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RESEARCH ARTICLE

On analyzing two dimensional fractional order brain tumor model based on orthonormal Bernoulli polynomials and Newton's method

Iman Masti,[a] Khosro Sayevand,[a*] Hossein Jafari[b]

[a] *Faculty of Mathematics and Statistics, Malayer University, Malayer Iran*
[b] *Department of Mathematical Sciences, University of South Africa, UNISA0003, South Africa*
iman.masty@gmail.com, ksayehvand@malayeru.ac.ir, jafariusern@gmail.com

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ABSTRACT

Recently, modeling problems in various field of sciences and engineering with the help of fractional calculus has been welcomed by researchers. One of these interesting models is a brain tumor model. In this framework, a two dimensional expansion of the diffusion equation and glioma growth is considered. The analytical solution of this model is not an easy task, so in this study, a numerical approach based on the operational matrix of conventional orthonormal Bernoulli polynomials (OBPs) has been used to estimate the solution of this model. As an important advantage of the proposed method is to obtain the fractional derivative in matrix form, which makes calculations easier. Also, by using this technique, the problem under the study is converted into a system of nonlinear algebraic equations. This system is solved via Newton's method and the error analysis is presented. At the end to show the accuracy of the work, we have examined two examples and compared the numerical results with other works.

1. Introduction

Modeling problems and natural phenomena in various sciences such as chemistry, physics, biology and even economics with the help of mathematics [1], and especially with using fractional calculus [2–8], has been welcomed by scientists. This has led to the emergence of various fractional equations, such as partial differential equations of fractional orders. Also, to improve the results and increase the accuracy of the presented models, different definitions of fractional derivatives were presented. For more details see [9–12]. Solving these problems exactly in many situations are impossible and it is necessary that some acceptable schemes are implemented to provide their approximate solution.

Glioblastoma tumor is an aggressive type of cancer that can develop in the brain, but the brain type is more common. The structure of this tumor is made up of cells called astrocytes that support the nerve cells of the brain. This cancer can occur at any age, but it is more common in the elderly. Therefore, the treatment of this tumor requires more detailed studies and better understanding of the tumor. One of these types of studies is the mathematical modeling of this brain tumor. For this reason, a model based on the two main components of cancer cell proliferation and dissemination for tumor growth and based on the Burgess equation was presented in [13] and [14]. For more details regarding the recent history of fractional calculus and brain models the reader is advised to consult the research works presented in [15] and [16].

The main two dimensional model provided for this issue is as follows [13]:

$$\frac{\partial B(e,r)}{\partial r} = D\nabla^2 B(e,r) + \rho B(e,r), \qquad (1)$$

*Corresponding Author

where D, $\partial_r B(e,r)$, $\nabla^2 B(e,r)$ and $B(e,r)$ denote the diffusion coefficient expressed as cm^2 per day, change of tumor cell density, diffusion of tumor cells and cell density at time t, respectively. Also ρ is rate of growth of cells. It should be noted that according to the two dimensional model presented in Eq. (1), the rate of change in tumor cell density is equal to the total rate of tumor cell proliferation and tumor cell growth.

In [13], Eq. (1) is given by

$$\partial_r B(e,r) = D\frac{1}{e^2}\partial_e \left(e^2\partial_e B(e,r)\right) + \rho B(e,r). \quad (2)$$

By adding a parameter based on killing to the Eq. (2), the following equation will be obtain [14]:

$$\partial_r B(e,r) = D\frac{1}{e^2}\partial_e \left(e^2\partial_e B(e,r)\right) + \rho B(e,r) - k_r B(e,r). \quad (3)$$

Eq. (3) can be written as follows:

$$\begin{aligned}\partial_r B(e,r) = D\Big(&\partial_{ee}B(e,r)\\ &+\frac{2}{e}\partial_e B(e,r)\Big) + (\rho - k_r)B(e,r).\end{aligned} \quad (4)$$

Suppose that $t = 2Dr, x = e$ and $\Phi(x,t) = uB(e,r)$. Therefore, we have

$$\frac{\partial r}{\partial t} = \frac{1}{2D}, \quad (5)$$

$$\partial_t\Phi(x,t) = x\partial_t B(x,r) = \frac{x}{2D}\partial_r B(x,r), \quad (6)$$

$$\partial_x\Phi(x,t) = x\partial_x B(x,r) + B(x,r), \quad (7)$$

$$\partial_{xx}\Phi(x,t) = x\partial_{xx}B(x,r) + 2\partial_x B(x,r). \quad (8)$$

From Eqs. (6-8), one will set

$$\partial_r B(x,r) = \frac{2D}{x}\partial_t\Phi(x,t),$$

$$\partial_x B(x,r) = \frac{1}{x}\left(\partial_x\Phi(x,t) - B(x,r)\right),$$

$$\partial_{xx}B(x,r) = \frac{1}{x}\left(\partial_{xx}\Phi(x,t) - 2\partial_x B(x,r)\right).$$

Thus, Eq. (4) becomes to

$$\partial_t\Phi(x,t) = \frac{1}{2}\partial_{xx}\Phi(x,t) + \frac{\rho - k_t}{2D}\Phi(x,t). \quad (9)$$

Let $S(x,t) = \frac{\rho - k_t}{2D}\Phi(x,t)$ and suppose that $\Phi(x,t_0)$ is initial growth profile. Then, the following model is achieved:

$$\begin{aligned}&\partial_t\Phi(x,t) = \frac{1}{2}\partial_{xx}\Phi(x,t) + S(x,t),\\ &\Phi(x,t_0) = \xi(x), \quad x,t \in (a,b).\end{aligned} \quad (10)$$

The fractional model of Eq. (10) can be expressed as follows:

$$\begin{aligned}&\partial_t^{\varsigma}\Phi(x,t) = \frac{1}{2}\partial_{xx}\Phi(x,t) + S(x,t),\\ &\Phi(x,t_0) = \xi(x),\end{aligned} \quad (11)$$

where ∂_t^{ς} denotes the fractional Caputo derivative of order $0 < \varsigma \leq 1$ which be defined in the follow-up.

Remark 1. *In Eq. (11), $S(x,t)$ can be linear or nonlinear. Here, the Newton's method is used to estimate the roots of the given equation.*

2. Preliminaries and notations

Definition 1. *The fractional Caputo derivative of order ς is defined by*

$$\begin{aligned}D_t^{\varsigma}\Phi(x,t) &= \frac{1}{\Gamma(n-\varsigma)}\int_0^t \frac{\Phi^{(n)}(x,s)}{(t-s)^{\varsigma-1+n}}ds,\\ n-1 &< \varsigma < n, \quad n \in \mathbb{N}.\end{aligned} \quad (12)$$

For more details about fractional derivatives, readers can refer to [11].

Definition 2. *[17]* The OBP of order M is defined as follows:

$$\mathcal{B}_M(t) = \sqrt{2M+1}\sum_{j=0}^{M}(-1)^j \binom{M}{j}\binom{2M-j}{M-j} t^{M-j}, \quad (13)$$

where $M = 0, 1, 2, \cdots$.

Therefore

$$\int_0^1 \mathcal{B}_r(u)\mathcal{B}_s(u)du = \begin{cases}1, & r = s,\\ 0, & otherwise.\end{cases} \quad (14)$$

Note: The function $\Phi(t) \in L^2(0,1)$ can be expanded in terms of OBPs as follows:

$$\Phi(t) = \sum_{i=0}^{M} q_i\mathcal{B}_i(t) = \mathcal{Q}^T B(t), \quad (15)$$

where $\mathcal{Q} = [q_0, q_1, \ldots, q_M]^T$ and $B(t) = [\mathcal{B}_0(t), \mathcal{B}_1(t), \ldots, \mathcal{B}_M(t)]^T$, with

$$q_r = \int_0^1 \Phi(u)\mathcal{B}_r(u)du. \quad (16)$$

Definition 3. *[17]* Two dimensional OBP of order M, N is defined in the following form

$$\mathcal{B}_{M,N}(x,t) = \mathcal{B}_M(x)\mathcal{B}_N(t),\ M,N = 0,1,2,\cdots. \tag{17}$$

Therefore

$$\int_0^1\int_0^1 \mathcal{B}_{m,n}(x,t)\mathcal{B}_{p,q}(x,t)dxdt = \begin{cases} 1, & m=p, n=q,\\ 0, & otherwise. \end{cases} \tag{18}$$

Consequently, a two variables function $\Phi(x,t) \in L^2\left((0,1)\times(0,1)\right)$ can be approximated in terms of OBPs as follows:

$$\Phi(x,t) \simeq \sum_{r=0}^{M}\sum_{s=0}^{M} q_{r,s}\mathcal{B}_r(x)\mathcal{B}_s(t) = \mathcal{B}^T(x)\mathcal{Q}\mathcal{B}(t), \tag{19}$$

where $\mathcal{Q} = [q_{ij}]_{(m+1)\times(m+1)}$, $i,j = 0,1,\cdots,m$ and

$$q_{ij} = \int_0^1\int_0^1 \mathcal{B}_i(u)Q(u,v)\mathcal{B}_j(v)dvdu. \tag{20}$$

Assume now that
$B(u) = [\mathcal{B}_0(u), \mathcal{B}_1(u), \cdots, \mathcal{B}_m(u)]^T$.
Now, as a direct consequence of Eq. (13), we get

$$B(s) = AT_M(s), \tag{21}$$

where

$$T_M(s) = [1, s, \cdots, s^M]^T, \tag{22}$$

and $A_{(m+1)\times(m+1)}$ is signified by

$$\begin{pmatrix} (-1)^0\binom{0}{0}\binom{0}{0} & 0 & \cdots & 0\\ (-1)^1\sqrt{3}\binom{1}{1}\binom{1}{0} & (-1)^0\sqrt{3}\binom{1}{0}\binom{2}{1} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ (-1)^m\sqrt{2m+1}\binom{m}{m}\binom{m}{0} & (-1)^{m-1}\sqrt{2m+1}\binom{m}{m-1}\binom{m+1}{1} & \cdots & (-1)^0\sqrt{2m+1}\binom{m}{0}\binom{2m}{m} \end{pmatrix}, \tag{23}$$

Since $det(A) \neq 0$, therefore

$$T_M(s) = A^{-1}B(s). \tag{24}$$

Taking the derivative of vector $\mathcal{B}(t)$, we will have

$$\frac{d}{dt}\mathcal{B}(t) = D\mathcal{B}(t), \tag{25}$$

where

$$\mathcal{R}^{-1}D\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 2 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & M & 0 \end{pmatrix},$$

and $\mathcal{R} = [r_{ij}], i,j = 0,1,\ldots,M$, and

$$r_{i,j} = \begin{cases} \binom{i}{j}B_{i-j}, & i \geq j,\\ 0, & i < j. \end{cases}$$

Also, for $s \geq 2$, we have

$$\frac{d^s}{dt^s}\mathcal{B}(t) = D^s\mathcal{B}(t). \tag{26}$$

Therefore

$$D_t^{\tau}\beta(t) \simeq D^s\beta(t), \tag{27}$$

where

$$\Lambda^{\zeta} = \begin{pmatrix} 0 & 0 & \cdots & 0\\ \Omega_{1,0,k} & \Omega_{1,1,k} & \cdots & \Omega_{1,M,k}\\ \Omega_{2,0,k} & \Omega_{2,1,k} & \cdots & \Omega_{2,M,k}\\ \vdots & \vdots & \ddots & \vdots\\ \Omega_{M,0,k} & \Omega_{M,1,k} & \cdots & \Omega_{M,M,k} \end{pmatrix},$$

and

$$\Omega_{i,j,k} = \sum_{k=1} w_{i,k}\sigma_{k,j}\frac{\Gamma(k+1)}{\Gamma(k-\zeta+1)},$$

$$\omega_{i,k} = \binom{i}{k}\beta_{i-k}, \tag{28}$$

Let $\mathcal{B}(t)$ be the orthonormal Bernoulli vector defined in Eq. (15) and suppose that $\varsigma > 0$. Thus, by using Eqs. (13) and the Caputo's fractional differentiation, $D^{\varsigma}B_m(t)$ is equal to

$$\sqrt{2m+1}\sum_{i=0}^{M}(-1)^i\binom{i}{m}\binom{2m-i}{m-i}D^{\varsigma}(t^{m-i})$$
$$= \sqrt{2m+1}\sum_{i=\lceil\varsigma\rceil}^{M}(-1)^i\binom{i}{m}\binom{2m-i}{m-i}\frac{\Gamma(m-i+1)}{\Gamma(m-i-\varsigma+1)}t^{m-i-\varsigma}, \tag{29}$$

where $p = 0,1,\cdots,m$.

Approximating t^j with $j = m-i-\varsigma$ by means of the orthonormal Bernoulli, leads to:

$$t^j \simeq \sum_{r=0}^{M} q_{r,j}B_r(t). \tag{30}$$

Hence, $q_{r,j} = \int_0^1 t^jB_r(t)dt$ and is equal to

$$\sqrt{2r+1}\sum_{k=0}(-1)^k\binom{r}{k}\binom{2r-k}{r-k}\frac{1}{j+r-k+1}.$$

Therefore $D^{\varsigma}B_m(t)$ can be approximated as

$$\sum_{i=\lceil\varsigma\rceil}^{m} \sqrt{2m+1}(-1)^i \binom{m}{i}\binom{2m-i}{m-i} \frac{\Gamma(m-i+1)}{\Gamma(m-i-\varsigma+1)} \sum_{r=0}^{m} q_{r,j} B_r(t)$$
$$= \sum_{r=0}^{m}\left(\sqrt{2m+1}\sum_{i=\lceil\varsigma\rceil}^{m}(-1)^i \binom{m}{i}\binom{2m-i}{m-i}\frac{\Gamma(m-i+1)}{\Gamma(m-i-\varsigma+1)} q_{r,j}\right) \times B_r(t)$$
$$= \sum_{r=0}^{m}\left(\sum_{i=\lceil\varsigma\rceil}^{m} \omega_{m,i,j,r}\right) B_r(t), \tag{31}$$

where $\omega_{m,i,j,r}$ is given by

$$\sqrt{2m+1}(-1)^i \binom{m}{i}\binom{2m-i}{m-i} \frac{\Gamma(m-i+1)}{\Gamma(m-i-\varsigma+1)} q_{r,j}. \tag{32}$$

Let us rewrite Eq. (31) in the vector form

$$D^{\varsigma}\mathcal{B}_p(t) \simeq \left[\sum_{i=\lceil\varsigma\rceil}^{p} \omega_{p,i,j,0}, \sum_{i=\lceil\varsigma\rceil}^{p} \omega_{p,i,j,1}, \ldots, \sum_{i=\lceil\varsigma\rceil}^{p} \omega_{p,i,j,m}\right] \phi(v), \tag{33}$$

where $p = 0, 1, \cdots, m$. In other words

$$D^{\varsigma}\mathcal{B}(v) \simeq D^{(\varsigma)}\mathcal{B}(v), \tag{34}$$

where $D^{(\varsigma)}$ is as follows

$$\begin{pmatrix} \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{0,i,j,0} & \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{0,i,j,1} & \cdots & \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{0,i,j,m} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{i,i,j,0} & \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{i,i,j,1} & \cdots & \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{i,i,j,m} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{m,i,j,0} & \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{m,i,j,1} & \cdots & \sum\limits_{i=\lceil\varsigma\rceil}^{m} w_{m,i,j,m} \end{pmatrix}. \tag{35}$$

Lemma 1. *Let $\varsigma > 0$, then $D_v^{\varsigma}\mathcal{B}(u,v)$ is approximated as $\mathbf{D}_v^{(\varsigma)}\mathcal{B}(u,v)$ where $\mathbf{D}_t^{(\varsigma)} = I \otimes D^{(\varsigma)}$, and $I_{(m+1)\times(m+1)}$ is the identity matrix. Here $\otimes$ is Kronecker product.*

Proof. Using Eq. (34) we take

$$\begin{aligned} \frac{\partial^{\varsigma}\mathcal{B}(u,v)}{\partial v^{\varsigma}} &= \frac{\partial^{\varsigma}(\mathcal{B}(u)\otimes\mathcal{B}(v))}{\partial v^{\varsigma}} \\ &= (I\mathcal{B}(u))\otimes(D^{(\varsigma)}\mathcal{B}(v)) \\ &= (I\otimes D^{(\varsigma)})(\mathcal{B}(u)\otimes\mathcal{B}(v)) \\ &:= \mathbf{D}_v^{(\varsigma)}\mathcal{B}(u,v). \end{aligned} \tag{36}$$

3. Error analysis

Theorem 1. *Suppose that $\phi_m(t) = \mathcal{Q}^T\mathcal{B}(t)$ where $\mathcal{Q} = [q_0, q_1, \cdots, q_m]^T$, is an approximation of a continuous function $\phi(t)$ on $[0,1]$ by the OBPs. Then, the coefficients q_n for $n = 0, 1, \cdots, m$ are bounded as follows:*

$$| q_n | \leq \Theta_n, \tag{37}$$

where

$$\Theta_n = \rho\sqrt{2n+1}\sum_{l=0}^{n}\binom{n}{l}\binom{2n-l}{n-l}, \tag{38}$$

here $\rho \in \mathbb{R}^+$ and $| \phi(t) | \leqslant \rho$.

Proof. Using the OBP, $\phi(t)$ can be approximated in the following form

$$\phi(t) = \phi_m(t) = \sum_{n=0}^{m} q_n B_r(t),$$

where q_n can be determined by

$$\begin{aligned} q_n &= \int_0^1 \phi(t)B_n(t)dt \\ &= \int_0^1 \phi(t)\sqrt{2n+1}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\binom{2n-l}{n-l}t^{n-l}dt \\ &= \sqrt{2n+1}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\binom{2n-l}{n-l}\int_0^1\phi(t)t^{n-l}dt. \end{aligned} \tag{39}$$

Since $\phi(t)$ is continuous on $[0,1]$, then based on the maximum-minimum Theorem one will set

$$\exists\ \rho > 0,\ \forall t \in [0,1],\ | \phi(t) | < \rho.$$

Thus, we will have

$$| q_n | \leq \rho\sqrt{2n+1}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\binom{2n-l}{n-l}. \tag{40}$$

Theorem 2. *Suppose that $\phi_m(t) = \mathcal{Q}^T\mathrm{B}(t)$ is an approximation of a continuous function $\phi(t)$ on $[0,1]$ by the OBPs. Then, the error bound is as follows:*

$$\| \phi(t) - \phi_m(t) \|_2 \leq \left(\sum_{i=m+1}^{\infty}\Theta_i^2\right)^{\frac{1}{2}}, \tag{41}$$

which Θ is presented in Eq. (38).

Proof. Suppose that $\phi(t) = \sum_{i=0}^{\infty} z_iB_i(t)$ and $\phi_m(t) = \sum_{i=0}^{m} z_iB_i(t)$. Therefore

$$\phi(t)-\phi_m(t)=\sum_{i=m+1}^{\infty} q_i B_i(t). \tag{42}$$

Since $\int_0^1 B_i(t)B_j(t)dt=\delta_{ij}$, then

$$\begin{aligned}\|\phi(t)-\phi_m(t)\|_2^2 &= \int_0^1 |\phi(t)-\phi_m(t)|^2 dt\\ &= \int_0^1 |\sum_{i=m+1}^{\infty} q_i B_i(t)|^2 dt\\ &= \sum_{i=m+1}^{\infty} q_i^2 \le \sum_{i=m+1}^{\infty} \Theta_i^2.\end{aligned} \tag{43}$$

Theorem 3. *[17]* Suppose $\phi_m(u,v)=\mathcal{Q}^T \mathsf{B}(u,v)$ is the best approximation for $\phi(u,v)$ by the two dimensional OBP. Then, we have

$$\|\phi(u,v)-\phi_m(u,v)\|_2 = \mathsf{O}\left(\frac{1}{(m+1)!2^{2m+1}}\right), \tag{44}$$

which means that if $m \longrightarrow \infty$, then $\phi_m(u,v) \longrightarrow \phi(u,v)$. Here, O is a big-O notation.

Corollary 1. *[17]* By an argument similar to Theorem (3), we have the convergence

$$\|\phi(u,v)-\phi_m(u,v)\|_2 \le \frac{\rho}{(m+1)!2^{2m+1}}, \tag{45}$$

where ρ is defined as above.

Now, according to the stated theorem, we give the numerical results section.

4. Method in action and numerical overviews

In this section, we propose a numerical scheme based on collocation method and operational matrices to approximate the solution of (11). At first, we can approximate the solution of (11) via OBPs as follows:

$$Q(x,t) \simeq \sum_{r=0}^{M}\sum_{s=0}^{M} q_{r,s}\mathcal{B}_r(x)\mathcal{B}_s(t) = \mathcal{B}^T(x)\mathcal{Q}\mathcal{B}(t). \tag{46}$$

Then, by using ∂_t^{ς} on both sides of (46), we have

$$\partial_t^{\varsigma} Q(x,t) \simeq \mathcal{B}^T(x)\mathcal{Q}\Delta^{\varsigma}\mathcal{B}(t). \tag{47}$$

In other words, we get

$$\partial_{xx} Q(x,t) \simeq \left(D^2\mathcal{B}(x)\right)^T \mathcal{Q}\mathcal{B}(t). \tag{48}$$

Now, by substituting (47) and (48) into (11), we will have

$$\mathcal{B}^T(x)\mathcal{Q}\Delta^{\varsigma}\mathcal{B}(t) = \frac{1}{2}\left(D^2\mathcal{B}(x)\right)^T \mathcal{Q}\mathcal{B}(t) + S(x,t). \tag{49}$$

From (46), the initial condition can be expressed as follows:

$$\mathcal{B}^T(x)\mathcal{Q}\mathcal{B}(0) \simeq \xi(x). \tag{50}$$

If we collocate (49) and (50) at the points $x_i=\frac{i}{M}, i=0,1,\ldots,M$, and $t_j=\frac{j}{M}$, $j=1,2,\ldots,M$, we have the following system of equations

$$\begin{cases}\mathcal{B}^T(x_i)\mathcal{Q}\Delta^{\varsigma}\mathcal{B}(t_j) = \frac{1}{2}\left(D^2\mathcal{B}(x_i)\right)^T\mathcal{Q}\mathcal{B}(t_j) + S(x_i,t_j),\\ \mathcal{B}^T(x_i)\mathcal{Q}\mathcal{B}(0) \simeq \xi(x_i).\end{cases} \tag{51}$$

By solving this system with the help of Matlab software and Newton's method, the values of $\mathcal{Q}$ can be obtained and then by inserting in Eq. (46), the approximate solution of this model is achieved.

In the following, to show the effectiveness of the proposed technique, numerical results for two examples are reported. Matlab software was used to obtain these results.

Example 1. *Consider the following equation [18]:*

$$\partial_t^{\varsigma} Q(x,t) = \partial_{xx} Q(x,t) + Q(x,t) + \Gamma(2+\varsigma)e^x t, \tag{52}$$

where

$$\begin{cases} Q(x,0)=0, \ \textit{for } 0\le x\le 1,\\ Q(0,t)=t^{1+\varsigma},\\ Q(1,t)=et^{1+\varsigma}, \ \textit{for } t>0,\\ 0<\varsigma\le 1,\end{cases} \tag{53}$$

and the exact solution is

$$Q(x,t)=e^x t^{1+\varsigma}. \tag{54}$$

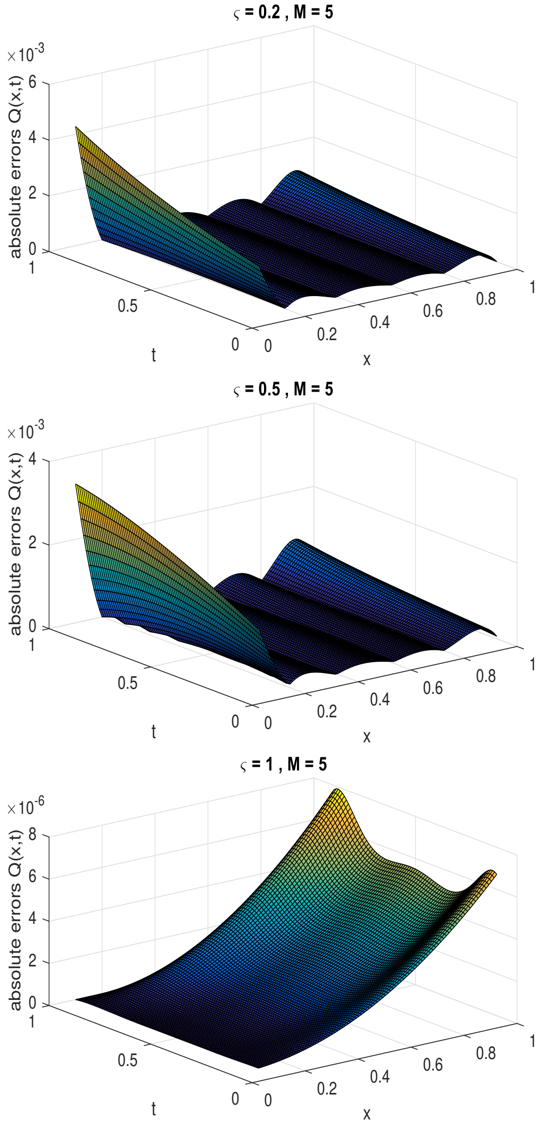

Figure 1. The absolute errors for $\varsigma = 0.2, 0.5, 1$ and $M = 5$ in Example 1.

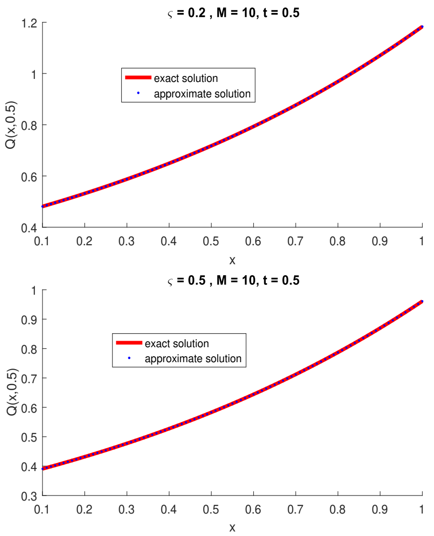

Figure 2. Comparison of analytical answer and approximate answer in Example 1 for different values of ς.

Table 1. Comparing L_∞-errors of Example 1 where $t = 0.5$ and $\varsigma = 0.5$ for different values of h.

h	[18]	Proposed method
0.25	1.2×10^{3}	3.4×10^{5}
0.125	3.9×10^{4}	1.3×10^{5}
0.0625	1.1×10^{4}	7.5×10^{6}

Fig. 1, shows the absolute errors for different values of ς and $M = 5$. A comparison of analytical answers and approximate answers for $\varsigma = 0.2, 0.5$, $M = 10$ and $t = 0.5$ is presented in Fig. 2. Also, in Table 1, a comparison is made between the B-spline wavelet operational method and our proposed method. The findings from this example demonstrate that the results are promising.

Example 2. *Consider the following equation [13]:*

$$\partial_t^\varsigma Q(x,t) = \frac{1}{2}\partial_{xx} Q(x,t) + \frac{1}{2} Q(x,t).$$

For this equation in the fractional state, no exact solution has been reported, but for $\varsigma = 1$, *the initial conditions have been considered so that its exact solution is* e^{x+t}. *The obtained results of the maximum absolute error for* $x = 1$ *and for different values of* t *and* ς *are reported in Table 2. Comparison of the maximum absolute error by different values of* ς *is presented in Fig. 3 and is sorted in Table 2, where* $M = 10$.

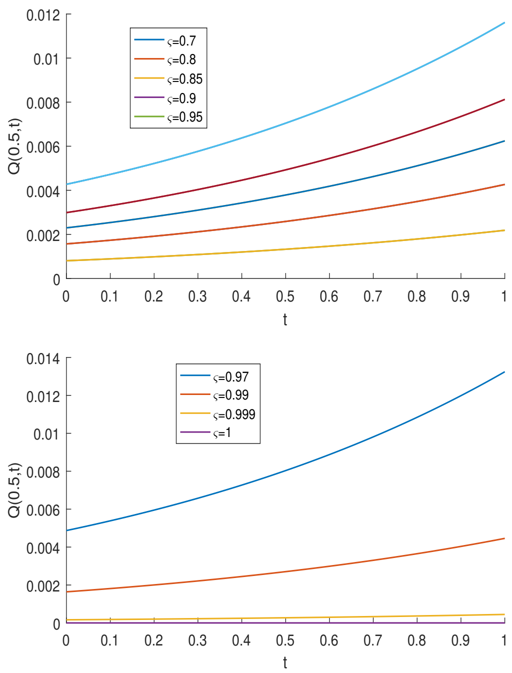

Figure 3. The absolute error of Example 2 for different values of ς where $M = 10$.

Table 2. Comparison of the maximum absolute error of Example 2 with setting $x = 1$ and for different values of t and ς.

ς	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$\varsigma = 0.7$	2.73768×10^{1}	6.29163×10^{1}	8.83953×10^{1}	0.11094×10^{1}	0.13240×10^{1}
$\varsigma = 0.8$	1.40599×10^{1}	3.63106×10^{1}	5.27557×10^{1}	6.75506×10^{1}	8.18971×10^{1}
$\varsigma = 0.85$	8.79442×10^{2}	2.48710×10^{1}	3.70498×10^{1}	4.80560×10^{1}	5.86404×10^{1}
$\varsigma = 0.9$	4.52923×10^{2}	1.48283×10^{1}	2.26811×10^{1}	2.98218×10^{1}	3.67948×10^{1}
$\varsigma = 0.95$	1.24112×10^{2}	6.11041×10^{2}	9.89665×10^{2}	1.33438×10^{1}	1.66726×10^{1}
$\varsigma = 0.97$	3.65980×10^{3}	3.20476×10^{2}	5.41044×10^{2}	7.41701×10^{2}	9.36426×10^{2}
$\varsigma = 0.99$	6.58061×10^{4}	8.44130×10^{3}	1.54557×10^{2}	2.17713×10^{2}	2.78533×10^{2}
$\varsigma = 0.999$	1.89218×10^{4}	6.98058×10^{4}	1.36302×10^{3}	1.91338×10^{3}	2.32859×10^{3}
$\varsigma = 0.9999$	2.08184×10^{5}	6.61829×10^{5}	1.20376×10^{4}	1.20065×10^{4}	9.60039×10^{6}
$\varsigma = 1$	1.70126×10^{8}	1.53014×10^{6}	1.58226×10^{5}	7.91219×10^{5}	2.73071×10^{4}

5. Concluding remarks

Glioblastoma tumor is an aggressive type of cancer that can develop in the brain but the brain type is more common. The structure of this tumor is made up of cells called astrocytes that support the nerve cells of the brain. This cancer can occur at any age, but it is more common in the elderly. Therefore, the treatment of this tumor requires more detailed studies and better understanding of the tumor. One of these types of studies is the mathematical modeling of this brain tumor. For this reason, a model based on the two main components of cancer cell proliferation and dissemination for tumor growth and based on the Burgess equation was presented (for more details see [13] and the references therein). In this study, using Caputo's derivative, a mathematical instrument was investigated to analysis this tumor case. The analytical solution of this model is not an easy task, so in this study, a numerical approach based on the operational matrix of conventional OBPs has been thoroughly used to estimate the solution of the proposed model. One of the advantages of this idea is to obtain the derivative of the fraction in matrix form, which makes calculations easier. Also, by using this technique, the problem under the study is converted to a system of nonlinear algebraic equations which was solved via Newton's method. In Table 1 and Table 2 we compare our method with Bernoulli polynomials operational method in Ref. [13] and B-spline wavelet operational method in Ref. [18]. Examination of these results show that the proposed method provides a more accurate answer than similar methods. In other words, the obtained results are interesting, promising and can be extended for other scientific models.

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Iman Masti *is a Ph.D. student at the Department of Mathematical Sciences, Malayer University. His research interests include bio-mathematics, fractional differential equations, and approximation methods.*
https://orcid.org/0009-0007-4454-5095

Khosro Sayevand *is a full professor in applied mathematics at the Department of Mathematical Sciences, Malayer University. His research interests include fractional calculus, modeling, perturbation theory, and linear programming.*
https://orcid.org/0000-0002-5397-1623

Hossein Jafari *is a full professor in applied mathematics. His research interests include bio-mathematics, fractional differential equations, Lie symmetry, and approximation methods.*
https://orcid.org/0000-0001-6807-6675