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Generalized (Φ,ρ)-convexity in nonsmooth vector optimization over cones

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Abstract. In this paper, new classes of cone-generalized (Φ,ρ)-convex functions are introduced for a nonsmooth vector optimization problem over cones, which subsume several known studied classes. Using these generalized functions, various sufficient Karush-Kuhn-Tucker (KKT) type nonsmooth optimality conditions are established wherein Clarke's generalized gradient is used. Further, we prove duality results for both Wolfe and Mond-Weir type duals under various types of cone-generalized (Φ,ρ)-convexity assumptions.

Keywords: Nonsmooth vector optimization over cones; cone-generalized (Φ,ρ)-convexity; nonsmooth optimality conditions; duality.
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1. Introduction

Convexity plays an important role in many aspects of optimization theory including sufficient optimality conditions and duality theorems. In a quest to weaken the convexity hypothesis various generalized convexity notions have been introduced. Hanson and Mond [8] introduced *F*-convexity and Vial [10] defined ρ-convexity. Preda [9] unified the two concepts and gave the notion of an (F,ρ)-convex function.

Another generalization of convexity is invexity, introduced by Hanson [7]. The concept of (Φ,ρ)-invexity has been introduced by Caristi et al. [3]. Sufficient optimality conditions and duality results have been studied under (Φ,ρ)-invexity for differentiable single-objective and multiobjective programs [3,6]. (Φ,ρ)-invexity notion has been extended to the nonsmooth case by Antczak and Stasiak [2].

In this paper, we use the concept of cones to define new classes of nonsmooth functions that we call *K*-generalized (Φ,ρ)-convex, *K*-generalized (Φ,ρ)-pseudoconvex and *K*-generalized (Φ,ρ)-quasiconvex functions, where *K* is a closed convex pointed cone with nonempty interior. Sufficient optimality conditions are proved for a nonsmooth vector optimization problem over cones using the above defined functions. Further, both Wolfe and Mond-Weir type duals are formulated and weak and strong duality results are established.

2. Definitions and preliminaries

Let *S* be a nonempty open subset of $\mathbf{R}^n$.

Definition 2.1. A function $\theta : S \to \mathbf{R}$ is said to be locally Lipschitz at a point $u \in S$ if for some $l_u > 0$,

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$$|\theta(x)-\theta(\bar{x})| \leq l_u \, \| x-\bar{x} \|$$

for all $x, \bar{x}$ in a neighborhood of *u*. We say that $\theta : S \to \mathbf{R}$ is locally Lipschitz on *S* if it is locally Lipschitz at each point of *S*.

Let $f = (f_1, f_2, \ldots, f_m)^t : S \to \mathbf{R}^m$ be a vector-valued function. Then *f* is said to be locally Lipschitz on *S* if each f_i is locally Lipschitz on *S*.

Definition 2.2. [4] Let $\theta : S \to \mathbf{R}$ be a locally Lipschitz function on *S*. The Clarke's generalized directional derivative of θ at $u \in S$ in the direction *v*, denoted as $\theta^0(u; v)$, is defined by

$$\theta^0(u;v) = \limsup_{\substack{y \to u \\ t \to 0^+}} \frac{\theta(y+tv)-\theta(y)}{t}$$

Definition 2.3. [4] The Clarke's generalized gradient of θ at $u \in S$, denoted as $\partial\theta(u)$, is given by

$$\partial\theta(u) = \{\xi \in \mathbf{R}^n : \theta^0(u;v) \geq \langle \xi, v \rangle, \ \forall \, v \in \mathbf{R}^n\}.$$

The generalized directional derivative of a locally Lipschitz function $f = (f_1, \ldots, f_m)^t : S \to \mathbf{R}^m$ at $u \in S$ in the direction *v* is given by

$$f^0(u;v) = (f_1^0(u;v),\ f_2^0(u;v), \ldots, f_m^0(u;v))^t.$$

The generalized gradient of f at *u* is the set $\partial f(u) = \partial f_1(u) \times \partial f_2(u) \times \ldots \times \partial f_m(u)$, where $\partial f_i(u)$ is the generalized gradient of f_i at *u* for *i* = 1, 2,...,*m*. An element $A = (A_1, \ldots, A_m)^t \in \partial f(u)$ is a continuous linear operator from $\mathbf{R}^n$ to $\mathbf{R}^m$ and

$Au = (A_1^t u, \ldots, A_m^t u)^t \in \mathbf{R}^m$ for all $u \in \mathbf{R}^n$.

Let $K \subseteq \mathbf{R}^m$ be a closed convex pointed cone with nonempty interior and let int*K* denote the interior of *K*. The positive dual cone K^* and the strict positive dual cone K^{s^*} of *K*, are respectively defined as

$$K^* = \{y^* \in \mathbf{R}^m : \langle y, y^* \rangle \geq 0 \text{ for all } y \in K\}, \text{and}$$

$$K^{s*} = \{y^* \in \mathbf{R}^m : \langle y, y^* \rangle > 0 \text{ for all } y \in K \setminus \{0\}\}.$$

Throughout the paper, we shall denote an element of $\mathbf{R}^{n+1}$ by the ordered pair (*a*, *r*), where $a \in \mathbf{R}^n$ and $r \in \mathbf{R}$. Consider a function $\varphi : S \times S \times \mathbf{R}^{n+1} \to \mathbf{R}$ such that $\varphi(x, u; \cdot)$ is convex on $\mathbf{R}^{n+1}$ and $\varphi(x, u; (0, r)) \geq 0$ for every *x* , $u \in S$ and any real number $r \in \mathbf{R}_+$. Let $f : S \to \mathbf{R}^m$ be a locally Lipschitz function , $u \in S$, $A = (A_1, \ldots, A_m)^t \in \partial f(u)$, $\rho = (\rho_1, \ldots, \rho_m)^t \in \mathbf{R}^m$ and $\Phi(x,u;(A,\rho))$ denote the vector $(\varphi(x,u;(A_1,\rho_1)), \ldots, \varphi(x,u;(A_m,\rho_m)))^t$.

We introduce the following definitions:

Definition 2.4. The function f is said to be *K*-generalized (Φ,ρ)-convex at *u* on *S* if for every $x \in S$

$$f(x) - f(u) - \Phi(x,u;(A,\rho)) \in K, \quad \forall A \in \partial f(u).$$

Definition 2.5. The function f is said to be *K*-generalized (Φ,ρ)-pseudoconvex at *u* on *S* if for every $x \in S$, $A \in \partial f(u)$

$$-\Phi(x,u;(A,\rho)) \notin \operatorname{int} K \Rightarrow -(f(x)-f(u)) \notin \operatorname{int} K.$$

Equivalently, if for every $x \in S$

$$f(x)-f(u) \in -\operatorname{int} K \Rightarrow \Phi(x,u;(A,\rho)) \in -\operatorname{int} K, \quad \forall A \in \partial f(u).$$

Definition 2.6. The function f is said to be *K*-generalized (Φ,ρ)-quasiconvex at *u* on *S* if for every $x \in S$

$$f(x)-f(u) \notin \operatorname{int} K \Rightarrow -\Phi(x,u;(A,\rho)) \in K, \quad \forall A \in \partial f(u).$$

If *f* is *K*-generalized (Φ,ρ)-convex (*K*-generalized (Φ,ρ)-pseudoconvex, *K*-generalized (Φ,ρ)-quasiconvex) at every $u \in S$ then *f* is said to be *K*-generalized (Φ,ρ)-convex (*K*-generalized (Φ,ρ)-pseudoconvex, *K*-generalized (Φ,ρ)-quasiconvex) on *S*.

Remark 2.7: 1) If $K = \mathbf{R}_+^m$ and $\varphi : S \times S \times \mathbf{R}^{n+1} \to \mathbf{R}$ is of the form

$$\varphi(x, u; (A, \rho)) = F(x, u, A) + \rho d(x, u)$$

where $F(x, u, \cdot)$ is sublinear, ρ is a constant and $d : S \times S \to \mathbf{R}_+$, then *K*-generalized (Φ,ρ)-convexity reduces to (F,ρ)-convexity introduced by Preda [9].

2) If *f* is a scalar valued function and $K = \mathbf{R}_+$, then Definition 2.4 becomes the definition of (Φ,ρ)-invexity given by Antczak and Stasiak [2].

3) If *f* is a differentiable function and $K = \mathbf{R}_+^m$, then the above definitions reduce to the corresponding definitions introduced in [6].

4) If $K = \mathbf{R}_+^m$ then Definition 2.4 becomes the definition of (Φ,ρ)-invexity introduced by Antczak [1].

Now we give an example of a *K*-generalized (Φ,*ρ*)-convex function.

Example 2.8. Let $S=\mathbf{R}^2$ and $K=\{(x,y): x\le 0,\ y\ge x\}$. Consider the following nonsmooth function $f:S\to\mathbf{R}^2$, $f(x)=(f_1(x), f_2(x))$.

$$f_1(x_1,x_2)=\begin{cases}-x_1, & x_1\ge 0\\ 2x_1x_2, & x_1<0\end{cases}$$

$$f_2(x_1,x_2)=\begin{cases}\frac{1}{2}x_1+\frac{1}{3}x_2^4, & x_1\ge 0\\ x_1^2+x_2^2, & x_1<0\end{cases}$$

Here,

$\partial f_1(0,0)=(A_{11},A_{12}), A_{11}\in[-1,0], A_{12}\in\{0\}$

and $\partial f_2(0,0)=(A_{21},A_{22}), A_{21}\in[0,\frac{1}{2}], A_{22}\in\{0\}$.

Define $\varphi: S\times S\times\mathbf{R}^3\to\mathbf{R}$ as

$$\varphi(x,u;(a,\rho))=\begin{cases}(x_1+x_2^4)\rho, & x_1\ge 0\\ (x_1^2+x_2^2)e^{-(a_1+a_2)}, & x_1<0\end{cases}.$$

Note that $\varphi(x,u;(.,.))$ is convex on $\mathbf{R}^3$, $\varphi(x,u;(0,r))\ge 0$, for every $(x,u)\in S\times S$ and any $r\in\mathbf{R}_+$.

Set $\rho=(0,\frac{1}{3})$. Then, at $u=(0,0)$ we have

$$f(x)-f(u)-\Phi(x,u;(A,\rho))$$
$$=\begin{cases}(-x_1,\frac{1}{6}x_1), & x_1\ge 0\\ (2x_1x_2-(x_1^2+x_2^2)e^{-(A_{11}+A_{12})}, & \\ (x_1^2+x_2^2)(1-e^{-(A_{21}+A_{22})})), & x_1<0\end{cases}$$

which gives that,

$f(x)-f(u)-\Phi(x,u;(A,\rho))\in K$, for every $x\in S$ and $A\in\partial f(0,0)$.

Hence, *f* is *K*-generalized (Φ,*ρ*)-convex at *u* on *S*.

It is clear that every *K*-generalized (Φ,*ρ*)-convex function is *K*-generalized (Φ,*ρ*)-pseudoconvex. Converse of this statement may not be true as shown by the following example.

Example 2.9. Let $S=\mathbf{R}^2$ and $K=\{(x,y): x\ge 0,\ y\ge x\}$. Consider the following nonsmooth function $f:S\to\mathbf{R}^2$, $f(x)=(f_1(x), f_2(x))$.

$$f_1(x_1,x_2)=\begin{cases}-x_1, & x_1\ge 0\\ 0, & x_1<0\end{cases}$$

$$f_2(x_1,x_2)=\begin{cases}-x_1-2x_2, & x_1\ge 0\\ x_1^2+x_2^2, & x_1<0\end{cases}$$

Here,

$\partial f_1(0,0)=(A_{11},A_{12}), A_{11}\in[-1,0], A_{12}\in\{0\}$

and $\partial f_2(0,0)=(A_{21},A_{22}), A_{21}\in[-1,0], A_{22}\in[-2,0]$.

Define $\varphi: S\times S\times\mathbf{R}^3\to\mathbf{R}$ as

$$\varphi(x,u;(a,\rho))=\begin{cases}(x_1+x_2^2)\rho, & x_1\ge 0\\ (x_1^2+x_2^2)e^{a_1+a_2}, & x_1<0\end{cases}.$$

Note that, $\varphi(x,u;(.,.))$ is convex on $\mathbf{R}^3$, $\varphi(x,u;(0,r))\ge 0$, for every $(x,u)\in S\times S$ and any $r\in\mathbf{R}_+$.

Set $\rho=(-\frac{1}{2},-1)$. Then, at $u=(0,0)$ we have

$$f(x)-f(u)\in -\operatorname{int}K\Rightarrow x_1>0, x_2>0$$
$$\Rightarrow \Phi(x,u;(A,\rho))\in -\operatorname{int}K,$$

for every $x\in S$ and $A\in\partial f(0,0)$.

Thus *f* is *K*-generalized (Φ,*ρ*)-pseudoconvex at *u* on *S*. But *f* fails to be *K*-generalized (Φ,*ρ*)-convex at *u* on *S* because for *x* = (4,1),

$$f(x)-f(u)-\Phi(x,u;(A,\rho))=\left(-\frac{3}{2},-1\right)\notin K.$$

3. Optimality conditions

Consider the following nonsmooth vector optimization problem over cones.

(NVOP) *K*-minimize *f*(*x*)

subject to $-g(x)\in Q$,

where $f: S\to\mathbf{R}^m$, $g: S\to\mathbf{R}^p$ are locally Lipschitz vector-valued functions and *S* is a nonempty open subset of $\mathbf{R}^n$. *K* and *Q* are closed convex pointed cones with nonempty interiors in $\mathbf{R}^m$ and $\mathbf{R}^p$ respectively.

Let $S_0=\{x\in S: -g(x)\in Q\}$ denote the set of feasible solutions of (NVOP).

Definition 3.1. A point $\bar{x}\in S_0$ is said to be

(i) a weak minimum of (NVOP) if for every $x \in S_0$

$$f(x) - f(\bar{x}) \notin -\text{int}\, K .$$

(ii) a minimum of (NVOP) if for every $x \in S_0$

$$f(x) - f(\bar{x}) \notin -K \setminus \{0\}.$$

The following constraint qualification and Karush-Kuhn-Tucker type necessary optimality conditions are a direct precipitation from Craven [5].

Definition 3.2. (Slater-type cone constraint qualification). The problem (NVOP) is said to satisfy Slater-type cone constraint qualification at $\bar{x}$ if, for all $B \in \partial g(\bar{x})$, there exists a vector $\Omega \in \mathbf{R}^n$ such that $B\Omega \in -\text{int}\, Q$.

Theorem 3.3. If a vector $\bar{x} \in S_0$ is a weak minimum for (NVOP) with $S = \mathbf{R}^n$ at which Slater-type cone constraint qualification holds, then there exist Lagrange multipliers $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$, such that

$$0 \in \partial(\bar{\lambda}^t f + \bar{\mu}^t g)(\bar{x})$$

$$\bar{\mu}^t g(\bar{x}) = 0 .$$

Note that, for $\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_m)^t \in \mathbf{R}^m$ and $\bar{\mu} = (\bar{\mu}_1, ..., \bar{\mu}_p)^t \in \mathbf{R}^p$,

$$\partial(\bar{\lambda}^t f + \bar{\mu}^t g)(\bar{x}) \subseteq (\partial f(\bar{x})^t \bar{\lambda} + \partial g(\bar{x})^t \bar{\mu}) .$$

Now we give the generalized form of nonsmooth KKT sufficient optimality conditions for (NVOP).

Theorem 3.4. Let f be K-generalized (Φ,ρ)-convex and g be Q-generalized (Φ,σ)-convex at $\bar{x} \in S_0$ on S_0. If there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$, such that

$$0 \in (\partial f(\bar{x})^t \bar{\lambda} + \partial g(\bar{x})^t \bar{\mu}) , \tag{1}$$

$$\bar{\mu}^t g(\bar{x}) = 0 , \tag{2}$$

$$\sum_{i=1}^{m} \bar{\lambda}_i + \sum_{j=1}^{p} \bar{\mu}_j > 0, \tag{3}$$

$$\bar{\lambda}^t \rho + \bar{\mu}^t \sigma \geq 0, \tag{4}$$

then $\bar{x}$ is a weak minimum for (NVOP).

Proof: Suppose to the contrary that $\bar{x}$ is not a weak minimum for (NVOP). Then there exists $\hat{x} \in S_0$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int}\, K. \tag{5}$$

By virtue of (1), there exist

$$\bar{A} = (\bar{A}_1, ..., \bar{A}_m)^t \in \partial f(\bar{x})$$

and $\quad \bar{B} = (\bar{B}_1, ..., \bar{B}_p)^t \in \partial g(\bar{x})$

such that,

$$\bar{A}^t \bar{\lambda} + \bar{B}^t \bar{\mu} = 0 . \tag{6}$$

Since f is K-generalized (Φ,ρ)-convex at $\bar{x}$ on S_0, we have

$$f(\hat{x}) - f(\bar{x}) - \Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) \in K . \tag{7}$$

Adding (5) and (7) we get,

$$\Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) \in -\text{int}\, K . \tag{8}$$

Since $\bar{\lambda} \in K^* \setminus \{0\}$, we have

$$\bar{\lambda}^t \Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) < 0 . \tag{9}$$

Also, since g is Q-generalized (Φ,σ)-convex at $\bar{x}$ on S_0 and $\bar{\mu} \in Q^*$, therefore

$$\bar{\mu}^t \{g(\hat{x}) - g(\bar{x}) - \Phi(\hat{x}, \bar{x}; (\bar{B}, \sigma))\} \geq 0 .$$

However, $\hat{x} \in S_0, \bar{\mu} \in Q^*$ and (2) together imply

$$\bar{\mu}^t \Phi(\hat{x}, \bar{x}; (\bar{B}, \sigma)) \leq 0 . \tag{10}$$

From (9) and (10), we have

$$\bar{\lambda}^t \Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) + \bar{\mu}^t \Phi(\hat{x}, \bar{x}; (\bar{B}, \sigma)) < 0. \tag{11}$$

Define $\tau = \dfrac{1}{\sum_{i=1}^{m} \bar{\lambda}_i + \sum_{j=1}^{p} \bar{\mu}_j}$,

$$\bar{\xi}_i = \tau \bar{\lambda}_i ,\ i = 1, 2, ..., m ,$$

$$\bar{\zeta}_j = \tau \bar{\mu}_j ,\ \ j = 1, 2, ..., p.$$

Let $\bar{\xi} = (\bar{\xi}_1, ..., \bar{\xi}_m)^t$ and $\bar{\zeta} = (\bar{\zeta}_1, ..., \bar{\zeta}_p)^t$.

(3), (4) and (6) respectively imply $\tau > 0$, $\bar{\xi}^t \rho + \bar{\zeta}^t \sigma \geq 0$ and $\bar{\xi}^t \bar{A} + \bar{\zeta}^t \bar{B} = 0$.

Also, by definition $\sum_{i=1}^{m} \bar{\xi}_i + \sum_{j=1}^{p} \bar{\zeta}_j = 1$.

Thus, using the properties of φ, we have

$$0 \le \varphi(\hat{x}, \bar{x}; (\bar{\xi}^t \bar{A} + \bar{\zeta}^t \bar{B},\ \bar{\xi}^t \rho + \bar{\zeta}^t \sigma))$$

$$= \varphi(\hat{x}, \bar{x}; (\sum_{i=1}^{m} \bar{\xi}_i \bar{A}_i + \sum_{j=1}^{p} \bar{\zeta}_j \bar{B}_j, \sum_{i=1}^{m} \bar{\xi}_i \rho_i + \sum_{j=1}^{p} \bar{\zeta}_j \sigma_j))$$

$$\le \sum_{i=1}^{m} \bar{\xi}_i \varphi(\hat{x}, \bar{x}; (\bar{A}_i, \rho_i)) + \sum_{j=1}^{p} \bar{\zeta}_j \varphi(\hat{x}, \bar{x}; (\bar{B}_j, \sigma_j))$$

$$= \tau(\bar{\lambda}^t \Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) + \bar{\mu}^t \Phi(\hat{x}, \bar{x}; (\bar{B}, \sigma))) < 0 \quad \text{(by (11))},$$

which is a contradiction.

Hence, $\bar{x}$ is a weak minimum for (NVOP).

Theorem 3.5. Let f be K-generalized (Φ,ρ)-pseudoconvex and g be Q-generalized (Φ,σ)-quasiconvex at $\bar{x} \in S_0$ on S_0 and suppose there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that (1), (2), (3) and (4) hold, then $\bar{x}$ is a weak minimum for (NVOP).

Proof. Let, if possible, $\bar{x}$ be not a weak minimum for (NVOP). Then there exists $\hat{x} \in S_0$ such that (5) holds.

In view of (1) there exist $\bar{A} \in \partial f(\bar{x})$ and $\bar{B} \in \partial g(\bar{x})$ such that (6) is satisfied.

Since f is K-generalized (Φ,ρ)-pseudoconvex at $\bar{x}$ on S_0, therefore from (5), we have

$$-\Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) \in \operatorname{int} K.$$

Now $\bar{\lambda} \in K^* \setminus \{0\}$ gives $\bar{\lambda}^t \Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) < 0.$

As $\hat{x} \in S_0$ and $\bar{\mu} \in Q^*$, we have $\bar{\mu}^t g(\hat{x}) \le 0$. On using (2), we get

$$\bar{\mu}^t \{g(\hat{x}) - g(\bar{x})\} \le 0. \tag{12}$$

If $\bar{\mu} \ne 0$, then (12) implies $g(\hat{x}) - g(\bar{x}) \notin \operatorname{int} Q$.

Since g is Q-generalized (Φ,σ)-quasiconvex at $\bar{x}$ on S_0, therefore

$$-\Phi(\hat{x}, \bar{x}; (\bar{B}, \sigma)) \in Q,$$

so that, $\bar{\mu}^t \Phi(\hat{x}, \bar{x}; (\bar{B}, \sigma)) \le 0.$ (13)

If $\bar{\mu} = 0$, then also (13) holds.

Now proceeding as in the last part of Theorem 3.4, we get a contradiction. Hence $\bar{x}$ is a weak minimum for (NVOP).

Theorem 3.6. Let f be K-generalized (Φ,ρ)-convex and g be Q-generalized (Φ,σ)-convex at $\bar{x} \in S_0$ on S_0. Suppose there exist $\bar{\lambda} \in K^{s*}$ and $\bar{\mu} \in Q^*$ such that (1), (2), (3) and (4) hold, then $\bar{x}$ is a minimum for (NVOP).

Proof. Let if possible $\bar{x}$ be not a minimum for (NVOP), then there exists $\hat{x} \in S_0$ such that

$$f(\bar{x}) - f(\hat{x}) \in K \setminus \{0\}. \tag{14}$$

As (1) holds, there exist $\bar{A} \in \partial f(\bar{x})$ and $\bar{B} \in \partial g(\bar{x})$ such that (6) holds.

Since f is K-generalized (Φ, ρ)-convex at $\bar{x}$ on S_0, therefore proceeding on the similar lines as in proof of Theorem 3.4 and using (14) we have

$$-\Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) \in K \setminus \{0\}.$$

As $\bar{\lambda} \in K^{s*}$, we have $\bar{\lambda}^t \Phi(\hat{x}, \bar{x}; (\bar{A}, \rho)) < 0.$

This leads to a contradiction as in Theorem 3.4. Hence $\bar{x}$ is a minimum for (NVOP).

4. Duality

We associate with the primal problem (NVOP), the following Wolfe-type dual problem (NWOD):

(NWOD)K-maximize $f(y) + \mu^t g(y) l$

$$\text{subject to } 0 \in (\partial f(y)^t \lambda + \partial g(y)^t \mu), \tag{15}$$

$$y \in S,\ l \in \operatorname{int} K,\ \lambda \in K^* \setminus \{0\},\ \mu \in Q^* \text{ and } \lambda^t l = 1.$$

We now establish duality results between (NVOP) and (NWOD).

Let W denote the set of feasible solutions of (NWOD) and Y_W be the subset of S given by $Y_W = \{y \in S : (y, \lambda, \mu) \in W\}$.

Theorem 4.1.(Weak Duality). Let x be feasible for (NVOP) and (y, λ, μ) be feasible for (NWOD). If f is K-generalized (Φ,ρ)-convex at y on $S_0 \cup Y_W$, g is Q-generalized (Φ,σ)-convex at y on $S_0 \cup Y_W$, $\sum_{i=1}^{m} \lambda_i + \sum_{j=1}^{p} \mu_j > 0$ and $\lambda^t \rho + \mu^t \sigma \ge 0$, then

$$f(y) + \mu^t g(y) l - f(x) \notin \operatorname{int} K. \tag{16}$$

Proof. Let if possible,

$$f(y) + \mu^t g(y) l - f(x) \in \operatorname{int} K. \tag{17}$$

Since (y, λ, μ) is feasible for (NWOD), therefore by (15), there exist

$\bar{A} \in \partial f(y)$ and $\bar{B} \in \partial g(y)$ such that

$$\bar{A}^t \lambda + \bar{B}^t \mu = 0. \quad (18)$$

Since f is K-generalized (Φ,ρ)-convex at y on $S_0 \bigcup Y_W$, therefore

$$f(x) - f(y) - \Phi(x, y; (\bar{A}, \rho)) \in K. \quad (19)$$

Adding (17) and (19), we get

$$\mu^t g(y)l - \Phi(x, y; (\bar{A}, \rho)) \in \text{int}\, K.$$

As $\lambda \in K^* \setminus \{0\}$ and $\lambda^t l = 1$, we have

$$\mu^t g(y) - \lambda^t \Phi(x, y; (\bar{A}, \rho)) > 0\,. \quad (20)$$

Again, since $x \in S_0$, g is Q-generalized (Φ,σ)-convex at y on $S_0 \bigcup Y_W$ and $\mu \in Q^*$, therefore

$$\mu^t [g(x) - g(y) - \Phi(x, y; (\bar{B}, \sigma))] \geq 0. \quad (21)$$

From (20) and (21), we have $\lambda^t \Phi(x, y; (\bar{A}, \rho)) + \mu^t \Phi(x, y; (\bar{B}, \sigma)) < \mu^t g(x)$.

Since x is feasible for (NVOP) and $\mu \in Q^*$, $\mu^t g(x) \leq 0$, so that we have

$$\lambda^t \Phi(x, y; (\bar{A}, \rho)) + \mu^t \Phi(x, y; (\bar{B}, \sigma)) < 0\,.$$

Now proceeding as in proof of Theorem 3.4, we obtain a contradiction. Hence (16) holds.

This weak duality result allows us to obtain strong duality result as follows.

Theorem 4.2. (Strong Duality). Let $\bar{x}$ be a weak minimum for (NVOP) at which Slater-type cone constraint qualification is satisfied. Then there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (NWOD). Moreover, if the conditions of Theorem 4.1, are satisfied for each feasible solution of (NWOD), then $\bar{x}$ is a weak maximum for (NWOD).

Proof. Since $\bar{x}$ is a weak minimum of (NVOP), therefore by Theorem 3.3, there exist $\bar{\lambda} \in K^* \setminus \{0\}$, $\bar{\mu} \in Q^*$ such that (1) and (2) hold.

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (NWOD). Now assume on the contrary that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is not a weak maximum for (NWOD), then there exists a feasible solution (y, λ, μ) for (NWOD) such that

$$\{f(y) + \mu^t g(y)l\} - \{f(\bar{x}) + \bar{\mu}^t g(\bar{x})l\} \in \text{int}\, K\,,$$

which on using (2) gives

$$f(y) + \mu^t g(y)l - f(\bar{x}) \in \text{int}\, K\,.$$

This contradicts Weak Duality Theorem 4.1. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak maximum for (NWOD).

Now we consider the following Mond-Weir type dual (NMOD) related to problem (NVOP):

(NMOD)K-maximize $f(y)$

$$\text{subject to } 0 \in \partial f(y)^t \lambda + \partial g(y)^t \mu \quad (22)$$

$$\mu^t g(y) \geq 0\,, \quad (23)$$

$$y \in S,\ \lambda \in K^* \setminus \{0\} \text{ and } \mu \in Q^*.$$

Let M denote the set of feasible solutions of (NMOD) and Y_M be the subset of S defined by $Y_M = \{y \in S : (y, \lambda, \mu) \in M\}$.

Theorem 4.3. (Weak Duality). Let x be feasible for (NVOP) and (y, λ, μ) be feasible for (NMOD). Suppose f is K-generalized (Φ,ρ)-pseudoconvex and g is Q-generalized (Φ,σ)-quasiconvex at y on $S_0 \bigcup Y_M$ such that $\sum_{i=1}^{m} \lambda_i + \sum_{j=1}^{p} \mu_j > 0$ and $\lambda^t \rho + \mu^t \sigma \geq 0$, then

$$f(y) - f(x) \notin \text{int}\, K\,. \quad (24)$$

Proof. Assume on the contrary,

$$f(y) - f(x) \in \text{int}\, K\,. \quad (25)$$

Since (y, λ, μ) is feasible for (NMOD), there exist $\bar{A} \in \partial f(y)$ and $\bar{B} \in \partial g(y)$ such that (18) holds.

As f is K-generalized (Φ,ρ)-pseudoconvex at y on $S_0 \bigcup Y_M$, therefore from (25), we have

$$-\Phi(x, y; (\bar{A}, \rho)) \in \text{int}\, K.$$

Since $\lambda \in K^* \setminus \{0\}$, we get $\lambda^t \Phi(x, y; (\bar{A}, \rho)) < 0$.

Also, $x \in S_0$ and $\mu \in Q^*$ so that $\mu^t g(x) \leq 0$. This together with (23) gives $\mu^t \{g(x) - g(y)\} \leq 0$.

Now proceeding on similar lines as in proof of Theorem 3.5 we get a contradiction. Hence (24) holds.

Theorem 4.4. (Strong Duality). Let $\bar{x}$ be a weak minimum of (NVOP) at which Slater-type cone constraint qualification is satisfied. Then there exist $\bar{\lambda} \in K^* \setminus \{0\}$ and $\bar{\mu} \in Q^*$ such that

$(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (NMOD). Moreover, if the conditions of Weak Duality Theorem 4.3 are satisfied for each feasible solution (y, λ, μ) of (NMOD), then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak maximum of (NMOD).

Proof. The proof is similar to that of Theorem 4.2 except that we invoke Theorem 4.3 instead of Theorem 4.1.

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References

[1] Antczak, T., On nonsmooth (Φ,ρ)-invexmultiobjective programming in finite-dimensional Euclidean spaces, Journal of Advanced Mathematical Studies, 7, 127-145 (2014).

[2] Antczak, T., Stasiak, A., (Φ,ρ)-invexity in nonsmooth optimization, Numerical Functional Analysis and Optimization, 32(1), 1-25 (2010).

[3] Caristi, G., Ferrara, M., Stefanescu, A., Mathematical programming with (Φ,ρ)-invexity. In: Konnov, I.V., Luc, D.T., Rubinov, A.M. (eds.) Generalized Convexity and Related Topics. Lecture Notes in Economics and Mathematical Systems, 583, 167-176. Springer, Berlin-Heidelberg-New York (2006).

[4] Clarke, F. H., Optimization and Nonsmooth Analysis, Wiley, New York (1983).

[5] Craven, B. D., Nonsmooth multiobjective programming, Numerical Functional Analysis and Optimization, 10(1-2), 49-64 (1989).

[6] Ferrara, M., Stefanescu, M.V., Optimality conditions and duality in multiobjective programming with (Φ,ρ)-invexity. Yugoslav Journal of Operations Research, 18, 153-165 (2008).

[7] Hanson, M.A., On sufficiency of the Kuhn-Tucker conditions, Journal of Mathematical Analysis and Applications, 80, 545-550 (1981).

[8] Hanson, M.A., Mond, B., Further generalization of convexity in mathematical programming. Journal of Information and Optimization Sciences, 3, 25-32 (1982).

[9] Preda, V., On efficiency and duality for multiobjective programs, Journal of Mathematical Analysis and Applications, 166, 365-377 (1992).

[10] Vial, J.P., Strong and weak convexity of sets and functions, Mathematics of Operations Research, 8, 231-259 (1983).

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