An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.14, No.2, pp.134-146 (2024)
http://doi.org/10.11121/ijocta.1418

RESEARCH ARTICLE

A Fractional-order mathematical model to analyze the stability and develop a sterilization strategy for the habitat of stray dogs

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ARTICLE INFO

Article History:
Received 24 June 2023
Accepted 22 December 2023
Available Online 30 March 2024

Keywords:
Fractional-order differential equation
Euler method
Stability analysis
Stray dog population and sterilization strategy
Semi-cycle solution

AMS Classification 2010:
60G25; 34B60; 68U01

ABSTRACT

Today, the socio-cultural lack of some countries with increased urbanization has led to the unconscious breeding of stray dogs. The failure to care for the offspring of possessive dogs or ignoring the responsibility to find a suitable family for the offspring increased the dog population on the streets and in the shelters. In this study, our main target is to analyze the habitat of stray dogs and the strategy of how to control the population without damaging the ecosystem of the species. For this aim, we establish a fractional-order differential equation system to investigate the fractal dimension with long-term memory that invovles two compartments; the non-sterilized dog population ($x(t)$) and the sterilized one ($y(t)$). Firstly, we analyze the stability of the equilibrium points using the Routh-Hurwitz criteria to discuss cases that should not affect the ecosystem of the dog population, but control the stray dog population in the habitat. Since the intervention to the stray dog population occurs at discrete time impulses, we use the Euler method's discretization process to analyse the local and global stability around the equilibrium points. Besides this, we show that the solutions of the system represent semi-cycle behaviors. At the end of the study, we use accurate data to demonstrate the sterilization rate of stray dogs in their habitat.

1. Introduction

The dog population is generally split into possessive and unattended dogs, which can be grouped under three main topics. This suggests that dogs have been lost while possessing an unclaimed population, dogs abandoned by the owner, and dogs already unclaimed. Dogs with masters are owned and maintained by one or more people [1]. Unattended dogs attempt to coalesce in a specific habitat and survive. This habitat consists of dog breeding, dog acceptance into the habitat from the outside, death, etc., and dog populations. Some factors affect the isolated dog population. This means that the attitude and behavior of the surrounding people, the ability of dogs to breed, and the ability of dogs to access resources such as food and water within a group determine the size of the population. The lack of control of the unclaimed dog population brings several risks as the population grows. This suggests that the number of stray dogs in a habitat increases the number of zoonotic diseases. Zoonotic diseases are common in individuals, including both groups, transmitted from animals to humans. Zoonosis includes bacterial, viral, fungal (fungal-induced), rickettsial

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(a type of parasitic microorganism), and parasitic infections. Unattended dogs attack people and cause traffic accidents in their habitats, rabies, and zoonosis. As a result, the lack of control over the unattended dog population threatens human health [2]. Keeping the breeding of dogs under control and preventing unwanted breeding will help strike a balance between the animal owners' demands and the dog population's size. Sterilizations are being made to prevent the growing number of stray dogs. Sterilization of dogs is a surgical operation that removes the ovaries and uteruses of female dogs and the testicles of male dogs. This will prevent the uncontrolled increase in the number of stray dogs. Another method of controlling the populations of stray dogs is to provide training for animal owners. Those who possess unconscious animals may leave them on the street after a while. Adopted dogs have been introduced to chips, increasing the population if they do not have chips for their dogs and newborn cubs [1, 2].

The main goal of mathematical modeling is to describe the process through a mathematical description of real-life problems. A.A. Thirthar et al. analysed in [3] an aquatic ecological model with aggregation of fear and harvesting effects. Similar mathematical models that investigates species in ecosystems can be seen in [4–6]. Thus, mathematical models were developed to help explain a system, examine the effects of its various components, and make predictions about its behavior [7–9].

Models containing fractional derivatives yield better results than integer-based models in the theory of control of dynamical systems through various physical and biological processes [10–12]. In [11], C.M. Nunes et al. considered a case study in Brazil, where the management of visceral leishmaniasis has primarily involved the spraying of residual insecticides to eliminate vectors and the identification and removal of infected dogs.

It is particularly appropriate to use fractional operators to explain the memory and inherited characteristics of many materials and processes, as such properties are ignored in the derivative of the integer digit. The future status of a population in population models depends on its past status. It is called the memory effect. The memory effect of the population can be examined by adding a delay term or using a fractional derivative in the model [13–15]. In time-variable events, fractional models show more realistic and accurate results than models of integers because they have memory. Therefore, many biological events are represented as fractional-order models. See, for example, [16–21].

This study develops a new mathematical model that deals with the parameters that affect the population of stray dogs. The most critical factor in the population, the neutering effect, was examined in different scenarios. Results have been released on the ideal sterilization strategy to prevent dog population extinction or uncontrolled escalation.

Let us say that the end limit value of the logistical growth $x(t)$ size over time is K (the maximum value that a population $x(t)$ can take, which is known as the maximum carrying capacity). The logistic model assumes that when $\frac{x(t)}{K} \sim 0$ exists, then the relative growth rate goes to zero, and when $\frac{x(t)}{K} \sim 1$ is reduced as a linear function, and the solution of the logistic equation tends to zero. A single species model (here, dog population) with carrying capacity K and growth rate r is given by [13], such as

$$D_t^\alpha x(t) = rx(t)(1 - \frac{x(t)}{K}). \tag{1}$$

In including a ratio of natural deaths in the unsterilized dog compartment to equation (1.1), we obtain

$$D_t^\alpha x(t) = rx(t)(1 - \frac{x(t)}{K}) - \mu x(t). \tag{2}$$

The overhead compartment decreases with a rate of β to build another compartment for sterilized dogs. Here, β denotes the rate of sterilization.

$$D_t^\alpha x(t) = rx(t)(1 - \frac{x(t)}{K}) - \mu x(t) - \beta x(t). \tag{3}$$

Since the unsterilized dog compartment would have deaths due to illness, starvation, or other causes, we include a rate of θ to denote the death of this class caused by non-natural reasons. Finally, the non-sterilized compartment can be defined as follows;

$$D_t^\alpha x(t) = rx(t)(1 - \frac{x(t)}{K}) - \mu x(t) - \beta x(t) - \theta x(t). \tag{4}$$

The sterilized compartment $y(t)$ exists from the rate β and compartment $x(t)$, which is given by

$$D_t^\alpha y(t) = \beta x(t). \tag{5}$$

The $y(t)$ population also involves the same parameters μ and θ, that denote natural and non-natural causes of death. Thus, the equation can be improved such as

$$D_t^\alpha y(t) = \beta x(t) - \mu y(t) - \theta y(t). \tag{6}$$

Now, the fractional dog population model can be expressed as a fractional differential equation system, where $(x(t))$ denotes the non-sterilized dogs and $(y(t))$ represents the sterilized dogs;

$$\begin{cases} D_t^{\alpha}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) \\ \qquad\qquad - \mu x(t) - \beta x(t) - \theta x(t) \\ D_t^{\alpha}y(t) = \beta x(t) - \mu y(t) - \theta y(t) \end{cases} \tag{7}$$

where D_t^{α} is the Caputo fractional derivative concerning time t and $0 < \alpha \leq 1$. Initial values are given as, $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$.
All compartments and parameters are given in Table 1 and Table 2.

Table 1. Variables used in the systems and their meanings.

Variables used in the systems	Meaning
$x(t)$	Not sterilized dog population
$y(t)$	Sterilized dog population
$N(t)$	Total population

Table 2. Parameters and their meanings.

Parameters	Meaning
β	Annual sterilization rate
μ	Annual natural mortality rate
θ	The annual death rate due to illness, hunger and other causes
K	Carrying capacity ratio
r	Annual growth rate

Definition 1. *[10] Let $f(t)$ be a function that can be continuously differentiable n times. The value of the function $f(t)$ for the value of α that satisfies the condition $n-1 < \alpha < n$. The Caputo fractional derivative of $\alpha-th$ order $f(t)$ is defined by*

$$D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^n (t-x)^{(n-\alpha-1)} f^n(x)dx. \tag{8}$$

Definition 2. *[23] Given a function a function $\varphi(t)$, the fractional integral with order $\alpha > 0$ is given by Abdel's formula as*

$$I_t^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{(\alpha-1)}\varphi(t)dt, x > 0. \tag{9}$$

Definition 3. *[24] The Mittag-Leffler function of one variable is $(\lambda \neq 0, z \in \mathbb{C}\colon \operatorname{Re}(\alpha) > 0)$:*

$$E_{\alpha}(\lambda, z) = E_{\alpha}(\lambda z^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{\alpha k}}{\Gamma(1+\alpha k)}. \tag{10}$$

2. Stability analysis

2.1. Perturbation and the Equilibrium Points

Let us consider the system.

$$\begin{cases} D_t^{\alpha}x\left(t\right) &= f\left(x\left(t\right), y\left(t\right)\right) \\ &= rx\left(t\right)\left(1 - \frac{x(t)}{K}\right) - \\ & \quad \left(\mu + \beta + \theta\right) x\left(t\right), \\ D_t^{\alpha}y\left(t\right) &= g\left(x\left(t\right), y\left(t\right)\right) = \beta x\left(t\right) - \\ & \quad \left(\mu + \theta\right) y\left(t\right). \end{cases} \tag{11}$$

To discuss the stability of system (11), we perturb the equilibrium point by adding $\varepsilon_i\left(t\right) > 0, \quad i = 1,\ 2$ such as

$$x\left(t\right) - \overline{x} = \varepsilon_1(t) > 0 \text{ and } y\left(t\right) - \overline{y} = \varepsilon_2(t) > 0. \tag{12}$$

Thus, we have

$$D_t^{\alpha}\varepsilon_1\left(t\right) \simeq f\left(\overline{x},\overline{y}\right) + \frac{\partial f\left(\overline{x},\overline{y}\right)}{\partial x}\varepsilon_1\left(t\right) + \frac{\partial f\left(\overline{x},\overline{y}\right)}{\partial y}\varepsilon_2\left(t\right),$$

and

$$D_t^{\alpha}\varepsilon_2\left(t\right) \simeq g\left(\overline{x},\overline{y}\right) + \frac{\partial g\left(\overline{x},\overline{y}\right)}{\partial x}\varepsilon_1\left(t\right) + \frac{\partial g\left(\overline{x},\overline{y}\right)}{\partial y}\varepsilon_2\left(t\right).$$

Using $f\left(\overline{x},\overline{y}\right) = g\left(\overline{x},\overline{y}\right) = 0$, we get a linearized system about $\left(\overline{x},\overline{y}\right)$ as

$$D_t^{\alpha}Z = JZ \tag{13}$$

where $Z = \left(\varepsilon_1\left(t\right), \varepsilon_2\left(t\right)\ \right)$ and J is the Jacobian matrix evaluated at $\left(\overline{x},\overline{y}\right)$,

$$J = \begin{pmatrix} \frac{\partial f(\overline{x},\overline{y})}{\partial x} & \frac{\partial f(\overline{x},\overline{y})}{\partial y} \\ \frac{\partial g(\overline{x},\overline{y})}{\partial x} & \frac{\partial g(\overline{x},\overline{y})}{\partial y} \end{pmatrix}. \tag{14}$$

C is the diagonal matrix of J given by

$$C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{15}$$

where from $B^{-1}JB = C$, B represents the eigenvectors of J and $\lambda_i,\ i = 1,\ 2$ are the eigenvalues. Therefore, we get

$$\begin{cases} D_t^{\alpha}\eta\left(t\right) = \lambda_1\eta_1 \\ D_t^{\alpha}\eta\left(t\right) = \lambda_2\eta_2 \end{cases}, \tag{16}$$

where $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ and $\eta = B^{-1}Z$,whose solutions are given by the following Mittag-Leffler functions:

$$\eta_1\left(t\right) = \sum_{n=0}^{\infty} \frac{\left(\lambda_1\right)^n t^{n\alpha}}{\Gamma\left(n\alpha+1\right)}\eta_1\left(0\right) = E_{\alpha}\left(\lambda_1 t^{\alpha}\right)\eta_1(0) \tag{17}$$

and

$$\eta_2\left(t\right) = \sum_{n=0}^{\infty} \frac{\left(\lambda_2\right)^n t^{n\alpha}}{\Gamma\left(n\alpha+1\right)}\eta_2\left(0\right) = E_{\alpha}\left(\lambda_2 t^{\alpha}\right)\eta_2\left(0\right) \tag{18}$$

Using the result of [25], if $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $i = 1, 2$, then $\eta_1(t)$ and $\eta_2(t)$ are decreasing; consequently, $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are decreasing. For the existence of $(\varepsilon_1(t), \varepsilon_2(t))$ in (13), if the solution of (13) increases, then the equilibrium point $(\overline{x}, \overline{y})$ is unstable; otherwise, if $(\varepsilon_1(t), \varepsilon_2(t))$ decreases, then $(\overline{x}, \overline{y})$ is locally asymptotically stable.

The equilibrium points of system (12) are

Extinction of the habitat:

$$\Lambda_1 = (0, 0),$$

Non-sterilized stray dog population:

$$\Lambda_2 = \left(\frac{K(r-\mu-\theta)}{r}, 0\right)$$

for $r > \mu + \theta$ and $\beta = 0$,

Sterilized dog population system (Co-existing):

$$\Lambda_3 = \left(\frac{K(r-\mu-\beta-\theta)}{r}, \frac{K\beta(r-\mu-\beta-\theta)}{r(\mu+\theta)}\right)$$

for $r > \mu + \theta + \beta$ and $\beta > 0$.

2.2. Stability analysis of the equilibrium points

The Jacobian matrix of system (11) around an equilibrium point is as follows;

$$J(\overline{x}, \overline{y}) = \begin{pmatrix} r - \frac{2r\overline{x}}{K} - (\mu+\beta+\theta) & 0 \\ \beta & -(\mu+\theta) \end{pmatrix}. \tag{19}$$

For $\Lambda_1 = (0, 0)$, we have the characteristic equation

$$\begin{aligned} &(r - (\mu+\beta+\theta) - \lambda_1)(-(\mu+\theta) - \lambda_2) = 0 \\ &\Longrightarrow \lambda_1 = r - (\mu+\beta+\theta) \text{ and } \lambda_2 = -(\mu+\theta). \end{aligned} \tag{20}$$

Theorem 1. *Let $\Lambda_1 = (0, 0)$ be the extinction point of system (11). Then, the following statements are true:*

- **(i)** *If $r < \mu+\beta+\theta$, then the equilibrium point Λ_1 is local asymptotic stable.*
- **(ii)** *If $r > \mu+\beta+\theta$, then the equilibrium point Λ_1 is an unstable saddle point.*

Proof. Since $\lambda_2 = -(\mu+\theta) < 0$, we consider the conditions of the eigenvalue λ_1, which is negative if $r < \mu+\beta+\theta$, and positive if $r > \mu+\beta+\theta$. This completes the proof.

Remark 1. *Humans should never act with the intent of eradicating a species. Here we noticed that if the sterilization rate and the non-natural causes of death for the stray dogs are more significant than the growth rate, then population extinction happens. The unstable case of the equilibrium point Λ_1 is when the growth rate of the population is greater than the sterilization and death rates.*

The following scenario considers the case that sterilization is not applied, which means that only one compartment would exist, which is the $x(t)$ compartment.

The characteristic equation of Λ_2 is

$$\begin{aligned} &(2\mu + 2\theta - r - \lambda_1)(-(\mu+\theta) - \lambda_2) = 0 \\ &\Longrightarrow \lambda_1 = 2\mu + 2\theta - r \text{ and } \lambda_2 = -(\mu+\theta). \end{aligned} \tag{21}$$

Theorem 2. *Let $\Lambda_2 = \left(\frac{K(r-\mu-\theta)}{r}, 0\right)$ be the equilibrium point of the non-serialized dog population of system (11). Then the following statements are true.*

- **(i)** *If $r > 2\mu+2\theta$, then Λ_2 is local asymptotic stable.*
- **(ii)** *If $\mu+\theta < r < 2\mu+2\theta$, then Λ_2 is an unstable saddle point.*

Proof. From the definition of the equilibrium point, we obtained that Λ_2 exists for $r > \mu+\theta$ and $\beta = 0$. Moreover, $\lambda_2 < 0$. Considering the conditions for the eigenvalue λ_1, we obtain that Λ_2 is locally stable if $r > 2\mu+2\theta$, and Λ_2 is a saddle point if $\mu+\theta<r < 2\mu+2\theta$. This completes the proof.

Remark 2. *It is seen from Theorem 2 that the stray dog population would exist as single species if there is no sterilization and the growth rate of the species is greater than the death rate. The instability of the population exists if non-natural death causes increase.*

We analyze the characteristic equation around the equilibrium point to consider the co-existing case of both sterilized and non-sterilized dog populations. Λ_3, which is obtained as follows;

$$\begin{aligned} &(2(\mu+\theta+\beta) - r - \lambda_1)(-(\mu+\theta) - \lambda_2) = 0 \\ &\Longrightarrow \lambda_1 = 2(\mu+\theta+\beta) - r \text{ and } \lambda_2 = -(\mu+\theta). \end{aligned} \tag{22}$$

Theorem 3. *Let Λ_3 be the co-existing (positive) equilibrium point of system (11). Then the following statements hold.*

- **(i)** *If $r > 2(\mu+\theta+\beta)$, then Λ_3 is local asymptotic stable.*
- **(ii)** *If $\mu+\theta+\beta < r < 2(\mu+\theta+\beta)$, then Λ_3 is an unstable saddle point.*

Proof. The co-existing (positive) equilibrium point exists for $r > \mu+\theta+\beta$ and $\beta > 0$. Moreover, we obtained $\lambda_2 = -(\mu+\theta) < 0$. Therefore, we have to analyze the eigenvalue λ_1 to discuss the stability conditions. It can be seen that if $r > 2(\mu+\theta+\beta)$, then both compartments show local stability in the system, and if

$\mu+\theta+\beta<r<2(\mu+\theta+\beta)$, then we have an unstable saddle point around the equilibrium point Λ_3. This completes the proof.

Remark 3. *A second compartment related to $x(t)$ exists if $\beta>0$. This is the rate of sterilization from Theorem 3 we obtained that both compartments represent stable behavior if the growth rate of the non-sterilized dog population is greater than the sterilization rate and the death of non-natural causes. At the same time, it is unstable if the sterilization rate and the non-natural causes increase.*

3. Existence and uniqueness

Considering system (11) and the initial conditions $x(0)>0$ and $y(0)>0$, the initial value problem can be written such as

$$D_t^\alpha V(t)=AV(t)+x(t)BV(t),\ \ t\in[0,\ T], \tag{23}$$

where, $V(t)=\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}$ and $V(0)=\begin{bmatrix} x(0)\\ y(0)\end{bmatrix}$.

Let us assume that $x(t)>0$ and $y(t)>0$, when $t>\sigma\geq 0$. In this case, the IVP can be written as

$$D_t^\alpha V(t)=\begin{bmatrix} r-(\mu+\beta+\theta) & 0\\ \beta & -(\mu+\theta)\end{bmatrix}\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}$$
$$+x(t)\begin{bmatrix} -\frac{r}{K} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}. \tag{24}$$

Definition 4. *[21] Assume that $C^*[0,\ T]$ is the class of continuous column vector $V(t)$, whose components $x(t),y(t)\in C[0,T]$ is the class of continuous functions on $[0,T]$. The norm of $V(t)\in C^*[0,T]$ is given by $\|V\|=sup_t\left|e^{-Nt}x(t)\right|+sup_t\left|e^{-Nt}y(t)\right|$. When $t>\sigma\geq 0$, we write $C^*{}_\sigma[0,\ T]$ and $C_\sigma[0,T]$.*

Definition 5. *[21] $V(t)\in C^*[0,\ T]$ is a solution of the IVP in system (11) if the following (i) and (ii) hold.*

- **(i)** $(t,V(t))\in\mathcal{M},\ t\in[0,T]$, *where* $\mathcal{M}=[0,\ T]\times\mathcal{H}$, $\mathcal{H}=\{(x(t),y(t):|x(t)|\leq a,|y(t)|\leq b)\}$
- **(ii)** $V(t)$ *satisfies (23).*

Theorem 4. *Let $V(t)\in C^*[0,T]$ be a solution of the IVP given in (23). Then $V(t)$ is has unique solution for (23) if $0\leq\frac{(A+aB)}{N^\alpha}<1$.*

Proof. Let us write

$$I^{1-\alpha}\frac{d}{dt}V(t)=AV(t)+x(t)BV(t). \tag{25}$$

Operating with I^α, we obtain

$$V(t)=V(0)+I^\alpha\{AV(t)+x(t)BV(t)\}. \tag{26}$$

Now, let $F:C^*[0,\ T]\to C^*[0,\ T]$ be defined by

$$FV(t)=V(0)+I^\alpha\{AV(t)+x(t)BV(t)\}. \tag{27}$$

Then,

$$\begin{aligned} & e^{-Nt}\|FV-FU\|\\ =\ & e^{-Nt}\{I^\alpha\{A(V(t)-U(t))\\ & +x(t)B(V(t)-U(t))\}\}\\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}e^{-N(t-s)}(V(s)-U(s))\\ & e^{-Ns}(A+aB)ds\\ \leq\ & \frac{A+aB}{N^\alpha}\|V-U\|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds.\end{aligned}$$

This implies that $\|FV-FU\|\leq\frac{(A+aB)}{N^\alpha}\|V-U\|$. If we choose N^α such that $N^\alpha>A+aB$, then we obtain $\|FV-FU\|\leq\|V-U\|$, and the operator F given in (27) has a unique fixed point.

Consequently, (26) has a unique solution $V(t)\in C^*[0,\ T]$. From (26), we have

$$V(t)=V(0)+\frac{t^\alpha}{\Gamma(\alpha+1)}(AV(0)+x(0)BV(0))$$
$$+I^{\alpha+1}\left\{AV'(t)+x'(t)BV(t)+x(t)BV'(t)\right\},$$

and

$$\frac{dV(t)}{dt}=\frac{t^{\alpha-1}}{\Gamma(\alpha)}(AV(0)+x(0)BV(0))$$
$$+I^\alpha\left\{AV'(t)+x'(t)BV(t)+x(t)BV'(t)\right\},$$
$$\Longrightarrow e^{-Nt}\frac{dV(t)}{dt}=e^{-Nt}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}(AV(0)\right.$$
$$+x(0)BV(0))+I^\alpha\left\{AV'(t)+x'(t)BV(t)\right.$$
$$\left.\left.+x(t)BV'(t)\right\}\right\},$$

from which, we can deduce that $V'(t)\in C^*{}_\sigma[0,\ T]$. Thus, we have

$$\frac{dV(t)}{dt}=\frac{d}{dt}I^\alpha(AV(t)+x(t)BV(t))$$
$$\Longrightarrow I^{1-\alpha}\frac{dV(t)}{dt}=I^{1-\alpha}\frac{d}{dt}I^\alpha(AV(t)+x(t)BV(t)),$$
$$\Longrightarrow D_t^\alpha V(t)=AV(t)+x(t)BV(t),$$

and $V(0)=V_0$. Therefore, this IVP is equivalent to the initial value problem in (23).

4. Generalized Euler method

In this study, we used the Generalized Euler method to solve the initial value problem with the Caputo fractional derivative. Many mathematical models are composed of nonlinear systems, and solutions to these systems can be challenging. Analytical solutions cannot be found in most cases, and a numerical approach should be considered. One such approach is the Generalized Euler method [25]. Let $D_t^\alpha y(t)=f(t,y(t),\ y(0)=y_0$, $0<\alpha\leq 1,\ 0<t<\alpha$ be the initial value problem. Let $[0,a]$ be the interval we want to find the problem's solution. For convenience, subdivide the $[0, a]$ into n sub-intervals $[t_j, t_{j+1}]$, where $h = \frac{a}{n}$ and $j = 0,\ 1,\ \ldots,\ n-1$.Suppose that $y(t)$, $D_t^{\alpha} y(t)$ and $D_t^{2\alpha} y(t)$ are continuous in the range $[0, a]$. Using the generalized Taylor's formula, the following equality is obtained [25, 26];

$$y(t_1) = y(t_0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_0,\ y(t_0)). \tag{28}$$

This process will be repeated to create an array. Let $t_{j+1} = t_j + h$ for $j = 0,\ 1,\ \ldots,\ n-1$. The generalized formula is given as follows;

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_j,\ y(t_j)). \tag{29}$$

For every $k = 0,\ 1,\ \ldots,\ n-1$ with step size h, we get

$$\left\{ \begin{array}{rcl} x_{k+1} & = & x_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \\ & & \left\{ r x_k \left(1 - \frac{x_k}{K}\right) - (\mu+\beta+\theta) x_k \right\}, \\ y_{k+1} & = & y_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \left\{ \beta x_k - (\mu+\theta) y_k \right\}. \end{array} \right. \tag{30}$$

4.1. Local and global stability of the discretisized system

System (11) is discretized as (30) to analyze the dynamical effect of the habitat for sterilized and non-sterilized stray dogs in discrete time. The equilibrium points are the same as obtained in Section 2.

The Jacobian matrix of system (30) around the equilibrium point $(\overline{x},\ \overline{y})$ is obtained, such as

$$J((\overline{x}, \overline{y})) = \begin{pmatrix} 1 + \frac{h^{\alpha}\left(r - (\mu+\beta+\theta) - \frac{2r\overline{x}}{K}\right)}{\Gamma(\alpha+1)} & 0 \\ \frac{h^{\alpha}\beta}{\Gamma(\alpha+1)} & 1 - \frac{h^{\alpha}(\mu+\theta)}{\Gamma(\alpha+1)} \end{pmatrix}. \tag{31}$$

The eigenvalues of (4.4) are local asymptotic stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The saddle point exists for the condition if the absolute value of one of the eigenvalues is greater than 1. Thus, Theorem 4.1-Theorem 4.3. is given without proof.

Theorem 5. *Let Λ_1 be the equilibrium point of system (30). The following statements hold:*

- **(i)** *If $r < \mu+\beta+\theta$, then the equilibrium point Λ_1 is local asymptotic stable.*
- **(ii)** *If $r > \mu+\beta+\theta$, then the equilibrium point Λ_1 is an unstable saddle point.*

Theorem 6. *Let $\Lambda_2 = \left(\frac{K(r-\mu-\theta)}{r}, 0\right)$ be the equilibrium point of the non-serialized dog population of system (30). Then the following statements are true.*

- **(i)** *If $r > 2\mu + 2\theta$, then Λ_2 is local asymptotic stable.*
- **(ii)** *If $\mu + \theta < r < 2\mu + 2\theta$, then Λ_2 is an unstable saddle point.*

Theorem 7. *Let Λ_3 be the co-existing (positive) equilibrium point of system (30). Then the following statements hold.*

- **(i)** *If $r > 2(\mu+\theta+\beta)$, then Λ_3 is local asymptotic stable.*
- **(ii)** *If $\mu+\theta+\beta < r < 2(\mu+\theta+\beta)$, then Λ_3 is an unstable saddle point.*

Theorem 8. *Let Theorem 5.-(i) holds and assume that $\beta x_k < (\mu+\theta) y_k$, $k = 0,\ 1,\ 2,\ \ldots$. If*

$$h_1 < \left\{ \frac{2 x_k \Gamma(\alpha+1)}{r K^{-1} {x_k}^2 + x_k (\mu+\theta+\beta-r)} \right\}^{\frac{1}{\alpha}}$$

and

$$h_2 < \left\{ \frac{2 y_k \Gamma(\alpha+1)}{(\mu+\theta) y_k - \beta x_k} \right\}^{\frac{1}{\alpha}},$$

then the Λ_1 is global asymptotic stable.

Proof. We define the Lyapunov functions $W(x_k)$ and $\tilde{W}(y_k)$ such as

$$W(x_k) = (x_k - \overline{x})^2 \text{ and } \tilde{W}(y_k) = (y_k - \overline{y})^2, \tag{32}$$

$k = 0,\ 1,\ 2,\ \ldots$ The change along the solution of the first equation in system (4.3) around the equilibrium point Λ_1 is;

$$\begin{aligned} \Delta W(x_k) &= W(x_{k+1}) - W(x_k) \\ &= (x_{k+1})^2 - (x_k)^2 \\ &= (x_{k+1} - x_k)(x_{k+1} + x_k). \end{aligned}$$

From Theorem 5.(i), we obtain that

$$\begin{aligned} -x_k = \ & \frac{h^{\alpha}}{\Gamma(\alpha+1)} \\ & \left\{ -\frac{r {x_k}^2}{K} + (r - (\mu+\beta+\theta)) x_k \right\} < 0. \end{aligned} \tag{33}$$

Moreover,

$$\begin{aligned} x_{k+1} + x_k = \ & 2x_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \\ & \left\{ r x_k \left(1 - \frac{x_k}{K}\right) - (\mu+\beta+\theta) x_k \right\} > 0 \end{aligned}$$

if $h_1 < \left\{ \frac{2 x_k \Gamma(\alpha+1)}{r K^{-1} {x_k}^2 + x_k(\mu+\theta+\beta-r)} \right\}^{\frac{1}{\alpha}}$. Thus, we obtain the condition for $\Delta W(x_k) < 0$.

Similarly,

$$\begin{aligned} \Delta \tilde{W}(y_k) &= \tilde{W}(y_{k+1}) - \tilde{W}(y_k) \\ &= (y_{k+1})^2 - (y_k)^2 \\ &= (y_{k+1} - y_k)(y_{k+1} + y_k). \end{aligned}$$

Computations showed that $\Delta \tilde{W}(y_k) < 0$, if

$$h_2 < \left\{ \frac{2 y_k \Gamma(\alpha+1)}{(\mu+\theta) y_k - \beta x_k} \right\}^{\frac{1}{\alpha}},$$

where $\beta x_k < (\mu+\theta) y_k$, for $k = 0,\ 1,\ 2, \ldots$ This completes the proof.

Theorem 9. and Theorem 10. have similar steps as Theorem 8 and therefore they will be omitted.

Theorem 9. *Let Theorem 6.-(i) holds and assume that* $x_k \in \left(\frac{K(r-\mu-\theta)}{r}, \frac{(\mu+\theta)y_k}{\beta}\right)$, $k = 0,\ 1,\ 2,\ \ldots$. *If*

$$h_1 < \left\{\frac{2\left(x_k - \frac{K(r-\mu-\theta)}{r}\right)\mathit{\Gamma}(\alpha+1)}{rK^{-1}{x_k}^2 + x_k(\mu+\theta+\beta-r)}\right\}^{\frac{1}{\alpha}}$$

and

$$h_2 < \left\{\frac{2y_k\mathit{\Gamma}(\alpha+1)}{(\mu+\theta)y_k - \beta x_k}\right\}^{\frac{1}{\alpha}},$$

then the Λ_2 *is global asymptotic stable.*

Theorem 10. *Let Theorem 7.-(i) holds and assume that* $x_k \in \left(\frac{K(r-\mu-\beta-\theta)}{r}, \frac{(\mu+\theta)y_k}{\beta}\right)$ *and* $y_k > \frac{K\beta(r-\mu-\beta-\theta)}{r(\mu+\theta)}$, $k = 0,\ 1,\ 2,\ \ldots$. *If*

$$h_1 < \left\{\frac{2\left(x_k - \frac{K(r-\mu-\beta-\theta)}{r}\right)\mathit{\Gamma}(\alpha+1)}{rK^{-1}{x_k}^2 + x_k(\mu+\theta+\beta-r)}\right\}^{\frac{1}{\alpha}}$$

and

$$h_2 < \left\{\frac{2\left(y_k - \frac{K\beta(r-\mu-\beta-\theta)}{r(\mu+\theta)}\right)\mathit{\Gamma}(\alpha+1)}{(\mu+\theta)y_k - \beta x_k}\right\}^{\frac{1}{\alpha}},$$

then the Λ_3 *is global asymptotic stable.*

4.2. Semi-cycle analysis of positive solutions

In this section, we analyze the conditions of the semi-cycle of every oscillatory solution of system (30).

The definition is given for a different equation

$$x_{n+1} = f(x_n,\ x_{n-1})$$

and will be proven for a system constructed in (30).

From [27], a *positive semi-cycle* of a solution $\{x_n\}_{n=-1}^{\infty}$ of $x_{n+1} = f(x_n,\ x_{n-1})$ consists of a "string" of terms $\{x_k,\ x_{k+1},\ \ldots,\ x_m\}$, all greater than or equal to the equilibrium point $\overline{x}$, with $k \geq -1$ and $m \leq \infty$, such that

either $k = -1$ or $k > -1$ and $x_{k-1} < \overline{x}$

and

either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$.

A *negative semi-cycle* of a solution $\{x_n\}_{n=-1}^{\infty}$ of $x_{n+1} = f(x_n,\ x_{n-1})$ consists of a "string" of terms $\{x_k,\ x_{k+1},\ \ldots,\ x_m\}$, all less than the equilibrium point $\overline{x}$, with $k \geq -1$ and $m \leq \infty$, such that

either $k = -1$ or $k > -1$ and $x_{k-1} \geq \overline{x}$

and

either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \overline{x}$.

Theorem 11. *[31] Assume that* $f \in C[(0,\ \infty)\times(0,\ \infty),(0,\ \infty)\ ]$ *and that* $f(x,\ y)$ *is decreasing in both arguments. Let* $\overline{x}$ *be a positive equilibrium point of* $x_{n+1} = f(x_n,\ x_{n-1})$. *Then every oscillatory solution of the difference equation* $x_{n+1} = f(x_n,\ x_{n-1})$ *has a semi cycle of the length at most two.*

Let system (30) be denoted as

$$\begin{cases} f(x,y) = x + \frac{h^\alpha}{\Gamma(\alpha+1)}\{rx(1-\frac{x}{K}) - (\mu+\beta+\theta)x\}, \\ g(x,\ y) = y + \frac{h^\alpha}{\Gamma(\alpha+1)}\{\beta x - (\mu+\theta)y\}. \end{cases} \tag{34}$$

Then, the following theorem can be obtained from Theorem 11.

Theorem 12. *Assume that* $\{(x_k, y_k)\}_{k=0}^{\infty}$ *is a positive solution to the system (30). If*

$$\begin{aligned} h &< \left(\frac{\mu+\theta}{\mathit{\Gamma}(\alpha+1)}\right)^{-\frac{1}{\alpha}} \\ &< \left(\frac{2r(\mu+\theta)+\beta K[r-(\mu+\beta+\theta)]}{\mathit{\Gamma}(\alpha+1)(K\beta+2r)}\right)^{-\frac{1}{\alpha}} \end{aligned}$$

and

$$\begin{aligned} \frac{\left(\mathit{\Gamma}(\alpha+1)h^{-\alpha}+r-(\mu+\beta+\theta)\right)K}{2r} &< x \\ &< \frac{\mu+\theta-\mathit{\Gamma}(\alpha+1)h^{-\alpha}}{\beta}, \end{aligned}$$

where $r > \mu+\beta+\theta > \frac{\beta K(\mu+\beta+\theta)}{\mu+\beta+\beta K}$, *every oscillatory solution of system (30) has a semi-cycle of length at most two.*

Proof. The first derivative of (30) to the first equation concerning x is decreasing, if

$$1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\left\{r - (\mu+\beta+\theta) - \frac{2rx}{K}\right\} < 0, \tag{35}$$

which holds for $\frac{\left(\Gamma(\alpha+1)h^{-\alpha}+r-(\mu+\beta+\theta)\right)K}{2r} < x$, where $r > \mu+\beta+\theta$.

On the other side, the first derivative of (30) to the second equation concerning y, while x is fixed is decreasing, if

$$1 + \frac{h^\alpha}{\Gamma(\alpha+1)}\{\beta x - (\mu+\theta)\} < 0, \tag{36}$$

which holds for $x < \frac{\mu+\theta-\Gamma(\alpha+1)h^{-\alpha}}{\beta}$, where

$$h < \left(\frac{\mu+\theta}{\Gamma(\alpha+1)}\right)^{-\frac{1}{\alpha}}.$$

Considering both conditions of (35) and (36), we obtain

$$\begin{aligned} \frac{\left(\Gamma(\alpha+1)h^{-\alpha}+r-(\mu+\beta+\theta)\right)K}{2r} &< x \\ &< \frac{\mu+\theta-\Gamma(\alpha+1)h^{-\alpha}}{\beta}, \end{aligned} \tag{37}$$

where

$$\begin{aligned} h &< \left(\frac{\mu+\theta}{\Gamma(\alpha+1)}\right)^{-\frac{1}{\alpha}} \\ &< \left(\frac{2r(\mu+\theta)+\beta K[r-(\mu+\beta+\theta)]}{\Gamma(\alpha+1)(K\beta+2r)}\right)^{-\frac{1}{\alpha}}, \end{aligned} \tag{38}$$

and $r > \mu+\beta+\theta > \frac{\beta K(\mu+\beta+\theta)}{\mu+\beta+\beta K}$. This completes the proof.

5. Numerical simulation of fractional dog population model

This section will show the numerical simulation and graphics of the fractional dog population model. Using the Generalized Euler method, we get the numerical simulation of the dynamical behavior of the constructed model [11]. Turkey has an average of 10 million dogs, of which 1,300,000 are sterilized [29]. The initial conditions and parameters a given, such as
$x(0) = 8.700.000, y(0) = 1.300.000, \beta = 0.15, \mu = 0.01, \theta = 0.05, r = 0.02, K = 10^7$ with a step size of $h = 0,1$. Hence the Euler method, we obtain the following;

Table 3. The values of $x(t)$, $y(t)$ and $N(t)$ at time t and $\alpha = 1$.

t	$x(t)$	$y(t)$	$N(t)$
0	8700000,00	1300000,00	10000000,00
1	8527300,00	1422700,00	9863572,00
2	8358226,70	1542073,30	9731314,11
3	8192703,93	1658194,26	9602943,31
4	8030657,15	1771135,65	9478200,81
5	7872013,35	1880968,69	9356849,72
6	7716701,07	1987763,08	9238672,96
7	7564650,35	2091587,02	9123471,50
8	7415792,69	2192507,25	9011062,55
9	7270061,04	2290589,10	8901278,24
10	7127389,76	2385896,48	8793964,14
11	6987714,58	2478491,95	8688978,14
12	6850972,57	2568436,71	8586189,35
13	6717102,15	2655790,68	8485477,12
14	6586043,00	2740612,47	8386730,12

Table 4. The values of $x(t)$, $y(t)$ and $N(t)$ at time t and $\alpha = 0.9$.

t	$x(t)$	$y(t)$	$N(t)$
0	8700000,00	1300000,00	10000000,00
1	8473941,27	1460610,33	9821420,15
2	8254096,53	1615520,71	9649946,47
3	8040294,94	1764897,91	9484953,02
4	7832370,41	1908904,04	9325883,28
5	7630161,37	2047696,68	9172241,56
6	7433510,71	2181428,98	9023585,59
7	7242265,66	2310249,85	8879520,20
8	7056277,61	2434303,97	8739691,67
9	6875402,06	2553732,01	8603782,95
10	6699498,47	2668670,68	8471509,53
11	6528430,18	2779252,86	8342615,67
12	6362064,27	2885607,71	8216831,25
13	6200271,47	2987860,76	8094068,93
14	6042926,09	3086134,01	7974021,67

Table 5. The values of $x(t)$, $y(t)$ and $N(t)$ at time t and $\alpha = 0.8$.

t	$x(t)$	$y(t)$	$N(t)$
0	8700000,00	1300000,00	10000000,00
1	8406124,65	1508792,73	9767846,98
2	8122750,85	1707952,60	9547619,77
3	7849503,32	1897846,02	9337967,56
4	7586020,22	2078826,05	9137731,77
5	7331952,60	2251232,94	8945915,68
6	7086964,01	2415394,56	8761659,43
7	6850730,02	2571626,81	8584219,21
8	6622937,77	2720234,13	8412950,08
9	6403285,62	2861509,83	8247291,47
10	6191482,66	2995736,55	8086755,09
11	5987248,42	3123186,59	7930914,72
12	5790312,43	3244122,35	7779397,56
13	5600413,89	3358796,62	7631876,92
14	5417301,32	3467452,95	7488065,96

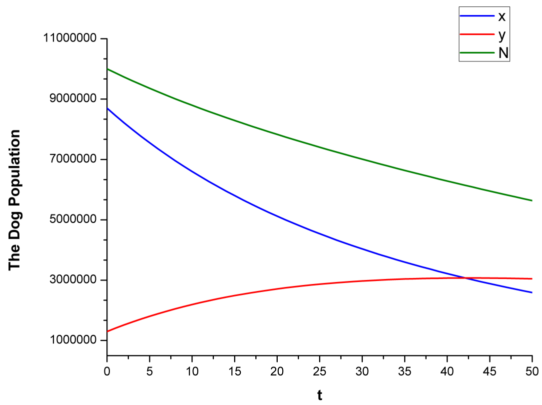

Figure 1. The graph of the change of the $x(t)$, $y(t)$ Compartment model and the N population with respect to time for $\alpha = 1$.

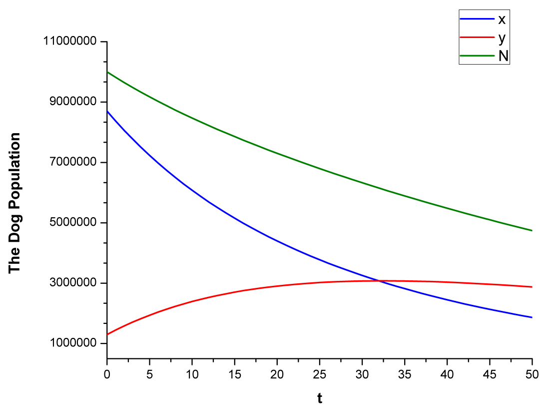

Figure 2. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 0.9$.

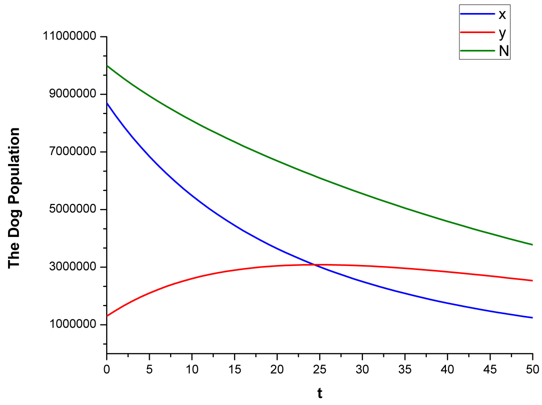

Figure 3. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 0.8$.

In Table 3, Table 4, and Table 5, the changes in the $x(t)$,$y(t)$ compartments and the $N(t)$ population are observed for different states of α. According to the graphs above, we can make the following comments. It is observed that According to the graphs above, we can make the following comments.

- The number of dogs that have not been neutered will decrease over time (Fig-1).
- The number of neutered dogs will increase over time (Fig-2).
- The number of dogs in the population will decrease slowly over time (Fig-3).

5.1. Cases According to the Sterilization Rate

Case1. The number of spayings performed in the dog population must be kept at an acceptable level. If the annual sterilization rate $\beta = 0.5$ is taken, the following graphs are obtained.

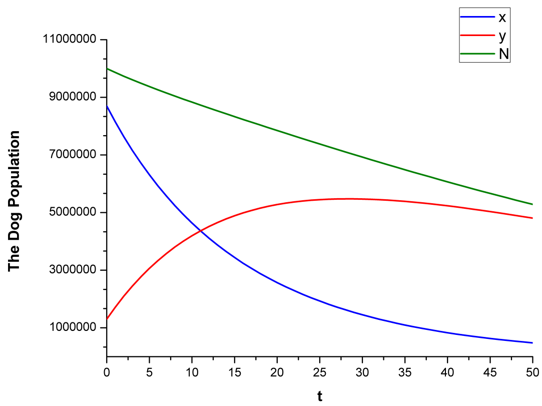

Figure 4. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 1$.

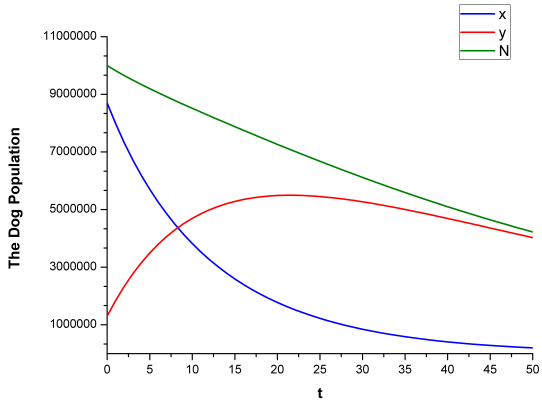

Figure 5. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 0.9$.

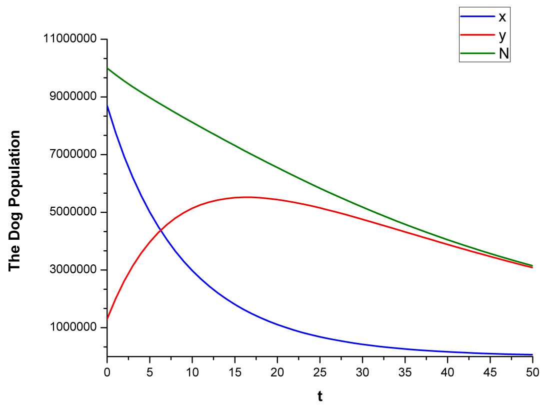

Figure 6. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 0.8$.

In the above figures, we observe the following highlights:

- It is observed that the number of non-neutered dogs will decrease over time and become less than the number of neutered dogs (Fig-4).
- It is observed that the number of neutered dogs will increase over time and become more significant than the number of non-neutered dogs (Fig-5).
- It is observed that the number of dogs in the population will decrease slowly over time (Fig-6).

Case2. If the annual sterilization rate $\beta = 0.05$ is taken, the following graphs are obtained.

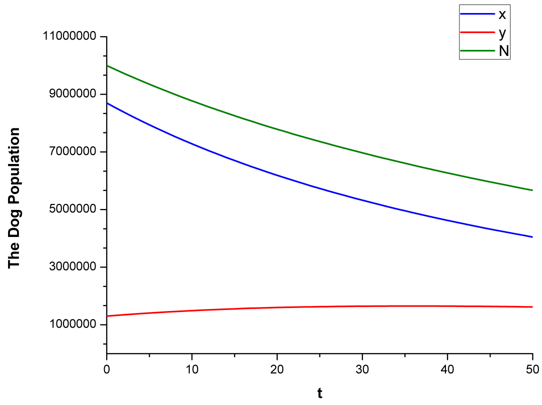

Figure 7. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 1$.

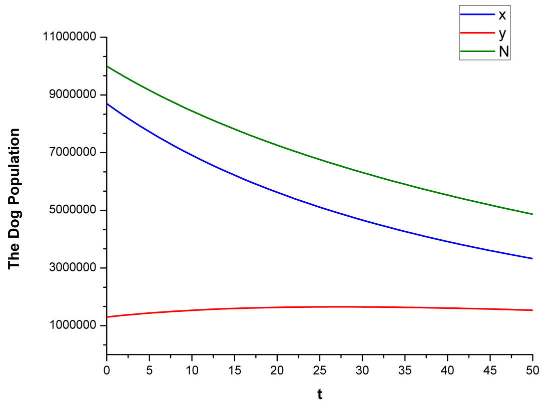

Figure 8. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 0.9$.

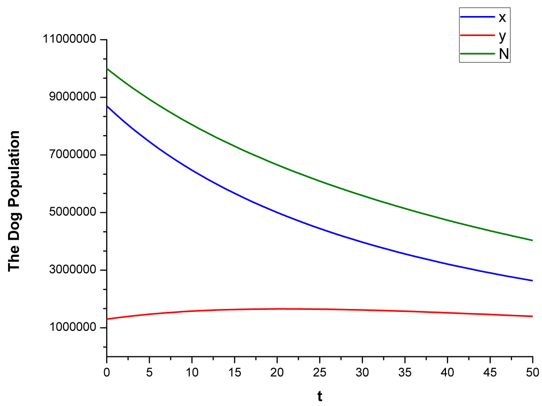

Figure 9. The graph of the change of the $x(t)$, $y(t)$ compartment model and the $N(t)$ population with respect to time for $\alpha = 0.8$.

In the above figures, we observe the following highlights:

- It is observed that the number of non-neutered dogs decreases slowly over time and progresses steadily (Fig-7).
- It is observed that the number of neutered dogs increases slowly over time and progresses steadily (Fig-8.
- It is observed that the number of dogs in the population progresses steadily, slowly decreasing over time (Fig-9).

6. Discussion

Over 10 million stray dogs are thought to exist in Turkey. Within ten years, it is predicted that there will be 60 million dogs without owners [29]. These numbers will inevitably lead to a rise in unattended dogs, dramatically raising the risk of unattended dog attacks that result in fatalities, serious injuries, and security issues. The attack on the unclaimed dogs resulted in numerous persons' death or severe injury. This study undertook the necessary analysis and demonstrated the significance of the sterilization method for managing the dog population. Our findings highlight the significance of managing the dog population and building up and implementing the mathematical model.

In order to promote the health and well-being of both dogs and people, dog population management tries to alter the indicators of population dynamics (cutting unintended births and abandonments, enhancing preventive treatment, and immigration control) [30]. Local sensitivity was examined by Baquero et al. (2016), who discovered that male and female sterilization have comparable effects [31]. Female animal sterilization is more effective than male animal sterilization, according to Amaku et al. (2010), and this activity depends on the size of the beginning population, the pace of growth, the rate of sterilization, and the timing of reproductive control [1].

The natural death rate, growth rate, sterilization rate, and carrying capacity of dogs were assessed while the dog population model was being developed. The logistical growth model was considered when creating the dog population model. Transportation capacity is one of the most valuable variables in simulations of dog population models. Neutering will no longer affect dogs as much as there will be more of them. Therefore, it is crucial to maintain population control.

Our findings highlight the significance of sterilization for managing the dog population. For extended periods, population dynamics-related factors will not have a single value. The model's parameters fall inside a reasonable range of values from a biological perspective and will become

more variable.
Concerning the interactions between populations of owned and stray dogs, the revised model allowed us to incorporate the effects of several factors on dog population dynamics. Thanks to our selected parameters, we could determine how the population number has changed. The impact of characteristics relating to natural mortality and transit capacity has also been considered in a more detailed model, and the impact of prospective interventions in the future sterilizing process has been quantified. The variance in the logistical growth model and the sterilization rate have demonstrated the possible effects of sterilization. The dynamics of the population will alter if the carrying capacity changes. This will impact the dynamics of leaving possessed dogs behind. Sterilization techniques and abandonment prevention will reduce population variance.
Our model enables us to get crucial knowledge regarding the dog population in the future through the neutering tactics of male and female dogs. Sustainable political, sanitary, moral, ethical, ecological, and human policies are needed to manage stray dog numbers on a social and environmental level. Such actions, like zoonoses like rabies and leishmaniasis control, should benefit animals and community members. Our findings show how crucial human variables are to the planning and execution of population control initiatives. The nation's governments benefit significantly from dog population models in their dog management initiatives.

7. Conclusions

In order to discuss the population of stray dogs, a new model of differential equations with fractal orders is constructed in this paper. The Routh-Hurwitz Criteria were used to analyze the equilibrium point's local stability. We applied the Generalized Euler Method to demonstrate the system's dynamical behavior in discrete time. This discretization technique proved the global and semicycle analyses. After the study, we use actual data to illustrate our findings and demonstrate how they align with our theoretical predictions. According to the graphs, the number of unsterilized dogs would gradually decline over time, the number of sterilized dogs would increase, and the overall number would gradually decline. The sterilization rate for the dog population was plotted and interpreted with $\beta = 0.05$ and $\beta = 0.5$. For the population of stray dogs, dog population models are particularly crucial. This was done using a mathematical model of controlling the dog population.
Figures 1-3 show that for $\beta = 0.15$, there is a regulated decline in the dog population due to neutering and natural causes. After around 43 years, if the sterilization rate stays constant, more neutered dogs will than unneutered dogs. After the fiftieth year, the population is anticipated to decline quickly. The drop in each compartment will be more pronounced with an increase in the parameter α. The number of dogs in each compartment reduced more quickly when the neutering rate was raised to $\beta = 0.5$ (see Fig. 4-6).
After 50 years, it is anticipated that the population will start to decline quickly in this circumstance. The drop in each compartment will be more pronounced with an increase in the parameter. The number of dogs in each compartment was observed to drop more quickly as the neutering rate rose (see Fig. 4-6). In this scenario, after roughly 31 years, there will be more neutered dogs than unneutered canines in the population, rapidly reducing the overall population. In this situation, the dog's extinction in the nation may one day be in jeopardy.
As a result, limiting the number of dogs in areas where people congregate is essential. Sterilization is the most ethical way to go about doing this. Despite this, sterilization must be carried out at a specific rate. In the absence of such measures, either the dog population will grow out of control or face extinction. The results of the model suggested in this article show that the value β should be maintained at or close to 0.1 to maintain the current dog population.

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