

A class of semilocal E -preinvex maps in Banach spaces with applications to nondifferentiable vector optimization

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Abstract. In this paper, a new class of semilocal E -preinvex and related maps in Banach spaces is introduced for a nondifferentiable vector optimization problem with restrictions of inequalities and some of its basic properties are studied. Furthermore, as its applications, some optimality conditions and duality results are established for a nondifferentiable vector optimization under the aforesaid maps assumptions.

Keywords: Semilocal E -preinvexity; E -type-I maps; vector optimization; optimality; duality.

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1. Introduction

In recent years, generalizations of convexity in connection with optimality conditions and duality theory have been of much interest and many contributions have been made to this development. See, e.g., [1–6] and the references therein.

Preda and Stancu-Minasian [7] derived optimality conditions for weak vector minima under semilocal preinvexity and η -semidifferentiability conditions and extended the Wolfe and Mond-Weir duals. Later, Preda [8] further established optimality conditions and duality results for a nonlinear fractional multiple objective programming problem with semilocal preinvex functions involving η -semidifferentiability. Additionally, Batista et al. [9] introduced the notions of preinvex maps in Banach spaces and obtained optimality conditions for vector problem. Subsequently, Yu and Liu [10] studied optimality and duality for the same vector problem involving the generalized type-I maps in Banach spaces.

On the other hand, Chen [11] proposed a class of semi- E -convex functions and discussed its basic properties. On the basis of this notion, Hu et al. [12] also brought forward the concept

of semilocal E -convexity, and studied its some characterizations, and established some optimality conditions and duality results for semilocal E -convex programming. Recently, Fulga and Preda [13] extended the E -convexity to E -preinvexity and local E -preinvexity, and studied some of their properties and an application. More recently, Luo and Jian [14] presented semi E -preinvex maps in Banach spaces and discussed their properties.

Motivated by research works of [10, 12, 14] and references therein, in present paper, I introduce the concepts of semilocal E -(pre)invexity, E - η -semidifferentiability and E -type-I maps in Banach spaces and study some of their important properties.

Additionally, I establish some optimality conditions for a nondifferentiable vector optimization problem with restrictions of inequalities under semilocal E -preinvexity, semilocal E -invexity and E -type-I assumptions, respectively. Furthermore, I formulate a dual type for this optimization problem and obtain weak and converse duality results using E -type-I maps. This work partially extends earlier works of [10, 14] to a wider class of maps.

2. Preliminaries and Definitions

Throughout this paper, let X, Y and $Z_j, j \in M = \{1, 2, \dots, m\}$ be real Banach spaces with topological duals X^*, Y^* and Z_j^* , respectively, $E : X \rightarrow X$ and $\eta : X \times X \rightarrow X$ be two fixed mappings.

Consider the following optimization problem:

$$(P) \begin{cases} \min f(x) \\ \text{s.t. } -g(x) = -(g_1(x), \dots, g_m(x)) \\ \in (D_1 \times \dots \times D_m), \\ x \in K \subset X, \end{cases} \quad (1)$$

where $f : X \rightarrow Y$ and $g_j : X \rightarrow Z_j$ are maps, K and D_j are subsets of X and Z_j . Denote the feasible set of (P) by $F = \{x \in K : -g_j(x) \in D_j, j \in M\}$. We assume that the spaces Y and Z_j are ordered by cones $C \subset Y, D_j \subset Z_j$ and that these cones are pointed, closed, convex, and with nonempty interior. The dual cone of C is denoted by

$$C^* = \{\mu^* \in Y^* : \langle \mu^*, x \rangle \geq 0, \forall x \in C\}.$$

The cone C induces a partial order \leq_C on Y given by:

$x, y \in Y, x \leq_C y$ if and only if $y - x \in C$;
 $x, y \in Y, x \leq_C y$ if and only if $y - x \in C \setminus \{0_Y\}$;
 $x, y \in Y, x <_C y$ if and only if $y - x \in \text{int}C$.

Analogously, D_j induces a partial order on Z_j .

Recall some definitions and results that will be used in the sequel.

Definition 1. ([14]) A set $K \subset X$ is said to be *E-invex with respect to η* if

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \forall x, y \in K, \lambda \in [0, 1].$$

Definition 2. ([14]) Let $K \subset X$ be an *E-invex set with respect to η* . A map $f : X \rightarrow Y$ is said to be *semi E-preinvex on K with respect to η* if

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \\ & \leq_C \lambda f(x) + (1 - \lambda)f(y), \\ & \forall x, y \in K, \lambda \in [0, 1]. \end{aligned}$$

Definition 3. ([10]) We say that $\bar{x} \in F$ is a *weakly efficient solution* [or, an *efficient solution*] of problem (P), if there exists no $x \in F$ such that

$$f(x) <_C f(\bar{x}) \quad [\text{or}, f(x) \leq_C f(\bar{x})].$$

Lemma 1. ([15]) Let $C \subset Y$ be a convex cone with $\text{int}C \neq \emptyset$ and C^* the dual cone of C . Then,
 (a) $\forall \mu^* \in C^* \setminus \{0_{Y^*}\}, x \in \text{int}C \Rightarrow \langle \mu^*, x \rangle > 0$;
 (b) $\forall \mu^* \in \text{int}C^*, x \in C \setminus \{0_Y\} \Rightarrow \langle \mu^*, x \rangle > 0$.

We below introduce the concepts of local star-shaped *E-convex set*, local *E-invex set*, semilocal *E-convex map* and local *E-preinvex map* in Banach spaces. Especially, if $X = R^n, Y = R$, these concepts were given by earlier research (see[12, 13]).

Definition 4. A set $K \subset X$ is said to be *local starshaped E-convex*, if there is a map E such that corresponding to each pair of points $x, y \in K$, there is a maximal positive number $a(x, y) \leq 1$ satisfying

$$\lambda E(x) + (1 - \lambda)E(y) \in K, \forall \lambda \in [0, a(x, y)]. \quad (2)$$

Definition 5. A map $f : X \rightarrow Y$ is said to be *semilocal E-convex on a local starshaped E-convex set $K \subset X$* if for each pair of $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2)), there exists a positive number $b(x, y) \leq a(x, y)$ satisfying

$$\begin{aligned} & f(\lambda E(x) + (1 - \lambda)E(y)) \\ & \leq_C \lambda f(x) + (1 - \lambda)f(y), \\ & \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

Definition 6. A set $k \subset X$ is said to be *local E-invex with respect to η* , if $\forall x, y \in K$, there exists $a(x, y) \in (0, 1]$ such that, $\forall \lambda \in [0, a(x, y)]$,

$$E(y) + \lambda\eta(E(x), E(y)) \in K. \quad (3)$$

Remark 1. Every local starshaped *E-convex set* is a local *E-invex set with respect to η* , where $\eta(x, y) = x - y, \forall x, y \in X$. Every *E-invex set with respect to η* is a local *E-invex set with respect to η* , where $a(x, y) = 1, \forall x, y \in X$. But their converses are not necessarily true.

Definition 7. A map $f : X \rightarrow Y$ is said to be *local E-preinvex on $k \subset X$ with respect to η* if for any $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (3)), there exists $0 < b(x, y) \leq a(x, y)$ such that K is a local *E-invex set* and

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \\ & \leq_C \lambda f(E(x)) + (1 - \lambda)f(E(y)), \\ & \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

3. Semilocal E-preinvex and Related Maps

In this section, we introduce the concepts of semilocal *E-preinvex* and related maps in Banach spaces and study some of their basic properties.

Definition 8. A map $f : X \rightarrow Y$ is said to be *semilocal E-preinvex on $k \subset X$ with respect to η* if for any $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (3)), there exists $0 < b(x, y) \leq a(x, y)$ such that K is a local *E-invex set* and

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \leq_C \\ & \lambda f(x) + (1 - \lambda)f(y), \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

If the inequality sign above is strict for any $x, y \in K$ and $x \neq y$, then f is called as a *strict semilocal E-preinvex map*.

Remark 2. Every semilocal E -convex map is a semilocal E -preinvex map, where $\eta(x, y) = x - y$, $\forall x, y \in X$. Every semi E -preinvex map with respect to η is a semilocal E -preinvex map, where $a(x, y) = b(x, y) = 1$, $\forall x, y \in X$. But their converses are not necessarily true.

See the following example.

Example 1. Let the map $E : R \rightarrow R$ be defined as

$$E(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } 1 < x \leq 2, \\ x, & \text{if } 0 \leq x \leq 1 \text{ or } x > 2, \end{cases} \quad (4)$$

and the map $\eta : R \times R \rightarrow R$ be defined as

$$\eta(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1 - x, & \text{if } x \neq y. \end{cases} \quad (5)$$

Obviously, R is a local starshaped E -convex set and a local E -invex set with respect to η . Let $f : R \rightarrow R$ be defined as

$$f(x) = \begin{cases} 0, & \text{if } 1 < x \leq 2, \\ 1, & \text{if } x > 2, \\ -x + 1, & \text{if } 0 \leq x \leq 1, \\ -x + 2, & \text{if } x < 0. \end{cases} \quad (6)$$

We can prove that f is semilocal E -preinvex on R with respect to η . However, when $x_0 = 2$, $y_0 = 3$, and for any $b \in (0, 1]$, there exists a sufficiently small $\lambda_0 \in (0, b]$ satisfying

$$\begin{aligned} f(\lambda_0 E(x_0) + (1 - \lambda_0)E(y_0)) &= f(3 - 2\lambda_0) \\ &= 1 > 1 - \lambda_0 = \lambda_0 f(x_0) + (1 - \lambda_0)f(y_0). \end{aligned}$$

That is, $f(x)$ is not a semilocal E -convex map on R .

Similarly, taking $x_1 = 1$, $y_1 = 4$, we have

$$\begin{aligned} f(E(y_1) + \lambda_1 \eta(E(x_1), E(y_1))) &= f(4) \\ &= 1 > 1 - \lambda_1 = \lambda_1 f(x_1) + (1 - \lambda_1)f(y_1), \end{aligned}$$

for some $\lambda_1 \in [0, 1]$.

Thus, $f(x)$ is not a semi E -preinvex map on R with respect to η .

Definition 9. A map f defined on a local E -invex set $k \subset X$ is said to be quasi-semilocal E -preinvex (with respect to η) if for all $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (3)) satisfying $f(x) \leq_C f(y)$, there is a positive number $b(x, y) \leq a(x, y)$ such that

$$\begin{aligned} f(E(y) + \lambda \eta(E(x), E(y))) &\leq_C f(y), \\ \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

Definition 10. A map f defined on a local E -invex set $K \subset X$ is said to be pseudo-semilocal E -preinvex (with respect to η) if for all $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (3)) satisfying $f(x) <_C f(y)$, there are

a positive number $b(x, y) \leq a(x, y)$ and a positive number $c(x, y)$ such that

$$\begin{aligned} f(E(y) + \lambda \eta(E(x), E(y))) &\leq_C f(y) - \lambda c(x, y), \\ \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

Remark 3. Every semilocal E -preinvex map on a local E -invex set K with respect to η is both a quasi-semilocal E -preinvex map and a pseudo-semilocal E -preinvex map.

Theorem 1. Let $f : X \rightarrow Y$ be a local E -preinvex map on a local E -invex set $K \subset X$ with respect to η , then f is a semilocal E -preinvex map if and only if $f(E(x)) \leq_C f(x)$, $\forall x \in K$.

Proof. Suppose that f is a semilocal E -preinvex map on set K with respect to η , then for each pair of points $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (3)), there exists a positive number $b(x, y) \leq a(x, y)$ satisfying

$$\begin{aligned} f(E(x) + \lambda \eta(E(y), E(x))) &\leq_C \lambda f(y) + (1 - \lambda)f(x), \lambda \in [0, b(x, y)]. \end{aligned}$$

By letting $\lambda = 0$, we have $f(E(x)) \leq_C f(x)$, $\forall x \in K$.

Conversely, assume that f is a local E -preinvex map on a local E -invex set K , then for any $x, y \in K$, there exist $a(x, y) \in (0, 1]$ satisfying (3) and $b(x, y) \in (0, a(x, y)]$ such that

$$\begin{aligned} f(E(y) + \lambda \eta(E(x), E(y))) &\leq_C \lambda f(E(x)) + (1 - \lambda)f(E(y)), \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

Since $f(E(x)) \leq_C f(x)$, $\forall x \in K$, then

$$\begin{aligned} f(E(y) + \lambda \eta(E(x), E(y))) &\leq_C \lambda f(x) + (1 - \lambda)f(y), \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

The proof is completed. \square

Remark 4. A local E -preinvex map on a local E -invex set with respect to η is not necessarily a semilocal E -preinvex map.

Example 2. Let $K = [-4, -1) \cup [1, 4]$,

$$E(x) = \begin{cases} x^2, & \text{if } |x| \leq 2, \\ -1, & \text{if } |x| > 2, \end{cases} \quad (7)$$

$$\eta(x, y) = \begin{cases} x - y, & \text{if } x \geq 0, y \geq 0, \text{ or } x \leq 0, y \leq 0, \\ -1 - y, & \text{if } x > 0, y \leq 0, \text{ or } x \geq 0, y < 0, \\ 1 - y, & \text{if } x < 0, y \geq 0, \text{ or } x \leq 0, y > 0, \end{cases} \quad (8)$$

and $f : R \rightarrow R$ be defined by $f(x) = x^2$, then f is local E -preinvex on K with respect to η .

Since $f(E(2)) = 16 > f(2) = 4$, from Theorem 1, it follows that f is not a semilocal E -preinvex map.

Definition 11. The set $G = \{(x, \alpha) : x \in K \subset X, \alpha \in Y\}$ is said to be a local E -invex set with respect to η corresponding to X if there are two maps η, E and a maximal positive number $a((x, \alpha_1), (y, \alpha_2)) \leq 1$, for each $(x, \alpha_1), (y, \alpha_2) \in G$ such that

$$(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha_1 + (1 - \lambda)\alpha_2) \in G, \forall \lambda \in [0, a((x, \alpha_1), (y, \alpha_2))].$$

Theorem 2. Let $K \subset X$ be a local E -invex set with respect to η . Then f is a semilocal E -preinvex map on K with respect to η if and only if its epigraph $G_f = \{(x, \alpha) : x \in K, f(x) \leq_C \alpha, \alpha \in Y\}$ is a local E -invex set with respect to η corresponding to X .

Proof. Assume that map f is semilocal E -preinvex on K with respect to η and $(x, \alpha_1), (y, \alpha_2) \in G_f$, then $x, y \in K, f(x) \leq_C \alpha_1, f(y) \leq_C \alpha_2$. Since K is a local E -invex set, there is a maximal positive number $a(x, y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \forall \lambda \in [0, a(x, y)].$$

In addition, in view of f being a semilocal E -preinvex map on K with respect to η , there is a positive number $b(x, y) \leq a(x, y)$ such that

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \\ & \leq_C \lambda f(x) + (1 - \lambda)f(y) \leq_C \lambda\alpha_1 + (1 - \lambda)\alpha_2, \\ & \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

That is, $(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha_1 + (1 - \lambda)\alpha_2) \in G_f, \forall \lambda \in [0, b(x, y)]$.

Therefore, $G_f = \{(x, \alpha) : x \in K, f(x) \leq_C \alpha, \alpha \in Y\}$ is a local E -invex set with respect to η corresponding to X .

Conversely, if G_f is a local E -invex set with respect to η corresponding to X , then for any points $(x, f(x)), (y, f(y)) \in G_f$, there is a maximal positive number $a((x, f(x)), (y, f(y))) \leq 1$ such that

$$(E(y) + \lambda\eta(E(x), E(y)), \lambda f(x) + (1 - \lambda)f(y)) \in G_f, \forall \lambda \in [0, a((x, f(x)), (y, f(y)))].$$

That is, $E(y) + \lambda\eta(E(x), E(y)) \in K$,

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \\ & \leq_C \lambda f(x) + (1 - \lambda)f(y), \end{aligned}$$

$$\forall \lambda \in [0, a((x, f(x)), (y, f(y)))].$$

Thus, K is a local E -invex set and f is a semilocal E -preinvex map on K . \square

Theorem 3. If f is a semilocal E -preinvex map on a local E -invex set $K \subset X$ with respect to η , then the level set $S_\alpha = \{x \in K : f(x) \leq_C \alpha\}$ is a local E -invex set for any $\alpha \in Y$.

Proof. For any $\alpha \in Y$ and $x, y \in S_\alpha$, then $x, y \in K$ and $f(x) \leq_C \alpha, f(y) \leq_C \alpha$. Since K is a local E -invex set, there is a maximal positive number $a(x, y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \forall \lambda \in [0, a(x, y)].$$

In addition, due to the semilocal E -preinvexity of f , there is a positive number $b(x, y) \leq a(x, y)$ such that

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \\ & \leq_C \lambda f(x) + (1 - \lambda)f(y) \leq_C \lambda\alpha + (1 - \lambda)\alpha = \alpha, \\ & \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

That is, $E(y) + \lambda\eta(E(x), E(y)) \in S_\alpha, \forall \lambda \in [0, b(x, y)]$.

Therefore, S_α is a local E -invex set with respect to η for any $\alpha \in Y$. \square

Theorem 4. Let $f : X \rightarrow Y$ be a map defined on a local E -invex set $K \subset X$. Then f is a semilocal E -preinvex map with respect to η if and only if for each pair of points $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (3)), there exists a positive number $b(x, y) \leq a(x, y)$ such that

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) <_C \lambda\alpha + (1 - \lambda)\beta, \\ & \forall \lambda \in [0, b(x, y)], \end{aligned}$$

whenever $f(x) <_C \alpha, f(y) <_C \beta$.

Proof. Let $x, y \in K$ and $\alpha, \beta \in Y$ such that $f(x) <_C \alpha, f(y) <_C \beta$. Due to the local E -invexity of K , there is a maximal positive number $a(x, y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \forall \lambda \in [0, a(x, y)].$$

In addition, owing to the semilocal E -preinvexity of f , there is a positive number $b(x, y) \leq a(x, y)$ such that

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \\ & \leq_C \lambda f(x) + (1 - \lambda)f(y) <_C \lambda\alpha + (1 - \lambda)\beta, \\ & \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

Conversely, let $(x, \alpha) \in G_f, (y, \beta) \in G_f$ (see epigraph G_f in Theorem 2), then $x, y \in K, f(x) \leq_C \alpha, f(y) \leq_C \beta$. Hence, $f(x) <_C \alpha + \epsilon$ and $f(y) <_C \beta + \epsilon$ hold for any $\epsilon >_C 0$. According to the hypothesis, for $x, y \in K$ (with a positive number $a(x, y) \leq 1$ satisfying (3)), there exists a positive number $b(x, y) \leq a(x, y)$ such that

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))), \\ & <_C \lambda\alpha + (1 - \lambda)\beta + \epsilon \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

Let $\epsilon \rightarrow 0^+$, then

$$\begin{aligned} & f(E(y) + \lambda\eta(E(x), E(y))) \leq_C \lambda\alpha + (1 - \lambda)\beta, \\ & \forall \lambda \in [0, b(x, y)]. \end{aligned}$$

That is, $(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha + (1 - \lambda)\beta) \in G_f, \forall \lambda \in [0, b(x, y)]$.

Therefore, G_f is a local E -invex set corresponding to X .

From Theorem 2, it follows that f is semilocal E -preinvex on K with respect to η . \square

Theorem 5. Assume that a map $f : X \rightarrow Y$ is semilocal E -preinvex on a local E -invex set $F \subset X$ with respect to η . If $E(\bar{x}) = \bar{x} \in F$ is a local weakly efficient solution for problem (P), then \bar{x} is a global weakly efficient solution for (P) on F .

Proof. By contradiction, suppose that $E(\bar{x}) = \bar{x} \in F$ is not a global weakly efficient solution for (P) on F , then there exists $y \in F$ such that

$$f(\bar{x}) - f(y) \in \text{int}C. \quad (9)$$

Since map f is semilocal E -preinvex on local E -invex set F , there exist positive number $b(\bar{x}, y) \leq a(\bar{x}, y) \leq 1$ such that $E(\bar{x}) + \lambda\eta(E(y), E(\bar{x})) \in F$ for any $\lambda \in (0, a(\bar{x}, y)]$ and

$$\begin{aligned} & f(E(\bar{x}) + \lambda\eta(E(y), E(\bar{x}))) \\ & \leq_C \lambda f(y) + (1 - \lambda)f(\bar{x}), \forall \lambda \in (0, b(\bar{x}, y)], \end{aligned}$$

or equivalently,

$$\lambda f(y) + (1 - \lambda)f(\bar{x}) - f(E(\bar{x}) + \lambda\eta(E(y), E(\bar{x}))) \in C,$$

that is,

$$\begin{aligned} & \lambda(f(y) - f(\bar{x})) + f(\bar{x}) \\ & - f(\bar{x} + \lambda\eta(E(y), \bar{x})) \in C, \forall \lambda \in (0, b(\bar{x}, y)]. \quad (10) \end{aligned}$$

Since C is a pointed cone, from (9) and (10), we obtain $\eta(E(y), \bar{x}) \neq 0$.

We observe

$$\begin{aligned} & f(\bar{x}) - f(\bar{x} + \lambda\eta(E(y), \bar{x})) \\ & = [\lambda(f(y) - f(\bar{x})) + f(\bar{x}) - f(\bar{x} + \lambda\eta(E(y), \bar{x}))] \\ & + \lambda[f(\bar{x}) - f(y)] \in C + \text{int}C \subset \text{int}C, \\ & \forall \lambda \in (0, b(\bar{x}, y)], \end{aligned}$$

which contradicts the local weakly efficiency of \bar{x} for problem (P).

Thus, \bar{x} is a global weakly efficient solution for problem (P) on F . \square

Remark 5. If $a(x, y) = b(x, y) = 1, \forall x, y \in X$, the results presented in this section reduce to the results given in [14].

4. Optimality Criteria

In this section, we establish some optimality conditions for the vector optimization problem (P) involving semilocal E -preinvex, semilocal E -invex and E -type-I maps, respectively.

First, we give an optimality condition for (P) involving semilocal E -preinvex maps.

Theorem 6. Assume that a map $f : X \rightarrow Y$ is semilocal E -preinvex on local E -invex set $F \subset X$ with respect to η and \bar{x} is a weakly efficient solution for the following optimization problem:

$$(P_E) \quad \min (f \circ E)(x), \quad \text{s.t. } x \in F = \{x \in K : -g_j(x) \in D_j, j \in M\}.$$

Then, $E(\bar{x})$ is a weakly efficient solution for problem (P).

Proof. Since \bar{x} is a weakly efficient solution for problem (P), then there exists no $x \in F$ such that

$$f(E(x)) = (f \circ E)(x) <_C (f \circ E)(\bar{x}) = f(E(\bar{x})).$$

Suppose to the contrary that $E(\bar{x})$ is not a weakly efficient solution for (P), then there exists a point $\hat{x} \in F$ such that

$$f(\hat{x}) <_C f(E(\bar{x})).$$

From necessity of Theorem 1, it follows that

$$f(E(x)) \leq_C f(x), \quad \forall x \in F.$$

Thus, we have

$$f(E(\hat{x})) <_C f(E(\bar{x})),$$

which is in contradiction with the weakly efficiency of \bar{x} for problem (P).

Hence, the theorem is proved. \square

Next, we introduce some concepts that will be used in the sequel.

Definition 12. Let $f : K \rightarrow Y$ be a map, where $K \subset X$ is a local E -invex set with respect to η . We say that f is E - η -semidifferentiable at $E(\bar{x}) \in K$ if $f'(E(\bar{x}); \eta(E(x), E(\bar{x})))$ exists for each $x \in K$, where

$$\begin{aligned} & f'(E(\bar{x}); \eta(E(x), E(\bar{x}))) \\ & = \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} [f(E(\bar{x}) + \lambda\eta(E(x), E(\bar{x}))) - f(E(\bar{x}))], \end{aligned}$$

(the right derivative at $E(\bar{x})$ along the direction $\eta(E(x), E(\bar{x}))$).

Remark 6. If E is an identity map, the E - η -semidifferentiability is the η -semidifferentiability notion [16]. If E is an identity map and $\eta(x, \bar{x}) = x - \bar{x}$, the E - η -semidifferentiability is the semidifferentiability notion. If a function is directionally differentiable, then it is semidifferentiable, but the converse is not true.

Definition 13. (a) A map $f : X \rightarrow Y$ is called semilocal E -invex at $\bar{x} \in k \subset X$ with respect to η on K , if f is E - η -semidifferentiable at \bar{x} , where $E(\bar{x}) = \bar{x}$ and

$$f(x) - f(\bar{x}) \geq_C f'(\bar{x}; \eta(E(x), \bar{x})), \quad \forall x \in K.$$

(b) A map $f : X \rightarrow Y$ is called quasi-semilocal E -invex at $\bar{x} \in k \subset X$ with respect to η on K , if

f is E - η -semidifferentiable at \bar{x} , where $E(\bar{x}) = \bar{x}$ and

$$f(x) \leq_C f(\bar{x}) \Rightarrow f'(\bar{x}; \eta(E(x), \bar{x})) \leq_C 0, \forall x \in K.$$

(c) A map $f : X \rightarrow Y$ is called pseudo-semilocal E -invex at $\bar{x} \in k \subset X$ with respect to η on K , if f is E - η -semidifferentiable at \bar{x} , where $E(\bar{x}) = \bar{x}$ and

$$f'(\bar{x}; \eta(E(x), \bar{x})) \geq_C 0 \Rightarrow f(x) \geq_C f(\bar{x}), \forall x \in K.$$

Remark 7. If a semilocal E -preinvex map f is E - η -semidifferentiable at \bar{x} , where $E(\bar{x}) = \bar{x}$, then f is a semilocal E -invex map at \bar{x} ; If f is a semilocal E -invex map at \bar{x} , then f is both a quasi-semilocal E -invex map and a pseudo-semilocal E -invex map at \bar{x} ; If f is a semilocal E -invex map at \bar{x} , then f is nondifferentiable at \bar{x} .

Below, we give a necessary and sufficient optimality condition for problem (P) involving semilocal E -invex maps.

Theorem 7. Assume that a map $f : X \rightarrow Y$ defined on a local E -invex set $K \subset X$ is semilocal E -invex at $u \in K$. Then u is a weakly efficient solution of map f on K if and only if u satisfies the inequality

$$f'(u; \eta(E(v), u)) \geq_C 0, \quad \forall v \in K. \quad (11)$$

Proof. Since the map f is semilocal E -invex at $u \in K$ on K with respect to η ,

$$f(v) - f(u) \geq_C f'(u; \eta(E(v), u)), \quad \forall v \in K.$$

If $u \in K$ satisfies the inequality (11), then

$$f(u) \leq_C f(v), \quad \forall v \in K,$$

which means $u \in K$ is the weakly efficient solution of map f on K .

Conversely, assume that u is a weakly efficient solution of map f on K , then,

$$f(u) \leq_C f(w), \quad \forall w \in K.$$

Since K is a local E -invex set, for any $v \in K$, there exists $a(u, v) \in (0, 1]$ such that

$$E(u) + \lambda \eta(E(v), E(u)) \in K, \quad \forall \lambda \in (0, a(u, v)].$$

Thus, from semilocal E -invexity of f at $u \in K$, it follows that

$$\begin{aligned} f(u) &\leq_C f(E(u) + \lambda \eta(E(v), E(u))) \\ &= f(u + \lambda \eta(E(v), u)), \forall \lambda \in (0, b(u, v)]. \end{aligned}$$

That is,

$$f(u + \lambda \eta(E(v), u)) - f(u) \geq_C 0, \quad \forall \lambda \in (0, b(u, v)].$$

Dividing the above inequality by λ and taking $\lambda \rightarrow 0^+$, we get

$$f'(u; \eta(E(v), u)) \geq_C 0, \quad \forall v \in K,$$

which is the desirable result (11).

Therefore, the proof is completed. □

By [9, Lemma 2.3], Definition 13 is also equivalent to the next definition.

Definition 14. (a) A map $f : X \rightarrow Y$ is called semilocal E -invex at $\bar{x} \in k \subset X$ with respect to η on K , if f is E - η -semidifferentiable at \bar{x} , where $E(\bar{x}) = \bar{x}$ and for any $x \in K$, $\mu^* \in C^*$ such that

$$\langle \mu^*, f(x) - f(\bar{x}) \rangle \geq (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x}));$$

(b) A map $f : X \rightarrow Y$ is called quasi-semilocal E -invex at $\bar{x} \in k \subset X$ with respect to η on K , if f is E - η -semidifferentiable at \bar{x} , where $E(\bar{x}) = \bar{x}$ and for any $x \in K$, $\mu^* \in C^*$ such that

$$\begin{aligned} \langle \mu^*, f(x) \rangle &\leq \langle \mu^*, f(\bar{x}) \rangle \\ &\Rightarrow (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})) \leq 0; \end{aligned}$$

(c) A map $f : X \rightarrow Y$ is called pseudo-semilocal E -invex at $\bar{x} \in k \subset X$ with respect to η on K , if f is E - η -semidifferentiable at \bar{x} , where $E(\bar{x}) = \bar{x}$ and for any $x \in K$, $\mu^* \in C^*$ such that

$$\begin{aligned} (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})) &\geq 0 \\ &\Rightarrow \langle \mu^*, f(x) \rangle \geq \langle \mu^*, f(\bar{x}) \rangle. \end{aligned}$$

Throughout the remainder of this paper, we always assume that $f : X \rightarrow Y$ and $g_j : X \rightarrow Z_j$, $j \in M$ are E - η -semidifferentiable.

Now, we extend the generalized type-I maps in [10] as follows.

Definition 15. (f, g) is said to be E -type-I at $\bar{x} \in K$ with respect to η , if $E(\bar{x}) = \bar{x}$ and for each $x \in K$, there exist two maps E and η , such that for all $\mu^* \in C^*$ and $v_j^* \in D_j^*$, $j \in M$

$$\langle \mu^*, f(x) - f(\bar{x}) \rangle \geq (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})); \quad (12)$$

$$-\sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle \geq \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(x), \bar{x})). \quad (13)$$

Definition 16. (f, g) is said to be quasi E -type-I at $\bar{x} \in K$ with respect to η , if $E(\bar{x}) = \bar{x}$ and for each $x \in K$, there exist two maps E and η , such that for all $\mu^* \in C^*$ and $v_j^* \in D_j^*$, $j \in M$

$$\begin{aligned} \langle \mu^*, f(x) \rangle &\leq \langle \mu^*, f(\bar{x}) \rangle \\ &\Rightarrow (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})) \leq 0; \quad (14) \end{aligned}$$

$$\begin{aligned} -\sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle &\leq 0 \\ &\Rightarrow \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(x), \bar{x})) \leq 0. \quad (15) \end{aligned}$$

Definition 17. (f, g) is said to be pseudo E -type-I at $\bar{x} \in K$ with respect to η , if $E(\bar{x}) = \bar{x}$ and for each $x \in K$, there exist two maps E and η , such that for all $\mu^* \in C^*$ and $v_j^* \in D_j^*$, $j \in M$

$$\begin{aligned} (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})) &\geq 0 \\ \Rightarrow \langle \mu^*, f(x) \rangle &\geq \langle \mu^*, f(\bar{x}) \rangle; \end{aligned} \quad (16)$$

$$\begin{aligned} \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(x), \bar{x})) &\geq 0 \\ \Rightarrow - \sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle &\geq 0. \end{aligned} \quad (17)$$

Definition 18. (f, g) is said to be quasipseudo E -type-I at $\bar{x} \in K$ with respect to η , if $E(\bar{x}) = \bar{x}$ and for each $x \in K$, there exist two maps E and η , such that for all $\mu^* \in C^*$ and $v_j^* \in D_j^*$, $j \in M$

$$\begin{aligned} \langle \mu^*, f(x) \rangle &\leq \langle \mu^*, f(\bar{x}) \rangle \\ \Rightarrow (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})) &\leq 0; \end{aligned} \quad (18)$$

$$\begin{aligned} \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(x), \bar{x})) &\geq 0 \\ \Rightarrow - \sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle &\geq 0. \end{aligned} \quad (19)$$

If in the above relation, we have

$$\begin{aligned} \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(x), \bar{x})) &\geq 0 \\ \Rightarrow - \sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle &> 0. \end{aligned} \quad (20)$$

Then, we say that (f, g) is quasistrictlypseudo E -type-I at $\bar{x} \in K$.

Definition 19. (f, g) is said to be pseudoquasi E -type-I at $\bar{x} \in K$ with respect to η , if $E(\bar{x}) = \bar{x}$ and for each $x \in K$, there exist two maps E and η , such that for all $\mu^* \in C^*$ and $v_j^* \in D_j^*$, $j \in M$

$$\begin{aligned} (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})) &\geq 0 \\ \Rightarrow \langle \mu^*, f(x) \rangle &\geq \langle \mu^*, f(\bar{x}) \rangle; \end{aligned} \quad (21)$$

$$\begin{aligned} - \sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle &\leq 0 \\ \Rightarrow \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(x), \bar{x})) &\leq 0. \end{aligned} \quad (22)$$

Remark 8. If (f, g) is E -type-I at $\bar{x} \in K$ with respect to η , then (f, g) is both quasi E -type-I and pseudo E -type-I at $\bar{x} \in K$ with respect to η . If E

is an identity map and $m = 1$, the above definitions reduce to the definitions of generalized type-I maps in [10].

Now, we establish the sufficient optimality conditions for (P) involving E -type-I maps.

Theorem 8. Assume that there exist $\bar{x} \in F$ and $\mu^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\mu^* \in \text{int}C^*$], $v_j^* \in D_j^*$, $j \in M$ such that the following two relations are satisfied,

$$\begin{aligned} (\mu^* \circ f)'(\bar{x}; \eta(E(x), \bar{x})) \\ + \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(x), \bar{x})) &\geq 0, \quad \forall x \in F, \end{aligned} \quad (23)$$

$$\sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle = 0. \quad (24)$$

Furthermore, if any one of the following conditions holds:

- (a) (f, g) is E -type-I at $\bar{x} \in F$ with respect to the same η ;
- (b) (f, g) is pseudoquasi E -type-I at $\bar{x} \in F$ with respect to the same η ;
- (c) (f, g) is quasistrictlypseudo E -type-I at $\bar{x} \in F$ with respect to the same η .

Then \bar{x} is a weakly efficient solution [or, an efficient solution] of (P) .

Proof. By contradiction, we assume that \bar{x} is not a weakly efficient solution [or, an efficient solution] of (P) . Then there is a feasible solution \check{x} of problem (P) such that

$$f(\check{x}) <_C f(\bar{x}) \quad [\text{or}, f(\check{x}) \leq_C f(\bar{x})].$$

From $\mu^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\mu^* \in \text{int}C^*$] and Lemma 1, we have

$$\langle \mu^*, f(\check{x}) - f(\bar{x}) \rangle < 0. \quad (25)$$

If condition (a) holds, then from relation (12), it follows that

$$(\mu^* \circ f)'(\bar{x}; \eta(E(\check{x}), \bar{x})) < 0. \quad (26)$$

According to relations (13) and (24), we obtain

$$\sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(\check{x}), \bar{x})) \leq 0. \quad (27)$$

Adding (26) and (27), we have

$$\begin{aligned} (\mu^* \circ f)'(\bar{x}; \eta(E(\check{x}), \bar{x})) \\ + \sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(\check{x}), \bar{x})) < 0, \end{aligned}$$

which is in contradiction with (23).

If condition (b) holds, then from relations (22) and (24), it follows that

$$\sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(\check{x}), \bar{x})) \leq 0.$$

Considering (23), we get

$$(\mu^* \circ f)'(\bar{x}; \eta(E(\check{x}), \bar{x})) \geq 0.$$

By (21), we have

$$\langle \mu^*, f(\check{x}) - f(\bar{x}) \rangle \geq 0,$$

which is a contradiction to (25).

If condition (c) holds, then from relations (18) and (25), it follows that

$$(\mu^* \circ f)'(\bar{x}; \eta(E(\check{x}), \bar{x})) \leq 0.$$

Combining the above inequality with (23), we get

$$\sum_{j=1}^m (v_j^* \circ g_j)'(\bar{x}; \eta(E(\check{x}), \bar{x})) \geq 0.$$

From (20), it leads to

$$-\sum_{j=1}^m \langle v_j^*, g_j(\bar{x}) \rangle > 0,$$

which contradicts (24).

Therefore, the theorem is proved. \square

Remark 9. If E is an identity map and $m = 1$, the results obtained in the above theorem become the results of Yu and Liu[10].

5. Duality

In this section, we provide weak and converse duality results utilizing E -type-I maps.

Consider the following dual type for problem (P):

$$(D) \begin{cases} \max & f(y) \\ \text{s.t.} & (\mu^* \circ f)'(y; \eta(E(x), y)) \\ & + \sum_{j=1}^m (v_j^* \circ g_j)'(y; \eta(E(x), y)) \geq 0, \\ & \forall x \in F, \\ & \sum_{j=1}^m \langle v_j^*, g_j(y) \rangle \geq 0, \\ & y \in K, \mu^* \in C^*, v_j^* \in D_j^*, j \in M. \end{cases} \quad (28)$$

Denote the feasible set of problem (D) by G , i.e.,

$$G = \{(y, \mu^*, v_j^*) : (\mu^* \circ f)'(y; \eta(E(x), y)) + \sum_{j=1}^m (v_j^* \circ g_j)'(y; \eta(E(x), y)) \geq 0, \sum_{j=1}^m \langle v_j^*, g_j(y) \rangle \geq 0, \forall x \in F, y \in K, \mu^* \in C^*, v_j^* \in D_j^*, j \in M\}.$$

Theorem 9. (Weak duality) Let $x \in F$, $(y, \mu^*, v_j^*) \in G$, $j \in M$, $\mu^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\mu^* \in \text{int}C^*$], and $v_j^* \in D_j^*$. Furthermore, if any one of the following conditions holds:

(a) (f, g) is E -type-I at $y \in F$ with respect to the same η ;

(b) (f, g) is pseudoquasi E -type-I at $y \in F$ with respect to the same η ;

(c) (f, g) is quasistrictlypseudo E -type-I at $y \in F$ with respect to the same η .

Then, $f(x) \not\prec_C f(y)$ [or, $f(x) \not\prec_C f(y)$].

Proof. Assume to the contrary that there exist $\check{x} \in F$, $(y, \mu^*, v_j^*) \in G$ such that

$$f(\check{x}) <_C f(y) \quad [\text{or, } f(\check{x}) \leq_C f(y)].$$

By $\mu^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\mu^* \in \text{int}C^*$] and Lemma 1, we have

$$\langle \mu^*, f(\check{x}) - f(y) \rangle < 0. \quad (29)$$

From $(y, \mu^*, v_j^*) \in G$, it follows that

$$-\sum_{j=1}^m \langle v_j^*, g_j(y) \rangle \leq 0. \quad (30)$$

According to the first inequality in (28) and $\check{x} \in F$, we get

$$(\mu^* \circ f)'(y; \eta(E(\check{x}), y)) + \sum_{j=1}^m (v_j^* \circ g_j)'(y; \eta(E(\check{x}), y)) \geq 0. \quad (31)$$

Utilizing relations (29), (30) and condition (a), we obtain

$$(\mu^* \circ f)'(y; \eta(E(\check{x}), y)) < 0,$$

$$\sum_{j=1}^m (v_j^* \circ g_j)'(y; \eta(E(\check{x}), y)) \leq 0.$$

Summing the above two inequalities, we have

$$(\mu^* \circ f)'(y; \eta(E(\check{x}), y)) + \sum_{j=1}^m (v_j^* \circ g_j)'(y; \eta(E(\check{x}), y)) < 0,$$

which is a contradiction to relation (31).

If condition (b) holds, then $-\sum_{j=1}^m \langle v_j^*, g_j(y) \rangle \leq 0$ implies that

$$\sum_{j=1}^m (v_j^* \circ g_j)'(y; \eta(E(\check{x}), y)) \leq 0.$$

Taking (31) into account, we obtain

$$(\mu^* \circ f)'(y; \eta(E(\check{x}), y)) \geq 0.$$

By condition (b) again, the above relation means that

$$\langle \mu^*, f(\check{x}) - f(y) \rangle \geq 0,$$

which contradicts (29).

If condition (C) holds, then (29) leads to

$$(\mu^* \circ f)'(y; \eta(E(\check{x}), y)) \leq 0.$$

On account of (31), we have

$$\sum_{j=1}^m (v_j^* \circ g_j)'(y; \eta(E(\check{x}), y)) \geq 0.$$

Using condition (C) again, we get

$$-\sum_{j=1}^m \langle v_j^*, g_j(y) \rangle > 0,$$

which is in contradiction with (30).

Therefore, the theorem is proved. \square

Theorem 10. (Converse duality) Let $(\bar{y}, \bar{\mu}^*, \bar{v}_j^*)$, $j \in M$ be a weakly efficient solution [or, an efficient solution] for problem (D). Assume that $\bar{\mu}^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\bar{\mu}^* \in \text{int}C^*$] and all conditions in Theorem 9 hold at \bar{y} . Then \bar{y} is a weakly efficient solution [or, an efficient solution] for (P).

Proof. We proceed by contradicting. If \bar{y} is not a weakly efficient solution [or, an efficient solution] for (P), there exists $\check{y} \in F$ such that

$$f(\check{y}) <_C f(\bar{y}) \quad [\text{or}, f(\check{y}) \leq_C f(\bar{y})].$$

From $\bar{\mu}^* \in C^* \setminus \{0_{Y^*}\}$ [or, $\bar{\mu}^* \in \text{int}C^*$] and Lemma 1, it follows that

$$\langle \bar{\mu}^*, f(\check{y}) - f(\bar{y}) \rangle < 0. \quad (32)$$

By $(\bar{y}, \bar{\mu}^*, \bar{v}_j^*) \in G$, $j \in M$, we have

$$\begin{aligned} & (\bar{\mu}^* \circ f)'(\bar{y}; \eta(E(\check{y}), \bar{y})) \\ & + \sum_{j=1}^m (\bar{v}_j^* \circ g_j)'(\bar{y}; \eta(E(\check{y}), \bar{y})) \geq 0, \end{aligned} \quad (33)$$

and

$$\sum_{j=1}^m \langle \bar{v}_j^*, g_j(\bar{y}) \rangle \geq 0. \quad (34)$$

On account of (32) and condition (a) of Theorem 9, we get

$$(\bar{\mu}^* \circ f)'(\bar{y}; \eta(E(\check{y}), \bar{y})) < 0. \quad (35)$$

Since condition (a) of Theorem 9 holds and $\check{y} \in F$, $(\bar{y}, \bar{\mu}^*, \bar{v}_j^*) \in G$, $j \in M$, it yields that

$$\sum_{j=1}^m (\bar{v}_j^* \circ g_j)'(\bar{y}; \eta(E(\check{y}), \bar{y})) \leq 0. \quad (36)$$

Summing (35) and (36), we obtain

$$\begin{aligned} & (\bar{\mu}^* \circ f)'(\bar{y}; \eta(E(\check{y}), \bar{y})) \\ & + \sum_{j=1}^m (\bar{v}_j^* \circ g_j)'(\bar{y}; \eta(E(\check{y}), \bar{y})) < 0 \end{aligned}$$

which is a contradiction to (33).

If condition (b) or (c) of Theorem 9 holds, by the analogous argument to that of Theorem 9, we obtain

$$\langle \bar{\mu}^*, f(\check{y}) - f(\bar{y}) \rangle \geq 0, \quad (37)$$

or

$$-\sum_{j=1}^m \langle \bar{v}_j^*, g_j(\bar{y}) \rangle > 0. \quad (38)$$

The inequalities (37) and (38) contradict (32) and (34), respectively.

Therefore, the theorem is proved. \square

Remark 10. If E is an identity map and $m = 1$, the results presented in this section reduce to the results given by Yu and Liu [10].

6. Conclusions

In this paper, we have introduced a new concept of semilocal E -preinvex maps in Banach spaces, which extend the semi E -preinvex maps presented by Luo and Jian (2011)[14]. Simultaneously, we have derived some of its basic properties. Next, we have defined an E - η -semidifferentiable map, which generalizes the η -semidifferentiable function introduced by Preda and Stancu-Minasian (1997)[7]. Based on this, we have proposed a new notion of nondifferentiable maps called semilocal E -invex maps and the concepts of E -type-I maps, which extend the generalized type-I maps brought forward by Yu and Liu (2007)[10]. In the framework of the new concepts, we have established some optimality conditions for the nondifferentiable vector optimization problem with inequalities constraints using semilocal E -preinvex, semilocal E -invex and E -type-I maps, respectively. Moreover, we have proved weak and converse duality results under various types of E -type-I maps requirements. The results presented in this paper extend and improve many results of [10, 14] and generalize results obtained in the literatures on this topic.

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