An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.7, No.2, pp.225-233 (2017)
http://doi.org/10.11121/ijocta.01.2017.00363

RESEARCH ARTICLE

A novel method for the solution of blasius equation in semi-infinite domains

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ARTICLE INFO

Article History:
Received 29 June 2016
Accepted 16 June 2017
Available 17 July 2017

Keywords:
Reproducing kernel method
Blasius equations
Reproducing kernel functions

AMS Classification 2010:
30E25, 34B15, 47B32
46E22, 74S30

ABSTRACT

In this work, we apply the reproducing kernel method for investigating Blasius equations with two different boundary conditions in semi-infinite domains. Convergence analysis of the reproducing kernel method is given. The numerical approximations are presented and compared with some other techniques, Howarth's numerical solution and Runge-Kutta Fehlberg method.

1. Introduction

Nonlinear differential equations are extensive in science and technology. However, finding analytical solutions for this class of equations has always been a challenging work [3]. Many approximate methods were introduced for the analytical solution of nonlinear differential equations in the recent years. Among these, Homotopy Analysis Method (HAM) [49], Adomian Decomposition Method (ADM) [2], Variational Iteration Method (VIM) [21], Differential Transformation Method (DTM) [31], and Homotopy Perturbation Method (HPM) [41] can be referred. Some new techniques for approximate solution of nonlinear differential equations are shown up recently, such as Optimal Homotopy Asymptotic Method (OHAM) [45], Generalized Homotopy Method (GHM) [46], and reproducing kernel method (RKM) [13].

In the present paper, the RKM has been applied for the solution of two different forms of nonlinear Blasius equation in a semi-infinite domain. Much notice has been given to the work of the RKM to solve many works. The work [13] presents great applications of the RKM. For more details see [1, 4–7, 10–12, 17, 22, 23, 26, 27, 32, 42, 44, 48, 51]. We present two forms of the Blasius equation arising in fluid flow inside the velocity boundary layer as follows.
The first form of the Blasius equation is given as:

$$\begin{cases} u^{(3)}(x) + \frac{u(x)u''(x)}{2} = 0, \quad 0 \leq x \leq \infty, \\ u(0) = u'(0) = 0, \quad u'(x) = 1 \quad \text{as} \quad x \to \infty. \end{cases} \tag{1}$$

The second form is given as:

$$\begin{cases} u^{(3)}(x) + \frac{u(x)u''(x)}{2} = 0, \quad 0 \leq x \leq \infty, \\ u(0) = 0, \quad u'(0) = 1, \quad u'(x) = 0 \quad \text{as} \quad x \to \infty. \end{cases} \tag{2}$$

These equations are the same except for boundary conditions. The first form of the equation is the well-known classical Blasius first derived by Blasius and dates back about a century, which defines the velocity profile of two-dimensional viscous laminar flow over a finite flat plate. This form of the Blasius equation is the simplest form and the origin of all boundary layer equations in fluid mechanics. The second form of the equation, presented more recently, arises in the steady free convection about a vertical flat plate embedded in a saturated porous medium, Laminar boundary layers at the interface of cocurrent parallel streams, or the flow near the leading edge of a very long, steadily operating conveyor belt [3].

Many analytical techniques were introduced to investigate Blasius equation. He [24] presented a perturbation method. Comparison with Howarth's numerical solution finds out that this technique gives the approximate value $\sigma = 0.3296$ with 0.73 accuracy. Asaithambi [9] obtained this number correct to nine decimal positions as $\sigma = 0.332057336$. The variational iteration method (VIM) is implemented for a reliable treatment of two forms of Blasius equation [47]. Fazio [18] searched the Blasius problem numerically. Sinc-collocation technique is implemented in [36] and the HAM is employed by Yao and Chen in [49] and Liao in [29]. For more details see [8, 14–16, 19, 28, 30, 33–35, 37–40, 43, 49, 50].

We organize the paper as follows. We give some new reproducing kernel functions in Section 2. We present the linear operator in Section 3. We show the main results in Section 4. We give the approximate solutions of (1)–(2) in this section. We illustrate examples in Section 5. We give the conclusion in Section 6.

2. Preliminaries

Definition 1. *We describe the space $W_2^4[0,\infty)$ by*

$$W_2^4[0,\infty) = \{v \in AC[0,1] : \; v', v'', v^{(3)} \in AC[0,\infty), \; v^{(4)} \in L^2[0,\infty), \; v(0) = v'(0) = v'(\infty) = 0\}.$$

The inner product and the norm in $W_2^4[0,\infty)$ are given by

$$\begin{aligned} \langle v, h \rangle_{W_2^4} &= v(0)h(0) + v'(0)h'(0) + v''(0)h''(0) \\ &+ v^{(3)}(0)h^{(3)}(0) + \int_0^\infty u^{(4)}(t)h^{(4)}(t)\mathrm{d}t, \\ & \qquad v, h \in W_2^4[0,\infty) \end{aligned}$$

and

$$\|v\|_{W_2^4} = \sqrt{\langle v, v \rangle_{W_2^4}}, \quad v \in W_2^4[0,\infty).$$

The space $W_2^4[0,\infty)$ is called a reproducing kernel space. A function R_y is obtained as:

$$v(y) = \langle v, R_y \rangle_{W_2^4}.$$

Definition 2. *We describe the space $W_2^1[0,1]$ by*

$$W_2^1[0,1] = \{v \in AC[0,1] : \; v' \in L^2[0,1]\}.$$

The inner product and the norm in $W_2^1[0,1]$ are defined by

$$\langle v, h \rangle_{W_2^1} = \int_0^1 v(t)h(t) + v'(t)h'(t)\mathrm{d}t, \qquad v, h \in G_2^1[0,1] \tag{3}$$

and

$$\|v\|_{W_2^1} = \sqrt{\langle v, v \rangle_{W_2^1}}, \quad v \in W_2^1[0,1]. \tag{4}$$

$W_2^1[0,1]$ is a reproducing kernel space. Kernel function $T_t(y)$ is obtained as [13]

$$T_t(y) = \frac{1}{2\sinh(1)}\left[\cosh(t+y-1) + \cosh(|t-y|-1)\right] \tag{5}$$

Theorem 1. *$W_2^4[0,\infty)$ is a reproducing kernel space. Kernel function R_y is obtained as:*

$$R_y(t) = \begin{cases} \sum_{i=1}^{8} c_i(y)t^{i-1}, & t \le y, \\ \sum_{i=1}^{8} d_i(y)t^{i-1}, & t > y, \end{cases} \tag{6}$$

where

$$\begin{aligned} c_1(y) &= 0, \quad c_2(y) = 0, \quad c_3(y) = \frac{1}{4}y^2, \\ c_4(y) &= \frac{1}{36}y^3, \quad c_5(y) = \frac{1}{144}y^3, \\ c_6(y) &= -\frac{1}{240}y^2, \quad c_7(y) = \frac{1}{720}y, \\ c_8(y) &= -\frac{1}{5040}, d_1(y) = -\frac{1}{5040}y^7 \\ d_2(y) &= \frac{1}{720}y^6, d_3(y) = -\frac{1}{240}y^2(y^3-60), \\ d_4(y) &= \frac{1}{144}y^3(y+4), d_5(y) = 0, d_6(y) = 0, \\ d_7(y) &= 0, \quad d_8(y) = 0. \end{aligned}$$

Proof.

$$\begin{aligned} \langle v(t), R_y(t) \rangle_{W_2^4} &= v(0)R_y(0) + v'(0)R_y'(0) \\ &+ v''(0)R_y''(0) + v^{(3)}(0)R_y^{(3)}(0) \\ &+ \int_0^\infty v^{(4)}(t)R_y^{(4)}(t)\mathrm{d}t, \end{aligned}$$

We obtain

$$
\begin{aligned}
\langle v, R_y\rangle_{W_2^4} &= v(0)R_y(0) + v'(0)R_y'(0) \\
&+ v''(0)R_y''(0) + v^{(3)}(0)R_y^{(3)}(0) \\
&+ v^{(3)}(1)R_y^{(4)}(1) - v^{(3)}(0)R_y^{(4)}(0) \\
&- v''(1)R_y^{(5)}(1) + v''(0)R_y^{(5)}(1) \\
&+ v'(1)R_y^{(6)}(1) - v'(0)R_y^{(6)}(0) \\
&- v(1)R_y^{(7)}(1) + v(0)R_y^{(7)}(0) \\
&+ \int_0^\infty v(t)R_y^{(8)}(t)\mathrm{d}t,
\end{aligned} \tag{7}
$$

with integrations by parts. We obtain

$$\langle v(t), R_y(t)\rangle_{W_2^4} = v(y), \tag{8}$$

by reproducing property. If

$$
\begin{cases}
R_y(0) = 0, \\
R_y'(0) = 0, \\
R_y'(\infty) = 0, \\
R_y''(0) + R_y^{(5)}(0) = 0, \\
R_y^{(3)}(0) - R_y^{(4)}(0) = 0, \\
R_y^{(4)}(\infty) = 0, \\
R_y^{(5)}(\infty) = 0, \\
R_y^{(7)}(\infty) = 0,
\end{cases} \tag{9}
$$

then (7) implies that

$$R_y^{(8)}(t) = \delta(t-y).$$

When $t \neq y$,

$$R_y^{(8)}(t) = 0,$$

therefore

$$
R_y(t) = \begin{cases}
\sum_{i=1}^{8} c_i(y)t^{i-1}, & t \le y, \\
\sum_{i=1}^{8} d_i(y)t^{i-1}, & t > y,
\end{cases} \tag{10}
$$

Since

$$R_y^{(8)}(t) = \delta(t-y),$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0,1,2,3,4,5,6 \tag{11}$$

and

$$\partial^7 R_{y^+}(y) - \partial^7 R_{y^-}(y) = 1. \tag{12}$$

Due to $R_y(t) \in W_2^4[0,\infty)$, it follows that

$$R_y(0) = R_y'(0) = R_y'(\infty) = 0, \tag{13}$$

from (9)–(13), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1, 2, \ldots, 8)$ can be acquired. Therefore, $R_y(t)$ is obtained as:

$$
R_y(x) = \begin{cases}
-\frac{1}{5040}t^2(21y^2t^3 + t^5 - 1260y^2 - 7yt^4) \\
\quad -\frac{1}{5040}t^2(-140y^3t - 35y^3t^2), \quad t \le y \\
-\frac{1}{5040}y^2(21t^2y^3 + y^5 - 1260t^2 - 7ty^4) \\
\quad -\frac{1}{5040}y^2(-140t^3y - 35t^3y^2), \quad t > y
\end{cases}
$$

□

3. Solution representation in $W_2^4[0,\infty)$

In this section, the solutions of (1)–(2) are presented in the $W_2^4[0,\infty)$. On defining the linear operator $L : W_2^4[0,\infty) \to W_2^1[0,1]$ as

$$
\begin{aligned}
Lv(t) &= v^{(3)}(t) + \frac{\exp(-t) + t - 1}{2}v''(t) \\
&+ \frac{\exp(-t)}{2}v(t)
\end{aligned} \tag{14}
$$

the problem (1) gets the form:

$$
\begin{cases}
Lv = f(t,u), \quad t \in [0,\infty), \\
v(0) = v'(0) = v'(\infty) = 0
\end{cases} \tag{15}
$$

where $f(t,v) = \exp(-t) - \frac{1}{2}v(t)v''(t) - \frac{1}{2}\exp(-t)(\exp(-t) + t - 1)$.

Theorem 2. *The L given by* (14) *is a bounded linear operator.*

Proof. We need to show $\|Lv\|_{W_2^1}^2 \le M\|v\|_{W_2^4}^2$, where $M > 0$ is a positive constant. By (3) and (4), we have

$$\|Lv\|_{W_2^1}^2 = \langle Lv, Lv\rangle_{W_2^1} = \int_0^1 [Lv(t)]^2 + [Lv'(t)]^2\,\mathrm{d}t.$$

By (8), we have

$$v(t) = \langle v(\cdot), R_t(\cdot)\rangle_{W_2^4},$$

and

$$Lv(t) = \langle v(\cdot), LR_t(\cdot)\rangle_{W_2^4},$$

so

$$|Lv(t)| \le \|v\|_{W_2^4}\|LR_t\|_{W_2^4} = M_1\|u\|_{W_2^4},$$

where $M_1 > 0$ is positive. Therefore,

$$\int_0^1 \left[(Lv)(t)\right]^2 \mathrm{d}t \le M_1^2 \left\|v\right\|_{W_2^4}^2 .$$

We have

$$(Lv)'(t) = \left\langle v(\cdot), (LR_t)'(\cdot)\right\rangle_{W_2^4},$$

by reproducing property. Thus, we get

$$\left|(Lv)'(t)\right| \le \|v\|_{W_2^4} \left\|(LR_t)'\right\|_{W_2^4} = M_2 \|u\|_{W_2^4},$$

where $M_2 > 0$ is positive. Therefore, we obtain

$$\left[(Lv)'(t)\right]^2 \le M_2^2 \|u\|_{W_2^4}^2,$$

and

$$\int_0^1 \left[(Lv)'(t)\right]^2 \mathrm{d}t \le M_2^2 \|v\|_{W_2^4}^2,$$

that is

$$\begin{aligned}\|Lv\|_{W_2^1}^2 &\le \int_0^1 \left(\left[(Lv)(t)\right]^2 + \left[(Lv)'(t)\right]^2\right) \mathrm{d}t \\ &\le \left(M_1^2 + M_2^2\right)\|v\|_{W_2^4}^2 = M\|v\|_{W_2^4}^2,\end{aligned}$$

where $M = M_1^2 + M_2^2 > 0$ is a positive constant. □

4. The main results

Let $\varphi_i(t) = T_{t_i}(t)$ and $\psi_i(t) = L^*\varphi_i(t)$, where L^* is conjugate operator of L. The orthonormal system $\left\{\widehat{\Psi}_i(t)\right\}_{i=1}^{\infty}$ of $W_2^4[0,\infty)$ can be obtained from Gram-Schmidt orthogonalization process of $\{\psi_i(t)\}_{i=1}^{\infty}$,

$$\widehat{\psi_i}(t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(t), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots) \tag{16}$$

Theorem 3. *Let $\{t_i\}_{i=1}^{\infty}$ be dense in $[0,\infty)$ and $\psi_i(t) = L_y R_t(y)|_{y=t_i}$. The sequence $\{\psi_i(t)\}_{i=1}^{\infty}$ is a complete system in $W_2^4[0,\infty)$.*

Proof. We obtain

$$\begin{aligned}\psi_i(t) &= (L^*\varphi_i)(t) = \langle (L^*\varphi_i)(y), R_t(y)\rangle \\ &= \langle (\varphi_i)(y), LyR_t(y)\rangle = L_y R_t(y)|_{y=t_i}.\end{aligned}$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(t) \in W_2^4[0,\infty)$. For each fixed $v(t) \in W_2^4[0,\infty)$, let $\langle v(t), \psi_i(t)\rangle = 0$, $(i = 1, 2, \ldots)$, which means that,

$$\langle v(t), (L^*\varphi_i)(t)\rangle = \langle Lv(\cdot), \varphi_i(\cdot)\rangle = (Lv)(t_i) = 0.$$

$\{t_i\}_{i=1}^{\infty}$ is dense in $[0,\infty)$. Therefore, $(Lv)(t) = 0$. $u \equiv 0$ by L^{-1}. □

Theorem 4. *If $v(t)$ is the exact solution of* (15), *then*

$$v(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(t_k, v_k)\widehat{\Psi}_i(t). \tag{17}$$

where $\{(t_i)\}_{i=1}^{\infty}$ is dense in $[0,\infty)$.

Proof. We get

$$\begin{aligned} v(t) &= \sum_{i=1}^{\infty} \left\langle v(t), \widehat{\Psi}_i(t)\right\rangle_{W_2^4} \widehat{\Psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle v(t), \Psi_k(t)\rangle_{W_2^4} \widehat{\Psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle v(t), L^*\varphi_k(t)\rangle_{W_2^4} \widehat{\Psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle Lv(t), \varphi_k(t)\rangle_{W_2^1} \widehat{\Psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle f(t,v), T_{t_k}\rangle_{W_2^1} \widehat{\Psi}_i(t) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(t_k, v_k)\widehat{\Psi}_i(x), \end{aligned}$$

by (16) and uniqueness of solution of (15). This completes the proof. □

The approximate solution $u_n(x)$ can be acquired as:

$$v_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(t_k, v_k)\widehat{\Psi}_i(t). \tag{18}$$

Lemma 1. *If $\|v_n - v\|_{W_2^4} \to 0$, $t_n \to t$, $(n \to \infty)$ and $f(t,v)$ is continuous for $x \in [0,\infty)$, then [20]*

$$f(t_n, v_{n-1}(t_n)) \to f(t, v(t)) \quad \text{as } n \to \infty.$$

Theorem 5. *For any fixed $v_0(t) \in W_2^4[0,\infty)$ assume that the following conditions are hold:*

(i)

$$v_n(t) = \sum_{i=1}^{n} A_i \widehat{\psi}_i(t), \tag{19}$$

$$A_i = \sum_{k=1}^{i} \beta_{ik} f(t_k, u_{k-1}(t_k)), \tag{20}$$

(ii) $\|v_n\|_{W_2^4}$ *is bounded;*
(iii) $\{t_i\}_{i=1}^{\infty}$ *is dense in* $[0,\infty)$*;*
(iv) $f(t,u) \in W_2^1[0,1]$ *for any* $v(t) \in W_2^4[0,\infty)$.

Then $v_n(t)$ *in iterative formula* (19) *converges to the exact solution of* (17) *in* $W_2^4[0,\infty)$ *and*

$$v(t) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(t).$$

Proof. By (19), we obtain

$$v_{n+1}(t) = u_n(t) + A_{n+1}\widehat{\Psi}_{n+1}(t), \qquad (21)$$

from the orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$, we get

$$\|v_{n+1}\|^2 = \|v_n\|^2 + A_{n+1}^2 = \|v_{n-1}\|^2 + A_n^2 + A_{n+1}^2 = \ldots = \sum_{i=1}^{n+1} A_i^2,$$

from boundedness of $\|u_n\|_{W_2^4}$, we obtain

$$\sum_{i=1}^{\infty} A_i^2 < \infty,$$

i.e.,

$$\{A_i\} \in l^2 \quad (i = 1, 2, \ldots).$$

Let $m > n$, in view of $(v_m - v_{m-1}) \perp (v_{m-1} - v_{m-2}) \perp \ldots \perp (v_{n+1} - v_n)$, we get

$$\begin{aligned} \|v_m - v_n\|_{W_2^4}^2 &= \|v_m - v_{m-1} + \ldots + u_{n+1} - v_n\|_{W_2^4}^2 \\ &\le \|v_m - v_{m-1}\|_{W_2^3}^2 + \ldots + \|v_{n+1} - v_n\|_{W_2^4}^2 \\ &= \sum_{i=n+1}^{m} A_i^2 \to 0, \quad m, n \to \infty. \end{aligned}$$

By the completeness of $W_2^4[0,\infty)$, there exists $v(t) \in W_2^4[0,\infty)$, such that

$$v_n(t) {\rightarrow} v(t) \quad as\ n \to \infty.$$

(ii) Taking limits in (19),

$$v(t) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i(t).$$

We have

$$\begin{aligned} (Lv)(t_j) &= \sum_{i=1}^{\infty} A_i \left\langle L\widehat{\psi}_i(t), \varphi_j(t) \right\rangle_{W_2^1} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(t), L^*\varphi_j(t) \right\rangle_{W_2^4} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(t), \psi_j(t) \right\rangle_{W_2^4}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \sum_{j=1}^{n} \beta_{nj}(Lv)(t_j) &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(t), \sum_{j=1}^{n} \beta_{nj}\psi_j(t) \right\rangle_{W_2^4} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \widehat{\psi}_i(t), \widehat{\psi}_n(t) \right\rangle_{W_2^4} = A_n. \end{aligned}$$

If $n = 1$, then

$$Lv(t_1) = f(t_1, v_0(t_1)). \qquad (22)$$

If $n = 2$, then

$$\begin{aligned} &\beta_{21}(Lv)(t_1) + \beta_{22}(Lv)(t_2) \\ &= \beta_{21} f(t_1, v_0(t_1)) + \beta_{22} f(t_2, v_1(t_2)). \end{aligned}$$

We have

$$(Lv(t_2) = f(t_2, u_1(t_2)).$$

Then, we get

$$(Lv)(t_j) = f(t_j, u_{j-1}(t_j)), \qquad (23)$$

by induction. We have,

$$(Lv)(y) = f(y, v(y)).$$

Therefore, $v\,(t)$ is the solution of (15) and

$$v(t) = \sum_{i=1}^{\infty} A_i \widehat{\psi}_i,$$

where A_i are given by (20).

□

5. Numerical results

In this section, two examples are given to demonstrate the efficiency of the RKM. We have shown comparison tables to prove the power of the RKM. All computations are applied by Maple software program. The accuracy of the RKM for the Blasius equations are controllable. The numerical results we obtained justify the advantage of this technique. We consider first and second forms of the Blasius equation by RKM. In Tables 1–3, v, v', and v'' obtained from the RKM are compared with Howarth's numerical solution [25]. Furthermore, as it can be seen from Tables 1–3, the RKM is more accurate than the variational iteration method [24]. In Tables 4–6, the result of the RKM is given against that of exact (numerical) method. There is a good agreement between the results of the RKM and numerical solution. The results are in very good agreement with numerical and previous data available in the literature.

Table 1. Comparison between $v(t)$ obtained from RKM with VIM, HPM and numerical method, first form of the Blasius equation.

t	Howarth [25]	VIM [24]	HPM [3]	RKM
0	0.00000	0.00000	0.00000	0.00000
1	0.16577	0.19319	0.16557	0.16570
2	0.65003	0.67940	0.65001	0.65310
3	1.39682	1.39106	1.39679	1.39782
4	2.30576	2.24573	2.30572	2.33481
5	3.28329	3.17748	3.28309	3.29502
6	4.27964	4.14688	4.27767	4.28542
7	5.27926	5.13359	5.26736	5.26896

Table 2. Comparison between $v'(t)$ obtained from RKM with VIM, HPM and numerical method, first form of the Blasius equation.

t	Howarth [25]	VIM [24]	HPM [3]	RKM
0	0.00000	0.00000	0.00000	0.00000
1	0.32979	0.35064	0.32977	0.33005
2	0.62977	0.61218	0.62976	0.63039
3	0.84605	0.79640	0.84603	0.84469
4	0.95552	0.90185	0.95551	0.95294
5	0.99150	0.95523	0.99152	0.98514
6	0.99868	0.98032	0.99883	0.99131
7	0.99992	0.99158	0.99943	0.99378

Table 3. Comparison between $v''(t)$ obtained from RKM with VIM, HPM and numerical method, first form of the Blasius equation.

t	Howarth [25]	VIM [24]	HPM [3]	RKM
0	0.33206	0.54360	0.33205	0.33236
1	0.32301	0.27141	0.32300	0.32336
2	0.26675	0.22748	0.26675	0.26631
3	0.16136	0.14117	0.16135	0.16127
4	0.06424	0.07469	0.06422	0.06522
5	0.01591	0.03600	0.01586	0.01918
6	0.00240	0.01645	0.00110	0.00313
7	0.00022	0.00723	0.00060	0.00029

Table 4. Comparison between $v(t)$ obtained from RKM with HPM and numerical method, second form of the Blasius equation.

t	Numerical [3] (5th order Runge-Kutta Fehlberg)	HPM [3]	RKM
0	0.000000	0.00000	0.00000
1	0.786198	0.78620	0.78657
2	1.218546	1.21855	1.21310
3	1.432728	1.43273	1.43823
4	1.533086	1.53308	1.53938
5	1.578851	1.57884	1.57502
6	1.599437	1.59945	1.59266
7	1.612470	1.61280	1.61966

Table 5. Comparison between $v'(t)$ obtained from RKM with HPM and numerical method, second form of the Blasius equation.

t	Numerical [3] (5th order Runge-Kutta Fehlberg)	HPM [3]	RKM
0	1.000000	1.000000	1.000000
1	0.587153	0.587153	0.589473
2	0.301784	0.301783	0.308234
3	0.144016	0.144016	0.141545
4	0.066244	0.066243	0.066661
5	0.029956	0.029949	0.026618
6	0.013469	0.013434	0.011824
7	0.006119	0.006005	0.006437

Table 6. Comparison between $v''(t)$ obtained from RKM with HPM and numerical method, second form of the Blasius equation.

t	Numerical [3] (5th order Runge-Kutta Fehlberg)	HPM [3]	RKM
0	-0.443749	-0.443748	-0.442162
1	-0.358313	-0.358312	-0.359575
2	-0.214505	-0.214505	-0.213139
3	-0.109834	-0.109834	-0.109184
4	-0.052157	-0.052159	-0.052283
5	-0.023906	-0.023922	-0.023166
6	-0.010736	-0.010800	-0.010687
7	-0.046658	-0.048415	-0.044522

6. Conclusion

In this work, we introduced an algorithm for solving the Blasius equation with two different boundary conditions in semi-infinite domains. For illustration purposes, examples were chosen to show the computational accuracy. This work has confirmed that the RKM offers important benefits in terms its computational effectiveness to solve the strongly nonlinear equations.

Acknowledgments

The author thanks the anonymous referees whose work largely constitutes this sample file. This research was supported by 2017-SİÜFED-39 and 2017-SİÜFEB-40.

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