 and Y_l^m are spherical harmonics, we get the below ordinary differential equation

$$\psi'' + (\lambda - q(r) - \frac{l(l+1)}{r^2})\psi = 0, \psi(1) = 0,\ l = 0,1,2. \quad (8)$$

$\psi,\ l = 0,1,2$ supplemented by the regularity condition

$$\psi(r) = O(r) \quad r \to 0. \quad (9)$$

The problem in Eq.(8), or Eq.(9) has a countable sequence of eigenvalues, denote for fixed l, by $\lambda_{l,n}, n = 1, 2, ...$

These are the natural frequencies $\{\lambda_{l,n}\}$ coming from an equation

$$\psi'' + \left(\frac{\lambda}{c^2(r)} - Q(r,l,\lambda) - \frac{l(l+1)}{r^2}\right)\psi = 0.$$

This corresponds to normal acoustic vibration modeling [9]. The two principal quantities are the propagation velocity for $c(r)$ and density $p(r)$ for acoustic waves. The radial part of the wave function ψ and Q is the function of c, p only, but

$Q(r,l,\lambda) = \frac{l(l+1)}{r^2} + C_1(p,c) + C_2(p)$ are mutually dependent on λ in the general form. It may be reasonable to keep the C_1 term equal to zero for certain purposes, thus it is giving the form a more conventional variant

$$\psi'' + \left(\frac{\lambda}{c^2(r)} - q(r) - \frac{l(l+1)}{r^2}\right)\psi = 0 \quad (10)$$

The simplest case is that we have to determine the spectral modulus q (or $q(r)$) when c is known and constant (if c is known then a motion-time conversion can be used). The effective accuracy of the data decreases with the increase of l and at the same time for very small l values. No other spectral information is available for sufficient accuracy. For this reason, we use more than one value to achieve reconstructions.

First let us recall some well-known results about Eq. (8) with $l = 0$ (e.g., [10,11]). If we define the map $q \to \lambda_n(q)$, the *nth* eigenvalue of Eq. (8) subject radially symmetric potential reconstruction to the Dirichlet boundary conditions

$\psi(0) = \psi(1) = 0$, then $\lambda_n(q) = \tilde{\lambda}_n(q)$ implies that

$$\int_0^1 (\tilde{q}(r) - q(r))\psi_n(r)\tilde{\psi}_n(r)dr = 0 \quad (11)$$

where ψ_n, $\tilde{\psi}_n$ are eigenfunctions for $q, \tilde{q}$. If $\zeta = \tilde{q} - q$ then in the limit as $\zeta \to 0$, we get

$$\int_0^1 \zeta(r)\psi_n^2(r)dr = 0 \quad (12)$$

or, what amounts to the same thing, the Frechet derivative of $\lambda_n(q)$ is with ψ_n now normalized in $L^2(0,1)$

$$Dq\lambda_n(q)\zeta = \int_0^1 \zeta(r)\psi_n^2(r)dr. \quad (13)$$

[11,Theorem 2.3]. From the following eigenvalue asymptotics

$$\lambda_n(q) = (n\pi)^2 + \int_0^1 q(r)dr + a_n \ , \ \{a_n\} \in l^2 \ , \quad (14)$$

we obtain the additional orthogonality relation

$$\int_0^1 \zeta(r)dr = 0 \quad (15)$$

where $\lambda_n(q) = \lambda_n(q + \zeta)$ for all n. The well-known theorem [11,Theorem 3.3] that a symmetric function *q* is uniquely determined by $\{\lambda_n(q)\}_{n=1}^{\infty}$ may be thought of as the fact that the set of functions $\{1, \psi_n^2\}$ is complete in the even subspace of $L^2(0,1)$.

If we add a second spectral sequence $\mu_n(q)$, the *nth* eigenvalue of Eq. (8) with boundary conditions $\psi(0) = 0$, $\psi'(1) + h\psi(1) = 0$ and eigenfunction ϕ_n, we get a second set of orthogonality relation,

$$\int_0^1 \zeta(r)\phi_n(r)dr = 0 \quad (16)$$

then it can be shown that

$$\{1, \psi_n^2, \phi_n^2\}_{n=1}^{\infty}. \quad (17)$$

is complete in $L^2(0,1)$; i.e., a linearized uniqueness result holds for the problem of determining $q \in L^2(0,1)$ from the eigenvalue data $\{\lambda_n(q), \mu_n(q)\}_{n=1}^{\infty}$.

In [1] a similar fact is derived and used in the singular case $l = 1$; namely, if $\lambda_{l,n}(q)$ denotes the *nth*

eigenvalue of Eq. (8) and Eq. (9), then following equality can be written

$$Dq\lambda_{l,n}(q)\zeta = \int_0^1 \zeta(r)\psi_{l,n}^2(r)dr \ , \tag{18}$$

where $\psi_{l,n}$ is the nth normalized eigenfunction. Now let Λ denote some subset of the indices (l,n). The condition $\lambda_{l,n}(q) = \lambda_{l,n}(\tilde{q})$ for all $(l,n) \in \Lambda$ implies, in the limit of small $\zeta = \tilde{q} - q$, that ζ is orthogonal to the subspace of $L^2(0,1)$ spanned by $\{\psi_{l,n}^2\}$. In the best case this subspace is all of $L^2(0,1)$, in which case we have a uniqueness result for the linearized inverse spectral problem.

Asymptotics of the eigenvalues are , (see [12] for $l=1$ or [13]- (9.5.12) for the specific case of $q=0$)

$$\lambda_{l,n} = (n+\frac{l}{2})^2\pi^2 + \int_0^1 q(r)dr - l(l+1) + r_{l,n}, \sum_{n=1}^{\infty} r_{l,n}^2 \le \infty \tag{19}$$

or

$$\sqrt{\lambda_{l,n}} = (n+\frac{l}{2})\pi + \frac{\int_0^1 q(r)dr - l(l+1)}{(2n+l)\pi} + \beta_{l,n} \sum_{n=1}^{\infty} \beta_{l,n}^2 \le \infty \ . \tag{20}$$

So, the mean value $\int_0^1 q(r)dr$ is determined uniquely by the eigenvalue sequence for any l. If the spectral data contains all of the eigenvalues for any fixed l we should really look at the span of $\{1, \psi_{l,n}^2\}_{l,n\in\Lambda}$ [14].

For technical convenience, we will actually work with a somewhat different mapping, defined as follows. For given q there exists a solution $\psi_1(x,\lambda,q)$ of Eq. (8) satisfying the normalization condition

$$\lim_{r\to 0} \frac{\psi_1(x,\lambda,q)}{r^{l+1}} = 1. \tag{21}$$

Now set

$$F_\Lambda(q,\mu_{l,n}) = \{\psi_1(1,\mu_{l,n},q)\}_{l,n\in\Lambda} \ . \tag{22}$$

For given spectral data $\{\lambda_{l,n}, (l,n)\in\Lambda\}$ the inverse spectral problem is equivalent to

$$F_\Lambda(q,\lambda_{l,n}) = 0. \tag{23}$$

The rest of this article is limited to the linearization of the small potentials, ie., F_Λ at $q=0$.

Proposition 1. If $\lambda_{l,n,0}$ denotes the eigenvalue of Eq. (8) and Eq. (9) with $q \equiv 0$ then

$$D_q F_\Lambda(0,\lambda_{l,n,0})\zeta = \left\{ c_{l,n} \int_0^1 r^2 j_l^2\left(\sqrt{\lambda_{l,n,0}}r\right)\zeta(r)dr \right\}_{(l,n)\in\Lambda} \tag{24}$$

for some $c_{l,n} \neq 0$. Here j_l, y_l are the spherical Bessel functions with standard normalizations [13]. The exact value of $c_{l,n}$ may be found in the proof. It is not hard to check that when $q \equiv 0$ we have

$$\psi_{l,n}(r) = Crj_l\left(\sqrt{\lambda_{l,n,0}}r\right) \tag{25}$$

So that (not surprisingly) uniqueness holds for $D_q F_\Lambda(0,\lambda_{l,n,0})\zeta = 0$ exactly if it holds for the system

$$D_q\lambda_{l,n}(0)\zeta = 0 \quad (l,n)\in\Lambda. \tag{26}$$

For any nonnegative integers l_1 and l_2 the set of functions

$$\left\{1, \phi_{l_1}, \left(\sqrt{\lambda_{l_1,n,0}}r\right), \phi_{l_2}, \left(\sqrt{\lambda_{l_2,n,0}}r\right)\right\} \tag{27}$$

is complete in $L^2(0,1)$. $(\phi_l(r) = xj_l(r))$.

3. Two spectra for the spherical Schrödinger equation

The spherical Schrödinger operator given by

$$L = \frac{-d^2}{dr^2} + \frac{l(l+1)}{r^2}, \quad r\in(0,1), \ l \ge -\frac{1}{2}$$

where l may not be an integer, arbitrary space size $n \ge 2$, or $l(l+1)$ has to be replaced by $l(l+n-2)+(n-1)(n-3)/4$ [15]. With the usual boundary conditions at $r=0$ and $r=1$

$$\lim_{r\to 0} r^l((l+1)\psi(r) - r\psi'(r)) = 0$$

for $l \in [-\frac{1}{2}, \frac{1}{2})$. It gives rise to a self-adjoint operator

in the Hilbert space $L^2(0,1)$.

Consider two singular Sturm--Liouville problems

$$\psi'' + (\lambda - q(r) - \frac{l(l+1)}{r^2})\psi = 0 \quad r \in (0,1) \tag{28}$$

$$\psi(0) = 0 \ , \ \psi'(1) + h\psi(1) = 0 \tag{29}$$

$$\psi'' + (\lambda - \tilde{q}(r) - \frac{l(l+1)}{r^2})\psi = 0 \quad r \in (0,1) \tag{30}$$

$$\psi(0) = 0 \ , \ \psi'(1) + \tilde{h}\,\psi(1) = 0 \tag{31}$$

where the functions $q(r)$, $\tilde{q}(r)$ are assumed to be real-valued and square integrable and h, $\tilde{h}$ are finite real numbers. Let us denote the spectrum of the first problem by $\{\lambda_n\}_0^\infty$ and the spectrum of the second by $\{\tilde{\lambda}_n\}_0^\infty$. And $\varphi(r,\lambda)$ is solution of Eq. (28) and $\tilde{\varphi}(r,\lambda)$ is the solution of Eq. (30) satisfying the initial condition respectively.

$$\varphi(r,\lambda) = -\lambda^{\frac{-2l+1}{4}}\sqrt{\frac{\pi r}{2}}.$$
$$\begin{Bmatrix} \frac{-1}{\sin((l+\frac{1}{2})\pi)} J_{-l-\frac{1}{2}}\sqrt{\lambda}r \ , \ l+\frac{1}{2} \in R^+ \setminus N_0 \\ Y_{l+\frac{1}{2}}\sqrt{\lambda}r - \frac{1}{\pi}\log(\lambda) J_{l+\frac{1}{2}}\sqrt{\lambda}r, l+\frac{1}{2} \in N_0 \end{Bmatrix} \tag{32}$$

where $J_{l+\frac{1}{2}}$ and $Y_{l+\frac{1}{2}}$ are the usual Bessel and Neumann functions. Note that, all branch cuts are selected along the negative real axis. If *l* is an integer, it is of course reduced to spherical Bessel and Neumann functions and can be expressed by trigonometric functions [16].

Using the power series for Bessel and Neumann functions, it can be writte uniquely in then form

$$\varphi(r,\lambda) = \frac{\Gamma(l+\frac{3}{2})2^{l+1}}{r^l\sqrt{\pi}}.$$
$$\begin{Bmatrix} \frac{1}{2l+1}\phi(\lambda r^2), \qquad l+\frac{1}{2} \in R^+ \setminus N_0 \\ \frac{1}{2l+1}\phi(\lambda r^2) - \frac{\log(\lambda)(\lambda r^2)^{l+\frac{1}{2}}}{\Gamma(l+\frac{3}{2})^2 2^{2l+1}}\psi(\lambda r^2), l+\frac{1}{2} \in N_0 \\ (\log(2) - \gamma)\phi(\lambda r^2) - \log(r)\psi(\lambda r^2), \quad l = \frac{1}{2} \end{Bmatrix}$$

where $\phi(\lambda), \psi(\lambda)$ are entire functions with $\phi(0) = \psi(0) = 1$ and γ is the Euler- Mascheroni constant.

It is well known that there exists a function $K(r,s)$ such that

$$\tilde{\varphi}(r,\lambda) = \varphi(r,\lambda) + \int_0^r K(r,s)\varphi(s,\lambda)ds. \tag{33}$$

Where $K(r,s)$ satisfies

$$\frac{\partial^2 K}{\partial r^2} + (\lambda - \tilde{q}(r) - \frac{l(l+1)}{r^2})K$$

$$= \frac{\partial^2 K}{\partial s^2} + (\lambda - q(s) - \frac{l(l+1)}{s^2})K \tag{34}$$

and

$$K(r,r) = \frac{1}{2}\int_0^r [\tilde{q}(t) - q(t)]dt \tag{35}$$

$$K(r,0) = 0 \tag{36}$$

This problem can be solved by using Riemann method.

$$c_n = \int_0^1 \varphi^2(r,\lambda_n)dr \ , \quad \tilde{c}_n = \int_0^1 \tilde{\varphi}^2(r,\tilde{\lambda}_n)dr$$

$$\rho(\lambda) = \sum_{\lambda_n} \frac{1}{c_n} \qquad \tilde{\rho}(\lambda) = \sum_{\tilde{\lambda}_n} \frac{1}{\tilde{c}_n}.$$

Let set the function $\rho(\lambda)$ ($\tilde{\rho}(\lambda)$) is called the spectral function of Eq. (28) and Eq. (29) (Eq. (30) and Eq. (31)). Eq. (28) and Eq. (29) is regarded as an unperturbed problem, while Eq. (30) and Eq. (31) is considered as a perturbation of Eq. (28) and Eq. (29) [17-18].

It is a well-known method in which two spectra for a given Sturm-Liouville equation make it possible to recover its spectral function, that is $\{c_n\}$ numbers. Assume that we know the spectrum $\{\mu_n\}$ of the problem. Which are given Eq. (28) and Eq. (29).

$$\psi'' + (\lambda - q(r) - \frac{l(l+1)}{r^2})\psi = 0, \quad (0 \le r \le 1) \tag{37}$$
$$\psi(0) = 0 \ , \quad \psi'(1) + h_1\psi(1) = 0, \qquad h_1 \ne h \ .$$

If we know the spectrums $\{\lambda_n\}$ and $\{\mu_n\}$, we can calculate the numbers $\{c_n\}$. Similarly, for Eq. (28), if in addition to $\{\tilde{\lambda}_n\}$, we also know the spectrum $\{\tilde{\mu}_n\}$ determined by the boundary conditions

$$\psi(0) = 0 \ , \quad \psi'(1) + \tilde{h}_1\ \psi(1) = 0, \qquad \tilde{h}_1 \ne \tilde{h},$$

thus we can determine the numbers $\{\tilde{c}_n\}$, asymptotics of the eigenvalues are Eq. (20). It is also shown that

[19],

$$\|\varphi_n\|^2 = \int_0^1 \varphi_n^2(r)dr = \frac{1}{2} + O\left(\frac{1}{n^2}\right).$$

Theorem 1. Consider the

$$L\psi = \psi'' + (\lambda - q(r) - \frac{l(l+1)}{r^2})\psi \qquad (38)$$

subject to the boundary conditions

$$\psi(0) = 0, \qquad (39)$$
$$\psi'(1) + h\psi(1) = 0, \qquad (40)$$

where q is square integrable on $(0,1]$. Let $\{\lambda_n\}$ be the spectrum of L subject to Eq. (39) and Eq. (40). If Eq. (40) is replaced by the new boundary condition

$$\psi'(1) + h_1\psi(1) = 0, \qquad (41)$$

then it yields a new operator and a new spectrum, as $\{\mu_n\}$, now consider the second operator

$$\tilde{L}\ \psi = \psi'' + (\lambda - \tilde{q}(r) - \frac{l(l+1)}{r^2})\psi, \qquad (42)$$

where $\tilde{q}$ is square integrable on $(0,1]$. Suppose that under the boundary conditions Eq. (39) and

$$\psi'(1) + \tilde{h}\ \psi(1) = 0. \qquad (43)$$

$\tilde{L}$ has the spectrum $\{\tilde{\lambda}_n\}$, with $\tilde{\lambda}_n = \lambda_n$ for all n. Here, $\tilde{L}$ with the boundary conditions Eq. (38) and

$$\psi'(1) + \tilde{h}_1\psi(1) = 0, \qquad (44)$$

is assumed to have the spectrum $\{\tilde{\mu}_n\}$. Denote the finite index set by Λ_0 for which $\tilde{\mu}_n \neq \mu_n$ and the infinite index set by Λ for which $\tilde{\mu}_n = \mu_n$. Under the above assumptions, it follows that the kernel $K(r,s)$ is degenerate in the extended sense;

$$K(r,s) = \sum_{\Lambda_0} c_n\ \tilde{\varphi}_n(r)\phi_n(s), \qquad (45)$$

where $\tilde{\phi}_n$ and φ_n are suitable solutions of Eq. (35) and Eq. (36).

Proof. It follows from Eq. (33)

$$\tilde{\phi}'(r,\lambda) = \phi'(r,\lambda) + K(r,r)\phi(r,\lambda) + \int_0^r \frac{\partial K}{\partial r}\phi(s,\lambda)ds, \qquad (46)$$

and

$$\tilde{\phi}'(r,\lambda) + \tilde{h}\ \tilde{\phi}(r,\lambda) = \phi'(r,\lambda) + \tilde{h}\ \tilde{\phi}(r,\lambda)$$

$$+K(r,r)\phi(r,\lambda) + \int_0^r \left(\frac{\partial K}{\partial r} + \tilde{h}\ K\right)\phi(s,\lambda)ds.$$

By substituting $r=1$ and $\lambda = \lambda_n$ into the last equation and using the boundary conditions Eq. (40) and Eq. (44), we obtain

$$(\tilde{h} - h)\varphi(1,\lambda_n) + K(1,1)\varphi(1,\lambda_n)$$

$$+\int_0^1 \left(\frac{\partial K}{\partial r} + \tilde{h}\ K\right)_{r=1} \varphi(s,\lambda_n)ds = 0. \qquad (47)$$

As $n \to \infty$ and $\varphi(1,\lambda_n) \to o(1)$, the integral on the right-hand side tends to zero. Therefore, we get

$$K(1,1) = h - \tilde{h}, \qquad (48)$$

$$\int_0^1 \left(\frac{\partial K}{\partial r} + \tilde{h}\ K\right)_{r=1} \phi(s,\lambda_n)ds = 0 \quad (n = 0,1,2,..), \qquad (49)$$

from Eq. (47). Since the system of functions $\varphi(s,\lambda_n)$ is complete, it follows from the last equation

$$\left(\frac{\partial K}{\partial r} + \tilde{h}\ K\right)_{r=1} = 0. \qquad (50)$$

Now we use the condition imposed on the second-mentioned spectrum. By using Eq. (40) again, we obtain

$$\tilde{\varphi}'(r,\lambda) + \tilde{h}_1\ \tilde{\varphi}(r,\lambda) =$$

$$= \phi'(r,\lambda) + \tilde{h}_1\ \tilde{\phi}(r,\lambda) + K(r,r)\phi(r,\lambda)$$

$$+\int_0^r \left(\frac{\partial K}{\partial r} + \tilde{h}_1 K\right)\phi(s,\lambda)ds. \qquad (51)$$

Setting $r=1$ and $\lambda = \mu_n$ ($n \in \Lambda$) and using Eq. (41) , Eq. (44), we get

$$\int_0^r \left(\frac{\partial K}{\partial r} + \tilde{h}_1 K \right) \phi(s,\mu_n) ds$$

$$+(\tilde{h}_1 - h\ )\phi(1,\mu_n) + K(1,1)\phi(1,\mu_n) = 0.$$

In the last equation, as $n \to \infty$, the left-hand side tends to zero and then $\varphi(1,\mu_n) \to o(1)$. Therefore, it can be written as follows:

$$K(1,1) = h_1 - \tilde{h}_1, \tag{52}$$

$$\int_0^1 \left(\frac{\partial K}{\partial r} + \tilde{h}\ K \right)_{r=1} \phi(s,\lambda_n) ds = 0 \quad n \in \Lambda. \tag{53}$$

Comparing Eq. (48) and Eq. (52), we obtain $h - \tilde{h} = h_1 - \tilde{h}_1$. For $n \in \Lambda_0$, relation Eq. (51) (for $r = 1$ and $\lambda = \mu_n$)

$$\int_0^1 \left(\frac{\partial K}{\partial r} + \tilde{h}_1 K \right)_{r=1} \varphi(s,\mu_n) ds =$$

$$= \tilde{\phi}'(1,\mu_n) + \tilde{h}_1\ \tilde{\phi}(1,\mu_n). \tag{54}$$

It follows from Eq. (53) and Eq. (54) that

$$\left(\frac{\partial K}{\partial r} + \tilde{h}_1 K \right)_{r=1} = \sum_{\Lambda_0} \frac{\tilde{\phi}'(1,\mu_n) + \tilde{h}_1\ \tilde{\phi}(1,\mu_n)}{\left\|\phi(s,\mu_n)\right\|^2} \phi(s,\mu_n). \tag{55}$$

We derive the following equations from Eq. (53) and Eq. (55);

$$K(1,s) = \frac{1}{\tilde{h}_1 - \tilde{h}} \sum_{\Lambda_0} \frac{\tilde{\phi}'(1,\mu_n) + \tilde{h}_1\ \tilde{\phi}(1,\mu_n)}{\left\|\phi(s,\mu_n)\right\|^2} \phi(s,\mu_n), \tag{56}$$

$$\frac{\partial K(r,s)}{\partial r}_{r=1} = -\frac{\tilde{h}}{\tilde{h}_1 - \tilde{h}} \cdot$$

$$\sum_{\Lambda_0} \frac{\tilde{\phi}'(1,\mu_n) + \tilde{h}_1\ \tilde{\phi}(1,\mu_n)}{\left\|\phi(s,\mu_n)\right\|^2} \phi(s,\mu_n). \tag{57}$$

The function $K(r,s)$ satisfies Eq. (45). Therefore, it follows from the initial conditions Eq. (56) and Eq. (57),

$$K(r,s) = \frac{1}{\tilde{h}_1 - \tilde{h}} \sum_{\Lambda_0} \frac{\tilde{\phi}'(1,\mu_n) + \tilde{h}_1\ \tilde{\phi}(1,\mu_n)}{\left\|\phi(s,\mu_n)\right\|^2}$$

$$\times \left[\tilde{c}(r,\mu_n) - \tilde{h}\ \tilde{s}(r,\mu_n) \right] \phi(s,\mu_n), \tag{58}$$

where $\tilde{c}(r,\lambda)$ and $\tilde{s}(r,\lambda)$ are solutions of Eq. (30) satisfying the initial conditions

$$\tilde{c}(1,\lambda) = \tilde{s}'(1,\lambda) = 1, \quad \tilde{c}'(1,\lambda) = \tilde{s}(1,\lambda) = 0.$$

Hence, we obtain Hochstadt's result in a some what more general formulation.

Theorem 2. If the spectras $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$ coincide and $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ differ in a finite number of their terms, i.e., $\mu_n = \tilde{\mu}_n$ for n [illegible], then

$$\tilde{q}(r) - q(r) = \sum_{\Lambda_0} \tilde{c}_n \frac{d}{dr} \left(\tilde{\phi}_n, \varphi_n \right)$$

where φ_n and $\tilde{\phi}_n$ are suitable solutions of Eq. (28) and Eq. (29).

Proof. Let

$$\tilde{q}(r) - q(r) = 2\frac{dK(r,r)}{dr}.$$

from Eq. (45). By differentiating Eq. (58) and setting $s = r$, we obtain

$$\tilde{q}(r) - q(r) = \frac{1}{\tilde{h}_1 - \tilde{h}} \sum_{\Lambda_0} \frac{\tilde{\phi}'(1,\mu_n) + \tilde{h}_1\ \tilde{\phi}(1,\mu_n)}{\left\|\phi(s,\mu_n)\right\|^2}$$

$$\times \frac{d}{dr} \left\{ \left[\tilde{c}(r,\mu_n) - \tilde{h}\ \tilde{s}(r,\mu_n) \right] \phi(1,\mu_n) \right\},$$

$$\tilde{q}(r) - q(r) = \sum_{\Lambda_0} \tilde{c}_n \frac{d}{dr} \left(\tilde{\phi}_n, \varphi_n \right).$$

Consequently, $\tilde{c}(r,\mu_n) - \tilde{h}\ \tilde{s}(r,\mu_n) = \tilde{\phi}_n$, $\varphi(r,\mu_n) = \varphi_n(r,\mu_n)$ and

$$\tilde{c}_n = \frac{\tilde{\varphi}'(1,\mu_n) + \tilde{h}_1\ \tilde{\varphi}(1,\mu_n)}{\left(\tilde{h}_1 - \tilde{h}\right)\left\|\varphi(s,\mu_n)\right\|^2}.$$

This completes the proof of Theorem 2.

4. Conclusion

Quantum mechanics and atomic structures can be explained very well. The theoretical explanations made for this purpose fit very well with the experimental observations. This new atom model covering all the atoms is the Wave Model. hydrogen atom is a fundamental and important example of wave space. Because the single electron hydrogen atom is the simplest atom and the Coulomb potential energy is spherically symmetric, the hydrogen atom is the simplest application of the wave model. However, because of the difficulty of the solutions of atoms with more than one electron, the Schrödinger equation can be solved using approximation methods. These results gave consistent results in experimental errors with the literature results. Accordingly, the energy potentials of the system, the interaction potentials with small contribution, are calculated by some methods. The energy shifts for hydrogen atoms can be handled independently for degenerate and non-degenerate situations.

In this study, we have applied this method to some

spherical symmetric potentials by obtaining a simple method for the general solution of time independent radials.Also, we have discussed the three-dimensional Schrödinger equation with bimetric potential. We have obtained new evidence for the difference of $\tilde{q}(r)-q(r)$ in the Hochstadt theorem and that the two spectra uniquely identify the potential function $q(r)$ in a Sturm-Liouville equation.

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