RESEARCH ARTICLE

Symmetry solution on fractional equation

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ARTICLE INFO	ABSTRACT
	As we know nearly all physical, chemical, and biological processes in nature can be described or modeled by dint of a differential equation or a system of differential equations, an integral equation or an integro-differential equation. The differential equations can be ordinary or partial, linear or nonlinear. So, we concentrate our attention in problem that can be presented in terms of a differential equation with fractional derivative. Our research in this work is to use symmetry transformation method and its analysis to search exact solutions to nonlinear fractional partial differential equations.

1. Introduction

The fractional derivatives that are about three centuries ago were presented [1, 2]. Fractional differential equations have been used successfully to describe many complex nonlinear phenomena and dynamic processes in physics, electromagnetics [3, 4], acoustics, astrophysics [5, 6], viscoelasticity, chemistry, electrochemistry, etc. [1, 2, 7, 8]. As for the methods for solving such equations, there is no effective general method. But there have been formulated and applied methods like Adomian decomposition method [9], differential transform method [10], modified simple equation method [11,12], Lie symmetry analysis [13,14] and so on.

Lie symmetry analysis is powerful and universal tool for searching solution of linear and nonlinear partial differential equations and it has been widely applied for studying the invariance properties of partial differential equation (PDE) [15]. A symmetry of a PDE is any transformation that each solution surface of the PDE is mapped to another solution surface of the same PDE, i.e. leaves invariant its solution space. So, by using the Lie

symmetry, the equation can be transformed into a nonlinear fractional ODE.

For construction a symmetry reductions of a fractional equation we investigated the symmetry properties by using the group analysis method and presented the vector fields the equation based on the point symmetry [13, 14]. It is shown that our equation could be transformed into a nonlinear fractional ordinary differential equation with the new independent variable.

In this work by using the Lie group, we investigate the symmetry properties of fractional partial differential equation (FPDE)

$$\frac{\partial^\alpha u}{\partial t^\alpha} = (g(u)u_{xx})_x, \qquad (1)$$

and find the correspondence infinitesimal operators and then construct some exact solution of these equations, in particulary the solution for fractional linear KdV equation.

The outline of this paper is as follows: in section 2 we will give general definitions and formulas of fractional derivative and symmetry analysis, also we show the application of symmetry group to

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fractional differential Eq. (1). In section 3 presented some exact solutions by using symmetry reductions.

2. Preliminaries

2.1. Lie symmetry analysis of fractional PDE

Consider a time FPDE with two independent variables and $0 < \alpha \leq 1$ is given as following:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (g(u)u_{xx})_x = 0, \qquad (2)$$

here fractional derivative are considered in the Riemann-Liouville terms.

Suppose f be integrable on $[a,b] \in \mathbb{R}$ and $n-1 < \alpha < n$, $n \in \mathbb{N}$. Then *Riemann-Liouville fractional derivative* is defined as

$$
\begin{aligned}
{}_0 D_t^\alpha f(t) &= \frac{\partial^\alpha f(t)}{\partial t^\alpha} \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau.
\end{aligned}
\qquad (3)
$$

Let f be integrable on $[0, \infty)$, and piecewise continuous function on $(0, \infty)$ and $Re\alpha > 0$, $t > 0$. Then *Riemann-Liouville fractional integral* is defined by

$$_0 I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx. \qquad (4)$$

One parameter Lie symmetry transformations are determined as

$$\bar{t} = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2),$$

$$\bar{x} = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \qquad (5)$$

$$\bar{u} = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$

where $\epsilon > 0$ is a infinitesimals parameter with $\xi = \frac{d\bar{x}}{d\epsilon}\mid_{\epsilon=0}$, $\tau = \frac{d\bar{t}}{d\epsilon}\mid_{\epsilon=0}$ and $\eta = \frac{d\bar{u}}{d\epsilon}\mid_{\epsilon=0}$ which will be determined.

After applying transformation (5) to usual partial derivatives u_x, u_{xx} and u_{xxx} it gives the following extensions [15]:

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial u}{\partial x} + \varepsilon\eta_1^x + O(\varepsilon^2),$$

$$\frac{\partial^2 \bar{u}}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta_2^x + O(\varepsilon^2), \qquad (6)$$

$$\frac{\partial^3 \bar{u}}{\partial x^3} = \frac{\partial^3 u}{\partial x^3} + \varepsilon\eta_3^x + O(\varepsilon^2).$$

Here η_1^x, η_2^x and η_3^x are defined by formulae

$$\eta_1^x = D_x\eta - u_x D_x\xi - u_t D_x\tau,$$

$$\eta_2^x = D_x\eta_1^x - u_{xx}D_x\xi - u_{xt}D_x\tau, \qquad (7)$$

$$\eta_3^x = D_x\eta_2^x - u_{xxx}D_x\xi - u_{xxt}D_x\tau,$$

where D_x is the total derivative

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots.$$

And the αth extended infinitesimal related to Riemann-Liouville fractional time derivative is [16] as

$$\frac{\partial^\alpha \bar{u}}{\partial t^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta_\alpha^t + O(\varepsilon^2). \qquad (8)$$

Here η_α^t has following form:

$$
\begin{aligned}
\eta_\alpha^t = D_t^\alpha(\eta) \;&+\; \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(uD_t\tau) \\
&-\; D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}u,
\end{aligned}
$$

and the operator D_t^α is the total fractional derivative operator. Using the generalized Leibnitz rule [2]

$$
\begin{aligned}
D_t^\alpha(f(t)g(t)) &= \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n}f(t)D_t^n g(t), \\
\binom{\alpha}{n} &= \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.
\end{aligned}
$$

Thus infinitesimal η_α^t is modified to

$$\eta_\alpha^t = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha(\tau_t + u_t\tau_u))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu$$

$$+ \sum_{n=1}^\infty \left[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{\alpha+1}\tau \right] D^{\alpha-n}u$$

$$- \sum_{n=1}^\infty \binom{\alpha}{n}(D_t^n\xi)(D_t^{\alpha-n}u_x). \qquad (9)$$

The corresponding vector field V associated with transformations (5) can be written as

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \quad (10)$$

Applying the third prolongation $pr^{(3)}V$ to Eq. (2), we will get

$$pr^{(3)}V(\Delta)\mid_{\Delta=0} = 0, \quad \Delta = \frac{\partial^\alpha u}{\partial t^\alpha} - (k(u)u_{xx})_x,$$

where the operator $pr^{(3)}V$ takes the following form:

$$pr^{(3)}V = V + \eta_\alpha^t \partial_{\partial_t^\alpha u} + \eta_1^x \partial_{u_x} + \eta_2^x \partial_{u_{xx}} + \eta_3^x \partial_{u_{xxx}}.$$

Our equation (2) can be written in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} - g'(u)u_{xx}u_x - g(u)u_{xxx} = 0. \qquad (11)$$

Substitution of transformations (5), (6) and (8) into (11) we get

$$\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} - g'(\bar{u})\bar{u}_{\bar{x}\bar{x}}\bar{u}_{\bar{x}} - g(\bar{u})\bar{u}_{\bar{x}\bar{x}\bar{x}}$$
$$= \frac{\partial^\alpha u}{\partial t^\alpha} - g'(u)u_{xx}u_x - g(u)u_{xxx}$$
$$+ \epsilon(\eta_\alpha^t - \eta(g''u_xu_{xx} + g'u_{xxx})$$
$$- g'u_x\eta_2^x - g'u_{xx}\eta_1^x - g\eta_3^x) + O(\epsilon^3).$$

So we find that the functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ must satisfy the symmetry condition

$$\eta_\alpha^t - \eta(g''u_xu_{xx} + g'u_{xxx}) - g'u_x\eta_2^x - g'u_{xx}\eta_1^x - g\eta_3^x = 0. \qquad (12)$$

Solving the Eq. (12) along with Eq. (2) and substituting the extended infinitesimal (7), (9) into the Eq. (12) we get following characteristic system:

$$\xi_u = \xi_t = \tau_u = \tau_x = \eta_{uu} = 0,$$

$$-\alpha g'\tau_t - g''\eta + 3g'\xi_x - g'\eta_u = 0,$$

$$-\alpha g\tau_t - g'\eta + 3g\xi_x = 0,$$

$$-g'\eta_{xx} - 3g\eta_{xxu} = 0,$$

$$-2g'\eta_{xu} + g'\xi_{xx} = 0, \qquad (13)$$

$$-g'\eta_x - 3g\eta_{xu} + 3g\xi_{xx} = 0,$$

$$\eta_{ut} - \frac{\alpha-1}{2}\tau_{tt} = 0,$$

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - g\eta_{xxx} = 0.$$

Solving these equations we investigate generating infinitesimal operators as following.

Case 1: For arbitrary $g(u)$ and $0 < \alpha \le 1$ there are three infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t}.$$

Case 2: For $g(u) = 1$ and $0 < \alpha \le 1$ there are two additional infinitesimal operators

$$X_4 = u\frac{\partial}{\partial u}, \quad X_\infty = h(t,x)\frac{\partial}{\partial u},$$

where the function $h(t,x)$ satisfies the linear fractional KdV equation $D_t^\alpha h = h_{xxx}$.

Case 3: For $g(u) = u^b$ with $b \ne 0$ and $0 < \alpha \le 1$ there are two additional infinitesimal operators

$$X_4 = x\frac{\partial}{\partial x} + \frac{3u}{b}\frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial t} - \frac{\alpha u}{b}\frac{\partial}{\partial u}.$$

Case 4: For $g(u) = u^b$ with $b = -3$ and $0 < \alpha \le 1$ there is one additional infinitesimal operator

$$X_6 = x\frac{\partial}{\partial x} - \frac{6t}{\alpha-3}\frac{\partial}{\partial t} - \frac{(3\alpha-3)u}{\alpha-3}\frac{\partial}{\partial u}.$$

Case 5: For $g(u) = e^u$ with integer α ($\alpha = 1$) there are two additional infinitesimal operators

$$X_4 = x\frac{\partial}{\partial x} + 3\frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial t} - \alpha\frac{\partial}{\partial u}.$$

Theorem 1. *The equation $D_t^\alpha u = (g(u)u_{xx})_x$ with $g(u) = e^u$ and $0 < \alpha < 1$ has no additional symmetries.*

Proof. For $g(u) = e^u$ and $0 < \alpha < 1$ the system (13) transforms to

(1) $\xi_u = \xi_t = \tau_u = \tau_x = \eta_{uu} = 0,$
(2) $-\alpha e^u\tau_t - e^u\eta + 3e^u\xi_x - e^u\eta_u = 0,$
(3) $-\alpha e^u\tau_t - e^u\eta + 3e^u\xi_x = 0,$
(4) $-e^u\eta_{xx} - 3e^u\eta_{xxu} = 0,$
(5) $-2e^u\eta_{xu} + e^u\xi_{xx} = 0,$
(6) $-e^u\eta_x - 3e^u\eta_{xu} + 3e^u\xi_{xx} = 0,$
(7) $\eta_{ut} - \frac{\alpha-1}{2}\tau_{tt} = 0,$
(8) $\frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - e^u\eta_{xxx} = 0.$

The first equation gives us that $\xi = A(x)$, $\tau = B(t)$ and $\eta = C(x,t)u + D(x,t)$. So from 2. and 3. equations we get $C = 0$, also from 5. and 7. equations we find $A = c_1 x + c_2$ and $B = c_3 t + c_4$. Thus by finding the corresponding derivatives and putting them to equation 2. we find that D is constant, but from 8. equation $\frac{\partial^\alpha D}{\partial t^\alpha} = 0$, which gives us $D = c_5 t^{\alpha-1}$ thereby we have obtained a contradiction. It means that for $0 < \alpha < 1$ and $g(u) = e^u$ there is not any additional symmetries. $\square$

3. Symmetry reductions and some exact solutions

3.1. The exact solution for $g(u) = 1$

For $g(u) = 1$ we have linear fractional KdV equation $D_t^\alpha u = u_{xxx}$ with infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t},$$

$$X_4 = u\frac{\partial}{\partial u}, \quad X_\infty = h(t,x)\frac{\partial}{\partial u}.$$

By composition of X_1 and X_4 we get generator

$$X_1 + kX_4 = \frac{\partial}{\partial x} + ku\frac{\partial}{\partial u},$$

where $k \in \mathbb{R}$. Then solution under the group has the form $u(t,x) = e^{kx}\phi(t)$, where $\phi(t)$ satisfies the equation

$$D_t^\alpha \phi(t) = k^3\phi(t),$$

and thus

$$u(t,x) = e^{kx}t^{\alpha-1}E_{\alpha,\alpha}(k^3t^\alpha).$$

Here $E_{\alpha,\beta}(x)$ is a Mittag-Leffler function

$$E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(\alpha m + \beta)}.$$

3.2. The exact solution for $g(u) = u^b$

For $g(u) = u^b$ we have $D_t^\alpha u = bu^{b-1}u_{xx}u_x + u^b u_{xxx}$ with infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t},$$

$$X_4 = x\frac{\partial}{\partial x} + \frac{3u}{b}\frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial t} - \frac{\alpha u}{b}\frac{\partial}{\partial u}.$$

With $X_4 = x\frac{\partial}{\partial x} + \frac{3u}{b}\frac{\partial}{\partial u}$ solution under the group has the form

$$u(t,x) = x^{3/b}\phi(t),$$

where $\phi(t)$ satisfies the equation

$$D_t^\alpha \phi(t) = \frac{3(9 - b^2)}{b^3}\phi^{b+1}(t).$$

If $b = 3$ then we derive $D_t^\alpha \phi(t) = 0$, which gives $\phi(t) = Ct^{\alpha-1}$, $C \in \mathbb{R}$. Therefore

$$u(x,t) = Cx^{3/b}t^{\alpha-1}.$$

Also if $b = -1$ then $D_t^\alpha \phi(t) = -24$, which gives

$$u(t,x) = -\frac{24x^{-3}t^\alpha}{\Gamma(\alpha + 1)}.$$

4. Conclusion

For construction a symmetry reductions of the fractional equation (1) we investigated the symmetry properties by using the symmetry analysis method and presented different infinitesimal operators. We obtained solutions for two particular equations with some generator operators. Also we showed that the equation $D_t^\alpha u = (e^u u_{xx})_x$ for $0 < \alpha < 1$ has only general symmetries with $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t}$ infinitesimal operators. The symmetry analysis or Lie group analysis is a very powerful method and is worthy of studying further to searching the solutions and symmetry properties of nonlinear partial differential equations and fractional nonlinear partial differential equations.

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