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RESEARCH ARTICLE

Global mathematical analysis of a patchy epidemic model

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ABSTRACT

The dissemination of a disease within a homogeneous population can typically be modeled and managed in a uniform fashion. Conversely, in non-homogeneous populations, it is essential to account for variations among subpopulations to achieve more precise predictive modeling and efficacious intervention strategies. In this study, we introduce and examine the comprehensive behavior of a deterministic two-patch epidemic model alongside its stochastic counterpart to assess disease dynamics between two heterogeneous populations inhabiting distinct regions. First, utilizing a specific Lyapunov function, we demonstrate that the disease-free equilibrium of the deterministic model is globally asymptotically stable. For the stochastic model, we establish that it is well-posed, meaning it possesses a unique positive solution with probability one. Subsequently, we ascertain the conditions necessary to ensure the total extinction of the disease across both regions. Furthermore, we explicitly determine a threshold condition under which the disease persists in both areas. Additionally, we discuss a scenario wherein the disease persists in one region while simultaneously becoming extinct in the other. The article concludes with a series of numerical simulations that corroborate the theoretical findings.

1. Introduction

Infectious diseases are defined as illnesses caused by pathogenic agents, transmitted from an infected person, animal, or contaminated inanimate object to a susceptible host [1]. They are the main cause death worldwide killing more people than all wars and natural disasters combined [2,3]. For instance, during the past three years, the world has been under enormous threat from the highly contagious coronavirus which first emerged in China and has spread rapidly to cover almost the entire globe leading to the death of more than six million people, according to the statistics of the World Health Organization [4]. In addition to the human casualties from the coronavirus, the economic and social disruption caused by this pandemic is devastating. Millions of people at risk of crossing the poverty line, thousands of companies face an existential threat and almost 50% of the global workforce, comprising 3.3 billion individuals, faces the threat of unemployment [5].

To understand how infectious diseases spread, the mathematical models are useful tools to describe and simulate concrete situations for anticipating their future behaviour. Most models for the transmission of infectious diseases descend from the classical SIR model of Kermack and McKendrick established in 1927 [6]. Such model is called a compartmental model where the population is divided into compartments of susceptible, infected, and recovered. A non-linear ordinary differential

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equations are used to model the dynamics between these compartments.

A major criticism for this model and the models that followed (for example [7–15]), is that the total population is assumed to be entirely homogenous and all individuals behave the same. Therefore, the model may not represent complex mobility and contact patterns for many real-world diseases. To overcome this inconvenience, Calvo et al. [16] have incorporated population heterogeneity to examine interactions between urban and rural populations on the dynamics of disease spreading by using a compartmental framework of susceptible–infected–susceptible dynamics with some level of immunity. The proposed model is as follows:

$$
\begin{aligned}
\frac{dS_1}{dt} =& \mu_1 N_1 + \beta_1 \frac{I_1}{N_1} S_1 - \mu_1 S_1 + \delta_{21} S_2 - \delta_{12} S_1, \\
\frac{dI_1}{dt} =& \beta_1 \frac{I_1}{N_1} S_1 - (\mu_1 + \gamma_1) I_1 + \rho_1 \frac{I_1}{N_1} R_1 + \delta_{21} I_2 - \delta_{12} I_1, \\
\frac{dR_1}{dt} =& \gamma_1 I_1 - \rho_1 \frac{I_1}{N_1} R_1 - \mu_1 R_1 + \delta_{21} R_2 - \delta_{12} R_1, \\
\frac{dS_2}{dt} =& \mu_2 N_2 + \beta_2 \frac{I_2}{N_2} S_2 - \mu_2 S_2 + \delta_{12} S_1 - \delta_{21} S_2, \\
\frac{dI_2}{dt} =& \beta_2 \frac{I_2}{N_2} S_2 - (\mu_2 + \gamma_2) I_2 + \rho_2 \frac{I_2}{N_2} R_2 + \delta_{12} I_1 - \delta_{21} I_2, \\
\frac{dR_2}{dt} =& \gamma_2 I_2 - \rho_2 \frac{I_2}{N_2} R_2 - \mu_2 R_2 + \delta_{12} R_1 - \delta_{21} R_2.
\end{aligned}
\tag{1}
$$

The subscript 1 is used to denote the urban parameters and variables, and the subscript 2 for the rural parameters and variables. For $i \in \{1,2\}$, S_i, I_i, R_i and N_i denote the numbers of susceptible, infected, post-recovery susceptible individuals and the total population, respectively. The parameter μ_i is the rate of birth and death. β_i is the infection transmission coefficient between susceptible and infected individuals. The post-recovery susceptible individuals are infected at rate ρ_i, while infected individuals become post-recovery susceptibles at rate γ_i. The motion between urban and rural populations is modeled by the function $\delta_{ij}(t)$ which denotes the fraction of individuals who travel from patch $i \in \{1,2\}$ to patch $j \in \{1,2\}$ (with $i \neq j$) at time t. To study the dynamics of system (1), the authors of [16] compute steady states, showing the local stability of the disease-free steady state, and identify conditions for the existence of the endemic steady states.

In the model above, infectious diseases can spread through interactions between the urban and rural populations. Therefore, infected individuals in urban area can infect rural population and the rural infected can transmit the disease to the urban dwellers. In this paper, we assume that there is no immunity. From this perspective, we propose another version of model (1) by introducing four infection transmission coefficients β_i $(i = 1,2,3,4)$, presented as follows:

$$
\begin{cases}
dx_1 = & (A_1 - \beta_1 x_1 y_1 - \beta_3 x_1 y_2 - \mu_1 x_1)dt, \\
dy_1 = & (\beta_1 x_1 y_1 + \beta_4 x_2 y_1 - \mu_1 y_1)dt, \\
dx_2 = & (A_2 - \beta_2 x_2 y_2 - \beta_4 x_2 y_1 - \mu_2 x_2)dt, \\
dy_2 = & (\beta_2 x_2 y_2 + \beta_3 x_1 y_2 - \mu_2 y_2)dt.
\end{cases}
\tag{2}
$$

For the variables x_i and y_i $(i = 1,2)$, we use the subscript 1 to denote the urban variable and the subscript 2 for the rural one. All the other parameters appearing in model (2) are assumed to be constant positives. The symbols involved in the model are described in Table 1.

On the other hand, the spread of diseases is characterized by randomness due to the unpredictability of the natural behavior [17]. A lot of scholars have introduced the white noise into the deterministic models to reveal the effect of the environmental fluctuations on the spread of diseases. For example, Cao et al. [18] considered a stochastic SEI epidemic model with saturation incidence and logistic growth. By constructing a suitable Lyapunov function, they established sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the solutions to the model. They also established sufficient conditions for the extinction of the disease. In [19], Pang et al. discussed the dynamics of a stochastic SIQS epidemic model and investigated the boundedness, extinction and the persistence of the stochastic system. Khan et al. [20] proposed a stochastic model to analyze the dynamics of the novel coronavirus disease. They studied the extinction and the persistence of the disease. For more details on the impact of environmental fluctuations on the spread of diseases and population dynamics, we refer the readers to [21–39].

Based on the aforementioned facts, we substitute $\beta_i dt$ in model (2) by $\beta_i dt + \sigma_i dB_i(t)$, where $B_i(t)$ are mutually independent standard Brownian motions and $\sigma_i > 0$ are the intensities of their corresponding white noises, $i = 1,2,3,4$. All these Brownian Motions are defined on a filtered probability space $\left(\Omega, \mathcal{F}_\Omega, (\mathcal{F}_{\{t\}})_{t\geq 0}, \mathbb{P}\right)$ endowed with a filtration that meets the usual criteria. Thus, we get a stochastic version of the deterministic model (2), defined as follows:

$$\begin{aligned}
dx_1(t) =&(A_1 - \beta_1 x_1(t)y_1(t) - \beta_3 x_1(t)y_2(t) - \mu_1 x_1(t))dt - \sigma_1 x_1(t)y_1(t)dB_1(t) - \sigma_3 x_1(t)y_2(t)dB_3(t),\\
dy_1(t) =&(\beta_1 x_1(t)y_1(t) + \beta_4 x_2(t)y_1(t) - \mu_1 y_1(t))dt + \sigma_1 x_1(t)y_1(t)dB_1(t) + \sigma_4 x_2(t)y_1(t)dB_4(t),\\
dx_2(t) =&(A_2 - \beta_2 x_2(t)y_2(t) - \beta_4 x_2(t)y_1(t) - \mu_2 x_2(t))dt - \sigma_2 x_2(t)y_2(t)dB_2(t) - \sigma_4 x_2(t)y_1(t)dB_4(t),\\
dy_2(t) =&(\beta_2 x_2(t)y_2(t) + \beta_3 x_1(t)y_2(t) - \mu_2 y_2(t))dt + \sigma_2 x_2(t)y_2(t)dB_2(t) + \sigma_3 x_1(t)y_2(t)dB_3(t).
\end{aligned} \tag{3}$$

Here, we assume that the urban susceptibles contamined by the rural infected individuals stay in the rural infectd compartment and rural susceptibles contamined by urban infected people stay in the urban infected class.

For convenience, the abbreviation "a.s." means "almost surely", while $\langle f(t)\rangle = t^{-1}\int_0^t f(r)dr$ is the time average of a continuous function f. For two numbers a and b, the symbols $a\wedge b$ and $a\vee b$ stand for the minimum and the maximum of a and b, respectively.

The rest of the paper proceeds as follows. In the next section, we study the stability of the equilibrium state $E = (\frac{A_1}{\mu_1}, 0, \frac{A_2}{\mu_2}, 0)$ for the deterministic model (2). Section 3 is devoted to verify if the stochastic model (3) has a unique positive solution with probability one. In section 4, the conditions ensuring the exponential extinction of the disease in both patches are established. Afterwards, we will carry out an analysis leading to defining a threshold for the disease to persist completely. There remains a case where the disease persists in one patch, and disappears in the other, which is the main theme of the section 6. The paper ends with the realization of numerical simulations using the software Matlab 2015b.

2. Stability of the deterministic model

The aim of mathematical modeling of the spread of epidemics is to know the conditions under which the epidemic dies out. The deterministic model (2) has one free-disease equilibrium $E = (\frac{A_1}{\mu_1}, 0, \frac{A_2}{\mu_2}, 0)$.

The following theorem gives sufficient conditions for local and global asymptotic stability of the free-disease equilibrium E.

Theorem 1. *If* $\Big(\big(\beta_1\frac{A_1}{\mu_1} + \beta_4\frac{A_2}{\mu_2} - \mu_1\big) \vee \big(\beta_3\frac{A_1}{\mu_1} + \beta_2\frac{A_2}{\mu_2} - \mu_2\big)\Big) < 0$, *then the equilibrium E is locally asymptotically stable.*

Proof. The Jacobian matrix related to model (2) at the equilibrium E is

$$J(E) = \begin{pmatrix} -\mu_1 & -\beta_1\frac{A_1}{\mu_1} & 0 & -\beta_3\frac{A_1}{\mu_1}\\ 0 & \beta_1\frac{A_1}{\mu_1} - \mu_1 + \beta_4\frac{A_2}{\mu_2} & 0 & 0\\ 0 & -\beta_4\frac{A_2}{\mu_2} & -\mu_2 & -\beta_2\frac{A_2}{\mu_2}\\ 0 & 0 & 0 & \beta_2\frac{A_2}{\mu_2} - \mu_2 + \beta_3\frac{A_1}{\mu_1} \end{pmatrix}.$$

According to the Hurwitz criterion, if $\Big(\big(\beta_1\frac{A_1}{\mu_1} + \beta_4\frac{A_2}{\mu_2} - \mu_1\big) \vee \big(\beta_3\frac{A_1}{\mu_1} + \beta_2\frac{A_2}{\mu_2} - \mu_2\big)\Big) < 0$, then the eigenvalues of matrix $J(E)$ are all negatives.

Thus, the equilibrium state E is locally asymptotically stable. □

Theorem 2. *If* $\Big(\big((\beta_1+\beta_4)\frac{A_1+A_2}{\mu_1\wedge\mu_2} - \mu_1\big) \vee \big((\beta_2+\beta_3)\frac{A_1+A_2}{\mu_1\wedge\mu_2} - \mu_2\big)\Big) < 0$, *then the equilibrium E is globally asymptotically stable.*

Proof. Consider the Lyapunov function $\mathcal{V}$ defined by

$$\mathcal{V}(x_1(t), y_1(t), x_2(t), y_2(t)) = \frac{1}{2}(y_1^2 + y_2^2).$$

The derivative of $\mathcal{V}$ along the trajectories of solution of model (2) is as follows

$$\begin{aligned}
&\frac{d\mathcal{V}(x_1(t), y_1(t), x_2(t), y_2(t))}{dt}\\
&= y_1(t)(\beta_1 x_1(t)y_1(t) + \beta_4 x_2(t)y_1(t) - \mu_1 y_1(t))\\
&+ y_2(t)(\beta_2 x_2(t)y_2(t) + \beta_3 x_1(t)y_2(t)\\
&- \mu_2 y_2(t))\\
&= y_1^2(t)(\beta_1 x_1(t) + \beta_4 x_2(t) - \mu_1)\\
&+ y_2^2(t)(\beta_2 x_2(t) + \beta_3 x_1(t) - \mu_2)\\
&\le y_1^2(t)\Big((\beta_1+\beta_4)\frac{A_1+A_2}{\mu_1\wedge\mu_2} - \mu_1\Big)\\
&+ y_2^2(t)\Big((\beta_2+\beta_3)\frac{A_1+A_2}{\mu_1\wedge\mu_2} - \mu_2\Big).
\end{aligned}$$

Assuming that $\Big(\big((\beta_1+\beta_4)\frac{A_1+A_2}{\mu_1\wedge\mu_2} - \mu_1\big) \vee \big((\beta_2+\beta_3)\frac{A_1+A_2}{\mu_1\wedge\mu_2} - \mu_2\big)\Big) < 0$, we get

$$\frac{d\mathcal{V}(x_1(t), y_1(t), x_2(t), y_2(t))}{dt} < 0 \quad \text{for any } t \ge 0,$$

which means that E is globally asymptotically stable. □

Table 1. Description of symbols in model (2).

Parameter	Meaning
x_i	The number of susceptible individuals to the disease, where $i = 1, 2$.
y_i	The number of infective members.
A_i	A constant input of new members into the population i per unit time.
μ_i	Natural death rate of x_i and y_i.
β_1	Transmission coefficient between x_1 and y_1.
β_2	Transmission coefficient between x_2 and y_2.
β_3	Transmission coefficient between x_1 and y_2.
β_4	Transmission coefficient between x_2 and y_1.

3. Well-posedeness of the stochastic model

Lemma 1. *The set* $\Gamma = \Big\{(x_1(t), y_1(t), x_2(t), y_2(t)) \in \mathbb{R}^4_+ : N(t) = x_1(t)+y_1(t)+x_2(t)+y_2(t) \le \frac{A_1+A_2}{\mu_1\wedge\mu_2}\Big\}$ *is positively invariant for the stochastic model (3).*

Proof. From system (3), we have

$$\begin{aligned} dN(t) =&\big(A_1 + A_2 - \mu_1(x_1(t) + y_1(t)) \\ &- \mu_2(x_2(t) + y_2(t))\big)dt \\ \le&(A_1 + A_2 - (\mu_1 \wedge \mu_2)N(t))dt. \end{aligned} \tag{4}$$

Thus

$$N(t) \le \frac{A_1 + A_2}{\mu_1 \wedge \mu_2} + \Big(N(0) - \frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)e^{-(\mu_1\wedge\mu_2)t}.$$

If $N(0) \le \frac{A_1+A_2}{\mu_1\wedge\mu_2}$, then $N(t) \le \frac{A_1+A_2}{\mu_1\wedge\mu_2}$ for all $t > 0$. □

Theorem 3. *For any* $(x_1(0), y_1(0), x_2(0), y_2(0)) \in \Gamma$, *the stochastic system (3) is mathematically well-posed in the sense that it has a unique solution* $(x_1(t), y_1(t), x_2(t), y_2(t)) \in \Gamma$ *with probability one.*

Proof. The coefficients of system (3) are locally Lipschitz continuous, for any given initial value $(x_1(0), y_1(0), x_2(0), y_2(0))$, then there is a unique local solution $(x_1(t), y_1(t), x_2(t), y_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time.

Let $k_0 > 0$ such that $x_1(0), y_1(0), x_2(0), y_2(0) > k_0$. For $k \le k_0$, we consider the stopping times

$$\begin{aligned} \tau_k =&\inf\big\{t \in [0, \tau_e) : x_1(t) \le k \text{ or } y_1(t) \le k \\ &\text{or } x_2(t) \le k \text{ or } y_2(t) \le k\big\}, \\ \tau =&\lim_{k\to 0}\tau_k = \inf\big\{t \in [0, \tau_e) : x_1(t) \le 0 \text{ or } y_1(t) \le 0 \\ &\text{or } x_2(t) \le 0 \text{ or } y_2(t) \le 0\big\}. \end{aligned}$$

Let

$$\begin{aligned} V(X(t)) &= V((x_1(t), y_1(t), x_2(t), y_2(t))) \\ &= 4\ln\frac{A_1 + A_2}{\mu_1 \wedge \mu_2} - \ln(x_1(t)y_1(t)x_2(t)y_2(t)). \end{aligned}$$

Applying Itô formula on V, we obtain

$$\begin{aligned} dV =& -\frac{dx_1}{x_1} - \frac{dx_2}{x_2} - \frac{dy_1}{y_1} - \frac{dy_2}{y_2} + \Big(\frac{1}{2}\sigma_1^2y_1^2 + \frac{1}{2}\sigma_3^2y_2^2 \\ &+ \frac{1}{2}\sigma_2^2y_2^2 + \frac{1}{2}\sigma_4^2y_1^2 + \frac{1}{2}\sigma_1^2x_1^2 + \frac{1}{2}\sigma_4^2x_2^2 + \frac{1}{2}\sigma_2^2x_2^2 + \frac{1}{2}\sigma_3^2x_1^2\Big)dt \\ \le&\Big[-\frac{A_1}{x_1} + \beta_1y_1 + \beta_3y_2 + \mu_1 - \beta_1x_1 - \beta_4x_2 + \mu_1 - \frac{A_2}{x_2} \\ &+ \beta_2y_2 + \beta_4y_1 + \mu_2 - \beta_2x_2 - \beta_3x_1 + \mu_2 + \Big(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \\ &+ \sigma_4^2\Big)\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2\Big]dt + \sigma_1(y_1 - x_1)dB_1 + \sigma_2(y_2 - x_2)dB_2 \\ &+ \sigma_3(y_2 - x_1)dB_3 + \sigma_4(y_1 - x_2)dB_4 \\ \le&\Big[2\mu_1 + 2\mu_2 + \Big(\beta_1 + \beta_2 + \beta_3 + \beta_4\Big)\frac{A_1 + A_2}{\mu_1 \wedge \mu_2} + \Big(\sigma_1^2 + \sigma_2^2 \\ &+ \sigma_3^2 + \sigma_4^2\Big)\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2\Big]dt + \sigma_1(y_1 - x_1)dB_1 \\ &+ \sigma_2(y_2 - x_2)dB_2 + \sigma_3(y_2 - x_1)dB_3 + \sigma_4(y_1 - x_2)dB_4. \end{aligned}$$

Integrating the both sides of the inequality above and then taking the expectation give

$$\mathbb{E}[V(X(t))] \le \lambda t + V(X(0)), \tag{5}$$

where

$$\begin{aligned} \lambda =& 2\mu_1 + 2\mu_2 + \Big(\beta_1 + \beta_2 + \beta_3 + \beta_4\Big)\frac{A_1 + A_2}{\mu_1 \wedge \mu_2} \\ &+ \Big(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2\Big)\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2. \end{aligned}$$

Using the stopping time τ_k, one has

$$\begin{aligned}\mathbb{E}[V(X(t\wedge\tau_k))] =&\mathbb{E}[V(X(t\wedge\tau_k))\,\mathbb{I}_{(\tau_k\le t)}]\\ &+\mathbb{E}[V(X(t\wedge\tau_k))\,\mathbb{I}_{(\tau_k>t)}]\\ \ge&\mathbb{E}[V(X(\tau_k))\,\mathbb{I}_{(\tau_k\le t)}],\end{aligned}$$

where $\mathbb{I}_A$ is the characteristic function of A. Notice that there is some component of $X(\tau_k)$ equals to k. Therefore

$$V(X(\tau_k)) \ge \ln\left(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\,\frac{1}{k}\right).$$

As a result, we have

$$\mathbb{E}[V(X(t\wedge\tau_k))] \ge \ln\left(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\,\frac{1}{k}\right)\times\mathbb{P}\Big(\tau_k\le t\Big).$$

Together with (5), we get

$$\mathbb{P}\Big(\tau_k\le t\Big)\le\frac{\lambda t+V(X(0))}{\ln\left(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\,\frac{1}{k}\right)}.$$

If we let $k\to 0$, we obtain for all $t\ge 0$: $\mathbb{P}\Big(\tau\le t\Big)=0$.
Hence

$$\mathbb{P}\Big(\tau=\infty\Big)=1.$$

As $\tau_e\ge\tau$, then $\tau_e=\infty$ a.s.
Finally, the solution is global. □

4. Decline of the disease

Theorem 4. *Let $(x_1(t),y_1(t),x_2(t),y_2(t))$ be the solution of system (3) with any initial value $(x_1(0),y_1(0),x_2(0),y_2(0))\in\Gamma$.*

(1) *If $\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_4^2}{2\sigma_4^2}<\mu_1$, then $\lim_{t\to\infty}y_1(t)=0$ a.s.*
(2) *If $\frac{\beta_2^2}{2\sigma_2^2}+\frac{\beta_3^2}{2\sigma_3^2}<\mu_2$, then $\lim_{t\to\infty}y_2(t)=0$ a.s.*

Proof. 1. Applying Itô formula to system (3), we get

$$\begin{aligned}d\ln y_1(t)=&\frac{1}{y_1(t)}dy_1(t)-\frac{1}{2}\frac{1}{y_1^2}(dy_1(t))^2\\ =&\Big(\beta_1x_1(t)+\beta_4x_2(t)-\mu_1-\frac{\sigma_1^2}{2}x_1^2(t)\\ &-\frac{\sigma_4^2}{2}x_2^2(t)\Big)dt+\sigma_1x_1(t)dB_1(t)\\ &+\sigma_4x_2(t)dB_2(t).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{t}\ln\frac{y_1(t)}{y_1(0)}=&\beta_1\langle x_1(t)\rangle+\beta_4\langle x_2(t)\rangle-\mu_1-\frac{\sigma_1^2}{2}\langle x_1^2(t)\rangle\\ &-\frac{\sigma_4^2}{2}\langle x_2^2(t)\rangle+\frac{M_1(t)}{t}\\ \le&\beta_1\langle x_1(t)\rangle+\beta_4\langle x_2(t)\rangle-\mu_1-\frac{\sigma_1^2}{2}\langle x_1(t)\rangle^2\\ &-\frac{\sigma_4^2}{2}\langle x_2(t)\rangle^2+\frac{M_1(t)}{t}\\ =&-\frac{\sigma_1^2}{2}\langle x_1(t)\rangle^2+\beta_1\langle x_1(t)\rangle-\frac{\sigma_4^2}{2}\langle x_2(t)\rangle^2\\ &+\beta_4\langle x_2(t)\rangle-\mu_1+\frac{M_1(t)}{t}\\ =&-\frac{\sigma_1^2}{2}\Big(\langle x_1(t)\rangle^2-2\frac{\beta_1}{\sigma_1^2}\langle x_1(t)\rangle\Big)\\ &-\frac{\sigma_4^2}{2}\Big(\langle x_2(t)\rangle^2-2\frac{\beta_4}{\sigma_4^2}\langle x_2(t)\rangle\Big)-\mu_1\\ &+\frac{M_1(t)}{t}\\ =&-\frac{\sigma_1^2}{2}\Big(\langle x_1(t)\rangle-\frac{\beta_1}{\sigma_1^2}\Big)^2+\frac{\beta_1^2}{2\sigma_1^2}\\ &-\frac{\sigma_4^2}{2}\Big(\langle x_2(t)\rangle-\frac{\beta_4}{\sigma_4^2}\Big)^2+\frac{\beta_4^2}{2\sigma_4^2}-\mu_1\\ &+\frac{M_1(t)}{t} \qquad (6)\\ \le&\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_4^2}{2\sigma_4^2}-\mu_1+\frac{M_1(t)}{t},\end{aligned}$$

where

$$M_1(t)=\sigma_1\int_0^t x_1(r)dB_1(r)+\sigma_4\int_0^t x_2(r)dB_2(r).$$

Bearing in mind the strong law of large numbers for martingales, we obtain

$$\limsup_{t\to\infty}\frac{\ln y_1(t)}{t}\le\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_4^2}{2\sigma_4^2}-\mu_1 \quad \text{a.s.}$$

Since $\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_4^2}{2\sigma_4^2}<\mu_1$, then $\limsup_{t\to\infty}\frac{\ln y_1(t)}{t}<0$ a.s.,
which implies

$$\lim_{t\to\infty}y_1(t)=0 \quad \text{a.s.}$$

2. Similarly, we get : $\lim_{t\to\infty}y_2(t)=0$ a.s., under the condition $\frac{\beta_2^2}{2\sigma_2^2}+\frac{\beta_3^2}{2\sigma_3^2}<\mu_2$.

This completes the proof of Theorem 4. □

Theorem 5. *Let $(x_1(t), y_1(t), x_2(t), y_2(t))$ be the solution of system (3) with any initial value $(x_1(0), y_1(0), x_2(0), y_2(0)) \in \Gamma$.*

(1) *If* $\frac{A_1+A_2}{\mu_1} \leq \frac{\beta_1}{\sigma_1^2}$, $\frac{A_1+A_2}{\mu_2} \leq \frac{\beta_4}{\sigma_4^2}$ *and* $\beta_1\frac{A_1+A_2}{\mu_1} + \beta_4\frac{A_1+A_2}{\mu_2} - \frac{\sigma_1^2}{2}\left(\frac{A_1+A_2}{\mu_1}\right)^2 - \frac{\sigma_4^2}{2}\left(\frac{A_1+A_2}{\mu_2}\right)^2 < \mu_1$, *then*

$$\lim_{t\to\infty} y_1(t) = 0 \quad a.s.$$

(2) *If* $\frac{A_1+A_2}{\mu_1} \leq \frac{\beta_3}{\sigma_3^2}$, $\frac{A_1+A_2}{\mu_2} \leq \frac{\beta_2}{\sigma_2^2}$ *and* $\beta_3\frac{A_1+A_2}{\mu_1} + \beta_2\frac{A_1+A_2}{\mu_2} - \frac{\sigma_3^2}{2}\left(\frac{A_1+A_2}{\mu_1}\right)^2 - \frac{\sigma_2^2}{2}\left(\frac{A_1+A_2}{\mu_2}\right)^2 < \mu_2$, *then*

$$\lim_{t\to\infty} y_2(t) = 0 \quad a.s.$$

Before giving the proof of Theorem 5, we will present the two following Lemmas.

Lemma 2. *Let $(x_1(t), y_1(t), x_2(t), y_2(t))$ be the solution of system (3). We have*

$$\lim_{t\to\infty} \frac{x_1(t)+y_1(t)+x_2(t)+y_2(t)}{t} = 0 \quad a.s.$$

Proof. Let $N(t) = x_1(t) + y_1(t) + x_2(t) + y_2(t)$.

From system (3), one has

$$\begin{aligned} dN(t) = &(A_1 + A_2 - \mu_1 (x_1(t) + y_1(t))) \\ &- (\mu_2 (x_2(t) - y_2(t)))\, dt. \end{aligned} \tag{7}$$

Then

$$\begin{aligned} &\Big(A_1 + A_2 - (\mu_1 \vee \mu_2)N(t)\Big)dt \leq dN(t) \\ &\leq \Big(A_1 + A_2 - (\mu_1 \wedge \mu_2)N(t)\Big)dt. \end{aligned} \tag{8}$$

Thus

$$\frac{A_1+A_2}{\mu_1 \vee \mu_2} + \Big(N(0) - \frac{A_1+A_2}{\mu_1 \vee \mu_2}\Big)e^{-(\mu_1\vee\mu_2)t} \leq N(t),$$

and

$$N(t) \leq \frac{A_1+A_2}{\mu_1 \wedge \mu_2} + \Big(N(0) - \frac{A_1+A_2}{\mu_1 \wedge \mu_2}\Big)e^{-(\mu_1\wedge\mu_2)t}.$$

Hence

$$\lim_{t\to\infty} \frac{x_1(t)+y_1(t)+x_2(t)+y_2(t)}{t} = 0 \quad \text{a.s.}$$

The proof is complete. □

Lemma 3. *Let $(x_1(t), y_1(t), x_2(t), y_2(t))$ be the solution of system (3). Then*

$$\limsup_{t\to\infty} \langle x_1(t)\rangle \leq \frac{A_1+A_2}{\mu_1} \quad a.s.,$$
$$\limsup_{t\to\infty} \langle x_2(t)\rangle \leq \frac{A_1+A_2}{\mu_2} \quad a.s.$$

Proof. By (7), we obtain

$$\langle x_1(t)\rangle \leq \frac{A_1+A_2}{\mu_1} - \frac{\phi(t)}{\mu_1},$$
$$\langle x_2(t)\rangle \leq \frac{A_1+A_2}{\mu_2} - \frac{\phi(t)}{\mu_2},$$

where

$$\begin{aligned} \phi(t) = &\frac{x_1(t)+y_1(t)+x_2(t)+y_2(t)}{t} \\ &- \frac{x_1(0)+y_1(0)+x_2(0)+y_2(0)}{t}. \end{aligned}$$

Bearing in mind Lemma 2, we get the seeked results. □

Proof of Theorem 5. 1. By Lemma 3, there is $T_1 > 0$ such that, for any $t \geq T_1$,

$$\langle x_1(t)\rangle \leq \frac{A_1+A_2}{\mu_1} \quad \text{and} \quad \langle x_2(t)\rangle \leq \frac{A_1+A_2}{\mu_2}.$$

For all $t \geq T_1$, we assume that

$$\langle x_1(t)\rangle \leq \frac{A_1+A_2}{\mu_1} \leq \frac{\beta_1}{\sigma_1^2},$$

and

$$\langle x_2(t)\rangle \leq \frac{A_1+A_2}{\mu_2} \leq \frac{\beta_4}{\sigma_4^2}.$$

Together with (6), we have

$$\begin{aligned} \frac{1}{t}\ln\frac{y_1(t)}{y_1(0)} \leq &-\frac{{\sigma_1}^2}{2}\Big(\frac{A_1+A_2}{\mu_1} - \frac{\beta_1}{\sigma_1^2}\Big)^2 + \frac{\beta_1^2}{2\sigma_1^2} \\ &- \frac{\sigma_4^2}{2}\Big(\frac{A_1+A_2}{\mu_2} - \frac{\beta_4}{\sigma_4^2}\Big)^2 + \frac{\beta_4^2}{2\,\sigma_4^2} - \mu_1 \\ =&\beta_1\frac{A_1+A_2}{\mu_1} + \beta_4\frac{A_1+A_2}{\mu_2} - \frac{\sigma_1^2}{2}\Big(\frac{A_1+A_2}{\mu_1}\Big)^2 \\ &- \frac{\sigma_4^2}{2}\Big(\frac{A_1+A_2}{\mu_2}\Big)^2 - \mu_1. \end{aligned}$$

Since

$$\beta_1\frac{A_1+A_2}{\mu_1}+\beta_4\frac{A_1+A_2}{\mu_2}-\frac{\sigma_1^2}{2}\Big(\frac{A_1+A_2}{\mu_1}\Big)^2 -\frac{\sigma_4^2}{2}\Big(\frac{A_1+A_2}{\mu_2}\Big)^2-\mu_1<0,$$

then

$$\limsup_{t\to\infty}\frac{\ln y_1(t)}{t}<0 \quad \text{a.s.}$$

Consequently

$$\lim_{t\to\infty} y_1(t)=0 \quad \text{a.s.}$$

2. Following the same method above, we get

$$\lim_{t\to\infty} y_2(t)=0 \quad \text{a.s.}$$

□

5. Disease prevalence

Theorem 6. *Let $(x_1(t),y_1(t),x_2(t),y_2(t))$ be the solution of system (3) with any initial value $(x_1(0),y_1(0),x_2(0),y_2(0))\in\Gamma$.*

(1) *If $(\beta_1\wedge\beta_4)\frac{A_1+A_2}{\mu_1\vee\mu_2}-\frac{\sigma_1^2+\sigma_4^2}{2}\Big(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\Big)^2 > (\beta_1\wedge\beta_4)+\mu_1$, then: $\liminf_{t\to\infty}\langle y_1(t)\rangle>0$ a.s.*

(2) *If $(\beta_2\wedge\beta_3)\frac{A_1+A_2}{\mu_1\vee\mu_2}-\frac{\sigma_2^2+\sigma_3^2}{2}\Big(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\Big)^2 > (\beta_2\wedge\beta_3)+\mu_2$, then: $\liminf_{t\to\infty}\langle y_2(t)\rangle>0$ a.s.*

Proof. 1. From (7), we have

$$(\mu_1\vee\mu_2)\langle x_1(t)+x_2(t)\rangle \ge A_1+A_2-\mu_1\langle y_1(t)\rangle -\mu_2\frac{A_1+A_2}{\mu_1\wedge\mu_2}-\phi(t).$$

Then, one can get

$$\begin{aligned}\langle x_1(t)+x_2(t)\rangle &\ge \frac{A_1+A_2}{\mu_1\vee\mu_2}-\frac{\mu_1}{\mu_1\vee\mu_2}\langle y_1(t)\rangle -\frac{\mu_2}{\mu_1\vee\mu_2}\frac{A_1+A_2}{\mu_1\wedge\mu_2}-\frac{\phi(t)}{\mu_1\vee\mu_2}\\ &\ge \frac{A_1+A_2}{\mu_1\vee\mu_2}-\langle y_1(t)\rangle-\frac{A_1+A_2}{\mu_1\wedge\mu_2} -\frac{\phi(t)}{\mu_1\vee\mu_2}.\end{aligned} \tag{9}$$

On the other hand, one can have

$$\frac{1}{t}\ln\frac{y_1(t)}{y_1(0)} \ge (\beta_1\wedge\beta_4)\langle x_1(t)+x_2(t)\rangle-\mu_1 -\frac{\sigma_1^2+\sigma_4^2}{2}\Big(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\Big)^2+\frac{M_1(t)}{t}. \tag{10}$$

Combining (9) and (10) yields

$$\begin{aligned}\frac{1}{t}\ln\frac{y_1(t)}{y_1(0)} &\ge (\beta_1\wedge\beta_4)\frac{A_1+A_2}{\mu_1\vee\mu_2}-(\beta_1\wedge\beta_4)\langle y_1(t)\rangle\\ &-(\beta_1\wedge\beta_4)\frac{A_1+A_2}{\mu_1\wedge\mu_2}-\frac{\beta_1\wedge\beta_4}{\mu_1\vee\mu_2}\phi(t)-\mu_1\\ &-\frac{\sigma_1^2+\sigma_4^2}{2}\Big(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\Big)^2+\frac{M_1(t)}{t}.\end{aligned}$$

As a consequence, we get

$$\begin{aligned}\liminf_{t\to\infty}(\beta_1\wedge\beta_4)\langle y_1(t)\rangle \ge&(\beta_1\wedge\beta_4)\frac{A_1+A_2}{\mu_1\vee\mu_2}\\ &-\frac{\sigma_1^2+\sigma_4^2}{2}\Big(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\Big)^2\\ &-\Big((\beta_1\wedge\beta_4)+\mu_1\Big) \quad \text{a.s.}\end{aligned}$$

Immediately, under the condition stated in the first part of Theorem 6, we deduce that

$$\liminf_{t\to\infty}\langle y_1(t)\rangle>0 \quad \text{a.s.}$$

2. Concerning the second part of Theorem 6, we get the desired result using the above method.

□

6. Simultaneous extinction and persistence

Theorem 7. *Let $(x_1(t),y_1(t),x_2(t),y_2(t))$ be the solution of system (3) with any initial value $(x_1(0),y_1(0),x_2(0),y_2(0))\in\Gamma$.*

(1) *If $(\beta_2\vee\beta_3)\frac{A_1+A_2}{\mu_1\vee\mu_2}-\frac{\sigma_2^2+\sigma_3^2}{2}\big(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\big)^2 > \mu_2$ and $\lim_{t\to\infty} y_1(t)=0$ a.s., then: $\liminf_{t\to\infty}\langle y_2(t)\rangle>0$ a.s.*

(2) *If $(\beta_1\vee\beta_4)\frac{A_1+A_2}{\mu_1\vee\mu_2}-\frac{\sigma_1^2+\sigma_4^2}{2}\big(\frac{A_1+A_2}{\mu_1\wedge\mu_2}\big)^2 > \mu_1$ and $\lim_{t\to\infty} y_2(t)=0$ a.s., then: $\liminf_{t\to\infty}\langle y_1(t)\rangle>0$ a.s.*

Proof. 1. In the case of the extinction of urban infected, we have: $\lim_{t\to\infty} y_1(t) = 0$ a.s.

Then, for any $\epsilon > 0$, there exist$T_2 > 0$ such that: $y_1(t) \le \epsilon$ for all $t \ge T_2$.

Together with (7), one can get

$$\begin{aligned}\phi(t) &\ge A_1 + A_2 - (\mu_1 \vee \mu_2)\langle x_1(t) + x_2(t)\rangle \\ &- \mu_1\langle y_1(t)\rangle - \mu_2\langle y_2(t)\rangle \\ &= A_1 + A_2 - (\mu_1 \vee \mu_2)\langle x_1(t) + x_2(t)\rangle \\ &- \mu_1\frac{1}{t}\int_0^{T_2} y_1(r)dr - \mu_1\frac{1}{t}\int_{T_2}^t y_1(r)dr \\ \mu_2\langle y_2(t)\rangle &\ge A_1 + A_2 - (\mu_1 \vee \mu_2)\langle x_1(t) + x_2(t)\rangle \\ &- \frac{T_2}{t}\mu_1 \sup_{r\in[0,T_2]} y_1(r) - \mu_1(1 - \frac{T_2}{t})\epsilon - \mu_1\langle y_2(t)\rangle.\end{aligned}$$

Then

$$\begin{aligned}\langle x_1(t) + x_2(t)\rangle \ge &\frac{A_1 + A_2}{\mu_1 \vee \mu_2} - \frac{T_2}{t}\sup_{r\in[0,T_2]} y_1(r) \\ &- \epsilon - \langle y_2(t)\rangle - \frac{\phi(t)}{\mu_1 \vee \mu_2}. \quad (11)\end{aligned}$$

Now, we apply Itô formula on system (3) to obtain

$$\begin{aligned}\frac{1}{t}\ln\frac{y_2(t)}{y_2(0)} \ge &(\beta_2 \wedge \beta_3)\langle x_1(t) + x_2(t)\rangle - \mu_2 \\ &- \frac{\sigma_2^2 + \sigma_3^2}{2}\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2 + \frac{M_2(t)}{t}, \quad (12)\end{aligned}$$

where

$$M_2(t) = \sigma_2\int_0^t x_2(r)dB_2(r) + \sigma_3\int_0^t x_1(r)dB_3(r).$$

Injecting (11) on (12) gives

$$\begin{aligned}\frac{1}{t}\ln\frac{y_2(t)}{y_2(0)} \ge &(\beta_2 \wedge \beta_3)\frac{A_1 + A_2}{\mu_1 \vee \mu_2} \\ &- (\beta_2 \wedge \beta_3)\frac{1}{t}\sup_{r\in[0,T_2]} y_1(r) \\ &- (\beta_2 \wedge \beta_3)\langle y_2(t)\rangle - (\beta_2 \wedge \beta_3)\epsilon \\ &- \frac{\beta_2 \wedge \beta_3}{\mu_1 \vee \mu_2}\phi(t) - \mu_2 \\ &- \frac{\sigma_2^2 + \sigma_3^2}{2}\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2 + \frac{M_2(t)}{t}.\end{aligned}$$

According to Lemma 2, we can have

$$\begin{aligned}\liminf_{t\to\infty}(\beta_2 \wedge \beta_3)\langle y_2(t)\rangle \ge &(\beta_2 \wedge \beta_3)\frac{A_1 + A_2}{\mu_1 \vee \mu_2} - \mu_2 \\ &- \frac{\sigma_2^2 + \sigma_3^2}{2}\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2 \quad \text{a.s.}\end{aligned}$$

2. Similarly, we get

$$\begin{aligned}\liminf_{t\to\infty}(\beta_1 \wedge \beta_4)\langle y_1(t)\rangle \ge &(\beta_1 \wedge \beta_4)\frac{A_1 + A_2}{\mu_1 \vee \mu_2} - \mu_1 \\ &- \frac{\sigma_1^2 + \sigma_4^2}{2}\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2 \quad \text{a.s.}\end{aligned}$$

The proof is complete.

□

7. Numerical results

The main goal of this section is to perform a numerical verification of the results obtained in the previous sections. First of all, we choose the initial value as $(x_1(0), y_1(0), x_2(0), y_2(0)) = (0.5, 0.7, 0.4, 0.9)$. The other parameters values are summarized in Table 2 split into 8 tests.

7.1. Deterministic stability

Based on the values of Test 0, we have

$$\Big(\big((\beta_1 + \beta_4)\frac{A_1 + A_2}{\mu_1 \wedge \mu_2} - \mu_1\big) \vee \big((\beta_2 + \beta_3)\frac{A_1 + A_2}{\mu_1 \wedge \mu_2} - \mu_2\big)\Big) = -0.41477.$$

According to Theorem 2, the equilibrium $E = (0.2308, 0, 0.1667, 0)$ is globally asymptotically stable which is depicted in Figure 1.

7.2. Stochastic extinction of the epidemic

By considering the values of Test 1, we have the following calculation

$$\frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_4^2}{2\sigma_4^2} - \mu_1 = -0.0036,$$

and

$$\frac{\beta_2^2}{2\sigma_2^2} + \frac{\beta_3^2}{2\sigma_3^2} - \mu_2 = -0.1375.$$

From Figure 2, we observe that: $\lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} y_2(t) = 0$, which conform to the Theorem 4. Second, we obtain for Test 2

$$\frac{A_1 + A_2}{\mu_1} - \frac{\beta_1}{\sigma_1^2} = -1.3626, \quad \frac{A_1 + A_2}{\mu_2} - \frac{\beta_4}{\sigma_4^2} = -2.441$$

and

$$\beta_1 \frac{A_1 + A_2}{\mu_1} + \beta_4 \frac{A_1 + A_2}{\mu_2} - \frac{\sigma_1^2}{2}\Big(\frac{A_1 + A_2}{\mu_1}\Big)^2 - \frac{\sigma_4^2}{2}\Big(\frac{A_1 + A_2}{\mu_2}\Big)^2 - \mu_1 = -0.017.$$

According to Theorem 5, $y_1(t)$ converges exponentially to zero (see Figure 2).

Next, based on the parameters values for Test 3, the numerical values are

$$\frac{A_1 + A_2}{\mu_1} - \frac{\beta_3}{\sigma_3^2} = -0.9437, \quad \frac{A_1 + A_2}{\mu_2} - \frac{\beta_2}{\sigma_2^2} = -2.2292$$

and

$$\beta_3 \frac{A_1 + A_2}{\mu_1} + \beta_2 \frac{A_1 + A_2}{\mu_2} \frac{\sigma_3^2}{2}\Big(\frac{A_1 + A_2}{\mu_1}\Big)^2 - \frac{\sigma_2^2}{2}\Big(\frac{A_1 + A_2}{\mu_2}\Big)^2 - \mu_2 = -0.0799.$$

From Figure 2, we see that $y_2(t)$ tends to zero, which agrees with Theorem 5.

7.3. Stochastic persistence of the epidemic

We choose values of Test 4 and Test 5 to get the following

$$(\beta_1 \wedge \beta_4)\frac{A_1 + A_2}{\mu_1 \vee \mu_2} - \frac{\sigma_1^2 + \sigma_4^2}{2}\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2 - (\beta_1 \wedge \beta_4) - \mu_1 = 0.1589$$

and

$$(\beta_2 \wedge \beta_3)\frac{A_1 + A_2}{\mu_1 \vee \mu_2} - \frac{\sigma_2^2 + \sigma_3^2}{2}\Big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\Big)^2 - (\beta_2 \wedge \beta_3) - \mu_2 = 0.1489.$$

By virtue of Theorem 6, the epidemic will be persistent in both urban and rural areas (see Figure 3).

7.4. Simultaneous extinction and persistence

Case 1. We have already considered that $\lim_{t\to\infty} y_1(t) = 0$. From values of Test 6, we obtain

$$(\beta_2 \vee \beta_3)\frac{A_1 + A_2}{\mu_1 \vee \mu_2} - \frac{\sigma_2^2 + \sigma_3^2}{2}\,\big(\frac{A_1 + A_2}{\mu_1 \wedge \mu_2}\big)^2 - \mu_2 = 0.0631.$$

Therefore, Theorem 7 yields

$$\liminf_{t\to\infty} \langle y_2(t)\rangle > 0,$$

which is well confirmed by Figure 4.

Case 2. Based on the values of Test 7 and , we get that

$$(\beta_1 \vee \beta_4)\tfrac{A_1+A_2}{\mu_1\vee\mu_2} - \tfrac{\sigma_1^2+\sigma_4^2}{2}\,\big(\tfrac{A_1+A_2}{\mu_1\wedge\mu_2}\big)^2 - \mu_1 = 0.0528.$$

If we consider $\lim_{t\to\infty} y_2(t) = 0$, then Theorem 7 implies that: $\liminf_{t\to\infty} \langle y_1(t)\rangle > 0$.

Therefore, Figure 4 reflects perfectly the statement of Theorem 7.

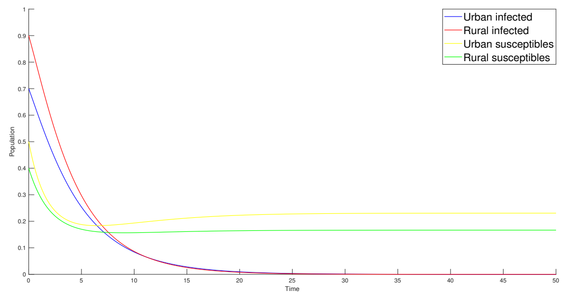

Figure 1. Computer simulation of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ for model (2), corresponding to Test 0.

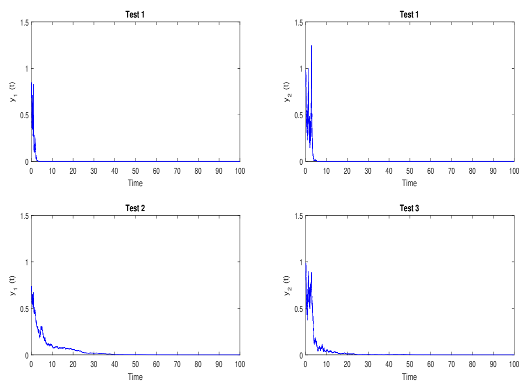

Figure 2. The paths of $y_1(t)$ and $y_2(t)$ for model (3), corresponding to Test 1, Test 2 and Test 3.

Table 2. Parameters values.

Parameters	Test 0	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7
A_1	0.06	0.3	0.06	0.04	0.1	0.1	0.06	0.04
A_2	0.05	0.25	0.05	0.035	0.1	0.2	0.05	0.035
μ_1	0.26	0.1	0.26	0.2	0.08	0.08	0.26	0.2
μ_2	0.3	0.2	0.1	0.2	0.09	0.09	0.1	0.2
β_1	0.14	0.2	0.14	—	0.5	—	0.14	0.6
β_2	0.1	0.15	0.1	0.15	—	0.5	0.5	0.15
β_3	0.2	0.2	0.2	0.211	—	0.4	0.4	0.211
β_4	0.08	0.3	0.211	—	0.4	—	0.211	0.7
σ_1	—	0.7	0.28	—	0.2	—	0.28	0.28
σ_2	—	0.6	—	0.24	—	0.2	0.2	0.24
σ_3	—	0.8	—	0.4	—	0.2	0.2	0.4
σ_4	—	0.9	0.244	—	0.2	—	0.244	0.244

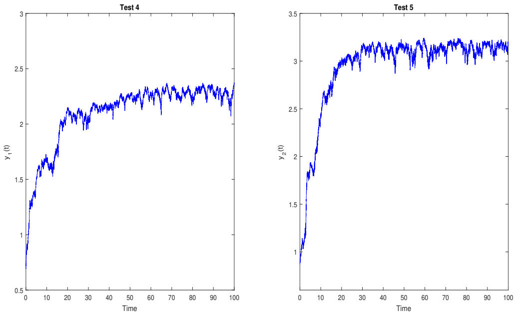

Figure 3. The paths of $y_1(t)$ and $y_2(t)$ for model (3), corresponding to Test 4 and Test 5.

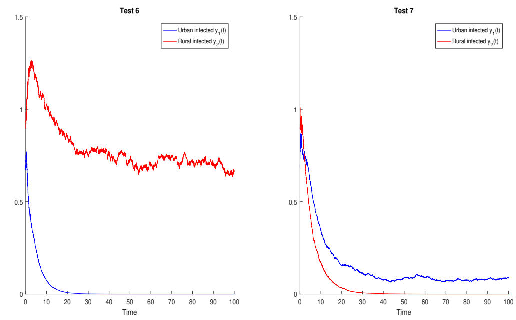

Figure 4. The paths of $y_1(t)$ and $y_2(t)$ for model (3), corresponding to Test 6 and Test 7.

8. Conclusion

In this paper, we elucidate the dynamics of disease transmission between two groups from distinct regions, operating under the assumption of comprehensive and unrestricted interaction. We consider both a deterministic two-patch epidemic model and its stochastic counterpart. For the deterministic model (2), we examine the global asymptotic stability of the equilibrium $E = (\frac{A_1}{\mu_1}, 0, \frac{A_2}{\mu_2}, 0)$. This result is illustrated in Figure 1. Regarding the stochastic version of model (2), we demonstrate the uniqueness of a positive solution for model (3). The thresholds that determine whether the disease will disappear are identified, as detailed in Theorems 4 and 5. Additionally, in Theorem 6, we establish conditions ensuring disease persistence. We also highlight a third scenario, distinct from those studied in sections 4 and 5, where the disease persists in one patch while simultaneously disappearing in the other. The accuracy of our theoretical findings is validated in the numerical simulation section.

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