Special issue of the 2nd International Conference on Computational Mathematics and Engineering Sciences (CMES2017)

An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.7, No.3, pp.288-292 (2017)
https://doi.org/10.11121/ijocta.01.2017.00503

RESEARCH ARTICLE

Numerical behavior of singular two-point boundary value problems in a comparative way

Selmahan Selim, Gözde Elver, Murat Sarı*

Department of Mathematics, Yıldız Technical University, Turkey
sselim@yildiz.edu.tr, elvergozde@gmail.com, sarim@yildiz.edu.tr

ARTICLE INFO

Article history:
Received: 29 June 2017
Accepted: 15 November 2017
Available Online: 23 November 2017

Keywords:
Singular boundary value problem
Differential quadrature method
Finite element method
Physical behavior

AMS Classification 2010:
90B23, 90B56

ABSTRACT

This article concentrates on discovering numerical behavior of the singular two-point boundary value problems through various numerical techniques. This is carried out in a comparative way by mainly using differential quadrature and finite element methods. Also a discussion has been done by means of advantages and disadvantages of the numerical methods of interest.To properly understand the behavior of the physical processes represented by the model equation, the calculated solutions have been discussed in detail.

1. Introduction

Singular two-point boundary value problems are encountered in many physical models such as electro-hydrodynamics and some thermal explosions, and thus, have been investigated by using a variety of numerical methods [1-5] .

We consider the class of singular two-point boundary value problems showing up frequently in applied mathematics,

$$(p(x)y')' = p(x)f(x,y), \quad 0 < x \le 1, \tag{1}$$

with the boundary conditions

$$y(0) = A, \quad \alpha y(1) + \beta y'(1) = B, \tag{2}$$

or

$$y'(0) = 0, \quad \alpha y(1) + \beta y'(1) = B, \tag{3}$$

where $\alpha > 0, \beta \ge 0$, and A, B are two finite constants.

The following conditions apply to the function $p(x)$:

1) $p(x) > 0$ on $(0,1]$, 2) $p(x) > 0 \in C^1(0,1]$, 3) $p(x) = x^{b_0} g(x)$ on $[0,1]$ and for some $r > 1, 1/g(x)$ is analytic in $\{z: |z| < r\}$.

Also, the function $f(x,y)$ have been satisfied the following conditions: 1) $f(x,y) \in \{[0,1] \times \mathbb{R}\}$ is continuous, 2) $\frac{\partial f}{\partial y}$ exists and is continuous, 3) $\frac{\partial f}{\partial y} \ge 0, \forall 0 \le x \le 1$ and all real y.

The problem (1) has a unique solution under the conditions (2) or (3) $(\alpha = 1, \beta = 0)$ [4-5].

In most cases, it is not possible to solve the singular boundary value problems analytically. However, there are some numerical/approximate methods used in the literature, for instance, finite difference methods[6-13], finite element methods [14-16], spline methods [17], differential quadrature methods [18-23] and series based methods [24-25].

2. Methods

2.1. Differential quadrature method (DQM)

The DQM was presented by Bellman at the beginning of the 1970s for solving differential equations [18]. In the DQM, derivatives of a function with respect to a coordinate direction are expressed as linear weighted sums of all the functional values at all grid points along that direction. In this study we used the polynomial-based differential quadrature (PDQ) but a Fourier expansion-based differential quadrature can also be used depending on the physical structure of the problem [19,22].

2.2. Finite difference method (FDM)

The finite difference approaches for derivatives are one of the simplest and oldest methods for solving differential equations in the early 18th century. To

* Corresponding author

solve differential equations numerically we can replace the derivatives in the equation with finite difference approximations on a discretized domain. A number of algebraic equations transformed from the differential equation can be solved by using a suitable method [26]. In this study, we used the second-order finite difference (FD2) approximation and the fourth-order finite difference (FD4) approximation for solving the model equations. The details can be found, for instance, in reference [11].

2.3. Finite element method (FEM)

The FEM is a numerical method that appeared at the beginning of the 1950s to solve various problems of science [27-28]. This method is based on the principle of mesh discretization of a continuous domain into a set of discrete subdomains, usually called elements. The process is to construct an integral of the inner product of the residual and the weight functions and set the integral to zero. In this study, we used the Galerkin FEM for solving the model problem. The process steps of the method can also be found in the literature [15,27].

In summary, as pointed out in the above references, the FDM can be considered to be simpler and easier to implement than the FEM. However, the FEM can be seen to be relatively more effective on nonlineartiy and irregular domains.

It is possible to find the results with sufficient accuracy by dividing the solution region into many elements in the FEM. If solution is achieved by separating the element into too many subregions, the required computational capacity and time will increase. However, the DQM requires less number of grids comparison to its rival methods. The FDM is easy to use and produce computer codes but is relatively less accurate.

In order to observe those advantages and disadvantages of the methods properly, here, we used comparatively the three methods in solving the singular two-point BVPs.

3. Numerical illustrations

To demonstrate the efficiency and accuracy of the DQM, the FDM and the FEM, we have solved the following two problems(the first is a linear and the second is a non-linear) whose exact solutions are known.

The performances of the approches are measured by the absolute and relative errors.

Problem 1 (Kumar [28])

$$(xy')' = -x\cos x - \sin x \ , \quad 0 < x \le 1$$

$$y'(0) = 0, \ \ y(1) = \cos 1,$$

with the exact solution $y(x) = \cos x$.

We solved this problem using the DQM, the FDM and the FEM with $p(x) = x, \ f(x,y) = f(x) = -\cos x - (\sin x / x)$in Equation (1) and $\alpha = 1, \ \beta = 0, B = \cos 1$ in Equation (3).

We used here the MATLAB code we produced for each method.

The relative and absolute errors are presented, for N=7 in Table 1 and for N=50 in Table 2 for uniform grids, respectively. The relative errors are plotted, for N=7 and N=30 in Figures 1,2 respectively.

Table 1. Comparison of the relative and absolute errors in *Problem 1* for N=7

(a) Sub-table 1.

x	FD2 Relative Error	FD4 Relative Error	FEM Relative Error	DQM Relative Error
0	7E-02	2E-04	8E-03	2E-07
0.166	3E-02	2E-04	3E-03	6E-08
0.333	2E-02	1E-04	2E-03	2E-08
0.5	1E-02	1E-04	1E-03	7E-08
0.666	9E-03	1E-04	8E-04	1E-07
0.833	4E-03	1E-04	3E-04	2E-07
1	0	0	0	0

(b) Sub-table 2.

x	FD2 Absolute Error	FD4 Absolute Error	FEM Absolute Error	DQM Absolute Error
0	7E-02	2E-04	8E-03	2E-07
0.166	3E-02	2E-04	3E-03	6E-08
0.333	2E-02	1E-04	2E-03	2E-08
0.5	1E-02	1E-03	1E-03	6E-08
0.666	7E-03	1E-04	6E-04	1E-07
0.833	3E-03	1E-04	2E-04	1E-07
1	0	0	0	0

Table 2. Comparison of the relative and absolute errors in *Problem 1* for N=50

(a) Sub-table 1.

x	FD2 Relative Error	FD4 Relative Error	FEM Relative Error	DQM Relative Error
0	1E-03	3E-09	1E-04	7E-03
0.16	5E-04	2.4E-09	5.5E-05	1E-03
0.34	3E-04	2.1E-09	3.1E-05	5E-04
0.53	2E-04	1.8E-09	1.8E-05	1E-04
0.65	1E-04	1.6E-09	1.2E-05	5E-04
0.85	6.4E-05	1E-09	4.9E-06	1E-03
1	0	0	0	0

(b) Sub-table 2.

x	FD2 Absolute Error	FD4 Absolute Error	FEM Absolute Error	DQM Absolute Error
0	1E-03	3E-09	1E-04	7E-03
0.16	5E-04	2.3E-09	2.4E-05	1E-03
0.34	3E-04	2E-09	2.9E-05	5E-04
0.53	1E-04	1.6E-09	1.6E-05	1E-04
0.65	1E-04	1.3E-09	1E-05	4E-04
0.85	4.2E-05	6.9E-10	3.2E-06	8E-04
1	0	0	0	0

Tables show the absolute and relative errors for the DQM, FD2, FD4 and FEM results. The error measurements stemmed from the DQM is less than the others, as long as less number of grids is used. When the number of grids increases, the most effective results obtained from the FD4 among the methods of interest.

Problem 2 (Kumar [29,30])

$$(x^{\alpha_0}y')' = \beta_0 x^{\alpha_0+\beta_0-2}e^{y}\left(\beta_0 x^{\beta_0}e^{y} - \alpha_0 - \beta_0 + 1\right),$$
$$0 < x \leq 1$$
$$y(0) = -\ln(4)\,, \quad y(1) = -\ln(5),$$

with the exact solution $y(x) = \ln\left(\frac{1}{4} + x^{\beta_0}\right)$ where $0 \leq \alpha_0 < 1$.

We solved this problem using the DQM and the FDM with $p(x) = x^{\alpha_0}, f(x,y) = \beta_0 x^{\alpha_0+\beta_0-2}e^{y}$ $\left(\beta_0 x^{\beta_0}e^{y} - \alpha_0 - \beta_0 + 1\right)$ in equation(1) and $\alpha = 1$, $\beta = 0, A = -\ln(4)\,, B = -\ln(5)$ in equation (3).

The relative and absolute errors are presented, for N=7 in Table 3 and for N=15 in Table 4 for uniform grids, respectively. The relative errors are plotted, for N=7 in Figure 3, for N=11 in Figure 4, respectively, with $\alpha_0 = 0.5,\ \beta_0 = 1$.

From the produced results both qualitatively and quantitatively, the DQM has been seen to be the most accurate one among the methods for the problems of interest.

Table 3. Comparison of the relative and absolute errors in *Problem 2* for N=7

(a) Sub-table 1.

x	FD2 Relative Error	FD4 Relative Error	DQM Relative Error
0	0	0	0
0.166	1E-05	3E-07	1.3E-09
0.333	1E-05	2.9E-07	9.3E-11
0.5	8.7E-06	2.6E-07	9.6E-10
0.666	5.9E-06	2.2E-07	1.6E-09
0.833	2.9E-06	1.8E-07	2.4E-09
1	0	0	0

(b) Sub-table 2.

x	FD2 Absolute Error	FD4 Absolute Error	DQM Absolute Error
0	0	0	0
0.166	1.4E-05	4.4E-07	1.8E-09
0.333	1.5E-05	4.3E-07	1.3E-10
0.5	1.3E-05	3.9E-07	1.4E-09
0.666	9.1E-06	3.5E-07	2.5E-09
0.833	4.6E-06	2.9E-07	3.7E-09
1	0	0	0

Table 4. Comparison of the relative and absolute errors in *Problem 2* for N=15 .

(a) Sub-table 1.

x	FD2 Relative Error	FD4 Relative Error	DQM Relative Error
0	0	0	0
0.142	2.3E-06	3.1E-09	1.1E-13
0.357	2.3E-06	1.8E-09	1.9E-13
0.5	1.9E-06	1E-09	2.3E-13
0.642	1.3E-06	2.5E-10	2.6E-13
0.857	5.4E-07	7.6E-10	2.9E-13
1	0	0	0

(b) Sub-table 2.

x	FD2 Absolute Error	FD4 Absolute Error	DQM Absolute Error
0	0	0	0
0.142	3.3E-06	4.4E-09	1.6E-13
0.357	3.4E-06	2.7E-09	2.8E-13
0.5	2.8E-06	1.5E-09	3.5E-13
0.642	2.1E-06	3.9E-10	4.1E-13
0.857	8.6E-07	1.2E-09	4.6E-13
1	0	0	0

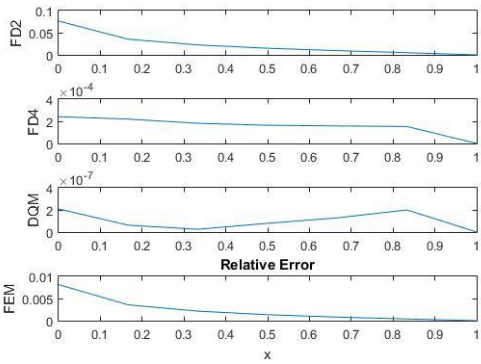

Figure 1. Comparison of relative error in *Problem 1* for N=7

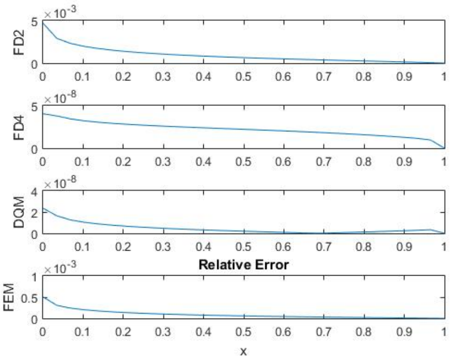

Figure 2. Comparison of relative error in *Problem* 1 for N=30

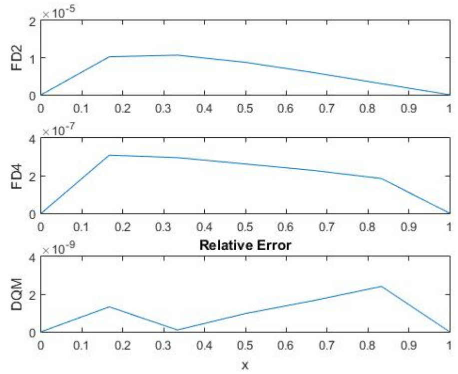

Figure 3. Comparison of relative error in *Problem 2* for N=7

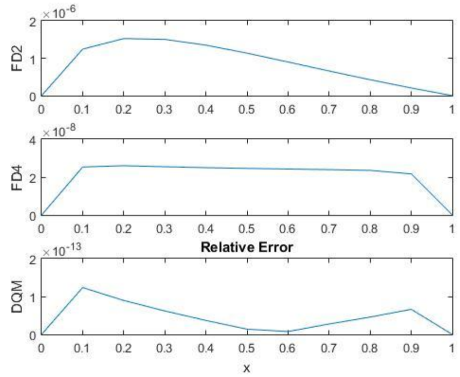

Figure 4. Comparison of relative error in *Problem 2* for N=11

4. Conclusion

This study has focused on the singular two-point BVPs with a linear or non-linear nature through different numerical methods. It has been concluded that the DQM is the most accurate one among the corresponding methods for this problem. However, the FDM and FEM can take opportunity to catch the same accuracy for very large number of grid points.

Note that for higher dimensional problems, the same discussion could be an important milestone in numerical modeling. In such a probable discussion, especially the advantages of the FEM and FDM may come out.

References

[1] Caglar, H., Caglar, N., & Ozer, M., (2009). B-spline solution of non-linear singular boundary value problems arising in physiology. *Chaos, Solitons and Fractals*, 39(3), 1232–1237.

[2] Argub, O., Hammour, Z.A., Momani, S., & Shawagfeh, N., (2012). Solving Singular Two-point Boundary Value Problems Using Genetic Algorithm. *Abstract and Applied Analysis*, 2012, 25 pages.

[3] Jain, R.K., & Jain, P., (1989). Finite difference methods for a class of singular two-point boundary value problems. *International Journal of Computer Mathematics*, 27(2), 113-120.

[4] Pandey, R.K., (1996). On a class of weakly regular singular two point boundary value problems-I. *Nonlinear Analysis*, 27(1), 1-12.

[5] Pandey, R.K., (1997). On a class of regular singular two point boundary value problems. *Journal of Mathematical Analysis and Applications*, 208(2), 388-403.

[6] Duffy, D. J., (2000). *Finite difference methods in financial engineering*. John Wiley-Sons, England.

[7] Burden, R.L., & Faires, J.D., (2001). *Numerical analysis*. Brooks-Cole.

[8] Causon, D.M. & Mingham, C.G., (2010). *Introductory finite difference methods for PDEs*. Bookboon.

[9] Yang, W.Y., Cao, W., Chung, T., & Morsis, J., (2005). *Applied numerical methods using Matlab*. Wiley-Interscience.

[10] Randall, J.LeVeque, (2007). *Finite difference methods for ordinary and partial differential equations*. Society for Industrial and Applied Mathematics.

[11] Fornberg, B., (1988). Generation of finite difference formulas on arbitrarily spaced grids, *Mathematics of* Computation, 51(184), 699-706.

[12] Chapra, S.C., & Canale R.P., (2015). *Numerical Methods for Engineering*. 7th Edition, Mc Graw Hill.

[13] Bergara, A., (2011). *Finite Difference Numerical Methods of Partial Differential Equations in Finance with MATLAB*. Master and Banca.

[14] Reddy, J.N., (2005). *An introduction to the finite element method*. 3rd Ed., McGraw-Hill Education (ISE Editions).

[15] Zienkiewicz, O.C., Taylor, R.L., & Zhu, J.Z., (2005). *The Finite Element Method: Its Basis and Fundamentals*. 6th ed., Butterworth-Heinemann.

[16] Bayraktar, M., (2017). *Solution of Burgers equation using Petrov-Galerkin Finite element method*. M.Sc. Thesis, Pamukkale University.

[17] Mohanty, R.K., & Evans, D.J., (2003). A fourth order cubic spline alternating group explicit method for non-linear singular two point boundary value problems, *International Journal of Computer Mathematics*, 80(4), 479-492.

[18] Bellman, R., Kashef, B.G., & Casti, J., (1972). Differential quadrature: a technique for the rapid solution of nonlinear partial differential equations. *Journal of Computational Physics*, 10(1), 40-52.

[19] Yücel, U., & Sari, M., (2009). Differential

quadrature method (DQM) for a class of singular two-point boundary value problems. *International Journal of Computer Mathematics*, 86(3), 465-475.

[20] Bert, C.W., & Malik, M., (1996). Differential quadrature method in computational mechanics: a review. *Applied Mechanics Reiews*, 49(1), 1-28.

[21] Chen, R.P., (2005). One-dimensional non-linear consolidation of multi-layered soil by differential quadrature method. *Computers and Geotechnics*, 32(5), 358-369.

[22] Shu, C., (2000). *Differential Quadrature and Its Application in Engineering*, Springer-Verlag, London.

[23] Shu, C., & Richards, B.E., (1992). Application of generalized differential quadrature to solve two-dimensional incompressible Navier-Stokes equations. *International Journal for Numerical Methods in Fluids*, 15(7), 791-798.

[24] Roul, P.& Biswal, D., (2017). A new numerical approach for solving a class of singular two-point boundary value problems. *Numerical Algorithms*, 75(3), 531–552.

[25] Roul, P., (2017). On the numerical solution of singular two-point boundary value problems: A domain decomposition homotopy perturbation approach..*Mathematical Methods in the Applied Sciences*, (accepted).

[26] Urroz, G.E., (2004). *Numerical Solution to Ordinary Differential Equations.* Holden Day, USA.

[27] Courant, R., (1943). Variational methods for the solution of problems of equilibrium and vibrations. *Bulletin of the American Mathematical Society*, 49(1), 1–23.

[28] Kumar, M., (2002). A three point finite difference method for a class of singular two-point boundary value problems. *Journal of Computational and Applied Mathematics*, 145(1), 89-97.

[29] Aziz, T., & Kumar, M., (2001). A fourth-order finite-difference method based on non-uniform mesh for a class of singular two-point boundary value problems , Journal of Computational and Applied Math., 136(1-2), 337-342.

[30] Singh, R., & Kumar, J., (2013). Solving a Class of Singular Two-point Boundary Value Problems Using New Modified Decomposition Method. *Computational Mathematics*, 2013, 11 pages.

Selmahan Selim *is currently Assistant Professor in the Department of Mathematics at Yildiz Technical University (YTU). She received her Ph.D., M.Sc. and B.Sc. degrees from YTU. Her research areas are differential equations, numerical analysis and computational methods.*

Gozde Elver *was born in Istanbul, Turkey in 1991. She received degrees of Bachelor of Science (B. Sc.) in 2015 at Yildiz Technical University. She is continuing her Master of Science at Yildiz Technical University since 2014. Her research interests are numerical analysis and applied mathematics.*

Murat Sari *was born in Adana, Turkey in 1969. He received degrees of Bachelor of Science in 1991 at Ondokuz Mayis University, and then Master of Philosophy (M.Phil-transferred) in 1997 and Ph.D. at the University of South Wales, in Mathematics, in 2000. Dr. Sari is currently an Associate Professor in the Department of Mathematics at Yildiz Technical University, Istanbul, Turkey. His research interests are computational methods, optimization algorithms, nonlinear behavior, mathematical modelling, biophysical modelling, medical modelling and economical modelling. He has over 40 international journal publications and nearly 40 international conference presentations with a couple of book chapters.*