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RESEARCH ARTICLE

A Numerical Treatment Based on Haar Wavelets for Coupled KdV Equation

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ABSTRACT

In this paper, numerical solutions of one dimensional coupled KdV equation has been investigated by Haar Wavelet method. Time derivatives given in this equation are discretized by finite differences and nonlinear terms appearing in the equations are linearized by some linearization techniques and space derivatives are discretized by Haar wavelets. For examining performance of the proposed method, single soliton solution and conserved quantities of some test problems are used. Also error analysis of numerical scheme is investigated and numerical results are compared with some results already existing in the literature.

1. Introduction

Analytical or numerical solutions of nonlinear problems has a crucial importance in all areas of physical, mathematical and engineering sciences. Nonlinear equations have interesting characteristics for physical systems and they can be understood by the solution of these problems either analytically or numerically. In general, finding the analytical solution of nonlinear problems is very hard or even impossible for some cases, because of that, numerical solutions of these equations are particularly important.

In this paper, we will consider coupled KdV (cKdV) equation which is an important nonlinear evolution equation and given in the following form

$$\begin{aligned} u_t - 6auu_y - 2bvv_y - au_{yyy} &= 0, \\ v_t + 3uv_y + v_{yyy} &= 0, \quad y_1 \le y \le y_2 \end{aligned} \tag{1}$$

with the initial conditions

$$u(y,0) = f(y), \quad v(y,0) = g(y), \quad y \in [y_1, y_2] \tag{2}$$

and the boundary conditions

$$\begin{aligned} u(y_1,t) = u(y_2,t) = u_y(y_2,t) = 0 \quad t \in [0,T] \\ v(y_1,t) = v(y_2,t) = v_y(y_2,t) = 0 \quad t \in [0,T] \end{aligned} \tag{3}$$

where a and b are constants [1]. These equations describe interaction of two long waves with different dispersion relations, it is introduced by Hirota and Satsuma [1] in 1981. A lot of long waves with weak dispersion such as internal, acoustic, and planetary waves in geophysical hydrodynamics are related with (cKdV) equation [2, 3].

Because of the importance of cKdV system among evolution equations it is studied by many researchers both analytically and numerically: A

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difference scheme given in [4] by Zhu for the periodic initial-boundary value problem of the cKdV Equation. Adomian decomposition method is used to solve this system by Kaya and Inan [5]. Tanh method is used to find solution of the system by Fan [6]. By using the Jacobian elliptic function expansion approach and Hermite transformation Ma and Zhu [7] have obtained some new exact solutions of the cKdV equations. cKdV equation is solved by Assas [8] by using variational iteration method. Homotopy analysis method is used by Abbasbandy [9] for solving the generalized cKdV system. Analytic solutions of nonlinear cKdV equations are studied by Al-Khaled et al. [10] by using tanh and the He's variational iteration methods. Mokhtari and Mohammadi [11] solved a coupled system of nonlinear partial differential equations by using Exp-function method. Ismail solved cKdV system by using finite difference and finite element methods [12–14]. Halim et al. [2, 3] introduced a numerical scheme for general cKdV systems. For the periodic initial boundary value problem of the cKdV system a finite difference scheme produced by Wazwaz [15]. By using collocation method and quintic splines Ismail [16] solved cKdV system. A quadratic B-spline Galerkin approach applied by Kutluay and Ucar [17] for solving cKdV system. Ismail and Ashi [18] used a Petrov-Galerkin method and product approximation technique to solve numerically the Hirota-Satsuma cKdV equation.

In this paper, for obtaining numerical solutions of systems (1), we have employed Haar wavelet method. The paper is organized as follows; In Section 2, an introduction about Haar wavelets is given. In Section 3, time and space discretizations are described and error analysis is given. Numerical results are given in Section 4 and finally the paper is concluded in Section 5.

2. Haar wavelets

The wavelet methods have been attracting more attention lately in solving differential equations numerically since they were first applied to solve differential equations in early 1990s. Before explaining the method, we will give basic information about Haar wavelets. They are special kind of wavelets, introduced in 1910 by Alfred Haar and they are the simplest of all possible wavelets with compact support. They are box shaped functions, defined in the interval [0,1). Together they form an orthonormal system in the space of square interable functions. In order to use these wavelets in differential equations one must solve the discontinuity problem of Haar wavelets. This problem was overcome by Chen and Hsiao [19], they used integral method in which the highest derivative of the function in the dierential equation is expanded into Haar series. After this achievement researchers have been using Haar wavelets to obtain numerical solutions of differential equations because of their simplicity and computational features. Recently, many authors have used Haar wavelet method for solving ordinary and partial differential equations [20–31]. Especially high order pdes like KdV and fractional coupled KdV equations are considered in [32, 33].

Here we give an explanation of the method, starting with the definition of the ith Haar wavelet as follows for $x \in [0, 1]$

$$h_i(x) = \begin{cases} 1, & \text{for } x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1, & \text{for } x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right] \\ 0, & \text{elsewhere} \end{cases} \tag{4}$$

where $m = 2^j$, $j = 0, 1, ..., J$ and $k = 0, 1, ..., m-1$ is dilation parameter and translation parameter, respectively. The index of h_i in Eq. (4) can be found by relation $i = m + k + 1$. For the lowest values of $m = 1$, $k = 0$, we have $i = 2$ and the greatest value of i will be $i = 2M = 2^{J+1}$; where J is the maximal resolution of the wavelet. For $i = 1$ we have Haar scaling function

$$h_1(x) = \begin{cases} 1, & \text{for } x \in [0, 1) \\ 0, & \text{elsewhere} \end{cases}$$

Any function $u(x) \in L^2[0, 1)$ can be expanded into Haar series as

$$u(x) = \sum_{i=1}^{\infty} c_i h_i(x),$$

where c_i can be found by

$$c_i = 2^j \int_0^1 u(x)h_i(x)dx,$$
$$i = 2^j + k,\ j \geq 0,\ 0 \leq k < 2^j.$$

Practically, for approximating a square integrable function $u(x)$ on interval $[0, 1)$ finite terms of Haar series are needed, hence one may write

$$u(x) = \sum_{i=1}^{2M} c_i h_i(x) = c^T_{(2M)} h_{(2M)}(x),$$

In the above relation $M = 2^j$, T denotes transpose and

$$c_{(2M)}^T = [c_1, c_2, ..., c_{(2M)}]$$
$$h_{(2M)}(x) = [h_1(x), h_2(x), ..., h_{(2M)}(x)]^T.$$

To employ Haar wavelet method for solving any order partial differential equation one needs the following integrals

$$p_{i,1}(x) = \int_0^x h_i(x)dx$$
$$p_{i,n+1}(x) = \int_0^x p_{i,n}(x)dx, \quad n = 1, 2, 3, ...$$

general form of the integral is given in [34]

$$p_{i,\alpha}(x) = \begin{cases} 0; \text{ for } x < \zeta_1 \\ \frac{1}{\alpha!}\left(x - \frac{k}{m}\right)^\alpha; \text{ for } x \in [\zeta_1, \zeta_2] \\ \frac{1}{\alpha!}\left[\left(x - \frac{k}{m}\right)^\alpha - 2\left(x - \zeta_2\right)^\alpha\right]; \\ \quad \text{for } x \in [\zeta_2, \zeta_3] \\ \frac{1}{\alpha!}\left[\left(x - \frac{k}{m}\right)^\alpha - 2\left(x - \zeta_2\right)^\alpha + \left(x - \zeta_3\right)^\alpha\right]; \\ \quad \text{for } x > \zeta_3 \end{cases}$$

For the first three integrals following expressions can be found from the above equation

$$p_{i,1}(x) = \begin{cases} x - \zeta_1, & \text{for } x \in [\zeta_1, \zeta_2] \\ \zeta_3 - x, & \text{for } x \in [\zeta_2, \zeta_3] \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

$$p_{i,2}(x) = \begin{cases} \frac{(x-\zeta_1)^2}{2}, & \text{for } x \in [\zeta_1, \zeta_2] \\ \frac{1}{4m^2} - \frac{(\zeta_3 - x)^2}{2}, & \text{for } x \in [\zeta_2, \zeta_3] \\ \frac{1}{4m^2}, & \text{for } x \in [\zeta_3, 1] \\ 0, & \text{elsewhere} \end{cases} \tag{6}$$

$$p_{i,3}(x) = \begin{cases} \frac{(x-\zeta_1)^3}{6}, & \text{for } x \in [\zeta_1, \zeta_2] \\ \frac{x-\zeta_2}{4m^2} - \frac{(\zeta_3 - x)^3}{6}, & \text{for } x \in [\zeta_2, \zeta_3] \\ \frac{x-\zeta_2}{4m^2}, & \text{for } x \in [\zeta_3, 1] \\ 0, & \text{elsewhere} \end{cases}. \tag{7}$$

where ζ_1, ζ_2 and ζ_3 defined as follow.

$$\zeta_1 = \frac{k}{m}, \quad \zeta_2 = \frac{k + 0.5}{m}, \quad \zeta_3 = \frac{k+1}{m}.$$

Once the above integrals are computed we can store the results in memory and we can use them wherever they are needed.

3. Discretization scheme for cKdV

Since we defined Haar wavelets for $x \in [0, 1]$. We have to transform the domain of Eq. (1) into unit interval. By using transformation $x = \frac{y-y_1}{L}$, $L = y_2 - y_1$ the interval $y_1 \leq y \leq y_2$ can be transformed into the unit interval $0 \leq x \leq 1$. Hence Eqs. (1) become

$$u_t - \frac{6}{L}auu_x - \frac{2}{L}bvv_x - \frac{1}{L^3}au_{xxx} = 0,$$
$$v_t + \frac{3}{L}uv_x + \frac{1}{L^3}v_{xxx} = 0.$$

Now we can start to discretization process

3.1. Time discretization for cKdV

To discretize the Eq. (1), we use forward finite differences for time derivatives and time averages of the other terms, as follows

$$\begin{aligned} \frac{u_{n+1} - u_n}{\Delta t} - \frac{6a}{2L}\left[(uu_x)_{n+1} + (uu_x)_n\right] \\ -\frac{2b}{2L}\left[(vv_x)_{n+1} + (vv_x)_n\right] \\ -\frac{a}{2L^3}\left[(u_{xxx})_{n+1} + (u_{xxx})_n\right] = 0, \\ \frac{v_{n+1} - v_n}{\Delta t} + \frac{3}{2L}\left[(uv_x)_{n+1} + (uv_x)_n\right] \\ +\frac{1}{2L^3}\left[(v_{xxx})_{n+1} + (v_{xxx})_n\right] = 0 \end{aligned}$$

For nonlinear term $(uu_x)_{n+1}$, we use $u_{n+1}\left(u_x\right)_n + u_n\left(u_x\right)_{n+1} - \left(uu_x\right)_n$ linearization [35] formula. We make similar linearization for $(vv_x)_{n+1}$ and $(uv_x)_{n+1}$. Hence we get

$$\begin{aligned} u_{n+1} - \frac{\Delta t}{L}3a\left[u_{n+1}(u_x)_n + u_n(u_x)_{n+1}\right] \\ -\frac{\Delta t}{L}b\left[v_{n+1}(v_x)_n + v_n(v_x)_{n+1}\right] \\ -\frac{a\Delta t}{2L^3}(u_{xxx})_{n+1} = u_n + \frac{a\Delta t}{2L^3}(u_{xxx})_n, \\ v_{n+1} + 3\frac{\Delta t}{2L}\left[u_{n+1}(v_x)_n + u_n(v_x)_{n+1}\right] \\ +\frac{\Delta t}{2L}(v_{xxx})_{n+1} = v_n - \frac{\Delta t}{2L^3}(v_{xxx})_n \end{aligned} \tag{8}$$

with the initial conditions

$$u_0 = f(x), \quad v_0 = g(x), \quad x \in [0, 1]$$

and boundary conditions

$$\begin{aligned} u_{n+1}(0) &= f_1(t_{n+1}), \quad u_{n+1}(1) = f_2(t_{n+1}), \\ (u_x)_{n+1}(1) &= f_3(t_{n+1}), \quad n = 0,1,...,N-1 \\ v_{n+1}(0) &= g_1(t_{n+1}), \quad v_{n+1}(1) = g_2(t_{n+1}), \\ (v_x)_{n+1}(1) &= g_3(t_{n+1}), \quad n = 0,1,...,N-1 \quad (9) \end{aligned}$$

where u_{n+1} and v_{n+1} are the solutions of the Eq. (8) at the $(n+1)$th time step.

3.2. Space discretization by Haar wavelets

In this subsection we show how to discretize space derivatives appeared in Eqs. (8), we start with the highest derivative by Haar wavelets. To do so we assume

$$(u_{xxx})_{n+1}(x) = \sum_{i=1}^{2M} c_i h_i(x) = c^T_{(2M)} h_{(2M)}(x) \quad (10)$$

where the row vector $c^T_{(2M)}$ is constant. Integrating Eq. (10) with respect to x from 0 to x, we get the following equation

$$(u_{xx})_{n+1}(x) = (u_{xx})_{n+1}(0) + \sum_{i=1}^{2M} c_i p_{i,1}(x). \quad (11)$$

In Eq. (11), $(u_{xx})_{n+1}(0)$ is unknown so to find it, we need to integrate Eq. (11) from 0 to 1. After that, using boundary conditions (9) we get

$$(u_x)_{n+1}(1) - (u_x)_{n+1}(0) = (u_{xx})_{n+1}(0) + \sum_{i=1}^{2M} c_i p_{i,2}(1)$$

$$(u_{xx})_{n+1}(0) = f_3(t_{n+1}) - (u_x)_{n+1}(0) - \sum_{i=1}^{2M} c_i p_{i,2}(1). \quad (12)$$

Substituting (12) into Eq. (11) results in the following equation

$$(u_{xx})_{n+1}(x) = \sum_{i=1}^{2M} c_i p_{i,1}(x) + f_3(t_{n+1}) - (u_x)_{n+1}(0) - \sum_{i=1}^{2M} c_i p_{i,2}(1). \quad (13)$$

Now, if we integrate Eq. (13) from 0 to x we get

$$(u_x)_{n+1}(x) = (u_x)_{n+1}(0) + \sum_{i=1}^{2M} c_i p_{i,2}(x) + x\left(f_3(t_{n+1}) - (u_x)_{n+1}(0)\right) - x\sum_{i=1}^{2M} c_i p_{i,2}(1). \quad (14)$$

In Eqs. (12), (13) and (14),$(u_x)_{n+1}(0)$ term is unknown. So to find $(u_x)_{n+1}(0)$ term we integrate Eq. (14) from 0 to 1 and use boundary conditions (9). Therefore we have

$$(u_x)_{n+1}(0) = 2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right]$$

Now by plugging the calculated value of $(u_x)_{n+1}(0)$ into Eq. (14) we obtain

$$(u_x)_{n+1}(x) = 2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right](1-x) + x\left(f_3(t_{n+1})\right) + \sum_{i=1}^{2M} c_i p_{i,2}(x) - x\sum_{i=1}^{2M} c_i p_{i,2}(1) \quad (15)$$

Finally, integrating (15) from 0 to x, we obtain

$$u(x) = 2\left[f_2(t_{n+1}) - f_1(t_{n+1}) - \frac{1}{2}f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) + \frac{1}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1)\right] \times \left(x - \frac{x^2}{2}\right) + \frac{x^2}{2}\left(f_3(t_{n+1})\right) + \sum_{i=1}^{2M} c_i p_{i,3}(x) - \frac{x^2}{2}\sum_{i=1}^{2M} c_i p_{i,2}(1) + f_1(t_{n+1}) \quad (16)$$

If we summarize, we have

$$\left.\begin{aligned}
(u_{xxx})_{n+1}(x) &= \sum_{i=1}^{2M} c_i h_i(x) \\
(u_{xx})_{n+1}(x) &= \sum_{i=1}^{2M} c_i p_{i,1}(x) + f_3(t_{n+1}) \\
&\quad - 2\left[f_2(t_{n+1}) - f_1(t_{n+1})\right. \\
&\quad - \frac{1}{2} f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) \\
&\quad \left. + \frac{1}{2} \sum_{i=1}^{2M} c_i p_{i,2}(1)\right] \\
&\quad - \sum_{i=1}^{2M} c_i p_{i,2}(1) \\
(u_x)_{n+1}(x) &= 2\left[f_2(t_{n+1}) - f_1(t_{n+1})\right. \\
&\quad - \frac{1}{2} f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) \\
&\quad \left. + \frac{1}{2} \sum_{i=1}^{2M} c_i p_{i,2}(1)\right](1-x) \\
&\quad + x\left(f_3(t_{n+1})\right) + \sum_{i=1}^{2M} c_i p_{i,2}(x) \\
&\quad - x \sum_{i=1}^{2M} c_i p_{i,2}(1) \\
(u)_{n+1}(x) &= 2\left[f_2(t_{n+1}) - f_1(t_{n+1})\right. \\
&\quad - \frac{1}{2} f_3(t_{n+1}) - \sum_{i=1}^{2M} c_i p_{i,3}(1) \\
&\quad \left. + \frac{1}{2} \sum_{i=1}^{2M} c_i p_{i,2}(1)\right] \\
&\quad \times \left(x - \frac{x^2}{2}\right) + \frac{x^2}{2}\left(f_3(t_{n+1})\right) \\
&\quad + \sum_{i=1}^{2M} c_i p_{i,3}(x) - \frac{x^2}{2} \sum_{i=1}^{2M} c_i p_{i,2}(1) \\
&\quad + f_1(t_{n+1})
\end{aligned}\right\} \tag{17}$$

Similarly, we have

$$\left.\begin{aligned}
(v_{xxx})_{n+1}(x) &= \sum_{i=1}^{2M} d_i h_i(x) \\
(v_{xx})_{n+1}(x) &= \sum_{i=1}^{2M} d_i p_{i,1}(x) + g_3(t_{n+1}) \\
&\quad - 2\left[g_2(t_{n+1}) - g_1(t_{n+1})\right. \\
&\quad - \frac{1}{2} g_3(t_{n+1}) - \sum_{i=1}^{2M} d_i p_{i,3}(1) \\
&\quad \left. + \frac{1}{2} \sum_{i=1}^{2M} d_i p_{i,2}(1)\right] \\
&\quad - \sum_{i=1}^{2M} d_i p_{i,2}(1) \\
(v_x)_{n+1}(x) &= 2\left[g_2(t_{n+1}) - g_1(t_{n+1})\right. \\
&\quad - \frac{1}{2} g_3(t_{n+1}) - \sum_{i=1}^{2M} d_i p_{i,3}(1) \\
&\quad \left. + \frac{1}{2} \sum_{i=1}^{2M} d_i p_{i,2}(1)\right](1-x) \\
&\quad + x\left(g_3(t_{n+1})\right) + \sum_{i=1}^{2M} d_i p_{i,2}(x) \\
&\quad - x \sum_{i=1}^{2M} d_i p_{i,2}(1) \\
(v)_{n+1}(x) &= 2\left[g_2(t_{n+1}) - g_1(t_{n+1})\right. \\
&\quad - \frac{1}{2} g_3(t_{n+1}) - \sum_{i=1}^{2M} d_i p_{i,3}(1) \\
&\quad \left. + \frac{1}{2} \sum_{i=1}^{2M} d_i p_{i,2}(1)\right] \\
&\quad \times \left(x - \frac{x^2}{2}\right) + \frac{x^2}{2}\left(g_3(t_{n+1})\right) \\
&\quad + \sum_{i=1}^{2M} d_i p_{i,3}(x) - \frac{x^2}{2} \sum_{i=1}^{2M} d_i p_{i,2}(1) \\
&\quad + g_1(t_{n+1})
\end{aligned}\right\} \tag{18}$$

Notice that for our problem

$$\begin{aligned}
f_1(t_{n+1}) &= 0, \quad g_1(t_{n+1}) = 0 \\
f_2(t_{n+1}) &= 0, \quad g_2(t_{n+1}) = 0 \\
f_3(t_{n+1}) &= 0, \quad g_3(t_{n+1}) = 0
\end{aligned}$$

Substituting Eqs. (17), (18) into Eq. (8) and discretizing the results at the collocation points $x_l = \frac{l-0.5}{2M}$, $l = 1, 2, ..., 2M$ we found following system of equations for cKdV system

$$\begin{aligned}
\mathbf{A}_{l,i}\mathbf{c}_i + \mathbf{B}_{l,i}\mathbf{d}_i &= u_n + \frac{a\Delta t}{2L^3}(u_{xxx})_n \\
\mathbf{D}_{l,i}\mathbf{c}_i + \mathbf{E}_{l,i}\mathbf{d}_i &= v_n - \frac{\Delta t}{2L^3}(v_{xxx})_n
\end{aligned} \tag{19}$$

where

$$
\begin{aligned}
\mathbf{A}_{l,i} =& \left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right)\left(1-\frac{\Delta t}{L}.3a.(u_x)_n\right)\\
&-\frac{\Delta t}{L}.3a.u_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right)-\frac{a\Delta t}{2L^3}h_i(x_l),\\
\mathbf{B}_{l,i} =& -\frac{\Delta t}{L}.b\left(\left[2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right](v_x)_n\right)\\
&-\frac{\Delta t}{L}.b\left(v_n\left[2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right]\right),\\
\mathbf{D}_{l,i} =& \left[3\frac{\Delta t}{2L}(v_x)_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\right)\right],\\
\mathbf{E}_{l,i} =& 2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right]\left(x_l-\frac{x_l^2}{2}\right)+p_{i,3}(x_l)-\frac{x_l^2}{2}p_{i,2}(1)\\
&+3\frac{\Delta t}{2L}u_n\left(2\left[-p_{i,3}(1)+\frac{1}{2}p_{i,2}(1)\right](1-x_l)+p_{i,2}(x_l)-x_l p_{i,2}(1)\right)+\frac{\Delta t}{2L^3}h_i(x_l).
\end{aligned}
$$

$\mathbf{c}_i$ and $\mathbf{d}_i$ are column vectors of wavelet coefficients and right hand side of Eqs. (19) is column vectors calculated at x_l collocation points for time steps n. By solving Eqs. (19) simultaneously, wavelet coefficients $\mathbf{c}_i$ and $\mathbf{d}_i$ can be calculated successively.

3.3. Error analysis

Convergence analysis of the Haar wavelets is taken from [28]. Using the asymptotic expansion of Eq. (16) as given in [28], one can write

$$
\begin{aligned}
u(x) = 2&\left[f_2(t_{n+1})-f_1(t_{n+1})-\frac{1}{2}f_3(t_{n+1})\right.\\
&\left.-\sum_{i=1}^{\infty}c_i p_{i,3}(1)+\frac{1}{2}\sum_{i=1}^{\infty}c_i p_{i,2}(1)\right]\\
&\times\left(x-\frac{x^2}{2}\right)+\frac{x^2}{2}\left(f_3(t_{n+1})\right)\\
&+\sum_{i=1}^{\infty}c_i p_{i,3}(x)-\frac{x^2}{2}\sum_{i=1}^{\infty}c_i p_{i,2}(1)+f_1(t_{n+1})
\end{aligned}
$$

Lemma 1. *Suppose that* $u(x) \in L^2(R)$ *with* $\left|\frac{\partial u(x)}{\partial x}\right| \leq K$, $\forall x \in (0,1)$*;* $K > 0$ *and* $u(x) = \sum_{i=1}^{\infty} c_i h_i(x)$*. Then* $|c_i| \leq K2^{(-3j-2)/2}$ *[37].*

Lemma 2. *Let* $u(x) \in L^2(R)$ *be a continuous function defined in* $(0,1)$*. Then the error norm at* J *th level satisfies the following inequality*

$$\|E_j\|^2 \leq \frac{K^2}{12}2^{-2J}$$

where $\left|\frac{\partial u(x)}{\partial x}\right| \leq K$, $\forall x \in (0,1)$; $K > 0$, M is a positive number related to the J th level resolution of the wavelet given by $M = 2^J$ [37].

Theorem 1. *Suppose that* $u(x)$ *is exact and* $u_{2M}(x)$ *is approximate solution of the Eq. (16), then*

$$\|E_j\| = \|u(x)-u_{2M}(x)\| \leq \frac{\sqrt{C}K2^{-3(2^J)-1}}{1-2^{-3/2}}$$

Proof. See Kumar et al. [28] □

Similar procedure is valid for the convegence of $v_{2M}(x)$. It is clear from above equation that by increasing the level of resolution J the error decreases.

4. Numerical Experiments

Numerical computations have been done with the free software package GNU Octave and graphical outputs were generated by Matplotlib package [36]. In order to measure the difference between the numerical and analytic solutions as the simulation proceeds we considered the error norms L_2 and L_∞ defined by

$$
\begin{aligned}
L_2 &= \sqrt{\Delta x\sum_{i=1}^{2M}|u_i^{\text{exact}}-u_i^{\text{num}}|^2}\\
L_\infty &= \max_i\left|u_i^{\text{exact}}-u_i^{\text{num}}\right|.
\end{aligned}
$$

We also check the conservation laws of the cKdV equation given by

$$I_1 = \int_{-\infty}^{\infty} u dy$$

$$I_2 = \int_{-\infty}^{\infty}\left(u^2+\frac{2}{3}bv^2\right)dy$$

$$I_3 = \int_{-\infty}^{\infty} \left[(1+a) \left(u^3 - \frac{1}{2} u_y^2 \right) + b \left(uv^2 - v_y^2 \right) \right] dy.$$

The invariants I_1,I_2 and I_3 [18] are monitored at the computations to check the conservation of the numerical scheme.

4.1. Single soliton

Firstly, we consider the following initial conditions for the single soliton problem for the Eq. (1)

$$u(y,0) = 2\lambda^2 \text{sech}^2(\xi), \qquad v(y,0) = \frac{1}{2\sqrt{w}} \text{sech}(\xi)$$

and the boundary conditions (3). This problem have the following exact solution [1].

$$u(y,t) = 2\lambda^2 \text{sech}^2(\xi), \qquad v(y,t) = \frac{1}{2\sqrt{w}} \text{sech}(\xi)$$

where

$$\xi = \lambda(y - \lambda^2 t) + \frac{1}{2\log(w)}, \qquad w = \frac{-b}{8(4a+1)\lambda^4}.$$

We solve the problem for $\Delta t = 0.01$, $\lambda = 0.5$, $a = -0.125$, $b = -3$ and different values of $2M$ at $t = 10$. Table 1 shows the L_2, L_∞ error norms for both u and v for increasing collocation points. We can easily see from the table that the error norms decrease with the increasing collocation points as expected. In Table 2 we tabulated the error norms with the invariants, for various values of time. We see that the error norms are sufficiently small and also the invariants are conserved with increasing time. Relative changes of invariants I_1, I_2 and I_3 between $t = 0$ and $t = 10$ are found as $\%9.5362 \times 10^{-6}$, $\%8.0525 \times 10^{-9}$, $\%3.5459 \times 10^{-6}$ respectively according to the formula $\frac{|I_i^{t=0} - I_i^{t=10}|}{I_i^{t=0}} \times 100$, $(i = 1, 2, 3)$.

Finally, for the single soliton problem we depicted the evolution of numerical solutions of u and v in Fig. 1 for $a = -0.125$, $b = -3$ and $\lambda = 0.5$.

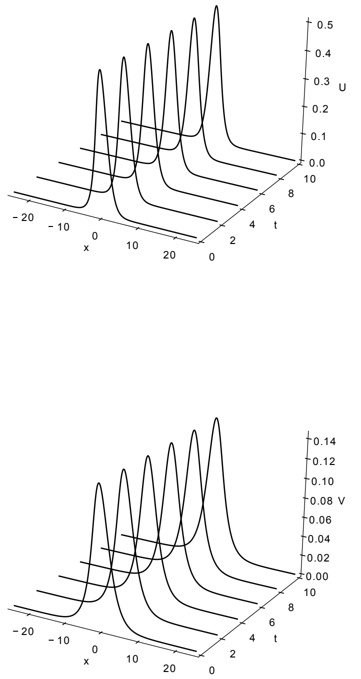

Figure 1. Numerical solutions for $\Delta t = 0.01$ and $2M = 1024$.

4.2. Birth of solitons

We consider Eq. (1) with the initial conditions

$$u(y,0) = e^{-0.01y^2}, \qquad v(x,0) = e^{-0.01y^2}$$

and the boundary conditions (3). Computer simulation of this problem are done for $a = 0.5$ and $b = -3$ in the interval $-50 \le y \le 150$. Numerical results of invariants and their comparison with Petrov-Galerkin method are tabulated in Table 4, as it can be seen from the table our results are agree with Ref. [18]. The positions and amplitudes of waves at $t = 25$ are given in Table 5. It is clearly seen from the table that for first three wave the positions are same for u and v. Finally, evolution of numerical solutions between $t = 0$ and $t = 25$ for $\Delta t = 0.01$ and $2M = 2048$ is depicted in Fig. 2.

In Table 3, we give a comparison of our results with ref. [18] for $\Delta t = 0.01$, $\lambda = 0.5$, $a = -0.125$, $b = -3$ and $2M = 1024$. Numerical results of the present method are comparable with the other methods.

Table 1. Numerical results for $\Delta t = 0.01$, $\lambda = 0.5$, $a = -0.125$, $b = -3$ and different values of $2M$ at $t = 10$.

$2M$	$L_2(u)$	$L_2(v)$	$L_\infty(u)$	$L_\infty(v)$
256	0.000951	0.000327	0.000583	0.000140
512	0.000240	0.000082	0.000147	0.000035
1024	0.000060	0.000021	0.000037	0.000009

Table 2. Numerical results for $\Delta t = 0.01$, $\lambda = 0.5$, $a = -0.125$, $b = -3$ and $2M = 1024$.

t	I_1	I_2	I_3	$L_2(u)$	$L_2(v)$	$L_\infty(u)$	$L_\infty(v)$
0	2.000000	0.500000	0.112500	0.000000	0.000000	0.000000	0.000000
2	2.000000	0.500000	0.112500	0.000025	0.000009	0.000019	0.000005
4	2.000000	0.500000	0.112500	0.000040	0.000014	0.000028	0.000007
6	2.000000	0.500000	0.112500	0.000050	0.000017	0.000033	0.000008
8	2.000000	0.500000	0.112500	0.000057	0.000019	0.000036	0.000008
10	2.000000	0.500000	0.112500	0.000060	0.000021	0.000037	0.000009

Table 3. A comparison for $\Delta t = 0.01$, $\lambda = 0.5$, $a = -0.125$, $b = -3$ and $2M = 1024$.

t	Present Method		Petrov-Galerkin [18]		Product Approx. Tech. [18]	
	$L_\infty(u)$	$L_\infty(v)$	$L_\infty(u)$	$L_\infty(v)$	$L_\infty(u)$	$L_\infty(v)$
2	0.000019	0.000005	0.000015	0.000004	0.000004	0.000005
4	0.000028	0.000007	0.000021	0.000005	0.000008	0.000007
6	0.000033	0.000008	0.000023	0.000006	0.000010	0.000009
8	0.000036	0.000008	0.000024	0.000007	0.000013	0.000012
10	0.000036	0.000009	0.000025	0.000008	0.000014	0.000013

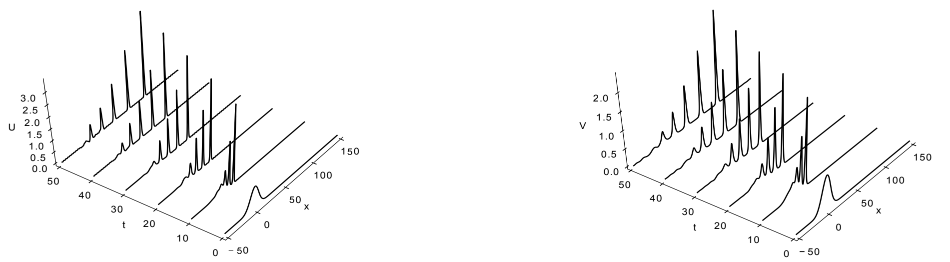

Figure 2. Numerical solutions for $\Delta t = 0.01$ and $2M = 1024$.

Table 4. Numerical results for $\Delta t = 0.01$ and $2M = 2048$ at various times.

t	I_1		I_2	
	Haar	Petrov-Galerkin [18]	Haar	Petrov-Galerkin [18]
0	17.7245385	17.724343	-12.5331414	-12.533142
5	17.7245385	17.723816	-12.5331169	-12.532956
10	17.7245385	17.723352	-12.5325963	-12.530116
15	17.7245385	17.722782	-12.5324374	-12.529239
20	17.7245020	17.722217	-12.5323640	-12.529013
25	17.7245998	17.721734	-12.5306828	-12.528983

Table 5. Amplitudes and positions of waves and their comparisons for $\Delta t = 0.01$ and $2M = 2048$ at $t = 25$.

	Position (y)	Amplitude (u)	Position (y)	Amplitude (v)
First wave	47.7	3.4508	47.7	2.4415
Second wave	32.4	2.4434	32.4	2.7298
Third wave	18.7	1.5601	18.7	1.1046
Fourth wave	7.0	0.6908	6.9	0.5236
Fifth wave	-2.8	0.2214	-3.6	0.1809

5. Conclusion

In conclusion, we have applied Haar wavelet method to coupled KdV equation in this study. Single soliton and birth of solitons have been used as test examples to see the efficiency of the Haar wavelet method. The error norms L_2 and L_∞ obtained by Haar wavelet method are compared with the exact solutions and with those numerical ones available in the literature. The comparisons of error norms as well as conservation of invariants during simulations clearly indicate that the present method is both reliable and competitive with other methods. As a conclusion, the proposed method can safely and quickly be employed to solve similar coupled partial differential equations.

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