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A multi-parametric programming algorithm for special classes of non-convex multilevel optimization problems

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Abstract. A global solution strategy for multilevel optimization problems with special non-convexity formulation in the objectives of the inner level problems is presented based on branch-and-bound and multi-parametric programming approach. An algorithm to such problems is proposed by convexifying the inner level problem while the variables from upper level problems are considered as parameters. The resulting convex parametric under-estimator problem is solved using multi-parametric programming approach. A branch-and-bound procedure is employed until a pre-specified positive tolerance is satisfied. Moreover, a $\epsilon-$convergence proof is given for the algorithm.

Keywords: Multi-level programming; multi-parametric programming; branch-and-bound; convexification.

AMS Classification: 90B50, 90C26, 90C31, 90C99

1. Introduction

In many real-world problems decisions have been made in a hierarchical order where individual decision makers have no direct control upon the decisions of the others, but their actions affect all other decision makers. Further, higher levels (or leaders) of the hierarchy have the power to strongly influence the performance and strategies of the decision makers at lower levels (or followers)[1]. Multilevel Optimization Problems (MLOP) are mathematical programs which have a subset of their variables constrained to be an optimal solution of other programs parameterized by their remaining variables [2]. It's implicitly determined by a series of optimization problems which must be solved in a predetermined sequence.

If we set a vector $x^i \in \mathbb{R}^{n_i}$ to represent the portion of the decision vector controlled by decision maker at level i, a function $f_i(x^1, x^2, \ldots, x^k)$ to represent the objective of the decision maker at level i and the inequality $g_i(x^1, x^2, \ldots, x^k) \leq 0$ to represent constraints at level i, a MLOP is mathematically formulated as:

$$\begin{aligned}
&\min_{x^1} f_1(x^1, x^2, \ldots, x^k) \\
&\quad \text{s.t. } g_1(x^1, x^2, \ldots, x^k) \leq 0 \\
&\qquad \text{where } [x^2, x^3, \ldots, x^k] \text{ solves} \\
&\qquad\quad \min_{x^2} f_2(x^1, x^2, \ldots, x^k) \\
&\qquad\quad\quad \text{s.t. } g_2(x^1, x^2, \ldots, x^k) \leq 0, \\
&\qquad\quad\quad\ \text{where } [x^3, x^4, \ldots, x^k] \text{ solves} \\
&\qquad\qquad\qquad\qquad \ddots \\
&\qquad\qquad\qquad\qquad \text{where } [x^k] \text{ solves}
\end{aligned} \tag{1}$$

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$$\min_{x^k} f_k(x^1, x^2, \ldots, x^k)$$
$$\text{s.t. } g_k(x^1, x^2, \ldots, x^k) \leq 0$$
$$x = (x^1, x^2, \ldots, x^k) \in X = (X_1, X_2, \ldots, X_k) \subseteq \mathbb{R}^n$$
$$X_i \subseteq \mathbb{R}^{n_i}, i = 1, \ldots, k, \quad n_1 + \cdots + n_k = n.$$

where k represents the number of levels in the hierarchy, and $\mathbf{x} \in \mathbb{R}^{\mathbf{n}} = \mathbb{R}^{\mathbf{n_1}} \times \mathbb{R}^{\mathbf{n_2}} \times \cdots \times \mathbb{R}^{\mathbf{n_k}}$ is the decision vector partitioned into the k decision levels.

Two-level problems (which are known as *bilevel programming problems*) and 3-level problems (usually referred to as *trilevel programming problems*) are common in applications [2]. A trilevel optimization problem, where $k = 3$ in equation (1), for instance, comprises of three subproblems, one at each optimization level with the following basic definitions of sets:

- The set

$$\Omega_3(x^1, x^2) = \{x^3 \in X_3 : g_3(x^1, x^2, x^3) \leq 0\} \quad (2)$$

  is called a feasible set for the third level.
- The set of parametric solutions defined as,

$$\Psi_3(x^1, x^2) = \{x^3 \in X_3 : x^3 \in \arg\min\{ f_3(x^1, x^2, x^3) : x^3 \in \Omega_3(x^1, x^2)\}\} \quad (3)$$

  is called the rational reaction set for the third level.
- The set

$$\Omega_2(x^1) = \{(x^2, x^3) \in X_2 \times X_3 : g_2(x^1, x^2, x^3) \leq 0, g_3(x^1, x^2, x^3) \leq 0, x^3 \in \Psi_3(x^1, x^2)\} \quad (4)$$

  is called a feasible set for the second level problem.
- The set of solutions

$$\Psi_2(x^1) = \{(x^2, x^3) \in X_2 \times X_3 : x^2 \in \arg\min\{ f_2(x^1, x^2, x^3) : (x^2, x^3) \in \Omega_2(x^1)\}\} \quad (5)$$

  is called the rational reaction set for the second level, for $X_i \subseteq \mathbb{R}^{n_i}, i = 1, 2, 3$.

One can easily see the parametric nature of the rational reaction sets, in equations (3) and (5), which describe the dependence of the decisions taken at the upper levels on the decisions taken at lower levels.

Because of this sequential nature of the rational reaction sets, MLOP are generally very complex and are difficult to solve. It has been shown that even the linear bilevel programming problem is strongly NP-Hard [2, 3]. Moreover, the complexity of the problem increases as significantly as the number of levels increase [4].

However, several algorithmic approaches have been developed that can solve convex bilevel and trilevel programming problems. For linear MLOPs vertex enumeration methods have been used in [5, 6]. For nonlinear MLOPs many of the methods try to transform the lower level problems by using the Karush-Kuhn-Tucker (KKT) conditions or penalty functions [7, 8]. However, the relaxed KKT conditions may result in non-optimal reactions from the lower levels or in ambiguity to choose one solution among the optimal reaction set. Moreover, it seems too difficult to extend such an approach beyond two levels, because of the non-convex constraints introduced due to complementarity conditions. Recently, Tilahun *et al.* [9] have proposed a meta-heuristic algorithm to solve a multilevel problem of general form. The algorithm takes the variables from other levels as fixed values and solve the problem at each level (from top to bottom) using (1+1)-evolutionary strategy. However, the comparison made between the previous values and the solution from each iteration, in the adaptation procedure, depends only on the objective value of each level without considering the reaction from other levels. This may create non optimal Stackelberg solutions, as the decision makers at each level optimizes its own objective function without taking the reaction from other levels into consideration.

By considering the upper level variables as parameters in the lower level problems Faísca *et. al.,* [10, 11] have proposed a Multi-parametric Programming (MPP) approach to solve MLOPs, when the lower level problems are convex. Using this approach, one can convert multilevel problems as sequential multi-parametric optimization problems. If the lower levels are convex, their parametric solutions are unique in each region, where the solution is stable. But many practical problems that are modeled using MLOP may contain non-convex terms in their lower level problems [8]. In this case, even though MPP approach produces a mathematical program without equilibrium constraints depending on the upper level problems, the rational reaction set of the inner level problem (with non-convexity formulation) may be a disconnected set. So, it turns out that the upper level optimization problem may not have a solution in the rational reaction set, even if the upper level is a linear programming problem. Moreover, the global optimal solution cannot be efficiently computed and the behavior of a local solution is hard to analyze for the inner level optimization problems.

In this paper we apply the process of convexification of the lower level problems to underestimate them by convex functions (if they are nonconvex) at each iteration and use MPP approach to propose a branch-and-bound algorithm to find a global approximate solution for multilevel problems with non-convexity at their inner levels. The paper is organized as follows; in Section 2, multiparametric nonlinear programming problems are described and the respective critical regions are defined. Section 3 presents the proposed algorithm for bilevel and trilevel programming problems and convergence proof is also presented. Illustrative examples are given in Section 4. We conclude the paper by giving a conclusive remark in Section 5.

2. Preliminary Concepts

In MLOPs, the lower level problems can be considered as parametric optimization problems, where the decision vectors for upper levels are the parameters. To solve such kind of problems the following notions are helpful.

2.1. Multi-parametric nonlinear programming

Consider the following multi-parametric nonlinear programming problem as the inner level problem of a bilevel programming problem, where the parameter vector θ represents the vector of upper level optimization variables and x is the optimization variable of the current level:

$$\begin{aligned} Z(\theta) = \min_x & f(x,\theta) \\ & \text{s.t.} \\ & g_i(x,\theta) \leq 0, \text{ for all } i = 1,2,\dots,p, \\ & h_j(x,\theta) = 0, \text{ for all } j = 1,2,\dots,q \\ & x \in X \subseteq \mathbb{R}^n, \theta \in \Theta \subseteq \mathbb{R}^m, \end{aligned} \tag{6}$$

where f, $g_1,\dots,g_p$ and $h_1,\dots,h_q$ are twice continuously differentiable functions in x and θ. Assume also that f is a convex function and the functions $g_1,\dots,g_p$, and $h_1,\dots,h_q$ define convex sets.

If x is a feasible solution to problem (6) for a given θ, we classify the constraint functions as: active constraints, where the set of constraints that satisfy the property $g_i(x,\theta) = 0$, for some i, and as inactive constraints, those that satisfy $g_i(x,\theta) < 0$ for some i.

Definition 1. *The active set $A(x,\theta)$ of the inequality constraints of problem (6) is the set of constraints' indices of the active constraints, that is,*

$$A(x,\theta) = \{i \in \{1,2,\dots,p\} | g_i(x,\theta) = 0\}$$

Definition 2. *For an active set $A(x,\theta)$, we say that the linear independence constraint qualification condition holds if the set of active constraint gradients are linearly independent.*

The first-order KKT optimality conditions for (6) are given for the Lagrangian:

$$L = f(x,\theta) + \sum_{i=1}^{p} \lambda_i g_i(x,\theta) + \sum_{j=1}^{q} \mu_j \; h_j(x,\theta)$$

by the following conditions.

$$\begin{aligned} \nabla_x L &= 0, \\ g_i(x,\theta) &\leq 0, \\ \lambda_i g_i(x,\theta) &= 0, \\ \lambda_i &\geq 0, \text{ for all, } i = 1,2,\dots,p \\ h_j(x,\theta) &= 0, \text{ for all, } j = 1,2,\dots,q \end{aligned} \tag{7}$$

where, λ and μ are Lagrange multiplier vectors. The triple (x_0,λ_0,μ_0) is called a KKT point if it satisfies the conditions in (7).

Definition 3. *Strict complementary slackness is said to hold at a KKT point (x_0,λ_0,μ_0) if and only if for $i = 1,2,\dots,p$, $\lambda_i > 0$ if $g_i(x_0,\theta_0) = 0$ and $\lambda_i = 0$ if $g_i(x_0,\theta_0) < 0$.*

The main sensitivity result for (6) is derived directly from system (7) and is given in the next theorem, taken from [12].

Theorem 1. *Let θ_0 be a vector of parameter values and (x_0,λ_0,μ_0) be a KKT triple corresponding to (7), where, λ_0 is nonnegative and x_0 is feasible in (6). Also assume that,*

1. *strict complementary slackness condition,*
2. *the gradients of the active constraints are linearly independent (LICQ: Linear Independence Constraint Qualification condition), and*
3. *the second-order sufficiency conditions*

hold. Then, in the neighborhood of θ_0, there exists a unique, once continuously differentiable function, $Z(\theta) = [x(\theta),\lambda(\theta),\mu(\theta)]$, satisfying (7) with $Z(\theta_0) = [x(\theta_0),\lambda(\theta_0),\mu(\theta_0)]$, where $x(\theta)$ is a unique isolated minimizer for (6), and

$$\begin{pmatrix} \frac{dx(\theta_0)}{d\theta} \\ \frac{d\lambda(\theta_0)}{d\theta} \\ \frac{d\mu(\theta_0)}{d\theta} \end{pmatrix} = -M_0^{-1} \cdot N_0 \tag{8}$$

where, M_0 and N_0 are the Jacobian of system (7) with respect to x and θ:

$$M_0 = \begin{bmatrix} \nabla_{xx}L & \nabla_x g_1 \dots & & \nabla_x g_p \\ \lambda_1 \nabla_x^T g_1 & -g_1 & & 0 \\ \vdots & & \ddots & \\ \lambda_p \nabla_x^T g_p & & -g_p & 0 \\ \nabla_x^T h_1 & 0 \cdots & & 0 \\ \vdots & & & \\ \nabla_x^T h_q & 0 \cdots & & 0 \end{bmatrix}$$

$$N_0 = \begin{bmatrix} \nabla_{\theta x}^{2T} L \\ -\lambda_1 \nabla_\theta g_1 \\ \vdots \\ -\lambda_p \nabla_\theta g_p \\ \nabla_\theta h_1 \\ \vdots \\ \nabla_\theta h_q \end{bmatrix}$$

Note that the assumptions in Theorem 1 ensure that the inverse of the Jaccobian of equation (7) exists [11, 13, 14]. In other words, when M_0 is not invertible any violation of the assumptions in Theorem 1 is easily detected.

In [11] Dua *et al.*, have proposed an algorithm to solve equation (8) in the entire range of the varying parameters for general convex problems. This algorithm is based on approximations of the nonlinear optimal expression, $x = \gamma^\star(\theta)$ by a set of first order approximations, as given by Corollary 1 below.

Corollary 1 (First order estimation of $x(\theta)$, $\lambda(\theta)$, $\mu(\theta)$, near $\theta = \theta_0$ [15]). *Under the considerations of Theorem 1, a first order approximation of $[x(\theta), \lambda(\theta), \mu(\theta)]$ in the neighborhood of θ_0 is given by,*

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - M_0^{-1} \cdot N_0 \cdot (\theta - \theta_0) \quad (9)$$

where $(x_0, \lambda_0, \mu_0) = (x(\theta_0), \lambda(\theta_0), \mu(\theta_0))$,
$M_0 = M(\theta_0), N_0 = N(\theta_0)$

The space of θ where this solution (9) remains optimal is defined as the *critical region* (CR), and can be obtained by using feasibility and optimality conditions. Feasibility is ensured by substituting $x(\theta)$ into the inactive inequalities given in problem (6), whereas the optimality condition is given by $\check{\lambda}(\theta) \geq 0$, where the multiplier $\check{\lambda}(\theta)$ corresponds to the vector of active inequalities, resulting in a set of parametric constraints. Each piecewise linear approximation is confined to regions defined by feasibility and optimality conditions [11]. If $\check{g}$ corresponds to the non-active constraints, and $\check{\lambda}$ to the Lagrangian multipliers of the active constraints, we have:

$$\begin{cases} \check{g}(x(\theta), \theta) \leq 0 & \text{Feasibility conditions} \\ \check{\lambda}(\theta) \geq 0 & \text{Optimality conditions.} \end{cases}$$

Consequently, the explicit expressions are given by a conditional piecewise linear function [11]:

$$\begin{cases} x = C_1 + K_1 \cdot \theta, & \theta \in CR_1^R \\ x = C_2 + K_2 \cdot \theta, & \theta \in CR_2^R \\ \vdots \\ x = C_p + K_p \cdot \theta, & \theta \in CR_p^R, \end{cases}$$

where C_i's are column vectors and K_i's are real matrices, whereas $CR_i^R \subseteq \mathbb{R}^m$ are critical regions and note that CR_i^R denotes the i^{th} critical region.

2.2. Convex relaxation of bilinear and concave terms

If the multi-parametric nonlinear optimization problem (6) contains bilinear and concave terms, the KKT conditions (7) may not produce parametric optimal solutions. To assure that KKT optimality conditions are both necessary and sufficient for obtaining the global parametric optimum of the inner level problem, the occurrence of any non-convex term must be underestimated by a convex envelope to approximate it by a convex function.

The convex envelope of bilinear terms $b_{ij}y_iy_j$ taken over the rectangle $R = \{(y_i, y_j) : y_i^L \leq yi \leq y_i^U, y_j^L \leq y_j \leq y_j^U\}$ is denoted by $VexR[b_{ij}y_iy_j]$ and can be obtained as follows:

Theorem 2 ([16]). *Let b_{ij}, for $i = 1, 2, \dots, n_2 - 1$ and $j = i + 1, \dots, n_2$, be a real number, then the convex envelope of a bilinear term $b_{ij}y_iy_j$ can be defined as: $VexR[b_{ij}y_iy_j] = max\{b_{ij}l_{ij}^1(y_i, y_j), b_{ij}l_{ij}^2(y_i, y_j)\}$*

where, $l_{ij}^1(y_i, y_j) = \begin{cases} y_j^L y_i + y_i^L y_j - y_i^L y_j^L, & b_{ij} > 0 \\ y_j^U y_i + y_i^L y_j - y_i^L y_j^U, & b_{ij} \leq 0 \end{cases}$

and $l_{ij}^2(y_i, y_j) = \begin{cases} y_j^U y_i + y_i^U y_j - y_i^U y_j^U, & b_{ij} > 0 \\ y_j^L y_i + y_i^U y_j - y_i^U y_j^L, & b_{ij} \leq 0 \end{cases}$

Theorem 2 gives us a tighter lower bound for each bilinear terms. On the other hand the over-estimator of such terms can be obtained by adding the maximum separation to the lower bound and can be stated in Corollary 2.

Corollary 2. *The maximum separation between $b_{ij}y_iy_j$ and $VexR[b_{ij}y_iy_j]$ inside the rectangle R is equal to $\delta_{ij} = |b_{ij}| \frac{[y_i^U - y_i^L][y_j^U - y_j^L]}{4}$*

Proof. See [17] □

Similarly, any occurrence of univariate concave functions can be trivially underestimated by their linearizations at the lower bounds of the variable ranges [18]. Thus the convex envelope of the concave function c(y) over the interval $[y^L, y^U]$ is the linear function of y obtained by:

$$Vex_c = c(y^L) + \frac{c(y^U) - c(y^L)}{y^U - y^L}(y - y^L)$$

The maximum separation between the concave function and its convex under-estimator can be found by minimization problem [19] as:

$$MaxSe = - \min_{y^L \le y \le y^U} \{-c(y) + Vex_c\},$$

where, Vex_c is the convex under-estimator of $c(y)$.

3. Algorithm for Multi-level Optimization with Special Non-convexity Formulation at Inner Levels

Applying the formulations in subsection 2.1, it is possible to solve convex MLOP as described in [10] and [11]. However, if the inner level problems are non-convex, we need to convexify them as described in subsection 2.2 and underestimate the given problem by an approximate convex problem. Using this approach we will propose an algorithm to solve nonlinear MLOP in terms of branch-and-bound procedure. In this paper, special form of non-convexity, where only bilinear and concave terms appear in the objective of the most inner level problem, will be considered and the algorithm will be tested for bilevel and trilevel problems of such a form.

3.1. Algorithm for bilevel programming problem with special non-convexity formulation at the inner problem

Consider the following non-convex bilevel problem:

$$\begin{aligned} &\min_x f_1(x,y) \\ &\quad \text{s.t. } g_1(x,y) \le 0 \\ &\min_y f_2(x,y) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} b_{ij} y_i y_j + c(y) + h_1(x) + h_2(y) \\ &\quad \text{s.t } g_2(x,y) \le 0 \\ &\quad y^L \le y \le y^U, \quad x^L \le x \le x^U, \end{aligned} \tag{10}$$

where $g_2(x,y)$ forms a convex polyhedron, $h_1(x)$ and $h_2(y)$ are linear with respect to x and y respectively, $c(y)$ is a concave function, $x \in X \subseteq \mathbb{R}^{n_1}$, $y \in Y \subseteq \mathbb{R}^{n_2}$ and X and Y are compact convex sets. Note that, the inner level is a non-convex minimization problem.

The approach begins with rewriting the inner level of problem (10) as a MPP:

$$\begin{aligned} &\min_y f_2(x,y) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} b_{ij} y_i y_j + c(y) + h_1(x) + h_2(y) \\ &\text{s.t } g_2(x,y) \le 0 \\ &\quad y^L \le y \le y^U \\ &\quad x^L \le x \le x^U \end{aligned} \tag{11}$$

As discussed in subsection (2.2) each bilinear and concave terms can be under-estimated by their respective tighter convex terms $VexR[b_{ij}y_iy_j]$ and Vex_c respectively. Thus, the under-estimator problem of (11) is formulated as:

$$\begin{aligned} &\min_y f_2(x,y) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} VexR[b_{ij}y_iy_j] + Vex_c + h_1(x) + h_2(y) \\ &\quad \text{s.t } g_2(x,y) \le 0 \\ &\quad y^L \le y \le y^U, \quad x^L \le x \le x^U \end{aligned} \tag{12}$$

Problem (12) is a linear MPP problem, which can be solved by using Linear MPP algorithm described in [14], resulting in:

$$y^i = m_2^i + k_2^i \cdot x, \; H_2^i \cdot x \le d_2^i, \tag{13}$$

where, $i = 1, 2, \ldots, I_2$, with I_2 being the number of critical regions. Substituting the expression given in (13) into the objective function of problem (12) we obtain the parametric lower bound $\check{Z}_i^1(x)$ for the solution of problem (11) within the corresponding critical regions, CR^i. Adding the maximum separations, δ_{ij} and $MaxSe$, for bilinear and concave terms respectively on $\check{Z}_i^1(x)$ gives the parametric over-estimator $\hat{Z}_i^1$.

Each of the lower bounds $\check{Z}_i^1(x)$ is then compared to $\hat{Z}_i^1$ and the region of x where $\hat{Z}_i^1(x) - \check{Z}_i^1(x) \le \epsilon$ are fathomed for a given small positive tolerance ϵ. If each of $\check{Z}_i^1(x)$ are within ϵ of $\hat{Z}_i^1(x)$ in the space of x, then the expressions in (13) can be incorporated into the leader's problem of (10). Otherwise, the initial region of y is partitioned into two by bisecting the variable that has the longest side from among those that resulted in non-convexity in the problem, as in [17]. After branching in y, the corresponding under-estimator subproblems in each subregion can be formulated as follows:

$$\begin{aligned}\check{Z}^{21}(x) = \min_y \sum_{i=1}^{n_2-1}\sum_{j=i+1}^{n_2} VexR1[b_{ij}y_iy_j] + \\ Vex_c1 + h_1(x) + h_2(y) \\ \text{s.t.} \quad g_2(x,y) \le 0 \qquad (14) \\ x^L \le x \le x^U, \quad y^L \le y \le y^U_{new},\end{aligned}$$

and,

$$\begin{aligned}\check{Z}^{22}(x) = \min_y \sum_{i=1}^{n_2-1}\sum_{j=i+1}^{n_2} VexR2[b_{ij}y_iy_j] + \\ Vex_c2 + h_1(x) + h_2(y) \\ \text{s.t.} \quad g_2(x,y) \le 0 \qquad (15) \\ x^L \le x \le x^U, \quad y^L_{new} \le y \le y^U\end{aligned}$$

where $VexR1[b_{ij}y_iy_j]$ and $VexR2[b_{ij}y_iy_j]$ are under-estimators of the bilinear terms over the region $y^L \le y \le y^U_{new}$ and $y^L_{new} \le y \le y^U$ respectively and Vex_c1 and Vex_c2 are convex under-estimators of the concave terms over the region $y^L \le y \le y^U_{new}$ and $y^L_{new} \le y \le y^U$ respectively.

Each of the problems (14) and (15) are again solved using linear MPP algorithm and one can substitute each of these parametric solutions into the objective functions of (14) and (15) respectively to obtain the parametric lower bounds ${\check{Z}_i}^{21}(x)$ and ${\check{Z}_i}^{22}(x)$ for the solution of problem (11). Similarly, one can get the parametric upper bound by adding the maximum separation to each of the lower bounds ${\check{Z}_i}^{21}(x)$ and ${\check{Z}_i}^{22}(x)$, to obtain the parametric over-estimators $\hat{Z}^{21}_i$ and $\hat{Z}^{22}_i$.

Now, by comparing each parametric upper bounds with the former upper bound for each subproblem we choose the least upper bound within the corresponding critical regions; and we shall denote it by $\overline{Z}_u(x)$. Similarly, comparing the lower bounds and taking the maximum of them, we update the lower bound as well and denote it by $\overline{Z}_l(x)$. Consequently, convergence test is performed by comparing the updated parametric upper bound with the updated lower bounds. If the convergence test is satisfied the branching procedure stops there.

Otherwise, the process continues in the same manner as discussed above until the lower bound and the upper bound are separated by sufficiently small positive tolerance ϵ. Following this procedure, the bilevel optimization problem will be converted to a single-level optimization problem:

$$\begin{aligned}\min_x f_1^*(x, y(x)) \\ \text{s.t} \\ g_1(x, y(x)) \le 0 \qquad (16) \\ x \in S,\end{aligned}$$

where $\quad S = \{x \in X : y \in Y, g_2(x, y(x)) \le 0\}$

Due to the partitioning of the parametric region, the solution of the first-level optimization problem depends on the number of critical regions obtained in the inner problem. Based upon the above discussion an algorithm for the solution of a bilevel programming problem is presented in Table 1

Table 1. A Parametric programming algorithm for bilevel programming problems with non-convexities in the inner problem

Step	Description
1	Consider the inner problem of (10) as a MPP problem, with the upper level variables are being considered as parameters;
2	Initialize the current parametric upper bound as $\bar{Z}_u(x) = \infty$, current parametric lower bound as $\bar{Z}_l(x) = -\infty$, a space of upper level optimization variable x, as a parameter space (CR) determined by the lower and upper bounds x^L and x^U respectively, a space of inner level optimization variable y determined by the lower and upper bounds y^L and y^U respectively, and tolerance, ϵ;
3	For a given region of x, CR, and the corresponding space of y, convexify the inner problem and solve using linear MPP algorithm and obtain the parametric lower bound, $\check{Z}(x)$ and the parametric upper bound, $\hat{Z}(x)$ for the solution of the inner problem as discussed in Section 3.1;
4	Compare $\bar{Z}_u(x)$ and $\hat{Z}(x)$ as described in Appendix B and update the current parametric upper bound as $\bar{Z}_u(x) = \min(\bar{Z}_u(x), \hat{Z}(x))$;
5	Compare $\bar{Z}_l(x)$ and $\check{Z}(x)$ as described in Appendix B and update the current parametric lower bound as $\bar{Z}_l(x) = \max(\bar{Z}_l(x), \check{Z}(x))$;
6	If, $\overline{Z}_u(x) - \overline{Z}_l(x) \le \epsilon$, and $\overline{Z}_u(x) \le \overline{Z}_l(x)$ fathom the corresponding space of parameters and the relaxed convex optimization problems are infeasible for some rectangle, then in such cases fathom those regions and the corresponding space of parameters.
7	If there is any more space of parameter x and a space of optimization variable y to be explored go to Step 8. Otherwise, go to Step 12.

Step	Description
8	Branch on y and formulate a convex under-estimator in each subrectangles of y;
9	Solve the convex under-estimator problems in each subrectangles using MPP approach and obtain the parametric lower and upper bounds for the solution in each subrectangles.
10	Compare the lower bounds of each subrectangles within the corresponding critical regions and take, $\check{Z}(x)$ to be the minimum within the corresponding critical region;
11	Compare the upper bounds of each subrectangle within the corresponding critical regions and take $\hat{Z}(x)$ to be the minimum within the corresponding critical region and go to Step (4).
12	Substitute each of the solutions of the inner problem into the leader's problem and formulate one-level optimization problem;
13	Solve each single-level problem using suitable global optimization method;
14	Compare the leader's optimal solutions and select the best as one needs.

In order to prove the convergence of the algorithm stated in Table 1 we need to justify the following statements.

Theorem 3. *Consider problem (10) and assume that the optimization variables are bounded. If the reformulated inner level problem has a solution $y(x) \in 2^{\mathbb{R}^n}$ at each iteration, then the functional sequence $\chi_k(x) = \hat{Z}_k(x) - \check{Z}_k(x)$, is a decreasing sequence, where $\hat{Z}_k(x)$ is the k^{th} least upper bound and $\check{Z}_k(x)$ is the k^{th} greatest lower bound.*

Proof. To prove the theorem we need the following two arguments.

(1) First we need to show that the sequence $\{\hat{Z}_k(x)\}_{k=1}^n$ is a decreasing sequence. To this end, let $\hat{Z}^p(x)$ be the upper bound at p^{th} iteration. Hence, from the algorithm for the p^{th} iteration we have, $\hat{Z}_p(x) = \min\{\min\{\hat{Z}_i(x)\}_{i=1}^{p-1}, \hat{Z}^p(x)\}$, and similarly, at $p+1$, we have $\hat{Z}_{p+1}(x) = \min\{\min\{\hat{Z}_i(x)\}_{i=1}^{p}, \hat{Z}^{p+1}(x)\} = \min\{\min\{\hat{Z}_i(x)\}_{i=1}^{p-1}, \hat{Z}_p(x), \hat{Z}^{p+1}(x)\}$, but $\hat{Z}_p(x) = \min\{\min\{\hat{Z}_i(x)\}_{i=1}^{p-1}, \hat{Z}^p(x)\}$. This implies that $\hat{Z}_{p+1}(x) \leq \hat{Z}_p(x)$, as desired.

(2) Secondly, we need to show that $\check{Z}_k(x)$ is an increasing sequence. Let $\check{Z}^p$ be the lower bound for p^{th} iteration. As a result of the algorithm, we have, $\check{Z}_p(x) = \max\{\max\{\check{Z}_i(x)\}_{i=1}^{p-1}, \check{Z}^p\}$ and similarly at $p+1$ we have,
$\check{Z}_{p+1}(x) = \max\{\max\{\check{Z}_i(x)\}_{i=1}^{p}, \check{Z}^{p+1}\} = \max\{\max\{\check{Z}_i(x)\}_{i=1}^{p-1}, \check{Z}_p, \check{Z}^{p+1}\}$, but,
$\check{Z}_p(x) = \max\{\max\{\check{Z}_i(x)\}_{i=1}^{p-1}, \check{Z}^p\}$
and as the size of the rectangular domain decreases, the maximum separation between the original non-convex function and its respective convex under-estimator function decreases. This implies that $\check{Z}_p(x) \leq \check{Z}_{p+1}(x)$.

From the discussion above we can realize that the parametric upper bound is found by adding the maximum separation between the under-estimator and the original non-convex function over the parametric lower bound. This shows that χ_k is independent of the parameter vector x.

Therefore, since $\hat{Z}_k(x)$ is a decreasing sequence and $\check{Z}_k(x)$ is an increasing sequence, we can conclude that the difference χ_k is a decreasing sequence and hence a convergent sequence.

□

Theorem 4. *Let $X \subseteq \mathbb{R}^m$ be a polyhedron and $CR^Q = \{x \in X : \tilde{g}_2(x) - \tilde{b} \leq 0\} \subseteq X$, be a critical region. Assume $CR^Q \neq \emptyset$. Also let $CR^i = \{x \in X : \tilde{g}_2^i(x) - \tilde{b}^i > 0, \tilde{g}_2^j(x) - \tilde{b}^j \leq 0, \forall j < i,\ i = 1, 2, \ldots, K\}$ where $K = size(b)$, and let $CR^{rest} = \bigcup_{i=1}^K CR^i$. Then*

(1) *$CR^{rest} \bigcup CR^Q = X$,*
(2) *$CR^Q \bigcap CR^i = \emptyset$,*
(3) *$CR^i \bigcap CR^j = \emptyset$, $\forall i \neq j$, i.e. $\{CR^Q, CR^1, \ldots, CR^K\}$ is a partition of X.*

Proof. (1) Since $CR^i \subseteq X$ for all i and $CR^Q \subseteq X$, it is clear that $CR^{rest} \bigcup CR^Q \subseteq X$. To show the backward inclusion let $x \in X$ and assume that $x \notin CR^Q$. Then, there exists an index i such that $\tilde{g}_2^i(x) - \tilde{b}^i > 0$. Let $i^* = \min_{i \leq K}\{i : \tilde{g}_2^i(x) > \tilde{b}^i\}$, by definition of i^* we have $\tilde{g}_2^{i^*}(x) > \tilde{b}^{i^*}$ and $\tilde{g}_2^j(x) < \tilde{b}^j$, $\forall j < i^*$. This implies that $x \in CR^{i^*}$, thus $x \in CR^{Rest} \bigcup CR^Q$. Hence $CR^{rest} \bigcup CR^Q = X$,.

(2) If $x \in CR^Q$ then by definition, there doesn't exist an index i that satisfy $\tilde{g}_2^i(x) - \tilde{b}^i > 0$. which implies that $x \notin CR^i$.

(3) Let $x \in CR^i$ and take $i > j$. Since $x \in CR^i$, by definition of $CR^i (i > j)$ $\tilde{g}_2^j(x) - \tilde{b}^j \leq 0$, which implies that $x \notin CR^j$.

□

As a direct consequence of the above two theorems we have the following corollary.

Corollary 3. *Let* $CR^{rest} \bigcup CR^Q = X$, $CR^Q \bigcap CR^i = \emptyset \; \forall i$ *and* $CR^i \bigcap CR^j = \emptyset, \forall i \neq j$ *be a partition of* X. *If* $\{\chi_k(x)\}_{k=1}^n$ *is a decreasing functional sequence, then the algorithm, given by Table 1, converges.*

Note that one can extend the idea discussed here above also for trilevel programming problems as discussed here below.

3.2. Algorithm for trilevel programming with non-convexity formulation in the inner problems

Consider the following non-convex trilevel optimization problem, where the second and third level objectives can have bilinear and concave terms.

$$\begin{aligned} &\min_x f_1(x,y,z) \\ &\quad \text{s.t. } g_1(x,y,z) \leq 0 \\ &\qquad \min_y f_2(x,y,z) \\ &\qquad \text{s.t. } g_2(x,y,z) \leq 0 \\ &\quad \min_z f_3(x,y,z) = \sum_{i=1}^{n_3-1} \sum_{j=i+1}^{n_3} b_{ij} z_i z_j + \\ &\qquad\qquad c(z) + h(x,y,z) \\ &\quad \text{s.t } \; g_3(x,y,z) \leq 0 \\ &x^L \leq x \leq x^U, y^L \leq y \leq y^U, z^L \leq z \leq z^U, \end{aligned} \tag{17}$$

where the constraint functions g_1, g_2 and g_3 each forms a convex polyhedron, h is linear with respect to all its variables, and $c(z)$ is a concave function.

Here again the solution procedure starts by convexifying the third-level problem and formulating a convex multi-parametric programming problem, where the variables determined by the two upper level decision makers are considered as parameters. Then the global parametric solution for the third level problem is found by the same approach as discussed in Section 3.1 for the solution of the inner level problem.

Substituting the solutions into the second-level problem, we obtain a multi-parametric programming problem, where the variables determined by the leader are the parameters. Note that since the parametric solutions for the third level problem are obtained in linear form, the complexity of the second level problem does not increase.

If the second level problem also has special non-convexity formulation, it is necessary to apply the same procedure as discussed in Section 3.1 for the inner level problem and substitute the solution into the leader's problem. Finally, the resulting single-level problem is solved using suitable global optimization methods.

On the other hand, if the second level problem is a convex multi-parametric problem, the KKT conditions are sufficient. Thus, the convexification process and the branch-and-bound procedure for the second level problem is not necessary in the above discussion.

Generally, the algorithm in Table 1 will be applied twice if the second level problem also has the same type of non-convexity as the third level. Otherwise, the relaxation step in Table 1 could be omitted for the second stage.

3.3. Algorithm for k-level programming with non-convexity formulation in the inner problems

Consider problem (1) with special non-convexity formulation at each (or some of the) optimization level(s). The k^{th}-level optimization problem can be rewritten as a multi-parametric programming problem where, the upper levels optimization variables are considered as parameters. The resulted problem can be solved parametrically using the algorithm described in Table 1 (from Step 2 - Step 11). The parametric solution can be incorporated into the $(k-1)^{th}$–level. In the same manner the $(k-1)^{th}$-level optimization problem can be reformulated as multi-parametric programming problem which also be solved globally using the algorithm described in Table 1 (from Step 2 - Step 11). Applying the same procedure recursively, one can arrive at the first level and solve the problem globally in each critical region obtained during branching.

4. Illustrative Examples

Example 1: A bilevel problem

Consider the following bilevel programming problem, where the inner problem contains a bilinear term:

$$\begin{aligned} &\min_x f_1 = -2x_1 + y_1 - y_2 - 1 \\ &\text{s.t } \frac{1}{2}x_1 + y_2 - 0.5 \leq 0 \text{ where } [y_1, y_2]^T \text{ solves} \\ &\qquad \min_y f_2 = y_1 y_2 \\ &\qquad\quad \text{s.t } \; -2y_1 - y_2 + x_1 \leq 0 \\ &\qquad\qquad -y_1 - 3y_2 + \frac{1}{2}x_2 \leq 0 \\ &\qquad\qquad -1 \leq x_1, x_2 \leq 1 \\ &\qquad\qquad -\frac{1}{6} \leq y_2 \leq \frac{7}{12}, \frac{1}{2} \leq y_1 \leq 1 \end{aligned} \tag{18}$$

First of all, consider the inner level problem of (18) as a MPP problem with x a parameter vector of the optimization problem:

$$\begin{aligned}\min_y\ & f_2 = y_1y_2 \\ \text{s.t}\ & -2y_1 - y_2 + x_1 \le 0 \\ & -y_1 - 3y_2 + \frac{1}{2}x_2 \le 0 \\ & -1 \le x_1, x_2 \le 1 \\ & -\frac{1}{6} \le y_2 \le \frac{7}{12}, \frac{1}{2} \le y_1 \le 1\end{aligned} \tag{19}$$

Under-estimate problem (19) by using the formula discussed in subsection 2.2 to get:

$$\begin{aligned}\min_y\ & f_2 = \frac{1}{2}y_2 - \frac{1}{6}y_1 + \frac{1}{12} \\ \text{s.t}\ & -2y_1 - y_2 + x_1 \le 0 \\ & -y_1 - 3y_2 + \frac{1}{2}x_2 \le 0 \\ & -1 \le x_1, x_2 \le 1 \\ & -\frac{1}{6} \le y_2 \le \frac{7}{12}, \frac{1}{2} \le y_1 \le 1\end{aligned} \tag{20}$$

Solving problem (20) using linear MPP algorithm one can get parametric solutions within each of the corresponding critical regions

$$CR_1^R = \begin{cases} y = \begin{cases} \frac{3}{5}x_1 - \frac{1}{10}x_2 - 0.4 \\ \frac{1}{5}x_1 - \frac{1}{5}x_2 + 0.83 \end{cases} \\ 0.5 - x_1 \le 0 \\ -\frac{1}{2}x_2 - 1.83 \le 0 \\ x_1 - 1 \le 0 \\ -x_2 - 1 \le 0 \\ -x_1 - 1 \le 0 \\ x_2 - 1 \le 0 \end{cases},$$

$$CR_2^R = \begin{cases} y = \begin{cases} \frac{3}{5}x_1 - \frac{1}{10}x_2 + 0.08 \\ \frac{1}{5}x_2 - \frac{1}{5}x_1 + 0.4 \end{cases} \\ -x_1 - 0.67 \le 0 \\ \frac{1}{2}x_2 - 8.5 \le 0 \\ x_1 - 0.5 \le 0 \\ x_2 - 1 \le 0 \\ -x_1 - 1 \le 0 \\ -x_2 - 1 \le 0 \end{cases} \quad \text{and}$$

$$CR_3^R = \begin{cases} y = \begin{cases} \frac{3}{5}x_1 - \frac{1}{10}x_2 + 0.4 \\ \frac{1}{5}x_2 - \frac{1}{5}x_1 + 0.167 \end{cases} \\ -x_1 - 1.83 \le 0 \\ -\frac{1}{2}x_2 - 0.5 \le 0 \\ x_1 + 0.67 \le 0 \\ x_2 - 1 \le 0 \\ -x_1 - 1 \le 0 \\ -x_2 - 1 \le 0 \end{cases}$$

Substitute the parametric solutions into the objective function of problem (20) to get the parametric lower bound of the solution with in the corresponding critical regions. These are $\check{Z}_1 = 0.56x_1 - 0.06x_2 - 1.6$ in CR_1^R, $\check{Z}_2 = 0.56x_1 - 0.06x_2 + 1.75$ in CR_2^R and $\check{Z}_3 = 0.56x_1 - 0.06x_2 + 0.5$ in CR_3^R.

The parametric upper bound can be found by adding the maximum separation (as described in Corollary 2) $\delta_{ij} = 0.125$ to the lower bounds and can be described as follows: $\hat{Z}_1 = 0.56x_1 - 0.06x_2 - 1.475$ in CR_1^R, $\hat{Z}_2 = 0.56x_1 - 0.06x_2 + 1.875$ in CR_2^R and $\hat{Z}_3 = 0.56x_1 - 0.06x_2 + 0.625$ in CR_3^R.

Compare upper bounds with lower bounds within the corresponding critical regions, CR_s^R. But since the difference, 0.125, is greater than the pre-specified tolerance $\epsilon = 2.8275 \times 10^{-6}$, branching on the optimization variable is performed. Hereby, the algorithm terminates after fifteen branching steps with parametric solutions

$$\begin{aligned} y_1(x_1, x_2) &= 0.6x_1 - 0.1x_2 + 0.6 \\ y_2(x_1, x_2) &= 0.2x_2 - 0.2x_1 + 1 \end{aligned}$$

and set of inequalities (what we call a critical region) $CR^R = \begin{cases} -1 \le x_1 \le 1 \\ -1 \le x_2 \le 1 \end{cases}$ for $\epsilon = 2.8275 \times 10^{-6}$ and there is no rest region to be explored. So, one can incorporate the rational reaction set into the upper level problem and problem (18) becomes a single level optimization problem:

$$\begin{aligned}\min_x\ & f_1 = -1.2x_1 - 0.3x_2 - 2.4 \\ & \text{s.t}\ 0.3x_1 + 0.2x_2 + 0.5 \le 0,\ -1 \le x_1, x_2 \le 1\end{aligned} \tag{21}$$

Solving this problem globally one obtains an optimal solution $(y_1, y_2, x_1, x_2) = (0.1, 1, -1, -1)$ and an optimal value as $f_2 = 0.1$ and $f_1 = -0.9$.

Example 2: A trilevel problem

Consider the following trilevel programming problem with the occurrence of a concave term in the third level:

$$\begin{aligned}\min_{x_1}\ & f_1 = -6x_1 + 2x_2 \\ \text{s.t}\ & \frac{1}{2}x_1 + x_2 \le \frac{1}{2}, \\ & \min_{x_2}\ f_2 = 2x_2 + z_2 \\ & \quad \text{s.t}\ -2x_1 + x_2 \le -z_1, \\ & \quad\quad \min_{z_1,z_2}\ f_3 = -z_1^2 + x_2 + z_1 \\ & \quad\quad \text{s.t}\ -2z_1 + z_2 + x_1 \le 0 \\ & \quad\quad\quad -z_1 - 3z_2 + \frac{1}{2}x_2 \le 0 \\ & \quad\quad\quad 0 \le x_1, x_2, z_1 \le 1, 0 \le z_2 \le \frac{1}{2}\end{aligned} \tag{22}$$

The solution procedure starts with reformulation of the third level problem as a MPP problem by considering the optimization variables of the two upper levels as parameters and can be described as follows:

$$\begin{aligned}\min_{z_1,z_2} f_3 &= -z_1^2 + x_2 + z_1 \\ \text{s.t } & -2z_1 + z_2 + x_1 \le 0 \\ & -z_1 - 3z_2 + \frac{1}{2}x_2 \le 0 \\ & 0 \le x_1, x_2, z_1 \le 1 \text{ , } 0 \le z_2 \le \frac{1}{2}\end{aligned} \tag{23}$$

Convexify the concave term as discussed in Subsection 2.2 and reformulate problem (23) as follows:

$$\begin{aligned}\min_{z_1,z_2} f_3 &= x_2 + \frac{1}{2}z_1 \\ \text{s.t } & -2z_1 + z_2 + x_1 \le 0 \\ & -z_1 - 3z_2 + \frac{1}{2}x_2 \le 0 \\ & 0 \le x_1, x_2, z_1 \le 1 \text{ , } 0 \le z_2 \le \frac{1}{2}\end{aligned} \tag{24}$$

After solving problem (24) using linear MPP algorithm, one gets a parametric solution $z = \begin{cases} \frac{3}{7}x_1 + \frac{1}{14}x_2 + 0.016 \\ \frac{1}{7}(x_2 - x_1) + 0.021 \end{cases}$ with the corresponding critical region $CR^R = \begin{cases} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{cases}$

Substitute the solution into the objective function of problem (24) to get the parametric lower bound $\check{Z}(x) = \frac{3}{14}x_1 + \frac{29}{28}$. Consequently, add the maximum separation ($MaxSe = 0.0625$) over $\check{Z}(x)$ to obtain the parametric upper bound as $\hat{Z}(x) = \frac{3}{14}x_1 + \frac{29}{28} + 0.0625$. Compare upper bounds with lower bounds within the corresponding CR^R. But the difference is greater than the pre-specified tolerance $\epsilon = 0.002$, then branching on the optimization variable is performed. Hereby, the algorithm terminates after three branching steps with parametric solution:

$$\begin{aligned} z_1(x_1, x_2) &= 0.4286x_2 + 0.071x_2 \\ z_2(x_1, x_2) &= 0.143(x_2 - x_1) + 0.01 \end{aligned}$$

and critical region $CR^R = \begin{cases} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{cases}$. Compare the upper bound $\hat{z}(x) = 0.357x_1 + 1.0595x_2 + 0.0019$ with lower bound $\check{Z}(x) = 0.357x_1 + 1.0595x_2$ and the difference (0.0019) is less than the pre-specified tolerance 0.002.

Now we can incorporate these results into the second level problem and obtain the second level problem as:

$$\begin{aligned}\min_{x_2} f_2 &= 1.857x_2 + 0.143x_1 + 0.01 \\ \text{s.t } & -1.5714x_1 + 1.4286x_2 \le 0 \\ & 0 \le x_1 \le 1\end{aligned} \tag{25}$$

Since problem (25) is a linear parametric programming problem, there is no need of convex relaxation and branching procedures. Hence, one can solve it using linear MPP algorithm for global optimality and get a parametric optimal solution for the second level, $x_2(x_1) = 2x_1 - 1$, and critical region $CR^R = \{0 \le x_1 \le 1$, lastly one can incorporate the result above into the leaders problem and solve it for x_1 to get the following results $(x_1, x_2, z_1, z_2) = (0.6, 0.2, 0.271, 0.047)$ with $(f_1, f_2, f_3) = (-3.2, 0.447, 0.16853)$

5. Conclusion

The difficulty and complexity of the solution approach for MLOP is easily confirmed by looking its simplest version, what we call it linear MLOP. Especially when nonconvexity appear in the inner level problem, most of the existing algorithms fail to work. However, in this paper we have described a global strategy for the solution of MLOP with non-convexity formulation in the inner problems based on the combination of MPP approach and a branch-and-bound algorithm. The proposed algorithm is suitable for problems involving only special non-convex and linear terms in the objective functions as well as in the constraint sets. The same procedure can be applied successively to solve any multilevel problem with the same form as discussed in subsection 3.3 for k-level case.

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Appendix A. Procedures to Define the Rest of Parametric Regions

Given an initial region, CR_{IG}, defined by $CR_{IG} = \{\theta_1^L \le \theta_1 \le \theta_1^U, \theta_2^L \le \theta_2 \le \theta_2^U\}$ and a region of optimality, CR^R such that $CR^R \subseteq CR_{IG}$, a procedure is described in this section to define the rest of the region,

$CR^{rest} = CR_{IG} - CR^R$. For the sake of simplicity we consider the case when only two parameters, θ_1 and θ_2, with CR^R defined by three inequalities, $\{g_1 \le 0, g_2 \le 0, \quad g_3 \le 0\}$ where g_1, g_2, g_3 are linear in θ. The procedure consists of considering one by one the inequalities which define CR^R. For example, consider inequality $g_1 \le 0$, the rest of the region can be addressed by reversing the sign of inequality $g_1 \le 0$ and removing redundant constraints in CR_{IG}, which is $CR_1^{rest} = \{g_1 \ge 0, \theta_1 \ge \theta_1^L, \theta_2 \le \theta_2^U\}$. Thus by considering the rest of the inequalities, the total of the rest region is given by, $CR^{rest} = \{CR_1^{rest} \cup CR_2^{rest} \cup CR_3^{rest}\}$, where CR_1^{rest}, CR_2^{rest} and CR_3^{rest} are given in Table 2.

Table 2. Definition of the rest regions

Region	Inequalities	
CR_1^{rest}	$g_1 \ge 0,$	$\theta_1 \ge \theta_1^L, \quad \theta_2 \le \theta_2^U$
CR_2^{rest}	$g_1 \le 0, \quad g_2 \ge 0,$	$\theta_1 \le \theta_1^U, \quad \theta_2 \le \theta_2^U$
CR_3^{rest}	$g_1 \le 0, g_2 \le 0, g_3 \ge 0,$	$\theta_1^L \le \theta_1 \le \theta_1^U, \theta_2^L \le \theta_2$

Appendix B: Comparison of parametric solutions

A method has been proposed in [14] for the comparison of two parametric solutions, $Z(\theta)^1$ and $Z(\theta)^2$ which are valid in the critical regions CR_1^R and CR_2^R respectively. The comparison process needs two steps. The first step is to define a region $CR^{int} = CR_1^R \bigcap CR_2^R$. In the second step, check whether CR^{int} is empty or not. If CR^{int} is empty, then there is no comparison to be performed, otherwise a new constraint $Z(\theta)^1 \le Z(\theta)^2$ is formulated and a constraint redundancy check is made for the new constraint in CR^{int}. This constraint redundancy test results in three cases which are analyzed as follows:

Case 1: If the new constraint is redundant (see [14]), then $Z(\theta)^1 \le Z(\theta)^2, \forall\theta \in CR^{int}$

Case 2: If the new constraint is infeasible (see [13]), then $Z(\theta)^1 \ge Z(\theta)^2, \forall\theta \in CR^{int}$

Case 3: If the new constraint is non-redundant, then

- $Z(\theta)^1 \le Z(\theta)^2, \forall\theta \in CR^{int} \cup \{\theta | Z(\theta)^1 - Z(\theta)^2 \le 0\}$ and
- $Z(\theta)^1 \ge Z(\theta)^2, \forall\theta \in CR^{int} \cup \{\theta | Z(\theta)^1 - Z(\theta)^2 \ge 0\}$.

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