An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.14, No.1, pp.74-80 (2024)
http://doi.org/10.11121/ijocta.1414

RESEARCH ARTICLE

An approximate solution of singularly perturbed problem on uniform mesh

Derya Arslan,[a] Ercan Celik[b*]

[a] *Department of Mathematics, University of Bitlis Eren, Turkey*
[b] *Department of Mathematics, Kyrgyz-Turkish Manas University, Kyrgyz*
d.arslan@beu.edu.tr, ercan.celik@manas.edu.kg

ARTICLE INFO

Article History:
Received 17 June 2023
Accepted 3 October 2023
Available Online 10 January 2024

Keywords:
Singularly perturbed equation
Integral boundary condition
Finite difference scheme
Uniform mesh

AMS Classification 2010:
65L10; 65L11; 65L12; 65L15; 65L20; 65L70; 34B10

ABSTRACT

In this study, we obtain approximate solution for singularly perturbed problem of differential equation having two integral boundary conditions. With this purpose, we propose a new finite difference scheme. First, we construct this exponentially difference scheme on a uniform mesh using the finite difference method. We use the quasilinearization method and the interpolating quadrature formulas to establish the numerical scheme. Then, as a result of the error analysis, we show that the method under study is convergent in the first order. Consequently, theoretical findings are supported by numerical results obtained with an example. Approximate solutions curves are compared on the chart to provide concrete indication. The maximum errors and convergence rates obtained are given on the table for different ε and N values.

1. Introduction

This study is concerned with the numerical solution the following singularly perturbed equation with integral boundary values for $0 \leq \ell_0 < \ell_1 \leq \ell$

$$\varepsilon^2 u''(t) + \varepsilon a(t) u'(t) - g(t,u) = 0, \quad 0 < t < \ell, \tag{1}$$

$$u(0) = \int_{\ell_0}^{\ell_1} u(t) f_0(t)\, dt + A, \tag{2}$$

$$u(\ell) = \int_{\ell_0}^{\ell_1} u(t) f_1(t)\, dt + B. \tag{3}$$

Here, ε is the perturbation parameter and is defined as $0 < \varepsilon \ll 1$. A and B are fixed. $a(t)$ and $g(t,u)$ are continuous functions in the interval $[0,\ell]$ and $[0,\ell] \times \mathbb{R}$, respectively. $f_0(t)$ and $f_1(t)$ are continuous functions on $[\ell_0, \ell_1]$.

When $\varepsilon = 0$ in the Eq. (1), the new equation is an algebraic equation. The boundary conditions will be unnecessary for the solution of this equation. In this case, there will be two boundary layers $t = 0$ and $t = 1$ of the problem (1-(3).

Equations with a positive parameter ε in the coefficient of the highest order derivative are called singularly perturbed equations. Solutions of these problems have thin boundary layers. In these layers, the solution changes abruptly and rapidly, while in other parts of the definition region it changes slowly and regularly. This irregularity causes the solution of singularly perturbed problems to have unlimited derivatives. Thus, serious difficulties arise in the operation of such problems. These difficulties are also evident in numerical solution. Because the approximate solution diverges from the exact solution as the mesh steps get smaller. For this reason, it is very important to establish appropriate numerical methods for the solution of problems with singular perturbations. Known classical numerical methods cannot give numerical results suitable for the exact solution. Especially in this study, an efficient numerical method such as the finite difference method, which gives uniform convergence according to ε

*Corresponding Author

is preferred for the solution of such problems [1]-[15]. Studies on problems with singular perturbations started in the 1900s. These problems are encountered in science, economics, sociology, engineering, medical science, fluids mechanics, aerodynamics, magnetic dynamics, emission theory, reaction diffusion, light emitting waves, communication lines, plasma dynamics, purified gas dynamics, motion of mass, plastics, chemical reactor theory, seismology, oceanography, meteorology, electric current, ion acoustic waves and some physical modeling [16]- [21]. Also, Bakhvalov used a special transformation in the numerical solution of boundary layer problems [22]. Bitsadze and Samarskii obtained some generalizations of linear elliptic boundary value problems [23]. Herceg and Surla found the numerical solution of the singularly perturbed problem with non-local boundary values using spline tension [24]. Gupta and Trofimchuk studied a sharper case for solving the second-order three-point boundary value problem [25]. Amiraliyev and Cakir found a uniformly convergent approach for the zeroth order reduced equation and convective singularly perturbed problem [2]. Cakir, for the three-point singularly perturbed problem, using the difference method, a second-order uniform convergence was obtained [3]. Cakir and Amiraliyev evaluated the three-point singularly perturbed boundary value problem using a Shishkin mesh [4]. In [1, 6–8, 24] and [26, 34] have been worked on the numerical solution of the singularly perturbed problem with nonlocal boundary and integral boundary conditions. The singularly perturbed problem has also been solved by different numerical methods [9, 10, 35]. There are also studies on existence and uniqueness of these problems in the literature [36]-[38].

The study of the linear state of our problem (1)-(3) is in [1], where the problem is solved with a layer-adapted mesh. The difference of this study from similar studies in the literature [6] is the use of a uniform mesh and the two integral boundary conditions of the problem (1)-(3). Although there are studies on singularly perturbed problems with two integral boundary conditions solved in Shishkin mesh as in [6], we have not come across any studies on singularly perturbed problems with two integral boundary conditions in uniform mesh. This gap, which is not in the literature, constitutes the motivation of our study.

In this paper, the difference scheme is obtained using the integral rules from [30]. In the second part, we investigated several important factors for the exact solution (1)-(3). The difference procedure for the uniform mesh of the problem (1)-(3) is given in part 3. In the fourth part, the convergence evaluation of the method is made. For the purpose of applying the theoretical procedure, an example whose exact solution is unknown is presented in Sect 5.

Throughout this study, C and C_0 will be used as positive constants that do not depend on ε and h. The norm $\|.\|$ is used to denote the maximum norm.

2. Some properties of the Exact Solution

Here we will give a Lemma and its proof, which will be needed for later parts of the study.

Lemma 1. *Let $u(t)$ be the solution of the (1)-(3), $a(t) \in C^1[0,\ell]$, $\gamma = \int_{\ell_0}^{\ell_1} (|f_0(t)| + |f_1(t)|)\, dt < 1$, $\partial g/\partial u - \varepsilon a'(t) \geq \beta_* >$ and $|\partial g/\partial t| \leqslant C$ for $t \in [0,\ell]$, then the estimations*

$$|u(t)| \leqslant C_0, \tag{4}$$

$$|u'(t)| \leqslant C\left\{1 + \frac{1}{\varepsilon}\left(e^{-\frac{c_0 t}{\varepsilon}} + e^{-\frac{c_1(\ell-t)}{\varepsilon}}\right)\right\}, \tag{5}$$

hold, where $0 \leqslant t \leqslant \ell$ and

$$C_0 = (1-\gamma)^{-1}\left(|A| + |B| + \beta^{-1}\|F\|_\infty\right),$$
$$c_0 = \frac{1}{2}\left(\sqrt{a^2(0) + 4\beta_*} + a(0)\right),$$
$$c_1 = \frac{1}{2}\left(\sqrt{a^2(\ell) + 4\beta_*} - a(\ell)\right).$$

Proof. Using the mean value theorem for $g(t,u)$ in (1), we have

$$g(t,u) = \frac{\partial g(t,\xi u)}{\partial u}u(t) + g(t,0), \quad 0 < \xi < 1,$$

supposing of

$$b(t) = \frac{\partial g(t,\xi u)}{\partial u} > 0,\ G(t) = g(t,0).$$

Let's rewrite the (1)-(3) problem as follows to get the proof of (4)

$$\varepsilon^2 u''(t) + \varepsilon a(t)u'(t) - b(t)\,u(t) = G(t), \tag{6}$$

$$u(0) = \int_{\ell_0}^{\ell_1} f_0(x)\,u(x)\,dx + A, \tag{7}$$

$$u(\ell) = \int_{\ell_0}^{\ell_1} f_1(t)\,u(t)\,dt + B. \tag{8}$$

Here, using the maximum principle [4,39] and (6)-(8) we arrive at the following inequality:

$$|u(t)| \leq |u(0)| + |u(\ell)| + \beta^{-1}\|G\|_\infty,\ t \in [0,l]. \tag{9}$$

Now, using the boundary values (7) and (8), let's obtain the inequality (4)

$$|u(0)| \leq |A| + \int_{\ell_0}^{\ell_1} |f_0(t)|\,|u(t)|\,dt, \qquad (10)$$

$$|u(\ell)| \leq |B| + \int_{\ell_0}^{\ell_1} |f_1(t)|\,|u(t)|\,dt. \qquad (11)$$

If we write the inequalities (10) and (11) in the inequality (9), we get the following result:

$$\begin{aligned}|u(t)| &\leq |A| + |B| + \int_{\ell_0}^{\ell_1} |f_0(t)|\,|u(t)|\,dt \\ &+ \int_{\ell_0}^{\ell_1} |f_1(t)|\,|u(t)|\,dt + \beta^{-1}\|G\|_\infty \\ &\leq |A| + |B| + \max_{[\ell_0,\ell_1]} |u(t)| \int_{\ell_0}^{\ell_1} |f_0(t)|\,dt \\ &+ \max_{[\ell_0,\ell_1]} |u(t)| \int_{\ell_0}^{\ell_1} |f_1(t)|\,dt + \beta^{-1}\|G\|_\infty \\ &\leq |A| + |B| + \|u\|_\infty \int_{\ell_0}^{\ell_1} |f_0(t)|\,dt \\ &+ \|u\|_\infty \int_{\ell_0}^{\ell_1} |f_1(t)|\,dt + \beta^{-1}\|G\|_\infty .\end{aligned}$$

Thus, the proof of (4) is completed. Also, the proof of (5) is almost the same to that of [39]. □

3. Uniform Mesh and Construction of the difference scheme

In this part, we will obtain the difference scheme for the (1)-(3) problem. For this we will work on the uniform mesh.

$$\omega_h = \left\{ t_i = ih, i = 1, 2, \ldots, N-1 : h = \frac{\ell}{N} \right\},$$
$$\bar{\omega}_h = \omega_h \cup \{t_0 = 0,\ t_N = \ell\}.$$

where N is the number of discretization points.

Let's give some notations for grid functions, where y_i is the approximate value for $u(t)$ at grid points t_i.

$$f_{\bar{t},i} := \frac{f_i - f_{i-1}}{h}, \quad f_{t,i} := \frac{f_{i+1} - f_i}{h},$$
$$f_{\overset{\circ}{t},i} := \frac{f_{t,i} + f_{\bar{t},i}}{2},$$
$$f_{\bar{t}t,i} := \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2},\ i = 1, 2, ..., N,$$
$$\|f\|_\infty \equiv \|f\|_{\infty,\bar{\omega}_h} := \max_{0 \leqslant i \leqslant N} |f_i|.$$

Let's start constructing the difference scheme with the following equation for $1 \leqslant i \leqslant N-1$:

$$\xi_i^{-1} h^{-1} \int\limits_{t_{i-1}}^{t_{i+1}} Lu(t)\varphi_i(t)dt = 0, \qquad (12)$$

where the functions $\{\varphi_i(t)\}_{i=1}^{N-1}$ have the form

$$\varphi_i(t) = \begin{cases} \varphi_i^{(1)}(t), & t_{i-1} < t < t_i, \\ \varphi_i^{(2)}(t), & t_i < t < t_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\varphi_i^{(1)}(t)$ and $\varphi_i^{(2)}(t)$, respectively, are the solution of the following problems:

$$\varepsilon\varphi_i^{(1)}(t) - a_i\varphi_i^{(1)}(t) = 0,\ t_{i-1} < t < t_i,$$
$$\varphi_i^{(1)}(t_{i-1}) = 0,\ \varphi_i^{(1)}(t_i) = 1,$$
$$\varepsilon\varphi_i^{(2)}(t) - a_i\varphi_i^{(2)}(t) = 0,\ t_i < t < t_{i+1},$$
$$\varphi_i^{(2)}(t_{i-1}) = 0,\ \varphi_i^{(2)}(t_i) = 1,$$

and the coefficient ξ_i^{-1} in (12) as

$$\xi_i^{-1} = \left(h^{-1} \int\limits_{t_{i-1}}^{t_{i+1}} \varphi_i(t)dt \right)^{-1}.$$

If we rearrange (12), we get the following system

$$-\varepsilon^2\xi_i^{-1}h^{-1} \int\limits_{t_{i-1}}^{t_{i+1}} \varphi_i'(t)u'(t)\,dt + \varepsilon a_i^{-1}h^{-1} \int\limits_{t_{i-1}}^{t_{i+1}} \varphi_i(t)u'(t)\,dt$$
$$-g(t_i, u_i) + R_i = 0, \qquad i = 1, 2, ..., N-1, \quad (13)$$

where

$$R_i = \varepsilon\xi_i^{-1}h^{-1} \int\limits_{t_{i-1}}^{t_{i+1}} [a(t) - a(t_i)]\,\varphi_i(t)u'(t)\,dt$$
$$-\xi_i^{-1}h^{-1} \int\limits_{t_{i-1}}^{t_{i+1}} \left[\varphi_i(t) \int\limits_{t_{i-1}}^{t_{i+1}} \frac{d}{dt} g(\xi, u(\xi))K_{0,i}^*(t,\xi)\,d\xi \right] dt, \qquad (14)$$
$$K_{0,i}^*(t,\xi) = T_0(t-\xi) - T_0(t_i - \xi).$$

If we benefit the formulas from [5], we have the following system for $1 \leq i \leq N-1$ from (13)

$$-\varepsilon^2\xi_i^{-1}h^{-1} \int\limits_{t_{i-1}}^{t_i} \varphi_i'(t)u'(t)\,dt - \varepsilon^2\xi_i^{-1}h^{-1} \int\limits_{t_i}^{t_{i+1}} \varphi_i'(t)u'(t)\,dt$$
$$+ \varepsilon a_i\xi_i^{-1}h^{-1} \int\limits_{t_{i-1}}^{t_i} \varphi_i(t)u'(t)\,dt$$
$$+ \varepsilon a_i\xi_i^{-1}h^{-1} \int\limits_{t_i}^{t_{i+1}} \varphi_i(t)u'(t)\,dt - g(t_i, u_i) + R_i$$

$$= -\varepsilon^2\xi_i^{-1}h^{-1}u_{\bar{t},i} \int\limits_{t_{i-1}}^{t_i} \varphi_i^{(1)'}(t)dt$$

$$-\varepsilon^2 \xi_i^{-1} h^{-1} u_{t,i} \int\limits_{t_i}^{t_{i+1}} \varphi_i^{(2)'}(t)dt$$

$$+\varepsilon a_i \xi_i^{-1} h^{-1} u_{\bar{x},i} \int\limits_{t_{i-1}}^{t_i} \varphi_i^{(1)}(t)dt$$

$$+\varepsilon a_i \xi_i^{-1} h^{-1} u_{t,i} \int\limits_{t_i}^{t_{i+1}} \varphi_i^{(2)}(t)dt - g(t_i, u_i) + R_i$$

$$= -\varepsilon^2 \xi_i^{-1} u_{\bar{t},i} \chi_{1,i} - \varepsilon^2 \xi_i^{-1} u_{t,i} \chi_{2,i} + \varepsilon a_i \xi_i^{-1} u_{\bar{x},i} \chi_{1,i}$$
$$+\varepsilon a_i \xi_i^{-1} u_{t,i} \chi_{2,i} - g(t_i, u_i) + R_i = 0.$$

After the $u_{\bar{t},i} = u_{\overset{\circ}{t},i} - \frac{h}{2} u_{\bar{t}t,i}$ and $u_{t,i} = u_{\overset{\circ}{t},i} + \frac{h}{2} u_{\bar{t}t,i}$ are substituted in the above equation, the following expression is obtained:

$$\varepsilon^2 \left\{ \xi_i^{-1} \left(1 + 0.5\varepsilon^{-1} h a_i (\chi_{2,i} - \chi_{1,i})\right) \right\} u_{\bar{t}t,i}$$
$$+\varepsilon a_i u_{\overset{\circ}{t},i} - g(t_i, u_i) + R_i = 0,$$

where

$$\chi_{1,i} = h^{-1} \int\limits_{t_{i-1}}^{t_i} \varphi_i^{(1)}(t)dt, \; \chi_{2,i} = h^{-1} \int\limits_{t_i}^{t_{i+1}} \varphi_i^{(2)}(t)dt.$$

So, from the above equations, the difference scheme is defined for $1 \leqslant i \leqslant N-1$:

$$\varepsilon^2 \theta_i u_{\bar{t}t,i} + \varepsilon a_i u_{\overset{\circ}{t},i} - g(t_i, u_i) + R_i = 0, \tag{15}$$

here

$$\theta_i = \xi_i^{-1} \left(1 + \frac{(\chi_{2,i} - \chi_{1,i})}{2\varepsilon} h a_i\right). \tag{16}$$

Now, the approximations for the first and second boundary conditions need to be determined. Let t_{N_0} and t_{N_1} be the grid points nearest to ℓ_0 and ℓ_1, respectively.

$$\int_{\ell_0}^{\ell_1} f_0(t) u(t) dt = \int_{\ell_0}^{t_{N_0}} f_0(t) u(t) dt$$
$$+ \int_{t_{N_0}}^{t_{N_1}} f_0(t) u(t) dt + \int_{t_{N_1}}^{\ell_1} f_0(t) u(t) dt, \tag{17}$$

and

$$\int_{t_{N_0}}^{t_{N_1}} f_0(t) u(t) dt = \sum_{i=N_0}^{N_1} \left[\int_{t_{i-1}}^{t_i} f_0(t) dt\right] u(t_i) + \bar{r}_0$$
$$= K_0(u) + \bar{r}_0, \tag{18}$$

where

$$K_0(u) = \sum_{i=N_0}^{N_1} \left[\int_{t_{i-1}}^{t_i} f_0(t) dt\right] u(t_i), \tag{19}$$

$$\bar{r}_0 = \sum_{i=N_0}^{N_1} \int_{t_{i-1}}^{t_i} \left[f_0(t) \int_{t_{i-1}}^{t_i} u'(\xi) (T_0(t-\xi) - 1) d\xi\right] dt.$$

Thus, we get the difference approximation corresponding to the first boundary value as:

$$u_0 - K_0(u) = A + r_0, \tag{20}$$

where

$$r_0 = \int_{\ell_0}^{t_{N_0}} f_0(t) u(t) dt + \int_{t_{N_1}}^{\ell_1} f_0(t) u(t) dt + \bar{r}_0. \tag{21}$$

Now we get the difference approtimation corresponding to the second boundary value as follows:

$$u_N - K_1(u) = B + r_1, \tag{22}$$

where

$$K_1(u) = \sum_{i=N_0}^{N_1} \left[\int_{t_{i-1}}^{t_i} f_1(t) dt\right] u(t_i), \tag{23}$$

$$r_1 = \int_{\ell_0}^{t_{N_0}} f_1(t) u(t) dt + \int_{t_{N_1}}^{\ell_1} f_1(t) u(t) dt + \bar{r}_1, \tag{24}$$

$\bar{r}_1 = \sum_{i=N_0}^{N_1} \int_{t_{i-1}}^{t_i} \left[f_1(t) \int_{t_{i-1}}^{t_i} u'(\xi) (T_0(t-\xi) - 1) d\xi\right] dt.$

If the error values in the (14), (21) and (24) are neglected, the following difference chart is found for $1 \leqslant i \leqslant N-1$:

$$\varepsilon^2 \theta_i y_{\bar{t}t,i} + \varepsilon a_i y_{\overset{\circ}{t},i} - g(t_i, y_i) = 0, \; , \tag{25}$$

$$y_0 = K_0(y) + A, \tag{26}$$

$$y_N = K_1(y) + B. \tag{27}$$

4. Stability of the Difference Scheme

Here, we give the stability of the finite difference method with the Theorem 1 and the evaluation of the error functions with the Lemma 2.

The error term z is $z = y - u$ for $1 < i < N$.

$$\varepsilon^2 \theta_i z_{\bar{t}t,i} + \varepsilon a_i z_{\overset{\circ}{t},i} - [g(t_i, y_i) - g(t_i, u_i)] = R_i, \tag{28}$$

$$z_0 = K_0(z) + r_0, \tag{29}$$

$$z_N = K_1(z) + r_1. \tag{30}$$

Lemma 2. *The estimates are valid for the terms R_i, r_0 and r_1 to be obtained with the help of results of Section 1 and Lemma 1*

$$\|R\|_{\infty,\omega_h} \le Ch, \tag{31}$$

$$|r_0| \le Ch, \tag{32}$$

$$|r_1| \le Ch. \tag{33}$$

Proof. We have from the expression (14) for R_i on an arbitrary mesh as follows

$$|R_i| \le C\left\{h + h + \int\limits_{t_{i-1}}^{t_{i+1}} \left(1+\left|u'(\xi)\right|\right)d\xi\right\}.$$

This inequality and (5) enable us to write the inequality as

$$|R_i| \le C\left\{h + \frac{1}{\varepsilon}\int\limits_{t_{i-1}}^{t_{i+1}} \left(e^{-\frac{c_0 t}{\varepsilon}} + e^{-\frac{c_1(\ell-t)}{\varepsilon}}\right)dx\right\}. \tag{34}$$

The mesh is uniform with $h = \ell N^{-1}$ for $1 \le i \le N$. So, from the above inequality, we get

$$\begin{aligned} |R_i| &\le C\left\{N^{-1} + \varepsilon^{-1}h\right\} \\ &\le Ch. \end{aligned}$$

Let us evaluate (32) using the expression (21) for r_0 as

$$\begin{aligned} |r_0| \le \\ \sum_{i=N_0}^{N_1}\int_{t_{i-1}}^{t_i}\left[|f_0(t)|\int_{t_{i-1}}^{t_i}\left|u'(\xi)\right||T_0(t-\xi)-1|\,d\xi\right]dt \\ +\int_{\ell_0}^{t_{N_0}}|f_0(t)||u(t)|\,dt + \int_{t_{N_1}}^{\ell_1}|f_0(t)||u(t)|\,dt \\ \le C\max_{[t_{i-1},\,t_i]}|f_0(t)|\int_0^{\ell}\left|u'(t)\right|dt + O(h) \\ \le Ch. \end{aligned} \tag{35}$$

The proof of (33) is similar to the proof of the inequality (32). All these complete the proof of Lemma 2. □

Lemma 3. *If z_i is the solution of (28)-(30) and*

$$\bar{\gamma} = \sum_{i=N_0}^{N}\int_{t_{i-1}}^{t_i}\left[|f_0(t)| + |f_1(t)|\right]dt < 1.$$

Then there is the following the estimate

$$\|z\|_{\infty,\bar{\omega}_h} \le C\left(\beta^{-1}\|R\|_{\infty,\omega_h} + |r_0| + |r_1|\right). \tag{36}$$

Proof. Using Lemma 2, we easily obtain (36). □

Theorem 1. *If u be the solution of (1)-(3) and y be the solution of (25)-(27). Then, the following estimate is satisfied.*

$$\|y-u\|_{\infty,\bar{\omega}_h} \le Ch.$$

This theorem gives the result of the convergence of the proposed method with the help of Lemma 2 and Lemma 3.

5. Numerical Illustrations

Here we provide some numerical results that exemplify the current method.

By using the quasilinearization technique, the scheme (25)-(27) can be arranged as:

$$\varepsilon^2\theta_i y^{(n)}_{\bar{t}t,i} + \varepsilon a_i y^{(n)}_{\overset{\circ}{t},i} - g\left(t_i, y_i^{(n-1)}\right) \tag{37}$$

$$-\frac{\partial g}{\partial y}\left(t_i, y_i^{(n-1)}\right)\left(y_i^{(n)} - y_i^{(n-1)}\right) = 0,$$

$$y_0^{(n)} = h\sum_{i=N_0}^{N_1} f_{0,i}y_i^{(n-1)} + A, \tag{38}$$

$$y_N^{(n)} = h\sum_{i=N_0}^{N_1} f_{1,i}y_i^{(n-1)} + B, \tag{39}$$

where $y_i^{(0)}$ for $1 \le i \le N$ and $n \ge 1$ is the initial guess.

Example 1. We apply the scheme (37)-(39) to the following singularly perturbed problem with integral boundary conditions:

$$\varepsilon^2 u'' + \varepsilon(1+t)u' = 2u - \arctan(t+u),\ 0 < t < 1,$$

$$u(0) = \int_{0.5}^{1}\cos(\pi t)\,u(t)\,dt + 1,$$

$$u(1) = \int_{0.5}^{1}\sin(\pi t)u(t)\,dt + 1.$$

The exact solution of the problem is unknown. For this reason, we have to use the double mesh as:

$$e_\varepsilon^N = \max_i\left|u_i^{\varepsilon,N} - \tilde{u}_{2i}^{\varepsilon,2N}\right|.$$

The rates of convergence are defined as

$$P_\varepsilon^N = \frac{\ln\left(e_\varepsilon^N/e_\varepsilon^{2N}\right)}{\ln 2}.$$

The e^N is the maximum errors as

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

Table 1. Convergence rates and maximum errors for ε and N.

$\varepsilon \downarrow \rightarrow N$	16	32	64	128
2^{-10}	0.084154	0.042753	0.021547	0.010816
	0.97	0.98	0.99	
2^{-11}	0.026562	0.015689	0.008531	0.004421
	0.75	0.87	0.94	
2^{-13}	0.008590	0.004484	0.002283	0.001144
	0.93	0.97	0.99	
2^{-15}	0.002299	0.001160	0.000580	0.000288
	0.98	0.99	1.00	
2^{-17}	0.000585	0.000292	0.000145	0.000072
	p=0.99	p=1.00	p=1.01	

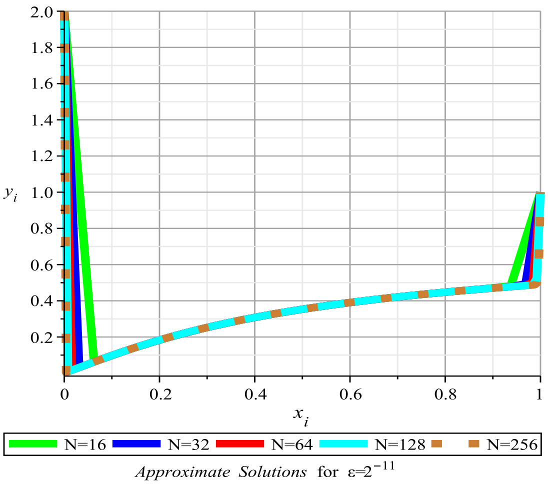

Figure 1. Numerical solution curves of Example 1.

Uniform convergence rates p and error values are given in Table 1. p values are around one. In Figure 1, the approximate solution curves have been plotted for each of the values $N = 16, 32, 64, 128, 256$ for $t = 0$. As N values increase, the approximate solution curves approach the coordinate axes around $t = 0$ and $t = 1$. Here it can be seen that the theoretical process is accurate and reliable.

6. Conclusion

In the study, the finite difference method is used to solve the problem with nonlocal conditions. The difference scheme has been established with the help of some integral forms on the uniform mesh. The difference problem was solved by the Gauss elimination method. Convergence analysis was performed. Uniform convergence was obtained from the first-order. The proposed method has been applied to a test problem. Numerical results show that the approaches described here contribute greatly to the understanding of singularly perturbed problem (see Table 1 and Figure 1). With the motivation given by this study, it is aimed to apply to nonlocal boundary condition and fuzzy problems with delay parameter with singularly perturbation feature.

Acknowledgments

The authors thanks the anonymous authors whose work largely constitutes this study.

References

[1] Cakir, M., Amiraliyev, G.M. (2005) A finite difference method for the singularly perturbed problem with nonlocal boundary condition. *Applied Mathematics and Computation*, 160, 539-549.

[2] Amiraliyev, G.M., Cakir, M. (2000). A uniformly convergent difference scheme for singularly perturbed problem with convective term and zeroth order reduced equation. *International Journal of Applied Mathematics*, 2(12), 1407-1419.

[3] Cakir, M. (2010). Uniform second-order difference method for a singularly perturbed three-point boundary value problem. *Advances in Difference Equations*, 13 pages.

[4] Cakir, M., Amiraliyev, G.M. (2010). A numerical method for a singularly perturbed three-point boundary value problem. *Journal of Applied Mathematics*, 17 pages.

[5] Amiraliyev, G.M., Mamedov, Y.D. (1995). Difference schemes on the uniform mesh for singular perturbed pseudo-parabolic equations. *Turkish Journal of Mathematics*, 19, 207-222.

[6] Arslan, D., Cakir, M. (2021). A new numerical approach for a singularly perturbed problem with two integral boundary conditions. *Computational and Applied ,mathematics*, 40(6).

[7] Arslan, D. (2020). An approximate solution of linear singularly perturbed problem with nonlocal boundary condition. *Journal of Mathematical Analysis*, 11(3), 46-58.

[8] Arslan, D. (2019). An effective numerical method for singularly perturbed nonlocal boundary value problem on Bakhvalov Mesh. *Journal of Informatics and Mathematical Sciences*, 11(3-4), 253-264.

[9] Arslan, D. (2019). A novel hybrid method for singularly perturbed delay differential equations. *Gazi University Journal of Science*, 32(1), 217-223.

[10] Arslan, D. (2019). Approximate solutions of singularly perturbed nonlinear Ill-posed and sixth-order Boussinesq equations with hybrid method. *Bitlis Eren Üniversitesi Fen Bilimleri Dergisi*, 8(2), 451-458.

[11] Negero, N.T, Duressa, G.F. (2021). A method of line with improved accuracy for singularly perturbed parabolic convection–diffusion problems with large temporal lag. *Results in Applied Mathematics*, 11, 100174.

[12] Negero, N.T. (2022). A uniformly convergent numerical scheme for two parameters singularly perturbed parabolic convection–diffusion problems with a large temporal lag. *Results in Applied Mathematics*, 16, 100338.

[13] Negero, N.T. (2023). A parameter-uniform efficient numerical scheme for singularly perturbed time-delay parabolic problems with two small parameters. *Partial Differential Equations in Applied Mathematics*, 7, 100518.

[14] Negero, N.T. (2023). A robust fitted numerical scheme for singularly perturbed parabolic reaction–diffusion problems with a general time delay. *Results in Physics*, 51, 106724.

[15] Negero, N.T. (2023). A fitted operator method of line scheme for solving two-parameter singularly perturbed parabolic convection-diffusion problems with time delay. *Journal of Mathematical Modeling*, 11(2).

[16] Nayfeh, A.H. (1985). *Perturbation Methods.* Wiley, New York.

[17] Nayfeh, A.H. (1979). *Problems in Perturbation.* Wiley, New York.

[18] Kevorkian, J., Cole, J.D. (1981). *Perturbation Methods in Applied Mathematics.* Springer, New York.

[19] O'Malley, R.E. (1991). *Singular Perturbation Methods for Ordinary Differential Equations*. Springer Verlag, New York.

[20] Miller, J.J.H., O'Riordan, E., Shishkin, G.I. (1996). *Fitted Numerical Methods for Singular Perturbation Problems*. World Scientific, Singapore.

[21] Roos, H.G., Stynes, M., Tobiska, L. (2008). *Robust Numerical Methods Singularly Perturbed Differential Equations*. Springer-Verlag, Berlin.

[22] Bakhvalov, N.S. (1969). On optimization of methods for solving boundary value problems in the presence of a boundary layer. The use of special transformation the numerical solution of boudary-layer problems. *Zhurnal Vychislitelnoi Matematiki i Matematicheskoi Fiziki*, 9(4) 841-859.

[23] Bitsadze, A.V., Samarskii, A.A. (1969). On Some Simpler Generalization of Linear Elliptic Boundary Value Problems. *Doklady Akademii Nauk SSSR*, 185, 739-740.

[24] Herceg, D., Surla, K. (1991). Solving a nonlocal singularly perturbed nonlocal problem by splines in tension. *Univ. u Novom Sadu Zb. Rad.Prirod.-Mat. Fak. Ser. Math.*, 21(2), 119-132.

[25] Gupta, C.P., Trofimchuk, S.I. (1997). A sharper condition for the solvability of a three-point second order boundary value problem. *Journal of Mathematical Analysis and Applications*, 205 586-597.

[26] Chegis, R. (1988). The numerical solution of singularly perturbed nonlocal problem (in Russian). *Lietuvas Matematica Rink*, 28, 144-152.

[27] Chegis, R. (1991). The difference scheme for problems with nonlocal conditions, *Informatica (Lietuva)*, 2, 155-70.

[28] Nahushev, A.M. (1985). On nonlocal boundary value problems (in Russian). *Differential Equations*, 21, 92-101.

[29] Sapagovas, M., Chegis, R. (1987). Numerical solution of nonlocal problems (in Russian). *Lietuvas Matematica Rink*, 27, 348-356.

[30] Xie, F., Jin, Z., Ni, M. (2010). On the step-type contrast structure of a second-order semilinear differential equation with integral boundary conditions. *Electronic Journal of Qualitative Theory of Differential Equations*, 62, 1-14.

[31] Kumar, D., Kumari, P. (2020). A parameter-uniform collocation scheme for singularly perturbed delay problems with integral boundary condition. *Journal of Applied Mathematics and Computing*, 63, 813-828.

[32] Sekar, E., Tamilselvan, A. (2019). Third order singularly perturbed delay differential equation of reaction diffusion type with integral boundary condition. *Journal of Applied Mathematics and Computational Mechanics*, 18(2), 99-110.

[33] Raja, V., Tamilselvan, A. (2019). Fitted finite difference method for third order singularly perturbed convection diffusion equations with integral boundary condition. *Arab Journal of Mathematical Science*, 25(2), 231-242.

[34] Khan, R.A. (2003). The generalized method of quasilinearization and nonlinear boundary value problems with integral boundary conditions. *Electronic Journal of Qualitative Theory of Differential Equations*, 10, 1-9.

[35] Rao, S.C.S., Kumar, M. (2007). B-spline collocation method for nonlinear singularly perturbed two-point boundary-value problems. *Journal of Optimization Theory and Applications*, 134(1), 91-105.

[36] Byszewski, L. (1991). Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *Journal of Mathematical Analysis and Applications*, 62, 494-505.

[37] Bougoffa, L., Khanfer, A. (2018). Existence and uniqueness theorems of second-order equations with integral boundary conditions. *Bulletin of the Korean Mathematical Society*, 55(3), 899-911.

[38] Benchohra, M., Ntouyas, S.K. (2000). Existence of solutions of nonlinear differential equations with nonlocal conditions. *Journal of Mathematical Analysis and Applications*, 252, 477-483.

[39] Samarskii, A.A. (1983). *Theory of Difference Schemes*. 2 nd ed., "Nauka", Moscow.

Derya Arslan *has obtained her PhD degree in Mathematics, Van Yuzuncu Yil University, Van, Turkey in 2016. She is currently working as an Associated Professor at the Department of Mathematics, Faculty of Arts and Sciences, University of Bitlis Eren, Bitlis, Turkey. Her fields of research are singularly perturbed problems, numerical methods, applied mathematics.*
https://orcid.org/0000-0001-6138-0607

Ercan Celik *has obtained her PhD degree in Mathematics, Atatürk University, Erzurum, Turkey in 2002. He is currently working as a Professor at the Department of Mathematics, Kyrgyz-Turkish ManasUniversity, Kyrgyz. Him fields of research are optimization, numerical analysis, applied mathematics.*
https://orcid.org/0000-0002-1402-1457