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# On $G$-invexity-type nonlinear programming problems

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**Abstract.** In this paper, we introduce the concepts of $KT$-$G$-invexity and $WD$-$G$-invexity for the considered differentiable optimization problem with inequality constraints. Using $KT$-$G$-invexity notion, we prove new necessary and sufficient optimality conditions for a new class of such nonconvex differentiable optimization problems. Further, the so-called $G$-Wolfe dual problem is defined for the considered extremum problem with inequality constraints. Under $WD$-$G$-invexity assumption, the necessary and sufficient conditions for weak duality between the primal optimization problem and its $G$-Wolfe dual problem are also established.

**Keywords:** Mathematical programming; $WD$-$G$-invexity; $G$-Karush-Kuhn-Tucker point; $G$-Wolfe dual problem.
**AMS Classification:** 90C26, 90C30, 90C46, 26A51.

## 1. Introduction

In the paper, we consider the following constrained optimization problem:

$$\begin{aligned} &\text{minimize } f(x) \\ \text{subject to } \quad & g_j(x) \leqq 0, \;\; j \in J = \{1, ...m\}\,, \quad \text{(P)} \\ & x \in X, \end{aligned}$$

where $f : X \to R$ and $g_j : X \to R$, $j \in J$, are differentiable functions defined on a nonempty open set $X \subset R^n$.

For the purpose of simplifying our presentation, we will next introduce some notation which will be used frequently throughout this paper.

Let

$$D := \{x \in X : g_j(x) \leqq 0, \;\; j \in J\}$$

be the set of all feasible solutions in problem (P).

Further, we denote an index set of active inequality constraints at point $\overline{x} \in X$ as follows:

$$J(\overline{x}) = \{j \in J : g_j(\overline{x}) = 0\}\,.$$

In recent years, attempts are made by several authors to define various classes of nonconvex functions and to study their optimality criteria and duality results in solving such types of optimization problems. One of a such generalization of a convex function is invexity notion introduced by Hanson [11] for differentiable mathematical programming problems. The term invex (which means invariant convex) was suggested later by Craven [10]. Over the years, many generalizations of this concept have been given in the literature (see, for instance, [1], [2], [3], [5], [6], [7], [8], [9], [12], [13], [14], [15], [16], and others).

In [14], Martin showed that elementary relaxations of the conditions defining invexity lead to

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modified invexity notions which are both necessary and sufficient for weak duality and Kuhn-Tucker sufficiency. Hence, Martin introduced the definition of Kuhn-Tucker invex ($KT$-invex) optimization problem and he proved that every Kuhn-Tucker point of the optimization problem with inequality constraints is a global minimizer if and only if this extremum problem is Kuhn-Tucker invex. Also Martin gave the necessary and sufficient conditions for weak duality to hold. Namely, he introduced the concept of $WD$-invexity and he showed that weak duality between the considered optimization problem with inequality constraints and its Wolfe dual problem holds if the primal extremum problem is $WD$-invex.

In [4], Antczak generalized Hanson's definition of a (differentiable) invex function and he introduced the concept of $G$-invexity for differentiable constrained optimization problems. He formulated and proved new necessary optimality conditions of $G$-F. John and $G$-Karush–Kuhn–Tucker type for differentiable constrained mathematical programming problems and, under $G$-invexity assumptions, he established the sufficiency of these necessary optimality conditions. Further, for the considered extremum problem with inequality constraints, Antczak [4] formulated the so-called $G$-Mond–Weir-type dual and he proved various duality results by assuming the functions involved to be $G$-invex with respect to the same function $\eta$ and with respect to, not necessarily, the same function $G$.

In this paper, following Martin [14] and Antczak [4], we introduce the definitions of $KT$-$G$-invexity and $WD$-$G$-invexity notions for the considered differentiable optimization problem (P) with inequality constraints. For such an extremum problem (P), we define the so-called $G$-Karush-Kuhn-Tucker point ($G$-$KKT$-point) and we prove that every $G$-Karush-Kuhn-Tucker point of problem (P) is its global minimizer if and only if problem (P) is $KT$-$G$-invex. Thus, we extend the result established by Ben-Israel and Mond [6] to the case of a new class of nonconvex optimization problems. Further, for the considered constrained optimization problem (P), we define a modified dual problem in the sense of Wolfe - we call it the $G$-Wolfe dual problem ($G$-$WD$). Under assumption that the primal problem (P) is $WD$-$G$-invex, we prove the necessary and sufficient conditions for weak duality to hold between problems (P) and ($G$-$WD$). Thus, the main purpose of this paper is to use the introduced concepts of $KT$-$G$-invexity and $WD$-$G$-invexity in proving the necessary and sufficient optimality conditions and the necessary and sufficient conditions for weak duality for a new class of nonconvex differentiable optimization problems.

## 2. Optimality

The following convention for equalities and inequalities will be used in the paper.

For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;

(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;

(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;

(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

**Definition 1.** *A function $f : R \to R$ is said to be increasing if and only if*

$$\forall x, y \in R \quad x < y \implies f(x) < f(y).$$

Now, for the considered constrained optimization problem (P), we define the concept of $KT$-$G$-invexity. Let $f : X \to R$ and $g : X \to R$ be defined as in the formulation of problem (P) and, moreover, $I_f(D)$ and $I_g(D)$ be the range of $f$ and $g$, that is, the image of $D$ under $f$ and the image of $D$ under $g$, respectively.

**Definition 2.** *The constrained optimization problem (P) is said to be Kuhn-Tucker-G-invex (shortly, $KT$-$G$-invex) at $u \in D$ on $D$ if there exist real-valued differentiable increasing functions $G_f : I_f(D) \to R$, $G_{g_j} : I_{g_j}(D) \to R$, $j \in J$, and a vector-valued function $\eta : D \times D \to R^n$ such that, the following relations*

$$\left.\begin{array}{l} x \in D \\ u \in D \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{c} G_f(f(x)) - G_f(f(u)) \geqq G'_f(f(u)) \nabla f(u) \eta(x,u) \\ \text{if } j \in J_{Max}(u), \text{ then } -G'_{g_j}(f(u)) \nabla g_j(u) \eta(x,u) \geqq 0 \end{array}\right. \tag{1}$$

*hold, where $J_{Max}(u) = \{j \in J : G_{g_j}(g_j(u)) = Max\{G_{g_j}(g_j(x)) : x \in D\}\}$.*

*If the relations (1) are satisfied at any point $u \in D$, then problem (P) is said to be $KT$-$G$-invex on $D$.*

**Remark 3.** *In the case when $G_f(a) \equiv a$ for any $a \in I_f(X)$, $G_{g_j}(a) \equiv a$, $j \in J$, for any $a \in I_{g_j}(X)$, it follows that $J_{Max}(u) = J(u)$ and we obtain the definition of $KT$-invexity introduced by Martin [14] for differentiable optimization problems.*

Now, we give the definition of a modified Kuhn-Tucker point in the considered optimization problem (P) and we call it a $G$-Karush-Kuhn-Tucker point.

**Definition 4.** [4] *A point $\overline{x} \in D$ (if it exists) is said to be a $G$-Karush-Kuhn-Tucker point in the considered optimization problem (P) if there exists $\overline{\xi} \in R^m$ such that the following relations*

$$G'_f\left(f(\overline{x})\right)\nabla f\left(\overline{x}\right) + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j\left(\overline{x}\right) = 0, \tag{2}$$

$$\overline{\xi}_j\left[G_{g_j}\left(g_j\left(x\right)\right) - G_{g_j}\left(g_j\left(\overline{x}\right)\right)\right] \leqq 0, \quad \forall j \in J,\ \forall x \in D, \tag{3}$$

$$\overline{\xi} \geqq 0 \tag{4}$$

*are satisfied, where $G_f$ is a real-valued differentiable increasing function defined on $I_f(D)$, and $G_{g_j}$, $j \in J$, is a real-valued differentiable increasing function defined on $I_{g_j}(D)$ such that $\sum_{j=1}^{m}\left[G'_{g_j}\left(g_j(\overline{x})\right)\right]^2 \neq 0$.*

**Remark 5.** *We call the relations (2)-(4) the $G$-Karush-Kuhn-Tucker necessary optimality conditions (see [4]) for the considered optimization problem (P).*

We now prove the necessary and sufficient optimality conditions for the considered optimization problem (P) under the assumption that it is $KT$-$G$-invex.

**Theorem 6.** *Every $G$-Karush-Kuhn-Tucker point is a global minimizer in problem (P) if and only if problem (P) is $KT$-$G$-invex.*

***Proof.*** (Sufficiency). Assume that problem (P) is $KT$-$G$-invex. Let $\overline{x}$ be a $G$-Karush-Kuhn-Tucker point in problem (P). Then, there exists a Lagrange multiplier $\overline{\xi} \in R^m$ such that the $G$-Karush-Kuhn-Tucker necessary optimality conditions (2)-(4) are satisfied. By (2), it follows

$$\begin{aligned} & G'_f\left(f(\overline{x})\right)\nabla f\left(\overline{x}\right)\eta\left(x,\overline{x}\right) \\ + & \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j\left(\overline{x}\right)\right)\nabla g_j\left(\overline{x}\right)\eta\left(x,\overline{x}\right) = 0, \\ & \forall x \in D. \end{aligned} \tag{5}$$

Using the first relation in (1) together with (5), we get

$$\begin{aligned} & G_f\left(f(x)\right) - G_f\left(f(\overline{x})\right) \\ \geqq & -\sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(f(\overline{x})\right)\nabla g_j\left(\overline{x}\right)\eta\left(x,\overline{x}\right). \end{aligned} \tag{6}$$

Since $\overline{x}$ is a $G$-Karush-Kuhn-Tucker point in problem (P), it is feasible in problem (P). As it follows from (3), if $\overline{\xi}_j \neq 0$ for some $j \in J$, then $G_{g_j}(g_j(\overline{x})) = Max\{G_{g_j}(g_j(x)) : x \in D\}$, that is, $j \in J_{Max}(\overline{x})$. Since $\overline{\xi} \geqq 0$, therefore, for $j \in J_{Max}(\overline{x})$, the second relation in (1) implies that the following relation

$$-\overline{\xi}_j G'_{g_j}\left(f(\overline{x})\right)\nabla g_j(\overline{x})\eta\left(x,\overline{x}\right) \geqq 0 \tag{7}$$

holds for all $x \in D$. Combining (6) and (7), we obtain that the inequality

$$G_f\left(f(x)\right) \geqq G_f\left(f(\overline{x})\right)$$

is satisfied for all $x \in D$. Since $G_f$ is an increasing function on its domain, the following inequality

$$f(x) \geqq f(\overline{x})$$

holds for all $x \in D$. This means that $\overline{x}$ is a global minimizer in problem (P).

(Necessity). Assume that every $G$-Karush-Kuhn-Tucker point of problem (P) is a global minimizer. For any pair of points $x, \overline{x} \in D$, we consider the following cases:

(i) Assume that $x$ and $\overline{x}$ are feasible points in problem (P) satisfying the inequality $f\left(x\right) < f\left(\overline{x}\right)$. Then, by definition, $\overline{x}$ is not a global minimizer in problem (P). By assumption, therefore, it is not a $G$-Karush-Kuhn-Tucker point for problem (P). This means that there exists no a set of multipliers such that (2)-(4) are fulfilled, that is, there exist no

$$\overline{\lambda} > 0,\ \overline{\xi}_j \geqq 0,\ j \in J_{Max}\left(\overline{x}\right)$$

such that

$$\begin{aligned} & \overline{\lambda} G'_f\left(f(\overline{x})\right)\nabla f\left(\overline{x}\right) \\ + & \sum_{j \in J_{Max}(\overline{x})} \overline{\xi}_j G'_{g_j}\left(g_j\left(\overline{x}\right)\right)\nabla g_j\left(\overline{x}\right) = 0, \end{aligned} \tag{8}$$

where $G_f$ is a real-valued differentiable increasing function defined on $I_f(D)$, and $G_{g_j}$, $j \in J$, is a real-valued differentiable increasing function defined on $I_{g_i}(D)$ with $\sum_{j=1}^{m}\left[G'_{g_j}\left(g_j(\overline{x})\right)\right]^2 \neq 0$.

Note that, if the equality (8) was satisfied, then the multipliers $\overline{\lambda}$, $\overline{\xi} = (\overline{\xi}_1, \ldots, \overline{\xi}_m)$ with $\overline{\xi}_j = 0$ for all $j \notin J_{Max}(\overline{x})$, would verify (2)-(4). According to Tucker's theorem of the alternative, it follows that there exists a vector $w \in R^n$, depending upon $\overline{x}$, such that

$$G'_f\left(f(\overline{x})\right)\nabla f\left(\overline{x}\right)w\left(\overline{x}\right) > 0 \tag{9}$$

and

$$G'_{g_j}\left(g_j\left(\overline{x}\right)\right)\nabla g_j\left(\overline{x}\right)w\left(\overline{x}\right) \geqq 0,\ j \in J_{Max}(\overline{x}). \tag{10}$$

Then, we set

$$\eta\left(x,\overline{x}\right) = \frac{G_f\left(f(x)\right) - G_f\left(f(\overline{x})\right)}{G'_f\left(f(\overline{x})\right)\nabla f(\overline{x})w\left(\overline{x}\right)}w\left(\overline{x}\right). \tag{11}$$

Hence, by (11), we have

$$G_f\left(f(x)\right) - G_f\left(f(\overline{x})\right) = G'_f\left(f(\overline{x})\right)\nabla f(\overline{x})\eta\left(x,\overline{x}\right). \tag{12}$$

By assumption, $f(x) < f(\overline{x})$ and $G_f$ is an increasing on its domain. Thus, we have

$$G_f\left(f(x)\right) < G_f\left(f(\overline{x})\right). \tag{13}$$

As it follows from (9) and (13), the scalar factor in (11) is negative. Then, multiplying (10) by this factor, we get

$$G'_{g_j}\left(g_j\left(\overline{x}\right)\right)\nabla g_j\left(\overline{x}\right)\eta\left(x,\overline{x}\right) \leqq 0, j \in J_{Max}\left(\overline{x}\right). \tag{14}$$

(ii) Now, assume that $x, \overline{x} \in D$ are feasible points in problem (P) satisfying the inequality $f(x) \geqq f(\overline{x})$. Since $G_f$ is an increasing on its domain, the above inequality implies

$$G_f\left(f(x)\right) \geqq G_f\left(f(\overline{x})\right).$$

In this case, therefore, it is sufficient to set that

$$\eta\left(x,\overline{x}\right) = 0 \tag{15}$$

to ensure that the inequality

$$G_f\left(f(x)\right) - G_f\left(f(\overline{x})\right) \geqq G'_f\left(f(\overline{x})\right)\nabla f(\overline{x})\eta\left(x,\overline{x}\right) \tag{16}$$

is satisfied. Moreover, by (15), for all $j \in J_{Max}\left(\overline{x}\right)$, we have

$$G'_{g_j}\left(g_j\left(\overline{x}\right)\right)\nabla g_j\left(\overline{x}\right)\eta\left(x,\overline{x}\right) = 0. \tag{17}$$

Thus, assuming only that every $G$-Karush-Kuhn-Tucker point is a global minimum in problem (P), we have shown the existence of a function $\eta : D \times D \to R^n$ that meets requirements of Definition 2. This is a conclusion of the necessity and completes the proof of theorem. □

**Remark 7.** *Note that to prove that every $G$-Karush-Kuhn-Tucker point is a global minimizer in problem (P) it is sufficient to assume that problem (P) is $KT$-$G$-invex at $\overline{x}$ on $D$.*

In order to illustrate this result, we present an example of $KT$-$G$-invex optimization problem.

**Example 8.** *Consider the following nonconvex optimization problem*

$$\begin{array}{c} f\left(x\right) = \ln\left(x^2 + x + 1\right) \to \min \\ g\left(x\right) = 1 - \exp\left(x\right) \leqq 0. \end{array} \qquad (P1)$$

*Note that $D = \{x \in R : x \geqq 0\}$ and $\overline{x} = 0$ is a feasible solution in the considered optimization problem (P1). We now show that $\overline{x} = 0$ is a $G$-Karush-Kuhn-Tucker point in problem (P1). In order to do it, we set $G_f\left(t\right) = \exp\left(t\right)$ and $G_g\left(t\right) = -\ln\left(1-t\right)$. Then, it is not difficult to show that there exist $\overline{\xi} = 1$ such that the conditions (2)-(4) are satisfied with the functions $G_f$ and $G_g$ defined above. Then, by Definition 4, $\overline{x} = 0$ is a $G$-Kuhn-Tucker point in problem (P1). Now, we show that the considered optimization problem (P1) is $KT$-$G$-invex at $\overline{x}$ on $D$ (with respect to functions $G_f$ and $G_g$ defined above). We set $\eta\left(x,\overline{x}\right) = x - \overline{x}$. Then, by Definition 2, it follows that the considered optimization problem (P1) is $KT$-$G$-invex at $\overline{x}$ on $D$ (with respect to $\eta$, $G_f$ and $G_g$ given above). Thus, $\overline{x} = 0$ is a global minimizer in the considered optimization problem (P1). Further, note that it is not possible to use the concept of invexity introduced by Hanson* [11] *to prove that $\overline{x} = 0$ is a global minimizer in the considered optimization problem (P1). It is not difficult to show that the functions constituting problem (P1) are not invex at $\overline{x}$ on $D$ with respect to the same function $\eta$ defined by $\eta : D \times D \to R$.*

In some cases of nonconvex optimization problems, it is easier to show that the considered optimization problem is $KT$-$G$-invex than $KT$-invex in the sense of definition introduced by Martin [14]. In some of such cases, the function $\eta$ has more complex form in the definition of $KT$-invexity than in the formulation of $KT$-$G$-invexity and, therefore, it is more difficult to find such a function $\eta$ satisfying the definition of $KT$-invexity. Now, we give an example of such a nonconvex optimization problem.

**Example 9.** *Consider the following nonconvex optimization problem:*

$$\begin{array}{c} f(x) = \arctan(\exp(x) + x - 1) \to \min \\ g(x) = \exp(-x) - 1 \leqq 0. \end{array} \qquad (P2)$$

*Note that* $D = \{x \in R : x \geqq 0\}$ *and* $\overline{x} = 0$ *is a feasible solution in the considered optimization problem (P2). Now, we show that* $\overline{x} = 0$ *is a G-Kuhn-Tucker point in problem (P2). In order to do it, we set* $G_f(t) = \tan(t)$ *and* $G_g(t) = \ln(t+1)$. *Then, it is not difficult to show that there exists* $\overline{\xi} = 2$ *such that the conditions (2)-(4) are satisfied for such defined functions* $G_f$ *and* $G_g$. *Then, by Definition 4,* $\overline{x} = 0$ *is a G-Karush-Kuhn-Tucker point in problem (P1). Now, we show that the considered optimization problem (P2) is KT-G-invex at* $\overline{x}$ *on D (with respect to functions* $G_f$ *and* $G_g$ *defined above). We set* $\eta(x, \overline{x}) = x - \overline{x}$. *Then, by Definition 2, it follows that the considered optimization problem (P1) is KT-G-invex at* $\overline{x}$ *on D (with respect to* $\eta$, $G_f$ *and* $G_g$ *given above). It is not difficult to show by the definition of KT-invexity given by Martin* [14] *that problem (P2) is not KT-invex* $\overline{x}$ *on D with respect to* $\eta$ *given above. In order to prove that it is KT-invex at* $\overline{x}$ *on D, we set* $\widetilde{\eta}(x, \overline{x}) = \frac{1}{2}(\arctan(x) - \arctan(\overline{x}))$. *Then, by definition, it is possible to show that problem (P2) is KT-invex* $\overline{x}$ *on D with respect to* $\widetilde{\eta}$ *given above. However, it is not difficult to see that the form of the function* $\widetilde{\eta}$ *with respect to which problem (P2) is KT-invex is more complex than the function* $\eta$ *with respect to which problem (P2) is KT-G-invex. The fact that the function* $\eta$ *with respect to which the given optimization problem is KT-G-invex is less complex than in the case of KT-invexity is an useful property from the practical point of view.*

## 3. Duality

In this section, for the considered optimization problem (P), we consider the modified Wolfe dual problem ($G$-$WD$), the so-called $G$-Wolfe dual problem. We give the necessary and sufficient conditions for weak duality between problems (P) and ($G$-$WD$). To do this, we use the concept of $WD$-$G$-invexity introduced in this section.

For the considered optimization problem (P), consider the $G$-Wolfe dual problem in the following form:

$$\begin{array}{c} G_f(f(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}(g_j(y)) \to \max \\ G'_f(f(y)) \nabla f(y) \\ + \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(y)) \nabla g_j(y) = 0, \\ y \in X,\ \xi_j \geqq 0,\ j \in J, \end{array} \qquad (G\text{-}WD)$$

where $G_f$ is a real-valued differentiable increasing function defined on $I_f(X)$, and $G_{g_j}$, $j \in J$, is a real-valued differentiable increasing function defined on $I_{g_j}(X)$. We denote by $W$ the set of all feasible solutions in the $G$-Wolfe dual problem ($G$-$WD$), that is, the set

$$W = \Big\{(y, \xi) \in X \times R^m : G'_f(f(y)) \nabla f(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(y)) \nabla g_j(y) = 0,\ \xi_j \geqq 0,\ j \in J\Big\}.$$

Now, we introduce the definition of $WD$-$G$-invexity for the considered optimization problem (P).

**Definition 10.** *Problem (P) is said to be weak duality G-invex (shortly, WD-G-invex) on X if there exists a vector valued function* $\eta : X \times X \to R^n$ *such that*

$$\left.\begin{array}{c} x \in D \\ u \in X \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{c} \textit{either } G_f(f(x)) - G_f(f(u)) \\ -G'_f(f(u)) \nabla f(u) \eta(x,u) \geqq 0, \\ -G_{g_j}(g_j(u)) - G'_{g_j}(g_j(u)) \nabla g_j(u) \eta(x,u) \geqq 0, \\ \textit{or } \ -G'_f(f(u)) \nabla f(u) \eta(x,u) > 0, \\ -G'_{g_j}(g_j(u)) \nabla g_j(u) \eta(x,u) \geqq 0. \end{array}\right.$$

**Remark 11.** *In the case when* $G_f(a) \equiv a$ *and* $G_{g_j}(b) \equiv b$, $j = 1, ..., m$, *we obtain the definition of of WD-invexity introduced by Martin* [14] *for differentiable optimization problems.*

**Definition 12.** *Weak duality is said to hold between problems (P) and (G-WD) if, for every feasible point x for the primal optimization problem (P) and every feasible pair* $(y, \xi) \in W$ *for its G-Wolfe dual problem (G-WD), we have*

$$G_f(f(x)) \geqq G_f(f(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}(g_j(y)).$$

Now, under assumption of $WD$-$G$-invexity, we prove the necessary and sufficient conditions for weak duality between problems (P) and ($G$-$WD$).

**Theorem 13.** *Weak duality holds between the primal optimization problem (P) and its G-Wolfe dual problem (G-WD) if and only if problem (P) is WD-G-invex on X.*

***Proof.*** (Necessity). Assume that the $G$-weak duality between problems (P) and ($G$-$WD$) holds. This means that for any feasible solutions $x$ and $(y, \xi)$ in problems (P) and ($G$-$WD$), the system

$$\begin{gathered} G_f'\left(f(y)\right)\nabla f\left(y\right)+\textstyle\sum_{j=1}^{m}\xi_j G_{g_j}'\left(g_j\left(y\right)\right)\nabla g_j\left(y\right)=0 \\ G_f\left(f(x)\right)-G_f\left(f(y)\right)-\textstyle\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j\left(y\right)\right)<0 \\ \xi=(\xi_1,...,\xi_m)\in R^m, \xi \geqq 0 \end{gathered}$$

is inconsistent in $\xi$. Equivalently, the homogeneous system

$$\begin{gathered} \left[\begin{array}{ccc} \alpha & \vdots & \beta \end{array}\right] \\ \times\left[\begin{array}{ccc} 0 & \vdots & 1 \\ G_f'\left(f(y)\right)\nabla f\left(y\right) & \vdots & G_f\left(f(x)\right)-G_f\left(f(y)\right) \end{array}\right] \\ +\xi\left[\begin{array}{ccc} G_g'\left(g\left(y\right)\right)\nabla g\left(y\right) & \vdots & -G_g\left(g\left(y\right)\right) \end{array}\right]=0, \\ \left[\begin{array}{ccc} \alpha & , & \beta \end{array}\right]>0, \quad \xi\geqq 0 \end{gathered}$$

is inconsistent in $(\alpha,\beta,\xi)$. By Tucker's theorem of the alternative, this, in turn, is equivalent to the consistency of the system

$$\left[\begin{array}{ccc} 0 & \vdots & 1 \\ G_f'\left(f(y)\right)\nabla f\left(y\right) & \vdots & G_f\left(f(x)\right)-G_f\left(f(y)\right) \end{array}\right] \times\left[\begin{array}{c} \eta \\ \vartheta \end{array}\right]\leq 0$$

$$\left[\begin{array}{ccc} G_g'\left(g\left(y\right)\right)\nabla g\left(y\right) & \vdots & -G_g\left(g\left(y\right)\right) \end{array}\right]\left[\begin{array}{c} \eta \\ \vartheta \end{array}\right]<0.$$

If the first component in the above system is strictly negative, that is, $\vartheta<0$, then we may take $\vartheta=-1$ to conclude that

$$\begin{gathered} G_f\left(f(x)\right)-G_f\left(f(y)\right)-G_f'\left(f(y)\right)\nabla f\left(y\right)\eta\geqq 0, \\ -G_g\left(g\left(y\right)\right)-G_g'\left(g\left(y\right)\right)\nabla g\left(y\right)\eta\geqq 0. \end{gathered} \tag{18}$$

If the first argument the above system is equal to zero, that is, $\vartheta=0$, then the second must be strictly negative. Therefore, we have

$$\begin{gathered} G_f'\left(f(y)\right)\nabla f\left(y\right)\eta\geqq 0, \\ -G_g'\left(g\left(y\right)\right)\nabla g\left(y\right)\eta\geqq 0. \end{gathered} \tag{19}$$

This means that there exists a vector valued function $\eta: X\times X\to R^n$ such that the inequalities (18) or (19) are satisfied. This means that (P) is $WD$-$G$-invex on $X$.

(Sufficiency). Let $x$ and $(y,\xi)$ be any feasible points in problems (P) and ($G$-$WD$), respectively. Assume that the considered optimization problem (P) is $WD$-$G$-invex on $X$. We consider the case when the following inequalities

$$\begin{aligned} & G_f\left(f(x)\right)-G_f\left(f(y)\right) \\ & \qquad -G_f'\left(f(y)\right)\nabla f(y)\eta\left(x,y\right)\geqq 0, \\ & -G_{g_j}\left(g_j\left(y\right)\right) \\ & \qquad -G_{g_j}'\left(g_j\left(y\right)\right)\nabla g_j\left(y\right)\eta\left(x,y\right)\geqq 0,\ j\in J. \end{aligned} \tag{20}$$

Multiplying the second inequality above by $\xi_j\geqq 0$ and then adding both sides of the obtained inequalities, we get

$$\begin{aligned} & -\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j\left(y\right)\right) \\ & -\sum_{j=1}^{m}\xi_j G_{g_j}'\left(g_j\left(y\right)\right)\nabla g_j\left(y\right)\eta\left(x,y\right)\geqq 0. \end{aligned} \tag{21}$$

Adding both sides of the first inequality in (20) and (21), we obtain

$$\begin{gathered} G_f\left(f(x)\right)-G_f\left(f(y)\right)-\textstyle\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j\left(y\right)\right)\geqq \\ \left[G_f'\left(f(y)\right)\nabla f(y)+\textstyle\sum_{j=1}^{m}\xi_j G_{g_j}'\left(g_j\left(y\right)\right)\nabla g_j\left(y\right)\right]\eta\left(x,y\right). \end{gathered}$$

From the feasibility of $(y,\xi)$ in $G$-Wolfe dual problem ($G$-$WD$), it follows that

$$G_f\left(f(x)\right)\geqq G_f\left(f(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j\left(y\right)\right).$$

Now, assume that the following inequalities in Definition 10

$$\begin{gathered} -G_f'\left(f(y)\right)\nabla f(y)\eta\left(x,y\right)>0, \\ -G_{g_j}'\left(g_j\left(y\right)\right)\nabla g_j\left(y\right)\eta\left(x,y\right)\geqq 0 \end{gathered}$$

are satisfied for all $x\in D$ and all $y\in X$. Multiplying the second inequality above by $\xi_j\geqq 0$ and then adding both sides of the obtained inequalities, we get

$$\left[G_f'\left(f(y)\right)\nabla f(y)+\sum_{j=1}^{m}\xi_j G_{g_j}'\left(g_j\left(y\right)\right)\nabla g_j\left(y\right)\right] \times\eta\left(x,y\right)<0.$$

By the above inequality, we conclude that $(y,\xi)$ is not feasible in $G$-Wolfe dual problem ($G$-$WD$) for any multiplier vector $\xi=(\xi_1,...,\xi_m)\in R^m$, $\xi\geqq 0$. This means that such a point plays no role in determining whether or not $G$-weak duality holds.

This completes the proof of theorem. □

## 4. Conclusions

In the paper, new concepts of generalized invexity have been defined for differentiable optimization problems. The so-called $KT$-$G$-invexity and $WD$-$G$-invexity notions defined for the considered differentiable optimization problem (P) with inequality constraints are generalizations the $G$-invexity notions introduced by Antczak [4] and the concepts of $KT$-invexity and $WD$-invexity introduced by Martin [14], respectively. It has turned out that the introduced $KT$-$G$-invexity notion is a necessary and sufficient condition for optimality in a new class of nonconvex differentiable optimization problems. Namely, it was proved that every so-called $G$-Kuhn-Tucker point in problem (P) is its global minimizer if and only if problem (P) is $KT$-$G$-invex. Moreover, as it follows from the proof of this result, some characterization of a function $\eta$ (with respect to which the given constrained optimization problem is $KT$-$G$-invex) is given. The property that the function $\eta$ could be less complex in the case of $KT$-$G$-invexity than in the case of $KT$-invexity for some nonconvex optimization problems is important from the practical point of view. Note that, for some nonconvex optimization problems, we are not in a position to establish the optimality of a feasible point satisfying necessary optimality conditions under invexity, but the concept of $KT$-$G$-invexity turned out to be useful in proving this result.

Further, for the considered differentiable optimization problem (P), the so-called $G$-Wolfe dual problem ($WD$-$G$) has been defined. The concept $WD$-$G$-invexity defined in the paper has turned out to be a necessary and sufficient condition to weak duality holds between problems (P) and ($WD$-$G$) In this way, this result was proved for a new class of nonconvex differentiable optimization problems.

Some interesting topics for further research remain. It would be of interest to investigate whether the results established in the paper are true also for a larger class of nonconvex constrained optimization problems, for instance, for a class of nonconvex nondifferentiable extremum problems. Thus, further research can focus on the usefulness of these concepts of generalized invexity in proving optimality conditions and duality results for other classes of nonconvex optimization problems. It seems that the techniques employed in this paper can be used in proving similarly results for the constrained vector optimization problems. We shall investigate these questions in subsequent papers.

## References

[1] Antczak, T., $(p, r)$-invex sets and functions, *Journal of Mathematical Analysis and Applications*, 263, 355-379 (2001).

[2] Antczak, T., A class of $B$-$(p, r)$-invex functions and mathematical programming, *Journal of Mathematical Analysis and Applications*, 286, 187-206 (2003).

[3] Antczak, T., $r$-pre-invexity and $r$-invexity in mathematical programming, *Computers and Mathematics with Applications*, 50, 551-566 (2005).

[4] Antczak, T., New optimality conditions and duality results of $G$-type in differentiable mathematical programming, *Nonlinear Analysis, Theory, Methods and Applications*, 66, 1617-1632 (2007).

[5] Arana Jiménez, M., Ruiz, G. and Rufian, A. eds., *Optimality conditions in vector optimization.* Bussum: Bentham Science Publishers, Ltd., (2010).

[6] Ben-Israel, A and Mond, B , What is invexity?, Journal of Australian Mathematical Society Ser.B, 28, 1-9 (1986).

[7] Bector, C.R. , Chandra, S., Gupta, S., and Suneja, S.K., Univex sets, functions and univex nonlinear programming, In: Komlosi, S., Rapcsak, T. and Schaible, S., eds. *Proceedings of Conference of Generalized Convexity*, Pecs, Hungary: Springer Verlag, 1-11 (1993).

[8] Bector, C.R. and Singh, C., $B$-vex functions, *Journal of Optimization Theory and Applications*, 71, 237-253 (1991).

[9] Caprari, E., $\rho$-invex functions and $(F, \rho)$-convex functions: properties and equivalences, *Optimization*, 52, 65-74 (2003).

[10] Craven, B.D., Invex functions and constrained local minima, *Bulletin of the Australian Mathematical Society*, 24, 357-366 (1981).

[11] Hanson, M.A., On sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Applications*, 80, 545-550 (1981).

[12] Hanson, M.A. and Mond, B., Further generalizations of convexity in mathematical programming, *Journal of Information and Optimization Sciences*, 3, 25-32 (1982).

[13] Jeyakumar, V., Equivalence of saddle-points and optima, and duality for a class of nonsmooth non-convex problems, *Journal of Mathematical Analysis and Applications*, 130, 334-343 (1988).

[14] Martin, D.H., The essence of invexity, *Journal of Optimization, Theory and Applications*, 47, 65-75 (1985).
[15] Suneja, S.K., Singh, C. and Bector, C.R., Generalizations of pre-invex functions and $B$-vex functions, *Journal of Optimization Theory and Applications*, 76, 577–587 (1993).
[16] Weir, T. and Jeyakumar, V., A class of nonconvex functions and mathematical programming, *Bulletin of the Australian Mathematical Society*, 38, 177–189 (1988).

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