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Solutions to Diffusion-Wave Equation in a Body with a Spherical Cavity under Dirichlet Boundary Condition

Yuriy Povstenko[a,b]

[a]Institute of Mathematics and Computer Science,
Jan Długosz University in Częstochowa-POLAND
Email: j.povstenko@ajd.czest.pl

[b]Department of Computer Science,
European University of Informatics and Economics (EWSIE)
in Warsaw-POLAND

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Abstract. Non-axisymmetric solutions to time-fractional diffusion-wave equation with a source term in spherical coordinates are obtained for an infinite medium with a spherical cavity. The solutions are found using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate φ, the Legendre transform with respect to the spatial coordinate μ, and the Weber transform of the order n+1/2 with respect to the radial coordinate r. In the central symmetric case with one spatial coordinate r the obtained results coincide with those studied earlier.

Keywords: Diffusion-wave equation, Laplace transform, Fourier transform, Legendre transform, Weber transform, Mittag-Leffler function

AMS subject classifications: 26A33, 35R11, 35K05, 45K05

1. Introduction

Fractional order partial differential equation, in particular, the time-fractional diffusion-wave equation are of great interest in studies of important physical phenomena in amorphous, colloid, glassy and porous materials, in fractals, percolation clusters, random and disordered media, in comb structures, dielectrics, semiconductors, polymers, biological systems, in geology, geophysics, medicine, economy, finance, etc. (see, for example, Bagley and Torvik [1], Carpinteri and Cornetti [2], Magin [3], Mainardi [4, 5], Metzler and Klafter [6, 7], Povstenko [8], Rabotnov [9, 10], Rossikhin and Shitikova [11], Uchaikin [12], West *et al.* [13], Zaslavsky [14] and references therein).

A survey of results in the field of fractional diffusion equation can be found in the book of Kilbas *et al.* [15] (see also [16]). Different formulations of the fractional order diffusion-wave equations were reviewed by Herzallah *et al.* [17]. The sequential fractional differential equations were considered by Miller and Ross [18], Podlubny [19], Klimek [20], Băleanu *et al.* [21]. The asymptotic behavior for the solution of fractional differential equations in the non-linear case was studied by Băleanu *et al.* [22].

At first, we recall the main ideas of fractional calculus [15, 19, 23]. It is common knowledge that by integrating $n-1$ times by parts the calculation of the n–fold primitive of a function $u(t)$ can be reduced to the calculation of a single integral

$$I^n u(t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} u(\tau)\, \mathrm{d}\tau, \qquad (1)$$

Corresponding author. Email: j.povstenko@ajd.czest.pl.

where n is a positive integer, $\Gamma(n)$ is the gamma function.

The notion of the Riemann–Liouville fractional integral is introduced as a natural generalization of the repeated integral $I^n u(t)$ written in a convolution type form:

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\,\mathrm{d}\tau, \quad \alpha > 0. \tag{2}$$

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^α, i.e.

$$\begin{aligned} D^\alpha_{RL} u(t) &= D^n I^{n-\alpha} u(t) \\ &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} u(\tau)\,\mathrm{d}\tau\right], \\ & n-1 < \alpha < n. \end{aligned} \tag{3}$$

There are other possibilities to introduce fractional derivatives. One of the alternative definitions was proposed by Caputo:

$$\begin{aligned} D^\alpha_C u(t) &= I^{n-\alpha} D^n u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n u(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \\ & n-1 < \alpha < n. \end{aligned} \tag{4}$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions [24]. In this paper we shall use the Caputo fractional derivative omitting the index C. The major utility of this type fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the governing equation is of fractional order [19, 25].

If care is taken, the results obtained using the Caputo formulation can be recast to the Riemann-Liouville version and vice versa according to the following relation [23]:

$$\begin{aligned} D^\alpha_{RL} u(t) &= D^\alpha_C u(t) \\ &+ \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}\, u^k(0^+), \\ & n-1 < \alpha < n. \end{aligned} \tag{5}$$

Previously, in studies concerning the time-fractional diffusion-wave equation in cylindrical or spherical coordinates only one or two spatial coordinates have been considered [8, 16, 26-43]. If the mass (or heat) exchange between a body and an environment is uniform over the whole surface, then the axisymmetric or central-symmetric problems are obtained. In reality, an assumption of uniformity of exchange with environment (an assumption of axis-symmetry or central-symmetry) is only a rough approximation. In this paper, we investigate solutions to equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\,\Delta u \tag{6}$$

in an infinite medium with a spherical cavity in spherical coordinate system in the case of three spatial coordinates r, μ, and φ.

Consider the time-fractional diffusion-wave Eq.(6) with a source term in spherical coordinates r, θ, and φ:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= a\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right)\right. \\ &\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\right] + Q(r,\theta,\varphi,t), \\ & R < r < \infty, \quad 0 \le \theta \le \pi, \\ & 0 \le \varphi \le 2\pi, \quad 0 < t < \infty, \\ & 0 < \alpha \le 2. \end{aligned} \tag{7}$$

Change of variable $\mu = \cos\theta$ in Eq.(7) leads to the following equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left\{\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial u}{\partial \mu}\right]\right.$$
$$\left. + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 u}{\partial \varphi^2}\right\} + Q(r,\mu,\varphi,t), \quad (8)$$

$$R < r < \infty, \quad -1 \leq \mu \leq 1,$$

$$0 \leq \varphi \leq 2\pi, \quad 0 < t < \infty,$$

$$0 < \alpha \leq 2.$$

For Eq.(8) the initial and boundary conditions are prescribed:

$$t = 0: \quad u = f(r,\mu,\varphi), \qquad 0 < \alpha \leq 2, \quad (9)$$

$$t = 0: \quad \frac{\partial u}{\partial t} = F(r,\mu,\varphi), \quad 1 < \alpha \leq 2, \quad (10)$$

$$r = R: \quad u = g(\mu,\varphi,t), \qquad 0 < \alpha \leq 2. \quad (11)$$

The solution to the initial-boundary-value problem Eqs.(8)-(11) can be written in the following form

$$u = \int_0^t \int_0^{2\pi} \int_{-1}^1 \int_R^\infty Q(\rho,\zeta,\phi,\tau)$$
$$\times\, \mathcal{G}_Q(r,\mu,\varphi,\rho,\zeta,\phi,t-\tau)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau$$
$$+ \int_0^t \int_0^{2\pi} \int_{-1}^1 g(\zeta,\phi,\tau)$$
$$\times\, \mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t-\tau)\,\mathrm{d}\zeta\,\mathrm{d}\phi\,\mathrm{d}\tau \qquad (12)$$
$$+ \int_0^{2\pi} \int_{-1}^1 \int_R^\infty f(\rho,\zeta,\phi)$$
$$\times\, \mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi$$
$$+ \int_0^{2\pi} \int_{-1}^1 \int_R^\infty F(\rho,\zeta,\phi)$$
$$\times\, \mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)\,\rho^2\,\mathrm{d}\rho\,\mathrm{d}\zeta\,\mathrm{d}\phi.$$

Further, we investigate the fundamental solutions $\mathcal{G}_Q(r,\mu,\varphi,\rho,\zeta,\phi,t)$ to the source problem (section 3), $\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)$ to the first Cauchy problem (section 5) and $\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)$ to the second Cauchy problem (section 6) under zero Dirichlet boundary condition as well as the fundamental solution $\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t)$ to the Dirichlet problem under zero source term and zero initial conditions (section 4).

2. Basic tools

Integral transforms technique allows us to remove the partial derivatives from the considered equation and to obtain the correspondent algebraic equation in a transform domain. In this section, we recall integral transforms used in the paper (for details see, e.g., books of Debnath and Bhatta [44], Doetsch [45], Galitsyn and Zhukovsky [46], Özişik [47], and Sneddon [48]). All the integral transforms are denoted by the asterisk.

2.1. Laplace transform

The Laplace transform is defined as

$$\mathcal{L}\left\{u(t)\right\} = u^*(s) = \int_0^\infty u(t)\,\mathrm{e}^{-st}\,\mathrm{d}t, \quad t \geq 0, \qquad (13)$$

where s is the transform variable.

The inverse Laplace transform is carried out according to the Fourier–Mellin formula

$$\mathcal{L}^{-1}\left\{u^*(s)\right\} = u(t)$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u^*(s)\,\mathrm{e}^{st}\,\mathrm{d}s, \quad t \geq 0, \qquad (14)$$

where c is a positive fixed number. The transform $u^*(s)$ is assumed analytical for $\mathfrak{Re}\, s > c$, all the singularities of $u^*(s)$ must lie to the left of the vertical line known as the Bromwich path of integration.

The Laplace transform rule for the fractional integral Eq.(2) has the following form:

$$\mathcal{L}\left\{I^\alpha u(t)\right\} = \frac{1}{s^\alpha}\,u^*(s). \qquad (15)$$

The Riemann-Liouville fractional derivative of the order $n-1 < \alpha < n$ for its Laplace transform requires knowledge of the initial values of the fractional integral $I^{n-\alpha}$ and its derivatives of the order $k = 1, 2, \ldots, n-1$ [15, 23]

$$\mathcal{L}\left\{D^{\alpha}_{RL}u(t)\right\} = s^{\alpha}u^{*}(s) - \sum_{k=0}^{n-1} D^{k} I^{n-\alpha} u(0^{+}) s^{n-1-k}. \tag{16}$$

The Caputo fractional derivative of the order $n-1 < \alpha < n$ for its Laplace transform rule requires the knowledge of the initial values of the function $u(t)$ and its integer derivatives of order $k = 1, 2, \ldots, n-1$

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha}u(t)}{\mathrm{d}t^{\alpha}}\right\} = s^{\alpha}u^{*}(s) - \sum_{k=0}^{n-1} u^{(k)}(0^{+}) s^{\alpha-1-k}. \tag{17}$$

Below the following formula for the inverse Laplace transform [15, 19, 23]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}+b}\right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^{\alpha}) \tag{18}$$

is used. Here $E_{\alpha,\beta}(z)$ is the generalized Mittag–Leffler function in two parameters α and β, which is described by the series representation [49]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\alpha n+\beta)}, \quad \alpha > 0, \quad \beta > 0. \tag{19}$$

2.2. Finite Fourier transform for 2π-periodic functions

Consider series development of the 2π-periodic function in the interval $[0, 2\pi]$

$$u(\varphi) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}\left(a_m \cos m\varphi + b_m \sin m\varphi\right), \tag{20}$$

where

$$a_m = \frac{1}{\pi}\int_0^{2\pi} u(\eta)\cos m\eta\, \mathrm{d}\eta,$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} u(\eta)\sin m\eta\, \mathrm{d}\eta, \tag{21}$$

$$m = 0, 1, 2, \ldots$$

Now we insert the coefficients (21) into the Eq.(20), thus obtaining

$$u(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} u(\eta)\,\mathrm{d}\eta + \frac{1}{\pi}\sum_{m=1}^{\infty}\int_0^{2\pi} u(\eta)\,\cos[m(\varphi-\eta)]\,\mathrm{d}\eta \tag{22}$$

or

$$u(\varphi) = \frac{1}{\pi}{\sum_{m=0}^{\infty}}{}' \int_0^{2\pi} u(\eta)\,\cos[m(\varphi-\eta)]\,\mathrm{d}\eta, \tag{23}$$

where the prime near the summation symbol denotes that the term with $m = 0$ should be multiplied by 1/2.

Eq.(23) can be considered as the integral transform

$$\mathcal{F}\{u(\varphi)\} = u^{*}(\varphi, m) = \int_0^{2\pi} u(\eta)\,\cos[m(\varphi-\eta)]\,\mathrm{d}\eta, \tag{24}$$

and its inverse

$$\mathcal{F}^{-1}\{u^{*}(\varphi, m)\} = u(\varphi) = \frac{1}{\pi}{\sum_{m=0}^{\infty}}{}' u^{*}(\varphi, m). \tag{25}$$

This transform is used for solving equations in polar, cylindrical, and spherical coordinates, as the following equation

$$\mathcal{F}\left\{\frac{\mathrm{d}^{2}u}{\mathrm{d}\varphi^{2}}\right\} = -m^{2}u^{*}(\varphi, m) \tag{26}$$

is fulfilled.

2.3. Legendre transform

The Legendre transform is applied to solve equations in spherical coordinates and reads:

$$\mathcal{P}\left\{u(\mu, m)\right\} = u^{*}(n, m) = \int_{-1}^{1} u(\mu, m)\, P_n^m(\mu)\,\mathrm{d}\mu, \tag{27}$$

where $P_n^m(\mu)$ is the associated Legendre function of the first kind of degree n and order m. The inverse Legendre transform has the form

$$\mathcal{P}^{-1}\left\{u^*(n,m)\right\} = u(\mu,m) = \sum_{n=0}^{\infty} \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)\,u^*(n,m), \quad n \geq m. \tag{28}$$

The importance of this integral transform results from the following equation:

$$\mathcal{P}\left\{\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial u}{\partial\mu}\right] - \frac{m^2}{1-\mu^2}\,u\right\} = -n(n+1)u^*(n,m). \tag{29}$$

2.4. Weber transform

The Weber integral transform of order ν is defined as

$$\mathcal{W}_\nu\{u(r)\} = u^*(\xi) = \int_R^\infty K_\nu(r,R,\xi)u(r)\,r\,\mathrm{d}r \tag{30}$$

having the inverse

$$\mathcal{W}_\nu^{-1}\{u^*(\xi)\} = u(r) = \int_0^\infty K_\nu(r,R,\xi)u^*(\xi)\,\xi\,\mathrm{d}\xi. \tag{31}$$

The significance of the Weber transform for problems in the domain $R \leq r < \infty$ is due to the following formula

$$\mathcal{W}_\nu\left\{\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{\nu^2}{r^2}\,u\right\} = -\xi^2u^*(\xi) + Ru(R)\left.\frac{\partial K_\nu(r,R,\xi)}{\partial r}\right|_{r=R} - RK_\nu(R,R,\xi)\left.\frac{\mathrm{d}u(r)}{\mathrm{d}r}\right|_{r=R}. \tag{32}$$

The specific expression of the kernel $K_\nu(r,R,\xi)$ depends on the boundary conditions at $r=R$. For Dirichlet boundary condition considered in the present paper, the kernel is chosen as

$$K_\nu(r,R,\xi) = \frac{J_\nu(r\xi)Y_\nu(R\xi) - Y_\nu(r\xi)J_\nu(R\xi)}{\sqrt{J_\nu^2(R\xi)+Y_\nu^2(R\xi)}}, \tag{33}$$

where $J_\nu(r)$ and $Y_\nu(r)$ are the Bessel functions of the first and second kind, respectively.

Since

$$\frac{\partial K_\nu(r,R,\xi)}{\partial r} = \frac{J_\nu'(r\xi)Y_\nu(R\xi) - Y_\nu'(r\xi)J_\nu(R\xi)}{\sqrt{J_\nu^2(R\xi)+Y_\nu^2(R\xi)}}\,\xi \tag{34}$$

and (see Galitsyn and Zhukovsky [46], Abramowitz and Stegun [50])

$$J_\nu(z)Y_\nu'(z) - Y_\nu(z)J_\nu'(z) = \frac{2}{\pi z}, \tag{35}$$

then

$$\mathcal{W}\left\{\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{\nu^2}{r^2}\,u\right\} = -\xi^2u^*(\xi) - \frac{2}{\pi}\,\frac{1}{\sqrt{J_\nu^2(R\xi)+Y_\nu^2(R\xi)}}\,u(R). \tag{36}$$

3. Fundamental solution to the source problem

Consider the time-fractional diffusion equation with a source term being the time and space delta pulse applied at a point with the spatial coordinates ρ, ζ, and ϕ

$$\frac{\partial^\alpha \mathcal{G}_Q}{\partial t^\alpha} = a \left\{ \frac{\partial^2 \mathcal{G}_Q}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{G}_Q}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial \mathcal{G}_Q}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 \mathcal{G}_Q}{\partial \varphi^2}\right\} + \frac{Q_0}{r^2}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\delta(\varphi-\phi)\,\delta_+(t), \tag{37}$$

$$R < r < \infty, \quad -1 \le \mu \le 1,$$

$$0 \le \varphi \le 2\pi, \quad 0 < t < \infty$$

$$0 < \alpha \le 2,$$

under zero initial and boundary conditions

$$t = 0: \quad \mathcal{G}_Q = 0, \qquad 0 < \alpha \le 2, \tag{38}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_Q}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{39}$$

$$r = R: \quad \mathcal{G}_Q = 0, \qquad 0 < \alpha \le 2. \tag{40}$$

It should be noted that the three-dimensional Dirac delta function in Cartesian coordinates $\delta(x)\,\delta(y)\,\delta(z)$ after passing to spherical coordinates takes the form $\frac{1}{4\pi r^2}\,\delta_+(r)$, but for the sake of simplicity we have omitted the factor 4π in the solution (12) as well as the factor $\frac{1}{4\pi}$ in the source term in Eq.(37).

In the source term, we have inserted the constant multiplier Q_0 to obtain the non dimensional quantity $\bar{\mathcal{G}}_Q$ (see Eq.(55)) which is displayed in Figures for non dimensional values of parameters describing the problem.

Let us introduce the new looked-for function $v = \sqrt{r}\,\mathcal{G}_Q$ for which we have the following initial-boundary-value problem:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = a \left\{ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{4r^2}\,v + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial v}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 v}{\partial \varphi^2}\right\} + \frac{Q_0}{r^{3/2}}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\delta(\varphi-\phi)\,\delta_+(t), \tag{41}$$

$$R < r < \infty, \quad -1 \le \mu \le 1,$$

$$0 \le \varphi \le 2\pi, \quad 0 < t < \infty$$

$$0 < \alpha \le 2,$$

and

$$t = 0: \quad v = 0, \qquad 0 < \alpha \le 2, \tag{42}$$

$$t = 0: \quad \frac{\partial v}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{43}$$

$$r = R: \quad v = 0, \qquad 0 < \alpha \le 2. \tag{44}$$

Now we shall use the integral transform technique. It should be emphasized that the order of integral transforms is important. Application of the Laplace transform (13) with respect to time t gives

$$s^\alpha v^* = a \left\{ \frac{\partial^2 v^*}{\partial r^2} + \frac{1}{r}\frac{\partial v^*}{\partial r} - \frac{1}{4r^2}\,v^* + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial v^*}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 v^*}{\partial \varphi^2}\right\} + \frac{Q_0}{r^{3/2}}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\delta(\varphi-\phi), \tag{45}$$

$$r = R: \quad v^* = 0. \tag{46}$$

The use of the finite Fourier transform (24) with respect to the angular coordinate φ allows us to remove the second derivative with respect to this coordinate according to Eq.(26)

$$
\begin{aligned}
s^\alpha v^{**} &= a\left\{\frac{\partial^2 v^{**}}{\partial r^2} + \frac{1}{r}\,\frac{\partial v^{**}}{\partial r} - \frac{1}{4r^2}\,v^{**}\right.\\
&+\frac{1}{r^2}\,\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\,\frac{\partial v^{**}}{\partial \mu}\right]\\
&\left.-\frac{m^2}{r^2\left(1-\mu^2\right)}\,v^{**}\right\} \qquad (47)\\
&+\frac{Q_0}{r^{3/2}}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\cos[m(\varphi-\phi)],
\end{aligned}
$$

$$
r=R: \quad v^{**}=0. \qquad (48)
$$

The Legendre transform (27) with respect to the spatial coordinate μ taking into account Eq.(29) leads to

$$
\begin{aligned}
s^\alpha v^{***} &= a\left[\frac{\partial^2 v^{***}}{\partial r^2} + \frac{1}{r}\,\frac{\partial v^{***}}{\partial r} - \frac{(n+1/2)^2}{r^2}\,v^{***}\right]\\
&+\frac{Q_0}{r^{3/2}}\,\delta(r-\rho)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)],
\end{aligned}
\qquad (49)
$$

$$
r=R: \quad v^{***}=0. \qquad (50)
$$

To eliminate the differentiation with respect to the radial coordinate r we apply the Weber transform (30) of the order $n+1/2$ with respect to this coordinate. Thus in the transforms domain we get

$$
\begin{aligned}
v^{****} &= \frac{Q_0}{\sqrt{\rho}}\;P_n^m(\zeta)\,\cos[m(\varphi-\phi)]\,\frac{1}{s^\alpha + a\xi^2}\\
&\times\frac{J_{n+1/2}(\rho\xi)Y_{n+1/2}(R\xi)-Y_{n+1/2}(\rho\xi)J_{n+1/2}(R\xi)}{\sqrt{J^2_{n+1/2}(R\xi)+Y^2_{n+1/2}(R\xi)}}.
\end{aligned}
\qquad (51)
$$

After inversion of integral transforms we gain

$$
\begin{aligned}
\mathcal{G}_Q &= \frac{Q_0}{\pi\sqrt{r\rho}}\sum_{n=0}^{\infty}{\sum_{m=0}^{n}}{}'\,\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\\
&\times P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]\\
&\times\int_0^\infty \frac{t^{\alpha-1}\,E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right)}{J^2_{n+1/2}(R\xi)+Y^2_{n+1/2}(R\xi)}\\
&\times\Big[J_{n+1/2}(\rho\xi)Y_{n+1/2}(R\xi)\\
&-Y_{n+1/2}(\rho\xi)J_{n+1/2}(R\xi)\Big]\\
&\times\Big[J_{n+1/2}(r\xi)Y_{n+1/2}(R\xi)\\
&-Y_{n+1/2}(r\xi)J_{n+1/2}(R\xi)\Big]\,\xi\,d\xi.
\end{aligned}
\qquad (52)
$$

In the case $m=0$, $n=0$, taking into account that the Bessel functions of the order one half can be represented as (see Abramowitz and Stegun [50])

$$
\begin{aligned}
J_{1/2}(r) &= \sqrt{\frac{2r}{\pi}}\,\frac{\sin r}{r},\\
Y_{1/2}(r) &= -\sqrt{\frac{2r}{\pi}}\,\frac{\cos r}{r},
\end{aligned}
\qquad (53)
$$

from (52) we get

$$
\begin{aligned}
\mathcal{G}_Q &= \frac{Q_0}{2\pi^2 r\rho}\int_0^\infty t^{\alpha-1}\,E_{\alpha,\alpha}\left(-a\xi^2 t^\alpha\right)\\
&\times\sin[(\rho-R)\xi]\,\sin[(r-R)\xi]\,\mathrm{d}\xi.
\end{aligned}
\qquad (54)
$$

The solution (54) coincides with the corresponding fundamental solution to the axisymmetric problem within the factor 4π which reflects integration with respect to μ and φ over the surface of the cavity.

Dependence of fundamental solution (52) on the coordinates r, μ, and φ is presented in Figures 1–3. In calculations we have introduced non dimensional quantities:

$$
\bar{\mathcal{G}}_Q = \frac{R^3}{Q_0}\,\mathcal{G}_Q, \qquad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{R}. \qquad (55)
$$

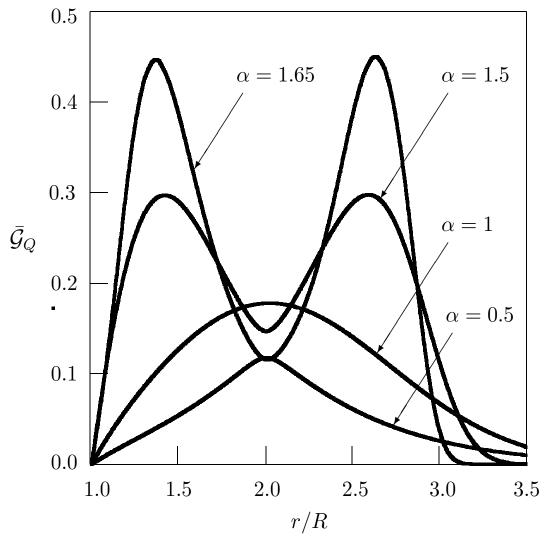

Figure 1. Dependence of the fundamental solution $\mathcal{G}_Q(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the radial coordinate r for $\mu = 0$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$.

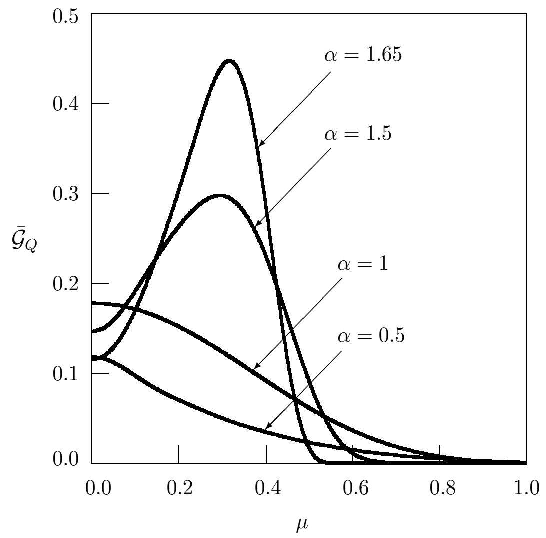

Figure 2. Dependence of the fundamental solution $\mathcal{G}_Q(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the coordinate μ for $r/R = 2$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$.

4. Fundamental solution to the Dirichlet problem

We study the time-fractional diffusion-wave equation

$$\frac{\partial^\alpha \mathcal{G}_g}{\partial t^\alpha} = a \left\{ \frac{\partial^2 \mathcal{G}_g}{\partial r^2} + \frac{2}{r} \frac{\partial \mathcal{G}_g}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial \mathcal{G}_g}{\partial \mu} \right] + \frac{1}{r^2 \left(1 - \mu^2\right)} \frac{\partial^2 \mathcal{G}_g}{\partial \varphi^2} \right\}, \tag{56}$$

$$R < r < \infty, \quad -1 \le \mu \le 1,$$

$$0 \le \varphi \le 2\pi, \quad 0 < t < \infty$$

$$0 < \alpha \le 2,$$

under zero initial conditions

$$t = 0: \quad \mathcal{G}_g = 0, \qquad 0 < \alpha \le 2, \tag{57}$$

$$t = 0: \quad \frac{\partial \mathcal{G}_g}{\partial t} = 0, \qquad 1 < \alpha \le 2, \tag{58}$$

and the prescribed boundary value of the sought-for function

$$r = R: \quad \mathcal{G}_g = g_0\, \delta(\mu - \zeta)\, \delta(\varphi - \phi)\, \delta_+(t). \tag{59}$$

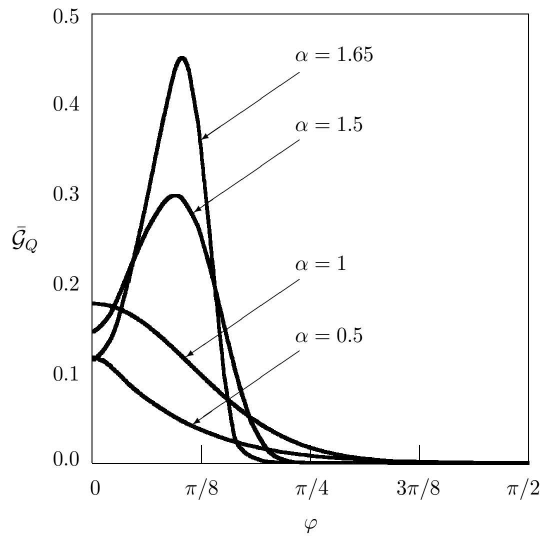

Figure 3. Dependence of the fundamental solution $\mathcal{G}_Q(r, \mu, \varphi, \rho, \zeta, \phi, t)$ on the angular coordinate φ for $r/R = 2$, $\mu = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$.

The integral transforms technique allows us to remove the partial derivatives and to get the expression for the auxiliary function v in the transforms domain (from here on we use only one asterisk for all the transforms):

$$v^* = -\frac{2a\sqrt{R} g_0}{\pi} P_n^m(\zeta) \cos[m(\varphi - \phi)] \frac{1}{s^\alpha + a\xi^2} \times \frac{1}{\sqrt{J_{n+1/2}^2(R\xi) + Y_{n+1/2}^2(R\xi)}}. \tag{60}$$

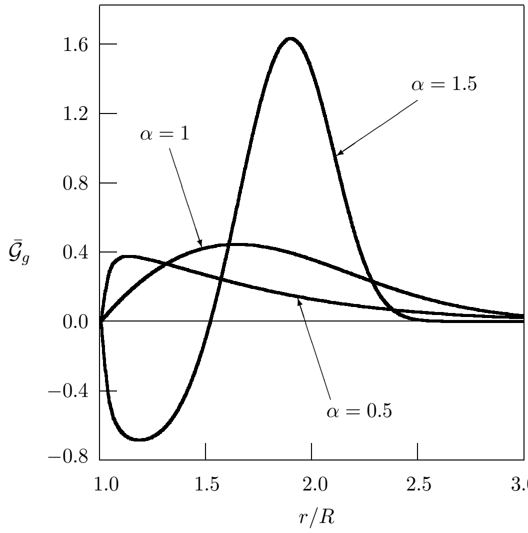

Figure 4. Dependence of the fundamental solution $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on the radial coordinate r for $\mu = 0$, $\varphi = 0$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$.

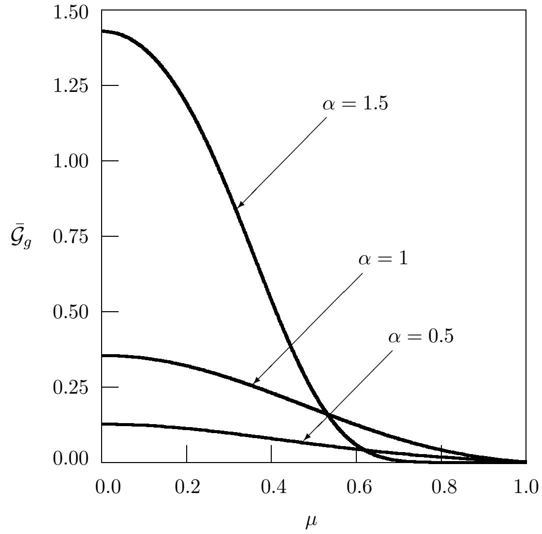

Figure 5. Dependence of the fundamental solution $\mathcal{G}_g(r, \mu, \varphi, \zeta, \phi, t)$ on the coordinate μ for $r/R = 2$, $\varphi = 0$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$.

Inversion of integral transforms gives

$$\begin{aligned}\mathcal{G}_g = &-\frac{a\sqrt{R}g_0}{\pi^2\sqrt{r}}\sum_{n=0}^{\infty}{\sum_{m=0}^{n}}'(2n+1)\,\frac{(n-m)!}{(n+m)!}\\ &\times P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]\\ &\times\int_0^{\infty}\frac{t^{\alpha-1}\,E_{\alpha,\alpha}\left(-a\xi^2t^{\alpha}\right)}{J_{n+1/2}^2(R\xi)+Y_{n+1/2}^2(R\xi)}\\ &\times\Big[J_{n+1/2}(r\xi)Y_{n+1/2}(R\xi)\\ &-Y_{n+1/2}(r\xi)J_{n+1/2}(R\xi)\Big]\,\xi\,d\xi.\end{aligned} \tag{61}$$

In the case $m = 0$, $n = 0$ from (61) we get

$$\begin{aligned}\mathcal{G}_g = &\frac{aRg_0}{2r\pi^2}\int_0^{\infty}t^{\alpha-1}\,E_{\alpha,\alpha}\left(-a\xi^2t^{\alpha}\right)\\ &\times\sin[(r-R)\xi]\,\xi\,\mathrm{d}\xi.\end{aligned} \tag{62}$$

Eq.(62) was obtained in [37] (with accuracy of the multiplier 4π, which reflects integration over the sphere surface).

Dependence of fundamental solution (61) on the coordinates r, μ, and φ is presented in Figures 4–6 with $\bar{\mathcal{G}}_g = t\mathcal{G}_g/g_0$.

5. Fundamental solution to the first Cauchy problem

In this case we have the equation

$$\begin{aligned}\frac{\partial^{\alpha}\mathcal{G}_f}{\partial t^{\alpha}} &= a\left\{\frac{\partial^2\mathcal{G}_f}{\partial r^2}+\frac{2}{r}\,\frac{\partial\mathcal{G}_f}{\partial r}\right.\\ &+\frac{1}{r^2}\,\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\,\frac{\partial\mathcal{G}_f}{\partial\mu}\right]\\ &\left.+\frac{1}{r^2\left(1-\mu^2\right)}\,\frac{\partial^2\mathcal{G}_f}{\partial\varphi^2}\right\},\\ &R<r<\infty,\quad -1\le\mu\le 1,\\ &0\le\varphi\le 2\pi,\quad 0<t<\infty\\ &0<\alpha\le 2,\end{aligned} \tag{63}$$

under delta pulse initial condition

$$t=0:\quad \mathcal{G}_f=\frac{f_0}{r^2}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\delta(\varphi-\phi),\quad 0<\alpha\le 2, \tag{64}$$

$$t=0:\quad \frac{\partial\mathcal{G}_f}{\partial t}=0,\qquad 1<\alpha\le 2, \tag{65}$$

and zero boundary value of the function

$$r=R:\quad \mathcal{G}_f=0. \tag{66}$$

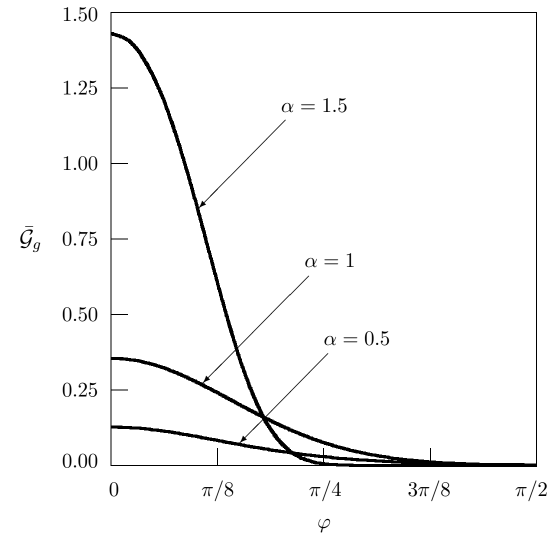

Figure 6. Dependence of the fundamental solution $\mathcal{G}_g(r,\mu,\varphi,\zeta,\phi,t)$ on the angular coordinate φ for $r/R=2$, $\mu=0$, $\zeta=0$, $\phi=0$, and $\kappa=0.5$.

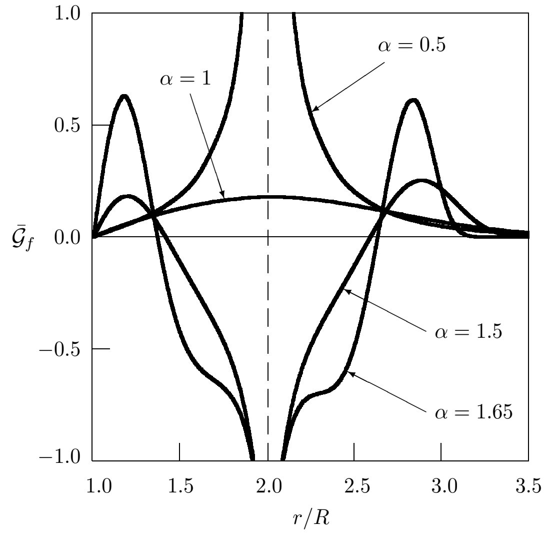

Figure 7. Dependence of the fundamental solution $\mathcal{G}_f(r,\mu,\varphi,\rho,\zeta,\phi,t)$ on the radial coordinate r for $\mu=0$, $\varphi=0$, $\rho/R=2$, $\zeta=0$, $\phi=0$, and $\kappa=0.5$.

The solution reads

$$\begin{aligned}\mathcal{G}_f &= \frac{f_0}{\pi\sqrt{r\rho}}\sum_{n=0}^{\infty}{\sum_{m=0}^{n}}' \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\\ &\times P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]\\ &\times\int_0^\infty \frac{E_\alpha\left(-a\xi^2t^\alpha\right)}{J_{n+1/2}^2(R\xi)+Y_{n+1/2}^2(R\xi)}\\ &\times\Big[J_{n+1/2}(\rho\xi)Y_{n+1/2}(R\xi)\\ &-Y_{n+1/2}(\rho\xi)J_{n+1/2}(R\xi)\Big]\\ &\times\Big[J_{n+1/2}(r\xi)Y_{n+1/2}(R\xi)\\ &-Y_{n+1/2}(r\xi)J_{n+1/2}(R\xi)\Big]\,\xi\,d\xi\end{aligned}\tag{67}$$

with the particular case corresponding to $m=0$, $n=0$:

$$\begin{aligned}\mathcal{G}_f &= \frac{f_0}{2\pi^2r\rho}\int_0^\infty E_\alpha\left(-a\xi^2t^\alpha\right)\\ &\times\sin[(\rho-R)\xi]\,\sin[(r-R)\xi]\,\mathrm{d}\xi.\end{aligned}\tag{68}$$

Figure 7 shows dependence of the fundamental solution (67) on the radial coordinate r. Here $\bar{\mathcal{G}}_f = R^3\mathcal{G}_f/f_0$.

6. Fundamental solution to the second Cauchy problem

In the case of the second Cauchy problem, which is considered for the order of time derivative $1<\alpha\le 2$, the initial value of the time derivative of the sought-for function is prescribed, and for the corresponding fundamental solution we have the equation

$$\begin{aligned}\frac{\partial^\alpha\mathcal{G}_F}{\partial t^\alpha} &= a\left\{\frac{\partial^2\mathcal{G}_F}{\partial r^2}+\frac{2}{r}\frac{\partial\mathcal{G}_F}{\partial r}\right.\\ &+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\mathcal{G}_F}{\partial\mu}\right]\\ &\left.+\frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2\mathcal{G}_F}{\partial\varphi^2}\right\},\\ &R<r<\infty,\quad -1\le\mu\le 1,\\ &0\le\varphi\le 2\pi,\quad 0<t<\infty\\ &1<\alpha\le 2,\end{aligned}\tag{69}$$

under zero initial condition for the function

$$t=0:\quad \mathcal{G}_F=0,\qquad 1<\alpha\le 2,\tag{70}$$

the delta pulse initial condition for its time derivative

$$t = 0: \quad \frac{\partial \mathcal{G}_F}{\partial t} = \frac{F_0}{r^2}\,\delta(r-\rho)\,\delta(\mu-\zeta)\,\delta(\varphi-\phi),$$
$$1 < \alpha \le 2, \tag{71}$$

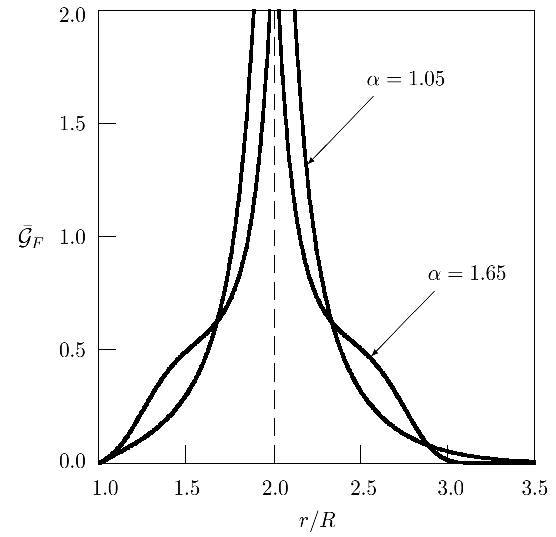

Figure 8. Dependence of the fundamental solution $\mathcal{G}_F(r,\mu,\varphi,\rho,\zeta,\phi,t)$ on the radial coordinate r for $\mu = 0$, $\varphi = 0$, $\rho/R = 2$, $\zeta = 0$, $\phi = 0$, and $\kappa = 0.5$.

and zero Dirichlet boundary condition

$$r = R: \quad \mathcal{G}_F = 0. \tag{72}$$

The integrals transform technique leads to

$$\mathcal{G}_F = \frac{F_0}{\pi\sqrt{r\rho}} \sum_{n=0}^{\infty} {\sum_{m=0}^{n}}' \frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}$$
$$\times P_n^m(\mu)\,P_n^m(\zeta)\,\cos[m(\varphi-\phi)]$$
$$\times \int_0^\infty \frac{t\,E_{\alpha,2}\left(-a\xi^2 t^\alpha\right)}{J_{n+1/2}^2(R\xi)+Y_{n+1/2}^2(R\xi)}$$
$$\times \Big[J_{n+1/2}(\rho\xi)Y_{n+1/2}(R\xi)$$
$$-Y_{n+1/2}(\rho\xi)J_{n+1/2}(R\xi)\Big]$$
$$\times \Big[J_{n+1/2}(r\xi)Y_{n+1/2}(R\xi)$$
$$-Y_{n+1/2}(r\xi)J_{n+1/2}(R\xi)\Big]\,\xi\,d\xi \tag{73}$$

with the particular case corresponding to the central symmetric case

$$\mathcal{G}_F = \frac{F_0}{2\pi^2 r\rho}\int_0^\infty tE_{\alpha,2}\left(-a\xi^2 t^\alpha\right)$$
$$\times\,\sin[(\rho-R)\xi]\,\sin[(r-R)\xi]\,\mathrm{d}\xi. \tag{74}$$

As above, the remark about the factor 4π concerns also Eqs. (68) and (74).

Figure 8 shows dependence of the fundamental solution (73) on the radial coordinate r. The non-dimensional quantity is introduced as $\bar{\mathcal{G}}_F = R^3\mathcal{G}_F/(tF_0)$.

7. Conclusions

The non-axisymmetric solutions to the source, Cauchy, and Dirichlet problems for time-fractional diffusion-wave equation have been found for a medium with a spherical cavity. The obtained solutions satisfy the appropriate initial and boundary conditions and reduce to the solutions of classical diffusion equation in the limit $\alpha = 1$. In the case $1 < \alpha < 2$, the time-fractional diffusion-wave equation interpolates the standard diffusion equation and the classical wave equation. For $1 < \alpha < 2$ the solutions to the fractional diffusion-wave equation feature propagating humps, underlining the proximity to the standard wave equation in contrast to the shape of curves describing the subdiffusion regime ($0 < \alpha < 1$).

In the case of the ballistic diffusion corresponding to the wave equation ($\alpha = 2$) the fundamental solution to the source problem contains wave fronts described by the Dirac delta function. Considering the radial direction for $0 < \kappa < (\rho - r)/R$, there are two wave fronts at $r/R = \rho/R - \kappa$ and $r/R = \rho/R + \kappa$ which are approximated by solutions to the diffusion-wave equation in the case $1 < \alpha < 2$ (Figure 1). The similar situation takes place for coordinates μ and φ (see Figures 2 and 3): the second wave front approximated by the solution in the case $1 < \alpha < 2$ is located symmetrically for negative values of these coordinates.

The behaviour of the solution towards the first Cauchy problem is very interesting (Figure 7). In the case of the ballistic diffusion there are also two wave fronts at $r/R = \rho/R - \kappa$ and $r/R = \rho/R + \kappa$ but only the solution to the classical diffusion equation ($\alpha = 1$) has no singularity at the point of application of the Dirac delta pulse. Such a singularity appears due to behavior of the Mittag-Leffler function

$E_\alpha(-x)$ for large values of the negative argument

$$E_\alpha(-x) \sim \frac{1}{\Gamma(1-\alpha)}\,\frac{1}{x}, \qquad 0<\alpha<2, \;\; \alpha \neq 1. \tag{75}$$

For $0 < \alpha < 1$ the solution tends to $+\infty$ when $r \to \rho$, and for $1 < \alpha < 2$ the solution approaches $-\infty$ when $r \to \rho$ (Figure 7).

For large values of the negative argument the asymptotic of the Mittag-Leffler function

$$E_{\alpha,2}(-x) \sim \frac{1}{\Gamma(2-\alpha)}\,\frac{1}{x}, \qquad 1<\alpha<2, \tag{76}$$

results in singularity of the fundamental solution to the second Cauchy problem at $r = \rho$ (Figure 8). It is seen from Figures that the fundamental solutions to the first and second Cauchy problems have singularities at the point of application of the delta pulses, whereas the fundamental solutions to the source and Dirichlet problems do not have such singularities.

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Yuriy Povstenko *received the M.S. degree in Mechanics from Lviv State University, Ukraine, in 1971; the Candidate degree (Ph.D) in Physics and Mathematics from Institute of Mathematics, Ukrainian Academy of Sciences, Lviv, Ukraine, in 1977; the Doctor degree in Physics and Mathematics from Saint Petersburg Technical University, Russia.*

He is currently a Professor at Institute of Mathematics and Computer Science, Jan Dlugosz University in Czestochowa, Poland. His current research interests include Fractional Calculus, Generalized Thermoelasticity, Nonlocal Elasticity, Surface Science, and Imperfections in Solid.