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Control of a hydraulic system by means of a fuzzy approach

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Abstract. Non linear models can be represented conveniently by Takagi-Sugeno fuzzy models when nonlinearities are bounded. This approach uses a collection of linear models which are interpolated by non linear functions. Then the global control law is the interpolation by the same functions of each feedback associated to each linear model. A Lyapunov approach enables to compute these feedback gains. The number of linear models depends directly on the number of nonlinearities the system has. The more models there are, the more difficult it is to guarantee the stability of the closed loop. This paper proposes a method to reduce the number of linear models by assuming a number of nonlinearities considered as uncertainties and to guarantee the global exponential stability of the system. This method is applied on a hydraulic system.

Keywords: Takagi Sugeno fuzzy models; stabilization; hydraulic system; LMIs.
AMS Classification: 93C95, 93C42, 93C10, 93D05

1. Introduction

Takagi-Sugeno fuzzy models enable to represent precisely a wide class of nonlinear systems in a bounded domain of state variables. In brief, each nonlinearity is replaced by two linear models. Then these latters are interpolated by nonlinear functions to get back the initial nonlinear system [1]. To guarantee the stability of the closed loop, a PDC (Parallel Distributed and Compensation) control law is often used: it assumes to link each linear model to a linear feedback. Using a Lyapunov approach, with a quadratic function, the whole control problem can be casted as Linear Matrix Inequalities problem (LMI) [1-3]. We can remark that if the interpolating functions depend on non-measurable outputs, the problem is much more complex. In the opposite case, the observer and the controller can be designed separately [4].

Since the number of linear models is equal to 2^n with n the number of nonlinearities, it is clear that the more nonlinearities the system has, the more difficult it is to get the feedback gains because the number of LMI conditions will be so important that the solvers do not find a solution. For example in [5], there was a hydraulic system aimed at mixing two liquids into one tank. There were three nonlinearities, and feedbacks gains could be computed to guarantee the stability of the closed loop. However this approach was applied to a system based on three liquids to mix, but cannot be applied on a system based on four nonlinearities and the stability of the closed loop cannot be proved with the same LMI problem as previously.

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This paper proposes to interpret some nonlinearities of the model as uncertainties and to guarantee the global exponential stability of the system.

However, the number of linear models is reduced to a half each time a non-linearity is suppressed. We hope to get a solution for that.

We will also show that this approach gives interesting results for our hydraulic system.

Section two recalls some basic principles of Takagi-Sugeno fuzzy models. Basic conditions to guarantee the stability of the closed loop will be evoked, as well as a few matrix properties used in the rest of the paper.

Section three presents the control problem when uncertainties in the model are taken into account.

Finally, section four exposes the results about our hydraulic system.

2. Fuzzy models stabilization

2.1 Fuzzy model

We begin by recalling what a Takagi-Sugeno (TS) model [2,6,7] is. Let be a nonlinear system:

$$\dot{x} = f(x) + g(x)u \; ; \quad y = Cx \tag{1}$$

A Takagi-Sugeno (TS) model is obtained, if it's possible, to find r partial models of the type:

$$\dot{x} = A_i x + B_i u \; ; \quad y = C_i x \tag{2}$$

and r functions $h_i(x)$, with the constraints

$$h_i(x) \geq 0 \text{ and } \sum_{i=1}^{r} h_i(x) = 1 \tag{3}$$

such that (1) is equal to :

$$\dot{x} = A_z x + B_z u \quad y = C_z x \tag{4}$$

With $A_z = \sum_{i=1}^{r} h_i(x) A_i$, $B_z = \sum_{i=1}^{r} h_i(x) B_i$ and

$$C_z = \sum_{i=1}^{r} h_i(x) C_i \tag{5}$$

Let us study the case of a single non linearity. To obtain a TS fuzzy model from a nonlinear model, we can use the following properties for every bounded function $f(x) \in \left[\underline{f} \quad \overline{f}\right]$:

If we put $h_1(x) = \dfrac{\overline{f} - f(x)}{\overline{f} - \underline{f}}$ and

$h_2(x) = \dfrac{f(x) - \underline{f}}{\overline{f} - \underline{f}}$ with $h_1(x) > 0$, $h_2(x) > 0$ and $h_1(x) + h_2(x) = 1$ we have :

$$f(x) = \frac{f(x) - \underline{f}}{\overline{f} - \underline{f}} \overline{f} + \frac{\overline{f} - f(x)}{\overline{f} - \underline{f}} \underline{f} \tag{6}$$

Hence, we obtain $r = 2^n$ partial models for a system having n nonlinearities.

For example: let be the nonlinear model $\dot{x}(t) = \sin(x(t))$, then the nonlinear function considered is $f(x) = \dfrac{\sin(x)}{x}$ which is bounded.

For $x \in [-x_0, x_0]$, $x_0 \leq \dfrac{\pi}{2}$ we obtain:

$$\begin{aligned} \frac{\sin(x)}{x} &= \frac{x_0 \sin(x) - x\sin(x_0)}{x(x_0 - \sin(x_0))} \cdot 1 + \\ &\frac{x_0(x - \sin(x))}{x(x_0 - \sin(x_0))} \cdot \frac{\sin(x_0)}{x_0} = h_1(x).1 + h_2(x).\frac{\sin(x_0)}{x_0} \end{aligned} \tag{7}$$

Then, the two rules TS model is:

if x is $h_1(x)$ then $\dot{x}(t) = x(t) \cdot 1$

if x is $h_2(x)$ then $\dot{x}(t) = x(t) \cdot \dfrac{\sin(x_0)}{x_0}$

2.2 Stabilization conditions

When dealing with TS fuzzy models the classical control law is PDC [8] :

$$u(t) = -\sum_{i=1}^{r} h_i(x) F_i x(t) \tag{8}$$

Determining the control law consists in finding r control gains F_i. For continuous fuzzy models, a quadratic Lyapunov function : $V(x(t)) = x^T(t) P x(t)$, with $P > 0$, is mainly used. This lead to a LMI problem which enables to compute the gains F_i [1-3, 8-10]. Let us show this:

Let be: $V(x) = x^T P x$, then we must have $\dot{V}(x) < 0$. This leads to:

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} < 0 \Leftrightarrow \tag{9}$$

$$x^T (A_z - B_z F_z)^T P x + x^T P (A_z - B_z F_z) x < 0 \Leftrightarrow \tag{10}$$

$$(A_z - B_z F_z)^T P + P(A_z - B_z F_z) < 0 \Leftrightarrow \tag{11}$$

$$X A_z^T + A_z X - B_j M_i - M_z^T B_z^T < 0 \tag{12}$$

with : $X = P^{-1}$ and $M_i = F_i P^{-1}$.

This last inequation (12) is verified if we have:

$$\forall i, \; \Upsilon_{ii} < 0 \tag{13}$$

$$\forall i, j \quad i < j, \; \Upsilon_{ij} + \Upsilon_{ji} < 0 \tag{14}$$

with $$\Upsilon_{ij} = X A_i^T + A_i X - B_j M_i - M_i^T B_j^T \tag{15}$$

These conditions are very conservative. Current researches try to lower more and more the conservativity of the LMIs. The next section will present a series of matrix properties in order to reduce this conservativity.

These properties can make conditions of stability into LMI if we obtain BMI (bilinear inequality of matrix) and/or maximize or minimize the number of free decisions variables to facilitate finding solution.

Please also note that in this paper only quadratic Lyapunov functions are used. They are currently researches trying to apply non quadratic Lyapunov functions for continuous fuzzy models. It is often required to use hypothesis about the derivative of the membership functions that must be checked a posteriori during experiments or simulations. Indeed terms $\dot{h}_i(z)$ appears in $\dot{P}$. But it is not possible to enforce such constraints in the control law through LMIs. This questions the whole validity of the approach because a valid simulation does not imply that the same validity would be obtained for other initial conditions. A thorough discussion on this issue can be found in [11], where the authors reject the whole approach due to this problem. Moreover non quadratic Lyapunov functions are often associated to bilinear matrix inequalities (BMIs).

2.3 Properties

In this section, some properties about matrices are given.

Lemma 1. *(Schur's complement [6])*

Matrices X, Y and R being of appropriate sizes, we have :

$$\begin{cases} Y - XR^{-1}X^T > 0 \\ R > 0 \end{cases} \Leftrightarrow \begin{bmatrix} Y & (*) \\ X^T & R \end{bmatrix} > 0 \quad (16)$$

((*) represents all terms induced by symmetry in a symmetric matrix).

Lemma 2. *[12]: The two next problems are equivalent:*

(i) Find $P > 0$, such that: $T + A^T P + PA < 0$ (17)

(ii) Find $P > 0$, L, G such that:

$$\begin{bmatrix} T + A^T L^T + LA & (*) \\ P - L^T + G^T A & -G - G^T \end{bmatrix} < 0 \quad (18)$$

Lemma 3. *(Relaxation [5,13,14])*

Let be the matrices Υ_{ij} and the condition:

$$\Upsilon_{zz} = \sum_{i=1}^{r} h_i^2(z)\Upsilon_{ii} + \sum_{i=1}^{r}\sum_{i=1}^{r} h_i(z)h_j(z)\left(\Upsilon_{ij} + \Upsilon_{ji}\right) < 0 \quad (19)$$

(19) is true if there exist Q_i and Q_{ij}, $(j > i)$ such as the following conditions are respected:

$$\forall i,\ \Upsilon_{ii} < Q_{ii} \quad (20)$$

$$\forall i, j \quad i < j,\ \Upsilon_{ij} + \Upsilon_{ji} \le Q_{ij} + Q_{ji} \quad (21)$$

$$\begin{bmatrix} Q_{11} & & & (*) \\ Q_{21} & Q_{22} & & \\ \vdots & & \ddots & \\ Q_{r1} & \dots & Q_{r(r-1)} & Q_{rr} \end{bmatrix} < 0 \quad (22)$$

Please remark that more advanced techniques have been proposed to obtain from (19) a finite sets of LMIs that are less restrictive than (20)-(22), [10,15-17]. They can be applied in our approach, since the Υ_{ij} terms are not modified. However the associated LMIs may become too large so that the solvers are unable to find solutions.

3. Rules reduction by uncertainties

3.1 Model

This approach consists of writing some nonlinearities as uncertainties [13,18].

$$\dot{x}(t) = \sum_{i=1}^{r} h_i \left\{ (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) \right\} \quad (23)$$

With:

$$\Delta A_i = H_a.\Delta a_i(t).Ea_i,\ \Delta B_i = H_b.\Delta b_i(t).Eb_i \quad (24)$$

Ha, Hb, Ea_i, Eb_i are constant matrixes and:

$$\Delta a_i.\Delta a_i^T < 1,\ \ \Delta b_i.\Delta b_i^T < 1 \quad (25)$$

Let us consider the following augmented state vector $\bar{x}^T(t) = \begin{pmatrix} x(t) & \int(y_c - y) \end{pmatrix}$, with y_c consign on the outputs.

$$\dot{\bar{x}}^T(t) = \begin{pmatrix} \dot{x}(t) & y_c - y \end{pmatrix} \quad (26)$$

$$\dot{\bar{x}}^T(t) = \left((A_z + \Delta A_z)x(t) + B_z u \quad -(C_z + \Delta C_z)x(t) \right) = \begin{pmatrix} A_z + \Delta A_z & 0 \\ -C_z - \Delta C_z & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \int(y_c - y) \end{pmatrix} + \begin{pmatrix} B_z + \Delta B_z \\ 0 \end{pmatrix} u \quad (27)$$

$$\dot{\bar{x}}(t) = \left(\bar{A}_z + \Delta\bar{A}_z\right)\bar{x}(t) + \left(\bar{B}_z + \Delta\bar{B}_z\right)u(t) \quad (28)$$

with $\bar{A}_z = \begin{pmatrix} A_z & 0 \\ -C_z & 0 \end{pmatrix}$, $\Delta\bar{A}_z = \begin{pmatrix} \Delta A_z & 0 \\ -\Delta C_z & 0 \end{pmatrix}$, $\bar{B}_z = \begin{pmatrix} B_z \\ 0 \end{pmatrix}$, $\Delta\bar{B}_z = \begin{pmatrix} \Delta B_z \\ 0 \end{pmatrix}$.

We suppose that the state is entirely accessible. In the case of the addition of an observer, there is no guarantee for convergence of the closed loop. We use an integral action in the forward path as indicated in figure 1.

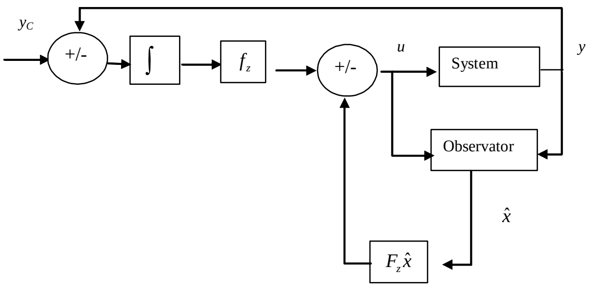

Figure 1. Sructure of observator/control.

3.2 Robust stabilization

$$u(t) = -\sum_{i=1}^{r} h_i\left(z(t)\right)\bar{F}_i\bar{x}(t) = -\bar{F}_z\bar{x}(t) \qquad (29)$$

The Lyapunov function is: $V(\bar{x}) = \bar{x}^T P\bar{x}$

A decay rate δ is chosen, so we have the condition:

$$\dot{V}(\bar{x}) \le -2\delta V(\bar{x}) \qquad (30)$$

The above inequality implies that the origin is globally exponentially stable.

Indeed,

$$\dot{V}(\bar{x}) \le -2\delta V(\bar{x})$$

Implies that, $V\left(\bar{x}(t)\right) \le V\left(\bar{x}(0)\right)e^{-2\delta t}$ (31)

Using the fact that:

$$\lambda_{\min}(P)\left\|\bar{x}(t)\right\|^2 \le V\left(\bar{x}(t)\right) = \bar{x}^T(t)P\,\bar{x}(t) \le \lambda_{\max}(P)\left\|\bar{x}(t)\right\|^2 \qquad (32)$$

$\lambda(.)_{\min}$ and $\lambda(.)_{\max}$ denotes the smallest and the largest eigenvalues.

So, $\left\|\bar{x}(t)\right\| \le \dfrac{1}{\lambda(P)^{1/2}_{\min}}.\lambda(P)^{1/2}_{\max}\left\|\bar{x}(0)\right\|.e^{-\delta t}$ (33)

$$\lambda(P)_{\min}\left\|\bar{x}(t)\right\|^2 \le V\left(\bar{x}(0)\right)e^{-2\delta t} \le \lambda(P)_{\max}\left\|\bar{x}(0)\right\|^2 e^{-2\delta t} \qquad (34)$$

$$\left\|\bar{x}(t)\right\| \le \frac{\lambda(P)^{1/2}_{\max}}{\lambda(P)^{1/2}_{\min}}.\left\|\bar{x}(0)\right\|.e^{-\delta t}, \qquad (35)$$

for all initial conditions $\bar{x}(0)$.

(28) and (29) give the closed loop:

$$\dot{\bar{x}}(t) = \left(\bar{A}_z - \bar{B}_z\bar{F}_z + \Delta\bar{A}_z - \Delta\bar{B}_z\bar{F}_z\right)\bar{x}(t) \qquad (36)$$

(4) is equivalent to :

$$\bar{x}^T \left\{ \begin{array}{l} \left(\bar{A}_z - \bar{B}_z\bar{F}_z + \Delta\bar{A}_z - \Delta\bar{B}_z\bar{F}_z\right)^T P + \\ P\left(\bar{A}_z - \bar{B}_z\bar{F}_z + \Delta\bar{A}_z - \Delta\bar{B}_z\bar{F}_z\right) + 2\delta P \end{array} \right\} \bar{x} < 0 \qquad (37)$$

After pre and post multiplication by $X = P^{-1}$ and $M_z = \bar{F}_z P^{-1}$, we obtain :

$$X\bar{A}_z^T + \bar{A}_z X - M_z^T\bar{B}_z^T - \bar{B}_z M_z + X\Delta\bar{A}_z^T + \Delta\bar{A}_z X - M_z^T\Delta\bar{B}_z^T - \Delta\bar{B}_z M_z + 2\delta X < 0 \qquad (38)$$

with $\Delta\bar{A}_z = H\Delta a_z(t)Ea_z$ and $\Delta\bar{B}_z = H\Delta b_z(t)Eb_z$ we have :

$$X\bar{A}_z^T + \bar{A}_z X - M_z^T\bar{B}_z^T - \bar{B}_z M_z + 2\delta X + XEa_z^T\Delta a_z^T H^T + H\Delta a_z Ea_z X - M_z^T Eb_z^T\Delta b_z^T H^T - H\Delta b_z Eb_z M_z < 0 \qquad (39)$$

By using this property

$$X^T Y + Y^T X \le \tau X^T X + \tau^{-1}Y^T Y \text{ and } \tau > 0 \qquad (40)$$

we obtain with $\tau a_z > 0$ and $\tau b_z > 0$ (i.e. $\forall i, \tau a_i > 0$ and $\tau b_i > 0$)

$$X\bar{A}_z^T + \bar{A}_z X - M_z^T\bar{B}_z^T - \bar{B}_z M_z + 2\delta X + \tau a_z^{-1} XEa_z^T\Delta a_z^T\Delta a_z Ea_z X + \tau a_z HH^T + \tau b_z^{-1}M_z^T Eb_z^T\Delta b_z^T\Delta b_z Eb_z M_z + \tau b_z HH^T < 0 \qquad (41)$$

By construction:

$\Delta a_z^T(t)\Delta a_z(t) \le I$ and $\Delta b_z^T(t)\Delta b_z(t) \le I$, then (41) is verified if :

$$X\bar{A}_z^T + \bar{A}_z X - M_z^T\bar{B}_z^T - \bar{B}_z M_z + 2\delta X + \tau a_z^{-1} XEa_z^T Ea_z X + \tau a_z HH^T + \tau b_z^{-1}M_z^T Eb_z^T Eb_z M_z + \tau b_z HH^T < 0 \qquad (42)$$

Or equivalently:

$$\begin{cases} X\bar{A}_z^T + \bar{A}_z X - M_z^T \bar{B}_z^T - \bar{B}_z M_z + 2\delta X + \\ XEa_z^T \tau a_z^{-1} Ea_z X + \tau a_z H \tau a_z^{-1} H^T \tau a_z + \\ M_z^T Eb_z^T \tau b_z^{-1} Eb_z M_z + \tau b_z H \tau b_z^{-1} H^T \tau b_z < 0 \end{cases} \quad (43)$$

So the use of Schur's complement gives :

$$\begin{bmatrix} X\bar{A}_z^T + \bar{A}_z X - M_z^T \bar{B}_z^T - \bar{B}_z M_z + 2\delta X & \tau a_z H & XEa_z^T & \tau b_z H & M_z^T Eb_z^T \\ \tau a_z H^T & -\tau a_z I & 0 & 0 & 0 \\ Ea_z X & 0 & -\tau a_z I & 0 & 0 \\ \tau b_z H^T & 0 & 0 & -\tau b_z I & 0 \\ Eb_z M_z & 0 & 0 & 0 & -\tau b_z I \end{bmatrix} < 0 \quad (44)$$

The next matrix defined

$$\Upsilon_{ij} = \begin{bmatrix} X\bar{A}_i^T + \bar{A}_i X - M_j^T \bar{B}_i^T - \bar{B}_i M_j + 2\delta X & \tau a_i H & XEa_i^T & \tau b_i H & M_j^T Eb_i^T \\ \tau a_i H^T & -\tau a_i I & 0 & 0 & 0 \\ Ea_i X & 0 & -\tau a_i I & 0 & 0 \\ \tau b_i H^T & 0 & 0 & -\tau b_i I & 0 \\ Eb_i M_j & 0 & 0 & 0 & -\tau b_i I \end{bmatrix} \quad (45)$$

Theorem 2. *Let be the uncertain TS model (28), the PDC law (29) and the Υ_{ij} defined in (45). If there exist a matrix $X > 0$, a scalars $\tau a_i > 0$, $\tau b_i > 0$, a matrixes M_i, Q_{ii}, Q_{ij}, $i, j \in \{1, \dots, r\}$, $i < j$ such that conditions (20), (21) and (22) are verified, then the closed loop is globally exponentially stable.*

Proposition. *Having Theorem 2 verified then, we can obtain an estimation as in (35), hence the system (28) is globally exponentially stable.*

Proof.
$\lim_{\infty} \bar{x}(t) = 0$ (since theorem 2 verified)

With $y_c \neq 0$, we obtain $\lim_{\infty}(y_c - y) = 0$ then the output of the system convergence to y_c.

Note that the Lemma 2 can be applied on the block (1,1) of Υ_{ij} to further enhanced the results. The application of these results will be done on a hydraulic system.

4. Application

4.1 System

The system that we are studying is represented in Figure 2. It's composed of k tanks interconnected with the k^{th} and used in chemical, pharmaceutical and agroalimentary industries. The goal is to obtain a mixture of $k-1$ liquids in the k^{th} tank exit with a previously fixed concentration of every liquid in the mixture. The outputs of the system to control are the flow q_{sk} and the relative concentrations $\lambda_i > 0$, $i \in \{2, \dots, k-1\}$ given related to the first tank.

The pumps supply the tanks respectively by variable flows. We suppose that these flows are proportional to the voltages applied to the Moto-pumps. The dynamic of these actioners are neglected.

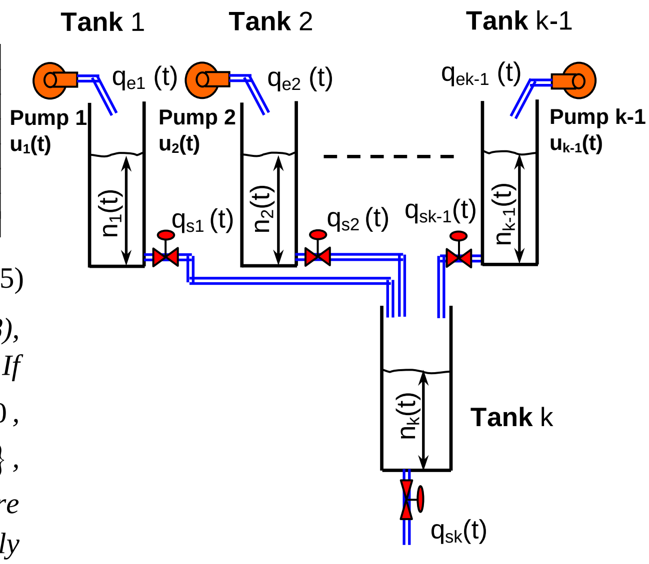

Figure 2. Diagram of the system of k tanks

4.2 Nonlinear model

The level of each tank depends on the difference of the liquid flowing into and off a tank.

Depending on the cross-cut of a valve S_i that can be considered constant, the amount of liquid flowing off by an outlet valve $q_{si}(t)$ according to Torricelli's law is :

$$q_{si}(t) = p_i S_i \sqrt{n_i(t)}, \; i \in \{1, \dots, k\}, \quad (46)$$

With $p_i = \rho_i \sqrt{2g}$

The amount of water $q_{ei}(t)$ flowing into tank i, $i \in \{1, \dots, k-1\}$ can be described by :

$$q_{ei}(t) = r_i S_i u_i(t), \; i \in \{1, \dots, k-1\} \quad (47)$$

r_i are constants.

From (46) and (47) we obtain the following differential equation:

$$\dot{n}_i(t) = -p_i\sqrt{n_i(t)} + r_i u_i(t) \qquad (48)$$

Based on these physical relations we get the k state equations to describe the nonlinear dynamic behaviour of the plant :

$$\begin{cases} \dot{n}_1(t) = -p_1\sqrt{n_1(t)} + r_1 u_1(t) \\ \vdots \\ \dot{n}_{k-1}(t) = -p_{k-1}\sqrt{n_{k-1}(t)} + r_{k-1}u_{k-1}(t), \\ \dot{n}_k(t) = \sum_{i=1}^{k-1} p_i \frac{S_i}{S_k}\sqrt{n_i(t)} - p_k\sqrt{n_k(t)} \end{cases} \qquad (49)$$

The goal is to control the flow $q_{sk}(t)$ on the outlet side of the tank k and the concentration of each liquid in the tanks i, $i \in \{1,\dots,k-1\}$ We introduce the parameters $\lambda_i > 0$, to indicate the values of the relative concentrations of each liquid compared to the first tank. The $k-1$ exits for the flow are so:

$$y_1(t) = q_{sk}(t) = p_k S_k\sqrt{n_k(t)} \qquad (50)$$

and for the concentrations: $i \in \{2,\dots,k-1\}$

$$y_i(t) = p_1 S_1\sqrt{n_1(t)} - \lambda_i S_i p_i\sqrt{n_i(t)} \qquad (51)$$

The goal is thus to make tracking on $y_1(t)$ (50) and regulation on $y_i(t)$, $i \in \{2,\dots,k-1\}$ (51).

If we consider the model described by equations (49), (50) and (51) and desire to obtain an exact TS model in a compact domain of variables $\left[n_1(t) \;\; n_2(t) \;\; \dots \;\; n_k(t)\right]^T$, it's necessary to consider the k nonlinearities $\sqrt{n_i(t)}$, $i \in \{1,\dots,k\}$. That leads to 2^k rules.

For example if we consider the case of 4 tanks, we obtain a 16 rules model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{16} h_i(z(t))\left(A_i x(t) + B_i u(t)\right) \\ y(t) = \sum_{i=1}^{16} h_i(z(t)) C_i x(t) \end{cases}$$

With matrices for the first model :

$$A_1 = \begin{bmatrix} \frac{-P_1}{\sqrt{\overline{n}_1}} & 0 & 0 & 0 \\ 0 & \frac{-P_2}{\sqrt{\overline{n}_2}} & 0 & 0 \\ 0 & 0 & \frac{-P_3}{\sqrt{\overline{n}_3}} & 0 \\ \frac{P_1}{\sqrt{\overline{n}_1}} & \frac{P_2}{\sqrt{\overline{n}_2}} & \frac{P_3}{\sqrt{\overline{n}_3}} & \frac{-P_4}{\sqrt{\overline{n}_4}} \end{bmatrix},$$

$$\forall i \in \{1,\dots,16\},\; B_i = \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{P_4}{\sqrt{\overline{n}_4}} \\ \frac{P_1}{\sqrt{\overline{n}_1}} & -\lambda_2\frac{P_2}{\sqrt{\overline{n}_2}} & 0 & 0 \\ \frac{P_1}{\sqrt{\overline{n}_1}} & 0 & -\lambda_3\frac{P_3}{\sqrt{\overline{n}_3}} & 0 \end{bmatrix}$$

The other 15 matrixes are obtained from those of the first by permuting the terms $\overline{n}_i$ and $\underline{n}_i$, for $i = 1 \dots 4$.

To avoid the problem of exponential augmentation of the number of rules according to the number of tanks we propose to reduce this number of rules by using a TS model with uncertainties.

We have said in the introduction that it's not possible to get a solution to LMI problem with four tanks.

We propose therefore to consider a single non linearity, $\sqrt{n_i(t)}$, all others being considered as uncertainties.

We can write for the first tank:

$$\dot{n}_k(t) = \sum_{i=1}^{k-1} p_i\frac{S_i}{S_k}\sqrt{n_i(t)} - \frac{p_k}{\sqrt{n_k(t)}} n_k(t) \qquad (52)$$

With the notation :

$\mu_k(t) = \dfrac{1}{\sqrt{n_k(t)}}$, we can write

$$\mu_k(t) = \frac{\mu_k(t) - \underline{\mu}_k}{\overline{\mu}_k - \underline{\mu}_k}\cdot\overline{\mu}_k - \frac{\overline{\mu}_k - \mu_k(t)}{\overline{\mu}_k - \underline{\mu}_k}\cdot\underline{\mu}_k$$

With $\mu_k(t) \in \left[\underline{\mu}_k, \bar{\mu}_k\right]$ that correspond to $n_k(t) \in \left[\underline{n}_k, \bar{n}_k\right]$: we take $\underline{\mu}_k = \dfrac{1}{\sqrt{\bar{n}_k}}$ and $\bar{\mu}_k = \dfrac{1}{\sqrt{\underline{n}_k}}$, we can write

if $\mu_k(t) = \dfrac{1}{\sqrt{n_k(t)}}$ and $\dfrac{\mu_k(t) - \underline{\mu}_k}{\bar{\mu}_k - \underline{\mu}_k}$ then

$$\dot{n}_k(t) = \sum_{i=1}^{k-1} p_i \frac{S_i}{S_k} \sqrt{n_i(t)} - \bar{\mu}_k \cdot p_k \cdot n_k(t)$$

if $\mu_k(t) = \dfrac{1}{\sqrt{n_k(t)}}$ and $\dfrac{\bar{\mu}_k - \mu_k(t)}{\bar{\mu}_k - \underline{\mu}_k}$ then

$$\dot{n}_k(t) = \sum_{i=1}^{k-1} p_i \frac{S_i}{S_k} \sqrt{n_i(t)} - \underline{\mu}_k \cdot p_k \cdot n_k(t)$$

all the other nonlinearities $\sqrt{n_i(t)}$, $i \in \{1, \ldots, k-1\}$ will be reported in the uncertainties. For this :

Let be a nonlinear function : $\alpha(t) \in [\underline{\alpha} \quad \bar{\alpha}]$, then :

$\alpha(t) = \alpha_m + \beta(t)\alpha_r$ with $\beta(t) \in [-1 \quad 1]$ and

$$\begin{cases} \alpha_m = 0,5 \cdot (\bar{\alpha}_i + \underline{\alpha}_i) \\ \alpha_r = 0,5 \cdot (\bar{\alpha}_i - \underline{\alpha}_i) \end{cases}$$

Then, we consider the bounded functions

$$\mu_i(t) = \frac{1}{\sqrt{n_i(t)}} \in \left[\underline{\mu}_i, \bar{\mu}_i\right],\ i \in \{1, \ldots, k-1\}$$

Corresponding on $n_i(t) \in \left[\underline{n}_i, \bar{n}_i\right]$. For each of them, we can write with $\beta_i(t) \in [-1, 1]$:

$$\mu_{im} = \frac{1}{2}\left(\bar{\mu}_i + \underline{\mu}_i\right), \mu_{ir} \frac{1}{2}\left(\bar{\mu}_i - \underline{\mu}_i\right),$$

$$\mu_i(t) = \mu_{im} + \mu_{ir}\beta_i(t)$$

Finally we obtain :

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} h_i \left\{ (A_i + \Delta A_i) x(t) + Bu(t) \right\} \\ y(t) = \sum_{i=1}^{2} h_i (C_i + \Delta C_i) x(t) \end{cases} \quad (53)$$

With:

$$A_1 = \begin{bmatrix} -p_1\mu_{1m} & 0 & \cdots & & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & & -p_{k-1}\mu_{(k-1)m} & 0 \\ \frac{S_1}{S_k} p_1\mu_{1m} & \cdots & \cdots & & \frac{S_{k-1}}{S_k} p_{k-1}\mu_{(k-1)m} & -\bar{\mu}_k \cdot p_k \end{bmatrix}$$

$$B = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & r_{k-1} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & & \cdots & & 0 & p_k S_k \bar{\mu}_k \\ p_1 S_1 \mu_{1m} & -\lambda_2 S_2 p_2 \mu_{2m} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ p_1 S_1 \mu_{1m} & 0 & \cdots & 0 & -\lambda_{k-1} S_{k-1} p_{k-1} \mu_{(k-1)m} & 0 \end{bmatrix}$$

To obtain (A_2, C_2) we replace $\bar{\mu}_k$ by. $\underline{\mu}_k$

$\Delta A_1 = \Delta A_2 =$

$$\begin{bmatrix} -p_1\mu_{1r}\beta_1(t) & 0 & \cdots & & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & & -p_{k-1}\mu_{(k-1)r}\beta_{k-1}(t) & 0 \\ \frac{S_1}{S_k} p_1\mu_{1r}\beta_1(t) & \cdots & \cdots & & \frac{S_{k-1}}{S_k} p_{k-1}\mu_{(k-1)r}\beta_{k-1}(t) & 0 \end{bmatrix} \quad (54)$$

$\Delta C_1 = \Delta C_2 =$

$$\begin{bmatrix} 0 & & \cdots & & 0 & 0 \\ p_1 S_1 \mu_{1r}\beta_1(t) & -\lambda_2 S_2 p_2 \mu_{2r}\beta_2(t) & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ p_1 S_1 \mu_{1r}\beta_1(t) & 0 & \cdots & 0 & -\lambda_{k-1} S_{k-1} p_{k-1} \mu_{(k-1)r}\beta_{k-1}(t) & 0 \end{bmatrix} \quad (55)$$

From the expression of uncertainties (54) we can write $\Delta A_1 = \Delta A_2 = H_a \Delta a(t) E_a$ with

$$\Delta a = \begin{bmatrix} \beta_1(t) & & 0 \\ & \ddots & \\ 0 & & \beta_{k-1}(t) \end{bmatrix} \text{ and}$$

$$H_a = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 \\ \frac{S_1}{S_k} & \cdots & \cdots & \frac{S_{k-1}}{S_k} \end{bmatrix}$$

$$E_a = \begin{bmatrix} p_1\mu_{1r} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p_{k-1}\mu_{(k-1)r} & 0 \end{bmatrix}$$

In the same way, from the expression (55), we can write $\Delta C_1 = \Delta C_2 = H_c \Delta c(t) E_c$ with

$$\Delta c = \begin{bmatrix} \beta_1(t) & & 0 \\ & \ddots & \\ 0 & & \beta_{k-1}(t) \end{bmatrix}$$

and

$$H_c = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & -1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix},$$

$$E_c = \begin{bmatrix} p_1 S_1 \mu_{1r} & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 p_2 S_2 \mu_{2r} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{k-1} S_{k-1} p_{k-1} \mu_{(k-1)r} & 0 \end{bmatrix}$$

Model TS with uncertainty is now explicit. Let us note that it is the "minimum" number of rules which prevailed with its obtaining, and which there exists obviously of other possible representatives of the nonlinear model (49).

Now we will compare this method with a classic method of linearization. For example we use the Taylor linearization.

Taylor linearization:

We recall that with : $\dot{n}_i = f(n_i, u_i)$

Taylor series can be writted as :

$$\begin{cases} f(n_i,u_i) = \underbrace{f(n_{si},u_{si})}_{0 \text{ in steady flow}} + \\ \left.\frac{\partial f}{\partial n_i}\right|_{n_{si},u_{si}} (n_i - n_{si}) + \left.\frac{\partial f}{\partial u_i}\right|_{n_{si},u_{si}} (u_i - u_{si}) \\ y(n_i) = \underbrace{y(n_{si})}_{y_{si} \text{ in steady flow}} + \left.\frac{\partial y}{\partial n_i}\right|_{n_{si}} (n_i - n_{si}) \end{cases} \quad (56)$$

n_{is}, u_{is}, y_{is} are the set points.

After linearisation and with: $n'_i = (n_i - n_{si}),\ u'_i = (u_i - u_{si});\ y'_i = (y_i - y_{si})$ we obtain:

$$\begin{cases} \dot{n}'_i(t) = -\frac{p_i}{2\sqrt{n_{si}(t)}} n'_i + r_i u'_i(t);\ i = 1,2,3; \\ \dot{n}'_k(t) = \sum_{i=1}^{k-1} \frac{p_i}{2\sqrt{n_{si}(t)}} (n_i - n_{si}) - \\ \frac{p_k}{2\sqrt{n_{sk}(t)}} (n_k - n_{sk});\ k = 4 \\ y'_1(t) = \frac{p_k}{2\sqrt{n_{sk}(t)}} \\ y'_i(t) = \frac{p_i}{2\sqrt{n_{si}(t)}} (n_i - n_{si}) - \\ \lambda_{k-2} \frac{p_{k-2}}{2\sqrt{n_{sk-2}(t)}} (n_{k-2} - n_{sk-2}) \end{cases} \quad (57)$$

From this equation the linear model is :

$$\begin{cases} \dot{n}'(t) = n' + Bu'(t);\ n' = (n - n_s),\ u' = (u - u_s) \\ y'(t) = n' \\ u'(t) = n',\ F \text{ are control gain} \end{cases}$$

Matrixes of the system are:

$$A = \begin{bmatrix} \frac{-P_1}{2\sqrt{n_{s1}}} & 0 & 0 & 0 \\ 0 & \frac{-P_2}{2\sqrt{n_{s2}}} & 0 & 0 \\ 0 & 0 & \frac{-P_3}{2\sqrt{n_{s3}}} & 0 \\ \frac{P_1}{2\sqrt{n_{s1}}} & \frac{P_2}{2\sqrt{n_{s2}}} & \frac{P_3}{2\sqrt{n_{s3}}} & \frac{-P_4}{2\sqrt{n_{s4}}} \end{bmatrix},$$

$$B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix},$$

$$C=\begin{bmatrix}0&0&0&\dfrac{P_4}{2\sqrt{n_{s4}}}\\ \dfrac{P_1}{2\sqrt{n_{s1}}}&-\lambda_2\dfrac{P_2}{2\sqrt{n_{s2}}}&0&0\\ \dfrac{P_1}{2\sqrt{n_{s1}}}&0&-\lambda_3\dfrac{P_3}{2\sqrt{n_{s3}}}&0\end{bmatrix} \quad (58)$$

4.3 Simulation

We deal with the case $k=4$. The results are obtained by using the software Matlab/Simulink. The max and the min values of liquid in tanks and λ_i are:

$$\underline{n}_1=\underline{n}_2=\underline{n}_3=0{,}1m,\ \overline{n}_1=\overline{n}_2=\overline{n}_3=4{,}3m,$$

$$\underline{n}_4=0{,}05m,\ \overline{n}_4=4m,\ \lambda_2=2,\ \lambda_3=3.$$

The uncertain model is written then with the equations (53) and (55) :

$$A_1=\begin{bmatrix}-0{,}36&0&0&0&0&0&0\\0&-0{,}18&0&0&0&0&0\\0&0&-0{,}18&0&0&0&0\\0{,}36&0{,}18&0{,}18&-0{,}25&0&0&0\\0&0&0&-0{,}25&0&0&0\\-0{,}36&0{,}31&0&0&0&0&0\\-0{,}36&0&0{,}27&0&0&0&0\end{bmatrix},$$

$$A_2=\begin{bmatrix}-0{,}36&0&0&0&0&0&0\\0&-0{,}18&0&0&0&0&0\\0&0&-0{,}18&0&0&0&0\\0{,}36&0{,}18&0{,}18&-2{,}24&0&0&0\\0&0&0&-2{,}24&0&0&0\\-0{,}36&0{,}31&0&0&0&0&0\\-0{,}36&0&0{,}27&0&0&0&0\end{bmatrix},$$

$$C_1=\begin{bmatrix}0&0&0&0{,}25&0&0&0\\0{,}6&-0{,}31&0&0&0&0&0\\0{,}36&0&-0{,}27&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},$$

$$C_2=\begin{bmatrix}0&0&0&2{,}23&0&0&0\\0{,}6&-0{,}31&0&0&0&0&0\\0{,}36&0&-0{,}27&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix},\ H=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\-1&-1&-1\end{bmatrix},$$

$$Ea_1=Ea2=\begin{bmatrix}-0{,}27&0&0&0&0&0&0\\0&-0{,}13&0&0&0&0&0\\0&0&-0{,}13&0&0&0&0\end{bmatrix} \quad (59)$$

The found solution gives the following gain:

$$F_1=\begin{bmatrix}44{,}1&2{,}1&5{,}7&113{,}1\\-4&3{,}5&-1{,}5&-7{,}2\\1{,}9&-0{,}8&3{,}9&9{,}1\end{bmatrix},$$

$$F_2=\begin{bmatrix}45{,}1&2{,}3&5{,}9&116{,}8\\-3{,}2&3{,}6&-1{,}3&-4{,}6\\2{,}8&-0{,}7&4&11{,}7\end{bmatrix},$$

$$K_1=\begin{bmatrix}2{,}65&2{,}18&-2{,}20\\-1{,}26&-0{,}6&2{,}13\\-0{,}73&-1{,}66&-0{,}44\\-0{,}66&-1{,}88&-0{,}84\end{bmatrix}$$

and

$$K_2=\begin{bmatrix}2{,}59&-37{,}9&38{,}2\\-1{,}35&0{,}14&-35\\-0{,}83&30{,}19&-2{,}34\\-0{,}83&-13{,}3&-14\end{bmatrix} \quad (60)$$

To guarantee fix relative concentrations, y_c2 and y_c3 must be equal to 0. We can see on the curves that with the control law we obtain exactly the desired set point, Figure 4.

We also notice that for the change of set point for $y_1(t)$ there is an important error in the relative concentrations $y_2(t)$. This corresponds to a transitional regime and the liquid produced in this lap of time will not be used.

To gain place, levels 1, 3 and 4 are suppressed. Level 2 are showed only with control variables, Figure 3.

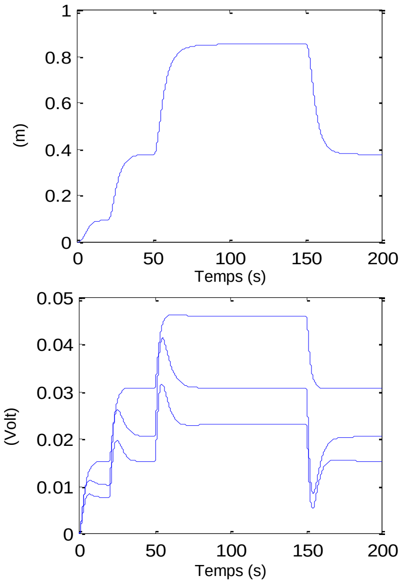

Figure 3. Level in tank 2 (m) and control variables $u_1(t)$, $u_2(t)$, $u_3(t)$.

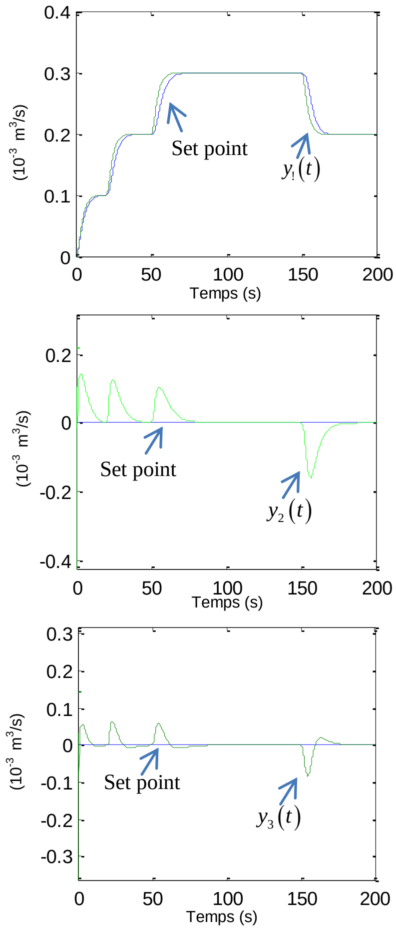

Figure 4. Evolutions of $y_1(t)$, $y_2(t)$ and $y_3(t)$.

The comparison with Taylor linearization shows that the latter does not give satisfied result: on Figure 5, $y_1(t)$ is not better than in our approach.

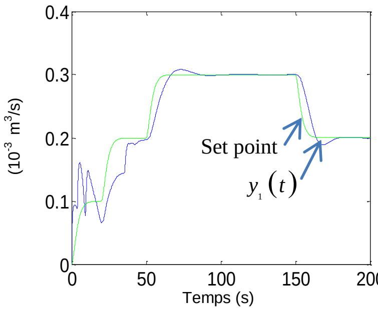

Figure 5. Evolutions of $y_1(t)$, (Taylor linearization)

5. Conclusion

The models of Takagi-Sugeno propose a rigorous approach allowing the treatment of nonlinear systems. In fact they allow their decomposition into linear systems, interpolated by nonlinear functions. It is possible with this approach to write a nonlinear system exactly as the convex sum of linear systems, according to weight determined in advance. The number of partial linear models determined from the number of nonlinearities is 2^k for a system having k nonlinearities.

Nevertheless, the study of stability of TS fuzzy models will be difficult in the case where the number of nonlinearities is important.

This study proposed a methodology to reduce the number of rules and to guarantee the global exponential stability of the system. The new model included only two rules. All the other nonlinearities were considered as uncertainties. In this paper LMIs conditions were proposed to guarantee the exponential stability of the closed loop. These results were applied to a hydraulic system which was described by a nonlinear model.

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