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RESEARCH ARTICLE

Theoretical and numerical analysis of a chaotic model with nonlocal and stochastic differential operators

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ABSTRACT

A set of nonlinear ordinary differential equations has been considered in this paper. The work tries to establish some theoretical and analytical insights when the usual time-deferential operator is replaced with the Caputo fractional derivative. Using the Caratheodory principle and other additional conditions, we established that the system has a unique system of solutions. A variety of well-known approaches were used to investigate the system. The stochastic version of this system was solved using a numerical approach based on Lagrange interpolation, and numerical simulation results were produced.

1. Introduction

Systems of nonlinear ordinary differential and integral equations make up a significant class of nonlinear equations because they have been discovered to be effective at simulating challenging real-world issues that come up in various branches of science, technology, and engineering [1–10]. We will emphasize that a variety of differential operators, including the most recent one proposed in the literature, piecewise derivatives, fractional derivatives, and classical derivatives, have been employed to reflect the intricacies of nature. In fact, no viable analytical solution that can be solved analytically has been proposed in recent years. Therefore, to arrive at numerical solutions to these nonlinear systems of equations, researchers frequently used numerical techniques. Conditions do exist, nevertheless, in which they acknowledge the need for exact solutions. However, it was also recently reported that certain of these differential equations may not be able to accurately depict complicated processes with crossover tendencies when only utilizing a single differential operator. A notion known as the piecewise differential operator was proposed as a solution and successfully applied in various significant applications [11,12]. In this study, we intend to investigate a model that has been studied in a number of significant works a modified system of nonlinear equations. Following that, we'll use various differential operator types and offer some numerical and stability analyses.

2. Definitions of derivatives

In this section, we summarized some basic fractional order definitions in the next section [11,13, 14].

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Definition 1. *Caputo fractional derivative of order $\gamma > 0$ of a function $f : (0, \infty) \to R$, according to Caputo, the fractional derivative of a continuous and differentiable function f is given as :*

$$ {}^{C}D_t^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-x)^{-\gamma} \frac{d}{dx} f(x) dx, \quad (1) $$

where $0 < \gamma \leq 1$.

Definition 2. *Let f be differentiable, then a piece-wise derivative with classical and fractional derivative with power-law kernel is given as*

$$ {}_0^{PC}D_t^{\gamma} f(t) = \begin{cases} f'(t), & \text{if } 0 \leq t \leq t_1 \\ {}_{t_1}^{C}D_t^{\gamma} f(t), & \text{if } t_1 \leq t \leq T \end{cases} \quad (2) $$

where ${}_0^{PC}D_t^{\gamma}$ *represents classical derivative on* $0 \leq t \leq t_1$ *and Caputo fractional derivative on* $t_1 \leq t \leq T$.

Definition 3. *The Riemann-Liouville fractional integral of order $\gamma > 0$ of a function $f : (0, \infty) \to R$, according to Riemann-Liouville, the fractional integral is considered as anti-fractional derivative of a function f is :*

$$ I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-x)^{\gamma-1} f(x) dx, \quad x > 0. \quad (3) $$

Definition 4. *Let f be continuous and $\gamma > 0$ then a piece-wise integral of f is given as*

$$ {}^{PPL}J_t^{\gamma} f(t) = \begin{cases} \int_0^t f(\tau) d\tau, & \text{if } 0 \leq t \leq t_1 \\ \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-\tau)^{\gamma-1} f(\tau) d\tau, & \text{if } t_1 \leq t \leq T \end{cases} . \quad (4) $$

where ${}^{PPL}J_t^{\gamma} f(t)$ *represents classical integral on* $0 \leq t \leq t_1$ *and the integral with power-law kernel on* $t_1 \leq t \leq T$.

3. Model derivation

Fractional order models are very important for studying natural problems. It is well known that the nature of the trajectory of the fractional order derivatives is non-local, which describes that the fractional order derivative has a memory effect, meaning that the future states depend on the present as well as the past states. With this motivation in 2012, Ozalp and Koca have considered Barley and Cherifs deterministic model as fractional order dynamic [15, 16]. In this work, we extended the fractional-order nonlinear model by adding λx_2^2 and λx_1^2 factors where λ is 1 or 0. We find these components sufficient to make relevant practical conclusions. The model can be more complex later, once that is shown to be necessary. With these assumptions, the complete model is given as

$$ \begin{aligned} {}_{t_1}^{C}D_t^{\alpha} x_1(t) &= -\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 \\ &\quad + \lambda x_2^2, \; 0 < \alpha \leq 1 \\ {}_{t_1}^{C}D_t^{\alpha} x_2(t) &= -\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + \lambda x_1^2, \\ x_1(0) &= 0 \;, \; x_2(0) = 0. \end{aligned} \quad (5) $$

Positive values for the model show positive conscious experience, while negative values show negative conscious experience. Other parameters are oblivion, reaction, and attraction constants. Stochastic modeling is used in many places, from statistics to biology, from economics to physics. We know that deterministic modeling is predictable, so we know the future for sure, while stochastic modeling is random, so we cannot predict the future for sure. So we say that stochastic models can give rise to deterministic behavior. In particular, we can construct a sequence of models with a decreasing level of detail, from a deterministic model to a stochastic model or vice versa. Stochastic modeling is random in nature, and uncertain factors are included in the model. So in this paper with a numerical part, we will consider the fractional-order deterministic interaction model as a fractional order stochastic model with an added noise piece.

$$ \begin{aligned} dx_1(t) &= \left(-\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 + \lambda x_2^2\right) dt \\ &\quad + \sigma_1 x_1 dB_1(t), \\ dx_2(t) &= \left(-\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + \lambda x_1^2\right) dt \\ &\quad + \sigma_2 x_2 dB_2(t), \end{aligned} \quad (6) $$

We believe that this nonlinear stochastic model will explain the stochastic rates and factors (ecological, historical, cultural and community conditions) better than its deterministic version

4. Chaotic number for modified nonlinear model

The concept of mathematical modeling is used to analyze the between at least two variables. People who are in communication are aware of each other, and their connection with each other is conscious. In this section, we search for the chaotic number (C_0), which has been worked on by some researchers recently [17]. So we can have an idea about the future of communication. The function F will be obtained from the nonlinear part of the model, and the function V will be obtained from the linear part of the model. Here we recall our nonlinear model including classical derivative.

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 + \lambda x_2^2,\\ \frac{dx_2(t)}{dt} &= -\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + \lambda x_1^2,\end{aligned} \tag{7}$$

with initial conditions $x_1(0) = 0$ and $x_2(0) = 0$. We note that in analysis we take $\lambda = 1$.

To begin, we divide the system into two sections.

$$\left[\begin{array}{c} x_1' \\ x_2' \end{array}\right] = f - v. \tag{8}$$

Here f is given as

$$f = \left[\begin{array}{c} -\beta_1 \varepsilon x_2^3 + x_2^2 \\ -\beta_2 \varepsilon x_1^3 + x_1^2 \end{array}\right] \tag{9}$$

and v is given as

$$v = \left[\begin{array}{c} \alpha_1 x_1 - \beta_1 x_2 \\ \alpha_2 x_2 - \beta_2 x_1 \end{array}\right]. \tag{10}$$

Let us take partial derivatives of f and v then we get F and V which are given as below

$$F = \left[\begin{array}{cc} 0 & -3\beta_1 \varepsilon x_2^2 + 2x_2 \\ -3\beta_2 \varepsilon x_1^2 + 2x_1 & 0 \end{array}\right], \tag{11}$$

and

$$V = \left[\begin{array}{cc} \alpha_1 & -\beta_1 \\ -\beta_2 & \alpha_2 \end{array}\right]. \tag{12}$$

To obtain Chaotic number (C_0), we have to calculate $N_G = F.V^{-1}$ matrice which is named as Next-Generation matrix of the system. Then (C_0) will be obtained from the spectral radius of the matrix of N_G.

First, we need to calculate V^{-1}. If V is

$$V = \left[\begin{array}{cc} \alpha_1 & -\beta_1 \\ -\beta_2 & \alpha_2 \end{array}\right], \tag{13}$$

then

$$V^{-1} = \frac{1}{\alpha_1\alpha_2 - \beta_1\beta_2}\left[\begin{array}{cc} \alpha_2 & \beta_1 \\ \beta_2 & \alpha_1 \end{array}\right]. \tag{14}$$

So we get

$$F.V^{-1} = \left[\begin{array}{cc} 0 & -3\beta_1 \varepsilon x_2^2 + 2x_2 \\ -3\beta_2 \varepsilon x_1^2 + 2x_1 & 0 \end{array}\right] \times \left[\begin{array}{cc} \frac{\alpha_2}{\alpha_1\alpha_2-\beta_1\beta_2} & \frac{\beta_1}{\alpha_1\alpha_2-\beta_1\beta_2} \\ \frac{\beta_2}{\alpha_1\alpha_2-\beta_1\beta_2} & \frac{\alpha_1}{\alpha_1\alpha_2-\beta_1\beta_2} \end{array}\right] \tag{15}$$

$$F.V^{-1} = \left[\begin{array}{cc} \frac{\beta_2\left(-3\beta_1\varepsilon x_2^2+2x_2\right)}{\alpha_1\alpha_2-\beta_1\beta_2} & \frac{\alpha_1\left(-3\beta_1\varepsilon x_2^2+2x_2\right)}{\alpha_1\alpha_2-\beta_1\beta_2} \\ \frac{\alpha_2\left(-3\beta_2\varepsilon x_1^2+2x_1\right)}{\alpha_1\alpha_2-\beta_1\beta_2} & \frac{\beta_1\left(-3\beta_2\varepsilon x_1^2+2x_1\right)}{\alpha_1\alpha_2-\beta_1\beta_2} \end{array}\right].$$

Now we calculate the eigenvalues by solving

$$\det\left(F.V^{-1} - \lambda I\right) = 0, \tag{16}$$

so we get

$$\begin{aligned}&\det\left(F.V^{-1} - \lambda I\right)\\ &= \det\left|\begin{array}{cc} \frac{\beta_2\left(-3\beta_1\varepsilon x_2^2+2x_2\right)}{\alpha_1\alpha_2-\beta_1\beta_2} - \lambda & \frac{\alpha_1\left(-3\beta_1\varepsilon x_2^2+2x_2\right)}{\alpha_1\alpha_2-\beta_1\beta_2} \\ \frac{\alpha_2\left(-3\beta_2\varepsilon x_1^2+2x_1\right)}{\alpha_1\alpha_2-\beta_1\beta_2} & \frac{\beta_1\left(-3\beta_2\varepsilon x_1^2+2x_1\right)}{\alpha_1\alpha_2-\beta_1\beta_2} - \lambda \end{array}\right|.\end{aligned} \tag{17}$$

Here we need simplification as

$$\begin{aligned} l_1 &= -3\beta_2\varepsilon x_1^2 + 2x_1,\\ l_2 &= -3\beta_1\varepsilon x_2^2 + 2x_2,\\ k &= \alpha_1\alpha_2 - \beta_1\beta_2.\end{aligned} \tag{18}$$

So start from forming a new matrix by subtracting λ from the diagonal entries of the given matrix we have

$$\begin{aligned}&\det\left(F.V^{-1} - \lambda I\right) = \det\left|\begin{array}{cc} \frac{\beta_2 l_2}{k} - \lambda & \frac{\alpha_1 l_2}{k} \\ \frac{\alpha_2 l_1}{k} & \frac{\beta_1 l_1}{k} - \lambda \end{array}\right| = 0,\\ &= \left(\frac{\beta_2 l_2}{k} - \lambda\right)\left(\frac{\beta_1 l_1}{k} - \lambda\right) - \frac{\alpha_1\alpha_2 l_2 l_1}{k^2} = 0,\\ &= \lambda^2 - \lambda\left(\frac{\beta_2 l_2}{k} + \frac{\beta_1 l_1}{k}\right) - \frac{l_2 l_1}{k^2}\left(\alpha_1\alpha_2 - \beta_1\beta_2\right) = 0.\end{aligned} \tag{19}$$

We can have two roots from the last equality

$$\lambda_1 = \frac{\beta_1 l_1 + \beta_2 l_2 + \sqrt{\beta_1^2 l_1^2 - 2 l_2 l_1 \beta_1\beta_2 + 4\alpha_1\alpha_2 l_2 l_1 + \beta_2^2 l_2^2}}{2\alpha_1\alpha_2 - \beta_1\beta_2} \tag{20}$$

and

$$\lambda_2 = \frac{\beta_1 l_1 + \beta_2 l_2 - \sqrt{\beta_1^2 l_1^2 - 2 l_2 l_1 \beta_1\beta_2 + 4\alpha_1\alpha_2 l_2 l_1 + \beta_2^2 l_2^2}}{2\alpha_1\alpha_2 - \beta_1\beta_2}. \tag{21}$$

We know that the maximum eigenvalue is the spectral radius of the matrix, so the chaotic number is found for this model as

$$C_0 = \frac{\beta_1 l_1+\beta_2 l_2+\sqrt{\beta_1^2 l_1^2-2l_2 l_1\beta_1\beta_2+4\alpha_1\alpha_2 l_2 l_1+\beta_2^2 l_2^2}}{2\alpha_1\alpha_2-\beta_1\beta_2}. \tag{22}$$

5. Global stability results for nonlinear model

Explicit solutions to a given differential equation are often difficult to find. In such cases, trying to understand how the solutions of the system behave as time goes to infinity can give a lot of information about the system. Equilibrium points are very important for systems because all solutions converge on these fixed points. To achieve this, we can use the Lyapunov method, which was introduced by Aleksandr Mikhailovich Lyapunov in 1982. So here, the Lyapunov function theory will be used to investigate the global stability of the system. Let us consider the model again.

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\alpha_1 x_1+\beta_1 x_2-\beta_1\varepsilon x_2^3+x_2^2, \\ \frac{dx_2(t)}{dt} &= -\alpha_2 x_2+\beta_2 x_1-\beta_2\varepsilon x_1^3+x_1^2, \end{aligned} \tag{23}$$

with initial conditions $x_1(0)=0$ and $x_2(0)=0$.

Theorem 1. *If $C_0 \geq 1$, the equilibrium point of model $E^*(x_1^*, x_2^*)$ is globally asymptotically stable.*

Proof. We prove this using the idea of the Lyapunov function. We start by defining the Lyapunov function associated with the system as below:

$$\begin{aligned} L(E^*(x_1^*, x_2^*)) = &\left(x_1 - x_1^* + x_1^*\log\frac{x_1^*}{x_1}\right) \\ &+\left(x_2 - x_2^* + x_2^*\log\frac{x_2^*}{x_2}\right). \end{aligned} \tag{24}$$

By the derivative of Lyapunov function with respect to t, we get

$$\frac{dL(t)}{dt} = \left(\frac{x_1-x_1^*}{x_1}\right)\frac{dx_1(t)}{dt} + \left(\frac{x_2-x_2^*}{x_2}\right)\frac{dx_2(t)}{dt}. \tag{25}$$

Now we put values in the above equation for derivatives

$$\begin{aligned} \frac{dL(t)}{dt} = &\left(1-\frac{x_1^*}{x_1}\right)\left(-\alpha_1 x_1+\beta_1 x_2-\beta_1\varepsilon x_2^3+x_2^2\right) \\ &+\left(1-\frac{x_2^*}{x_2}\right)\left(-\alpha_2 x_2+\beta_2 x_1-\beta_2\varepsilon x_1^3+x_1^2\right). \end{aligned} \tag{26}$$

Now we divide all items into positive and negative parts,

$$\frac{dL(t)}{dt} = L_1 - L_2, \tag{27}$$

Here

$$\begin{aligned} L_1 &= \beta_1 x_2 + x_2^2 + x_1^*\alpha_1 + \frac{x_1^*\beta_1\varepsilon x_2^3}{x_1} + \beta_2 x_1 + x_1^2 \\ &\quad + x_2^*\alpha_2 + \frac{x_2^*\beta_2\varepsilon x_1^3}{x_2}, \\ L_2 &= \alpha_1 x_1 + \beta_1\varepsilon x_2^3 + \frac{x_1^*\beta_1 x_2}{x_1} \\ &\quad + \frac{x_1^* x_2^2}{x_1} + \alpha_2 x_2 + \beta_2\varepsilon x_1^3 + \frac{x_2^*\beta_2 x_1}{x_2} + \frac{x_2^* x_1^2}{x_2}. \end{aligned} \tag{28}$$

Therefore if

$$\begin{aligned} &L_1 - L_2 > 0 \text{ then } \frac{dL(t)}{dt} > 0, \\ &L_1 - L_2 = 0 \text{ then } \frac{dL(t)}{dt} = 0, \\ &L_1 - L_2 < 0 \text{ then } \frac{dL(t)}{dt} < 0. \end{aligned} \tag{29}$$

□

5.1. Second derivative of Lyapunov

The Lyapunov function is used for reporting the global stability of systems. The sign of the first derivative of the Lyapunov function may not be enough to say whether we are talking about the local maximum or the local minimum. So we can proceed with analysis to determine the sign of the second derivative of the Lyapunov function. With the following inequality, we obtain the second derivative of the Lyapunov function for our model:

$$\begin{aligned} \frac{d}{dt}\left(\frac{dL(t)}{dt}\right) &= \frac{d}{dt}\left(\left(\frac{x_1-x_1^*}{x_1}\right)\frac{dx_1(t)}{dt} + \left(\frac{x_2-x_2^*}{x_2}\right)\frac{dx_2(t)}{dt}\right), \\ &= \left(\frac{x_1'}{x_1}\right)^2 x_1^* + \left(\frac{x_2'}{x_2}\right)^2 x_2^* + \left(\frac{x_1-x_1^*}{x_1}\right)\frac{d^2x_1(t)}{dt^2} \\ &\quad + \left(\frac{x_2-x_2^*}{x_2}\right)\frac{d^2x_2(t)}{dt^2} \end{aligned} \tag{30}$$

Here we need first and second-order derivative counterparts of equations.

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\alpha_1 x_1+\beta_1 x_2-\beta_1\varepsilon x_2^3+x_2^2, \\ \frac{dx_2(t)}{dt} &= -\alpha_2 x_2+\beta_2 x_1-\beta_2\varepsilon x_1^3+x_1^2, \\ \frac{d^2x_1(t)}{dt^2} &= -\alpha_1\frac{dx_1(t)}{dt}+\beta_1\frac{dx_2(t)}{dt}-3\beta_1\varepsilon x_2^2\frac{dx_2(t)}{dt}+2x_2\frac{dx_2(t)}{dt}, \\ \frac{d^2x_2(t)}{dt^2} &= -\alpha_2\frac{dx_2(t)}{dt}+\beta_2\frac{dx_1(t)}{dt}-3\beta_2\varepsilon x_1^2\frac{dx_1(t)}{dt}+2x_1\frac{dx_1(t)}{dt}. \end{aligned} \tag{31}$$

If we arrange the last two derivatives

$$\frac{d^2x_1(t)}{dt^2} = \alpha_1^2 x_1 + \alpha_1\beta_1\varepsilon x_2^3 + \beta_1\beta_2 x_1 + \beta_1 x_1^2 + 3\beta_1\varepsilon\alpha_2 x_2^3 + 3\beta_1\beta_2\varepsilon^2 x_2^2 x_1^3 + 2\beta_2 x_1 x_2 + 2x_2 x_1^2 - (\alpha_1\beta_1 x_2 + \alpha_1 x_2^2 + \beta_1\alpha_2 x_2 + \beta_1\beta_2\varepsilon x_1^3 + 3\beta_1\beta_2\varepsilon x_2^2 x_1 + 3\beta_1\varepsilon x_2^2 x_1^2 + 2\alpha_2 x_2^2 + 2x_2\beta_2\varepsilon x_1^3), \tag{32}$$

and

$$\frac{d^2x_2(t)}{dt^2} = \alpha_2^2 x_2 + \alpha_2\beta_2\varepsilon x_1^3 + \beta_1\beta_2 x_2 + \beta_2 x_2^2 + 3\beta_2\varepsilon\alpha_1 x_1^3 + 3\beta_1\beta_2\varepsilon^2 x_1^2 x_2^3 + 2x_2 x_1\beta_1 + 2x_1 x_2^2 - (\alpha_2\beta_2 x_1 + \alpha_2 x_1^2 + \beta_2\alpha_1 x_1 + \beta_1\beta_2\varepsilon x_2^3 + 3\beta_1\beta_2\varepsilon x_1^2 x_2 + 3\beta_2\varepsilon x_2^2 x_1^2 + 2\alpha_1 x_1^2 + 2x_1\beta_1\varepsilon x_2^3). \tag{33}$$

Let us consider

$$\frac{d^2x_1(t)}{dt^2} = A_1 + A_2, \quad \frac{d^2x_2(t)}{dt^2} = B_1 + B_2. \tag{34}$$

Here A_1 and B_1 are positive part and taken as

$$\begin{aligned} A_1 = {} & \alpha_1^2 x_1 + \alpha_1\beta_1\varepsilon x_2^3 + \beta_1\beta_2 x_1 + \beta_1 x_1^2 + 3\beta_1\varepsilon\alpha_2 x_2^3 \\ & +3\beta_1\beta_2\varepsilon^2 x_2^2 x_1^3 + 2\beta_2 x_1 x_2 + 2x_2 x_1^2, \\ B_1 = {} & \alpha_2^2 x_2 + \alpha_2\beta_2\varepsilon x_1^3 + \beta_1\beta_2 x_2 + \beta_2 x_2^2 + 3\beta_2\varepsilon\alpha_1 x_1^3 \\ & +3\beta_1\beta_2\varepsilon^2 x_1^2 x_2^3 + 2x_2 x_1\beta_1 + 2x_1 x_2^2 \end{aligned} \tag{35}$$

and A_2 and B_2 are negative part and taken as

$$\begin{aligned} A_2 = {} & -(\alpha_2\beta_2 x_1 + \alpha_2 x_1^2 + \beta_2\alpha_1 x_1 + \beta_1\beta_2\varepsilon x_2^3 + 3\beta_1\beta_2\varepsilon x_1^2 x_2 \\ & +3\beta_1\varepsilon x_2^2 x_1^2 + 2\alpha_2 x_2^2 + 2x_2\beta_2\varepsilon x_1^3), \\ B_2 = {} & -(\alpha_2\beta_2 x_1 + \alpha_2 x_1^2 + \beta_2\alpha_1 x_1 + \beta_1\beta_2\varepsilon x_2^3 + 3\beta_1\beta_2\varepsilon x_1^2 x_2 \\ & +3\beta_2\varepsilon x_2^2 x_1^2 + 2\alpha_1 x_1^2 + 2x_1\beta_1\varepsilon x_2^3). \end{aligned} \tag{36}$$

So we have

$$\begin{aligned} \frac{d^2L(t)}{dt^2} = {} & \left(\frac{x_1'}{x_1}\right)^2 x_1^* + \left(\frac{x_2'}{x_2}\right)^2 x_2^* \\ & +A_1 + A_2 - \frac{x_1^*}{x_1}A_1 - \frac{x_1^*}{x_1}A_2 \\ & +B_1 + B_2 - \frac{x_2^*}{x_2}B_1 - \frac{x_2^*}{x_2}B_2. \end{aligned} \tag{37}$$

Now we divide normalsize all items with positive and negative parts

$$\frac{d^2L(t)}{dt^2} = \Phi_1 - \Phi_2, \tag{38}$$

Here the positive part of equality is given as

$$\Phi_1 = \left(\frac{x_1'}{x_1}\right)^2 x_1^* + \left(\frac{x_2'}{x_2}\right)^2 x_2^* + A_1 + B_1 + \frac{x_1^*}{x_1}A_2 + \frac{x_2^*}{x_2}B_2, \tag{39}$$

and the negative part of equality is given as

$$\Phi_2 = A_2 + B_2 - \frac{x_1^*}{x_1}A_1 - \frac{x_2^*}{x_2}B_1. \tag{40}$$

Therefore if

$$\begin{aligned} & \Phi_1 - \Phi_2 > 0 \text{ then } \frac{d^2L(t)}{dt^2} > 0, \\ & \Phi_1 - \Phi_2 = 0 \text{ then } \frac{d^2L(t)}{dt^2} = 0, \\ & \Phi_1 - \Phi_2 < 0 \text{ then } \frac{d^2L(t)}{dt^2} < 0. \end{aligned} \tag{41}$$

6. Existence and uniqueness of system solution

In the last past years, several authors have devoted their attention to developing conditions under which nonlinear differential equations admit unique solutions, in particular for the case of classical derivatives. Several extensions have been done within the framework of fractional derivation with singular and non-singular kernels. We shall state one of the important on here, which will be used.

Theorem 2. *Let $I_T = [0,T]$, the function $f : \underset{(t,y)\to}{I \times R} \to \underset{f(t,y)}{R}$ is such that, $f(t,y)$ is measurable for $y \in R$ and $y \to f(t,y)$ is continuous for each $t \in I_T$. If there exists on $M \in L^2[I_T, R]$ such that*

$$|f(t,y)|^2 \leq M\left(1 + |y|^2\right), \ \forall (t,y) \in I_T \times R \tag{42}$$

then there exists a continuous $u(t)$ such that

$$u(t) = \int_0^t f(\tau, u(\tau))\, d\tau. \tag{43}$$

If in addition, one have

$$|f(t,y) - f(t,\overline{y})| < K|y - \overline{y}|^2, \ \forall y, \overline{y} \in R \tag{44}$$

then the solution is unique.

Indeed the existence can be achieved via sequence by constructing the Picard, Tonelli other sequences [18, 19]. The main task is to show that under the above condition, the sequence is equicontinuous uniformly and bounded uniformly. The Peano-Cauchy theorem helps us to secure the existence [20]. The Gronwall inequality helps us obtain uniqueness within the framework of fractional calculus, there is an extra condition on the fractional order. It's required that $\alpha > \frac{1}{2}$ since

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau\right|^2 \quad (45)\\
\leq\ & \frac{1}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2} |f(\tau, y(\tau))|^2 d\tau \\
\leq\ & \frac{1}{\Gamma^2(\alpha)}\int_0^t (t-\tau)^{2\alpha-2} d\tau \int_0^t |f(\tau, y(\tau))|^2 d\tau \\
\leq\ & \frac{t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)} \|f(.,y(.))\|^2_{L^2[0,T]}
\end{aligned}$$

Thus $\alpha > \frac{1}{2}$.

The existence and uniqueness of the solution of a differential equation are the most important parts of the theory of differential equations. There are various proofs on this subject. Here we will do our proof by obtaining the necessary conditions via Linear growth and Lipschitz for our model [21].

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 + x_2^2, \quad (46)\\
\frac{dx_2(t)}{dt} &= -\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + x_1^2,
\end{aligned}$$

with initial conditions $x_1(0) = 0$ and $x_2(0) = 0$. Let us find the necessary conditions for the existence and uniqueness, we must prove that $\forall\ [0, T_1]$ and $f_i(x_1, x_2)$ for $i = 1, 2$ satisfy

1)Linear growth condition

$$|f_i(x_i, t)|^2 \leq s_i(1 + |x_i|^2) \quad \text{for } i = 1, 2. \quad (47)$$

2)The Lipschitz condition

$$|f_i(x_i, t) - f_i(\overline{x}_i, t)|^2 \leq \overline{s}_i |x_i - \overline{x}_i|^2 \quad \text{for } i = 1, 2. \quad (48)$$

Now we define the norm $\|\varphi\|_\infty = \sup_{t\in D_\varphi} |\varphi(t)|$. Now we put the existence and uniqueness of the solution for $[0, T_1]$. For $[0, T_1]$, there exist 2 positive constant M_1 and $M_2 < \infty$ such that

$$\begin{aligned}
\|x_1\|_\infty &< M_1, \quad (49)\\
\|x_2\|_\infty &< M_2.
\end{aligned}$$

Let us write the system as below:

$$\begin{cases} \dot{x_1} = f_1(x_1, x_2), \\ \dot{x_2} = f_2(x_1, x_2), \end{cases} \text{if } 0 \leq t \leq T_1. \quad (50)$$

For proof, we consider the function

$$\begin{aligned}
|f_1(x_1,x_2)|^2 &= \left|-\alpha_1 x_1 + \beta_1 x_2 - \beta_1\varepsilon x_2^3 + x_2^2\right|^2,\\
&\leq 4\alpha_1^2|x_1|^2 + 4\beta_1^2|x_2|^2 \\
&+ 4\beta_1^2\varepsilon^2\left|x_2^3\right|^2 + 4\left|x_2^2\right|^2\\
&\leq 4\alpha_1^2|x_1|^2 + 4\beta_1^2 \sup_{t\in[0,T_1]}|x_2|^2\\
&+ 4\beta_1^2\varepsilon^2 \sup_{t\in[0,T_1]}\left|x_2^3\right|^2 + 4\sup_{t\in[0,T_1]}\left|x_2^2\right|^2\\
&\leq 4\alpha_1^2|x_1|^2 + 4\beta_1^2\|x_2\|_\infty^2 + 4\beta_1^2\varepsilon^2\left\|x_2^3\right\|_\infty^2\\
&+ 4\left\|x_2^2\right\|_\infty^2,\\
&\leq 4\beta_1^2\|x_2\|_\infty^2 + 4\beta_1^2\varepsilon^2\left\|x_2^3\right\|_\infty^2\\
&+ 4\left\|x_2^2\right\|_\infty^2 \times\\
&\left(1 + \frac{4\alpha_1^2|x_1|^2}{4\beta_1^2\|x_2\|_\infty^2 + 4\beta_1^2\varepsilon^2\left\|x_2^3\right\|_\infty^2 + 4\left\|x_2^2\right\|_\infty^2}\right)\\
&\leq s_1(1 + |x_1(t)|^2)
\end{aligned} \quad (51)$$

Here

$$s_1 = 4\beta_1^2\|x_2\|_\infty^2 + 4\beta_1^2\varepsilon^2\left\|x_2^3\right\|_\infty^2 + 4\left\|x_2^2\right\|_\infty^2 \quad (52)$$

and under the condition that

$$\frac{\alpha_1^2}{\beta_1^2\|x_2\|_\infty^2 + \beta_1^2\varepsilon^2\left\|x_2^3\right\|_\infty^2 + \left\|x_2^2\right\|_\infty^2} < 1, \quad (53)$$

then we have

$$|f_1(x_1, x_2)|^2 \leq s_1(1 + |x_1(t)|^2). \quad (54)$$

Using the same routine

$$\begin{aligned}
|f_2(x_1,x_2)|^2 &= \left|-\alpha_2 x_2 + \beta_2 x_1 - \beta_2\varepsilon x_1^3 + x_1^2\right|^2,\\
&\leq 4\alpha_2^2|x_2|^2 + 4\beta_2^2|x_1|^2 + 4\beta_2^2\varepsilon^2\left|x_1^3\right|^2 + 4\left|x_1^2\right|^2,\\
&\leq 4\alpha_2^2|x_2|^2 + 4\beta_2^2\sup_{t\in[0,T_1]}|x_1|^2\\
&+ 4\beta_2^2\varepsilon^2\sup_{t\in[0,T_1]}\left|x_1^3\right|^2 + 4\sup_{t\in[0,T_1]}\left|x_1^2\right|^2\\
&\leq 4\alpha_1^2|x_2|^2 + 4\beta_2^2\|x_1\|_\infty^2 + 4\beta_1^2\varepsilon^2\left\|x_1^3\right\|_\infty^2\\
&+ 4\left\|x_1^2\right\|_\infty^2,\\
&\leq 4\beta_2^2\|x_1\|_\infty^2 + 4\beta_1^2\varepsilon^2\left\|x_1^3\right\|_\infty^2 + 4\left\|x_1^2\right\|_\infty^2 \times\\
&\left(1 + \frac{4\alpha_1^2|x_2|^2}{4\beta_2^2\|x_1\|_\infty^2 + 4\beta_1^2\varepsilon^2\left\|x_1^3\right\|_\infty^2 + 4\left\|x_1^2\right\|_\infty^2}\right)\\
&\leq s_2(1 + |x_2(t)|^2)
\end{aligned} \quad (55)$$

Here

$$s_2 = 4\beta_2^2 \|x_1\|_\infty^2 + 4\beta_1^2\varepsilon^2 \left\|x_1^3\right\|_\infty^2 + 4 \left\|x_1^2\right\|_\infty^2 \quad (56)$$

and under the condition

$$\frac{\alpha_1^2}{\beta_2^2 \|x_1\|_\infty^2 + \beta_1^2\varepsilon^2 \left\|x_1^3\right\|_\infty^2 + \left\|x_1^2\right\|_\infty^2} < 1 \quad (57)$$

Therefore the condition of linear growth is verified if

$$\max\left\{ \begin{array}{l} \frac{\alpha_1^2}{\beta_1^2\|x_2\|_\infty^2+\beta_1^2\varepsilon^2\left\|x_2^3\right\|_\infty^2+\left\|x_2^2\right\|_\infty^2}, \\ \frac{\alpha_1^2}{\beta_2^2\|x_1\|_\infty^2+\beta_1^2\varepsilon^2\left\|x_1^3\right\|_\infty^2+\left\|x_1^2\right\|_\infty^2}, \end{array} \right\} < 1. \quad (58)$$

The first part of proof is completed. Now we have to verify Lipschitz condition for equations. If we have $\forall x_1, \overline{x}_1 \in R^2$ and $t \in [0, T_1]$, for the function $f_1(x_1, x_2)$,

$$\begin{aligned} |f_1(x_1, x_2) - f_1(\overline{x}_1, x_2)| &\le \alpha_1^2 |x_1 - \overline{x}_1|, \quad (59) \\ &\le \overline{s}_1 |x_1 - \overline{x}_1|. \end{aligned}$$

If we have $\forall x_2, \overline{x}_2 \in R^2$ and $t \in [0, T_1]$ for the function $f_2(x_1, x_2)$,

$$\begin{aligned} |f_2(x_1, x_2) - f_2(x_1, \overline{x}_2)| &\le \alpha_2^2 |x_2 - \overline{x}_2|, \quad (60) \\ &\le \overline{s}_2 |x_2 - \overline{x}_2|. \end{aligned}$$

We verified the Lipschitz condition, which completes the proof.
Finally, we consider the following fractional order model as below;

$$\begin{aligned} &{}_{t_0}^{C}D_t^\alpha x_1(t) = f_1(t, x_1(t)), \quad \text{if } t > 0 \quad (61) \\ &{}_{t_0}^{C}D_t^\alpha x_2(t) = f_2(t, x_2(t)), \\ &x_1(t_0) = x_{10},\ x_2(t_0) = x_{20} \quad \text{if } t = 0. \end{aligned}$$

We can write the system above as

$$\begin{aligned} {}_{t_0}^{C}D_t^\alpha X(t) &= F(t, X(t)), \quad (62) \\ X(t_0) &= X_0, \end{aligned}$$

where

$$X(t) = \left\{ \begin{array}{l} x_1(t), \\ x_2(t) \end{array} \right., \ X(t_0) = \left\{ \begin{array}{l} x_1(t_0), \\ x_2(t_0) \end{array} \right.,$$

$$F(t, X(t)) = \left\{ \begin{array}{l} f_1(t, x_1(t)), \\ f_2(t, x_2(t)) \end{array} \right. . \quad (63)$$

Now applying the fractional integral on both sides

$$X(t) = \frac{1}{\Gamma(\alpha)} \int_0^t F(\tau, X(\tau)) (t-\tau)^{\alpha-1} d\tau. \quad (64)$$

At the previous section we showed that $f_1(t, x_1(t))$ and $f_2(t, x_2(t))$ satisfy the Lipschitz condition and are bounded in $[a, b]$. Using the Picard iteration for above , then we have that

$$X_{n+1}(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t F(\tau, X_n(\tau)) (t-\tau)^{\alpha-1} d\tau. \quad (65)$$

For the existence theory, we define Banach space $\Phi = X \times X$ where $X = C[0, T_1]$ under the following norm

$$\|X\| = \max_{t \in [0, T_1]} |x_1(t), x_2(t)|. \quad (66)$$

So we have

$$\begin{aligned} \|X_{n+1}\| &= \max_{t \in [0, T_1]} \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^t F(\tau, X_n(\tau)) (t-\tau)^{\alpha-1} d\tau \right| \\ &\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^t s(1 + \|X_n\|) (t-\tau)^{\alpha-1} d\tau \\ &\le \frac{s(1 + \|X_n\|)}{\Gamma(\alpha)} \frac{(t-t_0)^\alpha}{\alpha}. \end{aligned} \quad (67)$$

So we have that $\forall t \in [a, b]$

$$\|X_{n+1}\| \le \frac{s(1 + \|X_n\|)}{\Gamma(\alpha+1)} (b - t_0)^\alpha \quad (68)$$

But $\forall n > 0$, $\exists c \in [x_0 - c, x_0 + c]$ then

$$\begin{aligned} &\frac{s(1 + \|X_n\|)}{\Gamma(\alpha+1)} (b - t_0)^\alpha < c, \\ &b < \left(\frac{c\Gamma(\alpha+1)}{s(1 + \|X_n\|)} \right)^{\frac{1}{\alpha}} + t_0. \end{aligned} \quad (69)$$

Under the above condition $X_n(t)$ for $n \ge 0$ is uniformly bounded and well-defined. For equicontinuity of X, let us take $t_1 < t_2 < T_1$, then consider

$$\|X_n(t_1) - X_n(t_2)\|$$

$$= \frac{1}{\Gamma(\alpha)} \max \left| \begin{array}{l} \int_{t_0}^{t_1} F(\tau, X_{n-1}(\tau)) (t_1-\tau)^{\alpha-1} d\tau \\ -\int_{t_0}^{t_2} F(\tau, X_{n-1}(\tau)) (t_2-\tau)^{\alpha-1} d\tau \end{array} \right|$$

$$= \frac{1}{\Gamma(\alpha)} \max \left| \begin{array}{l} \int_{t_0}^{t_2} F(\tau, X_{n-1}(\tau)) (t_1-\tau)^{\alpha-1} d\tau \\ -\int_{t_0}^{t_2} F(\tau, X_{n-1}(\tau)) (t_2-\tau)^{\alpha-1} d\tau \\ +\int_{t_2}^{t_1} F(\tau, X_{n-1}(\tau)) (t_1-\tau)^{\alpha-1} d\tau \end{array} \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} \| F(\tau, X_{n-1}(\tau)) \| \left\{ (t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1} \right\} d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t_1} \| F(\tau, X_{n-1}(\tau)) \| (t_1-\tau)^{\alpha-1} d\tau$$

$$\leq \frac{s(1+\|X_n\|)}{\Gamma(\alpha)} \left\{ \frac{(t_1-t_0)^{\alpha}}{\alpha} - \frac{(t_2-t_0)^{\alpha}}{\alpha} - \frac{(t_1-t_2)^{\alpha}}{\alpha} \right\}$$

$$+ \frac{s(1+\|X_n\|)}{\Gamma(\alpha)} \left\{ \frac{(t_1-t_2)^{\alpha}}{\alpha} \right\}$$

$$\leq \frac{s(1+\|X_n\|)}{\Gamma(\alpha+1)} \left\{ (t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha} \right\}. \tag{70}$$

Noting that the $(t-t_0)^{\alpha}$ is differentiable, by the Mean Value theorem we can find $c \in [t_1 - t_0, t_2 - t_0]$ such that

$$\alpha (c-t_0)^{\alpha-1} (t_1-t_2) = (t_1-t_0)^{\alpha} - (t_2-t_0)^{\alpha}. \tag{71}$$

So we have

$$\begin{aligned} \|X_n(t_1) - X_n(t_2)\| &\leq \frac{s(1+\|X_n\|)}{\Gamma(\alpha+1)} \alpha (c-t_0)^{\alpha-1} (t_1-t_2) \\ &\leq \frac{s(1+\|X_n\|)}{\Gamma(\alpha+1)} \alpha (c-t_0)^{\alpha-1} \|t_1-t_2\| \\ &< \varepsilon \end{aligned} \tag{72}$$

then $\forall \varepsilon > 0$, we must find $\exists \delta > 0$ such that

$$\delta < \frac{\varepsilon \Gamma(\alpha)}{s(1+\|X_n\|) \alpha (c-t_0)^{\alpha-1}}. \tag{73}$$

So under the condition above $X_n(t)$ is uniformly equicontinuous.

Beside the Caratheodory principle verified above, one can demonstrate the existence and uniqueness of the system solutions of the considered system.

We have that

$$\begin{aligned} {}_0^C D_t^{\alpha} x_1(t) &= f_1(t, x_1(t)), \quad \text{if } t > 0 \\ {}_0^C D_t^{\alpha} x_2(t) &= f_2(t, x_2(t)). \end{aligned} \tag{74}$$

It is sufficient to show that $\forall t \in I_b = [0, b]$ the associate Jacobian matrix is differentiable continuous. The Jacobian associated to this system is given as

$$J(x_1, x_2) = \begin{bmatrix} -\alpha_1 & \beta_1 - 3\varepsilon\beta_1 x_2^2 + 2\lambda x_2 \\ \beta_2 - 3\varepsilon\beta_2 x_1^2 + 2\lambda x_1 & -\alpha_2 \end{bmatrix}. \tag{75}$$

The above is continuous for $\forall (x, y)$ which completes the proof.

7. Model with piecewise concept

It indeed above model can be used to replicate some interpersonal interaction, one will notice that the current mathematical model show only one process, for example with the Caputo one can only describe the relation following the power-law behavior. Whereas in normal situations, interpersonal interaction undergoes piecewise behaviors, where the relation change as function of time in the case of ordinary differential equation and space time in the case of partial differential equation. In this section, we shall consider the model with two to three processes, including classical behaviors, then power law behaviors or power law and stochastic with piecewise idea [11]. In these cases, the following mathematical systems are constructed

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 + \lambda x_2^2, \text{ if } 0 \leq t \leq t_1 \\ \frac{dx_2(t)}{dt} &= -\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + \lambda x_1^2, \\ {}_{t_1}^{C} D_t^{\alpha} x_1(t) &= -\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 + \lambda x_2^2, \text{ if } t_1 \leq t \leq T \\ {}_{t_1}^{C} D_t^{\alpha} x_2(t) &= -\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + \lambda x_1^2, \end{aligned} \tag{76}$$

or

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 + \lambda x_2^2, \text{ if } 0 \leq t \leq t_1 \\ \frac{dx_2(t)}{dt} &= -\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + \lambda x_1^2, \\ dx_1(t) &= \left(-\alpha_1 x_1 + \beta_1 x_2 - \beta_1 \varepsilon x_2^3 + \lambda x_2^2\right) dt + \sigma_1 x_1 dB_1(t), \\ &\quad \text{if } t_1 \leq t \leq T \\ dx_2(t) &= \left(-\alpha_2 x_2 + \beta_2 x_1 - \beta_2 \varepsilon x_1^3 + \lambda x_1^2\right) dt + \sigma_2 x_2 dB_2(t). \end{aligned} \tag{77}$$

Obviously the above system can not be solved analytically indeed due to non linearity, therefore we will present some existence and uniqueness conditions for the two systems. Indeed by putting

$$\begin{aligned} \frac{dx_1(t)}{dt} &= f_1(t, x_1, x_2), \text{ if } 0 \le t \le t_1 \\ \frac{dx_2(t)}{dt} &= f_2(t, x_1, x_2), \\ {}_{t_1}^{C}D_t^{\alpha} x_1(t) &= f_1(t, x_1, x_2), \text{if } t_1 \le t \le T \\ {}_{t_1}^{C}D_t^{\alpha} x_2(t) &= f_2(t, x_1, x_2). \end{aligned} \tag{78}$$

The following Picard system of sequence can be defined

$$\begin{aligned} x_{1n+1}(t) &= x_1(0) + \int_0^t f_1(\tau, x_{1n}, x_{2n})\, d\tau, \text{ if } 0 \le t \le t_1 \\ x_{2n+1}(t) &= x_2(0) + \int_0^t f_2(\tau, x_{1n}, x_{2n})\, d\tau, \\ x_{1n+1}(t) &= x_1(t_1) + \int_{t_1}^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f_1(\tau, x_{1n}, x_{2n})\, d\tau, \\ &\quad \text{if } t_1 \le t \le T \\ x_{2n+1}(t) &= x_2(t_1) + \int_{t_1}^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f_2(\tau, x_{1n}, x_{2n})\, d\tau, \end{aligned} \tag{79}$$

and

$$\begin{aligned} x_{1n+1}(t) &= x_1(0) + \int_0^t f_1(\tau, x_{1n}, x_{2n})\, d\tau, \text{ if } 0 \le t \le t_1 \\ x_{2n+1}(t) &= x_2(0) + \int_0^t f_2(\tau, x_{1n}, x_{2n})\, d\tau, \\ x_{1n+1}(t) &= x_1(t_1) + \int_{t_1}^t f_1(\tau, x_{1n}, x_{2n})\, d\tau + \sigma_1 \int_{t_1}^t x_{1n} dB_1(t), \\ &\quad \text{if } t_1 \le t \le T \\ x_{2n+1}(t) &= x_2(t_1) + \int_{t_1}^t f_2(\tau, x_{1n}, x_{2n})\, d\tau + \sigma_2 \int_{t_1}^t x_{2n} dB_2(t), \\ &\quad \text{if } t_1 \le t \le T. \end{aligned} \tag{80}$$

The above sequences are Picard sequences that indeed satisfying indeed under some conditions uniform equicontinuity and bounded, this lead to the existence of a unique system of solutions. The detailed proof will not be presented here. However, a numerical scheme will be used to solve numerically the above equation. For the classical case, we shall adopt Heun's method

$$\begin{aligned} \widetilde{x}_{1n+1} &= x_{1n} + h\left[f_1(t_n, x_{1n}, x_{2n})\right], \\ \widetilde{x}_{2n+1} &= x_{2n} + h\left[f_2(t_n, x_{1n}, x_{2n})\right], \\ x_{1n+1} &= x_{1n} + \frac{h}{2}\left[f_1(t_n, x_{1n}, x_{2n}) + f_1(t_{n+1}, \widetilde{x}_{1n+1}, \widetilde{x}_{2n+1})\right], \\ x_{2n+1} &= x_{2n} + \frac{h}{2}\left[f_2(t_n, x_{1n}, x_{2n}) + f_2(t_{n+1}, \widetilde{x}_{1n+1}, \widetilde{x}_{2n+1})\right], \end{aligned} \tag{81}$$

replacing $\widetilde{x}_{1n+1}$ and $\widetilde{x}_{2n+1}$, we get

$$\begin{aligned} x_{1n+1} &= x_{1n} + \frac{h}{2}[f_1(t_n, x_{1n}, x_{2n}) \\ &\quad + f_1(t_{n+1}, x_{1n} + hf_1(t_n, x_{1n}, x_{2n}))], \\ x_{2n+1} &= x_{2n} + \frac{h}{2}[f_2(t_n, x_{1n}, x_{2n}) \\ &\quad + f_2(t_{n+1}, x_{2n} + hf_2(t_n, x_{1n}, x_{2n}))]. \end{aligned}$$

For the Caputo type, to avoid confusion, we define

$$\begin{aligned} x_1(t_{n+1}) &= x_{1n+1}, \\ x_2(t_{n+1}) &= x_{2n+1}, \\ x_1(t_0) &= x_{10}, \\ x_2(t_0) &= x_{20}. \end{aligned} \tag{82}$$

$$\begin{aligned} x_{1n+1} &= x_{10} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} f_1(\tau, x_1, x_2)(t_{n+1} - \tau)^{\alpha-1} d\tau, \\ x_{2n+1} &= x_{20} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} f_2(\tau, x_1, x_2)(t_{n+1} - \tau)^{\alpha-1} d\tau, \\ x_{1n+1} &= x_{10} + \frac{1}{2\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} [f_1(t_j, x_{1j}, x_{2j}) \\ &\quad + f_1(t_{j+1}, x_{1j+1}, x_{2j+1})](t_{n+1} - \tau)^{\alpha-1} d\tau, \\ x_{2n+1} &= x_{20} + \frac{1}{2\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} [f_2(t_j, x_{1j}, x_{2j}) \\ &\quad + f_2(t_{j+1}, x_{1j+1}, x_{2j+1})](t_{n+1} - \tau)^{\alpha-1} d\tau \end{aligned} \tag{83}$$

$$\begin{aligned} x_{1n+1} &= x_{10} + \frac{h^\alpha}{2\Gamma(\alpha+1)} \sum_{j=0}^{n-1} [f_1(t_j, x_{1j}, x_{2j}) + f_1(t_{j+1}, x_{1j+1}, x_{2j+1})] \\ &\quad \{(n-j+1)^\alpha - (n-j)^\alpha\} \\ &\quad + \frac{h^\alpha}{2\Gamma(\alpha+1)} \left[f_1(t_n, x_{1n}, x_{2n}) + f_1(t_{n+1}, \widetilde{x}_{1n+1}, \widetilde{x}_{2n+1})\right], \\ x_{2n+1} &= x_{20} + \frac{h^\alpha}{2\Gamma(\alpha+1)} \sum_{j=0}^{n-1} [f_2(t_j, x_{1j}, x_{2j}) + f_2(t_{j+1}, x_{1j+1}, x_{2j+1})] \\ &\quad \{(n-j+1)^\alpha - (n-j)^\alpha\} \\ &\quad + \frac{h^\alpha}{2\Gamma(\alpha+1)} \left[f_2(t_n, x_{1n}, x_{2n}) + f_2(t_{n+1}, \widetilde{x}_{1n+1}, \widetilde{x}_{2n+1})\right], \end{aligned} \tag{84}$$

where

$$\widetilde{x}_{1n+1} = x_{10} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} f_1(t_j, x_{1j}, x_{2j}) \{(n-j+1)^{\alpha} - (n-j)^{\alpha}\},$$
$$\widetilde{x}_{2n+1} = x_{20} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} f_2(t_j, x_{1j}, x_{2j}) \{(n-j+1)^{\alpha} - (n-j)^{\alpha}\}. \tag{85}$$

Finally for the stochastic part, the following numerical solution can be obtained

$$\begin{aligned}
\widetilde{x}_{1n+1} &= x_{1n} + hf_1(t_n, x_{1n}, x_{2n}) + \sigma_1 x_{1n}[B_{1n+1} - B_{1n}],\\
\widetilde{x}_{2n+1} &= x_{2n} + hf_2(t_n, x_{1n}, x_{2n}) + \sigma_2 x_{2n}[B_{2n+1} - B_{2n}],\\
x_{1n+1} &= x_{1n} + \frac{h}{2}[f_1(t_n, x_{1n}, x_{2n}) + f_1(t_{n+1}, \widetilde{x}_{1n+1}, \widetilde{x}_{2n+1})]\\
&\quad + \sigma_1 x_{1n}[B_{1n+1} - B_{1n}],\\
x_{2n+1} &= x_{2n} + \frac{h}{2}[f_2(t_n, x_{1n}, x_{2n}) + f_2(t_{n+1}, \widetilde{x}_{1n+1}, \widetilde{x}_{2n+1})]\\
&\quad + \sigma_2 x_{2n}[B_{2n+1} - B_{2n}].
\end{aligned} \tag{86}$$

8. Numerical simulations

In this section, we will deal with the numerical simulation of the interpersonal model with the piecewise differential operators and the numerical scheme where the Lagrange polynomial interpolation is used [22]. In the numerical scheme, the first part is classical, the second part is stochastic and the last part is fractional. The numerical simulations are shown in Fig. 1 for $\alpha = 1$, Fig. 2 for $\alpha = 0.97$, Fig. 3 for $\alpha = 0.98$, and Fig. 4 is obtained for chaos for $\alpha = 1$, Fig. 5 is obtained for chaos for $\alpha = 0.97$, Fig. 6 is obtained for chaos for $\alpha = 0.98$. For all figures, density of randomness are taken as $\sigma_1 = 0.09$, and $\sigma_2 = 0.09$. Also figures including the initial conditions as $x_1(0) = -0.1$, $x_2(0) = 0.8$. Also, for the numerical simulations of the system we consider the values of the parameters as follows:

$$\alpha_1 = 0.1, \alpha_2 = 0.01, \beta_1 = 5.7, \beta_2 = -1, \varepsilon = 0.01$$

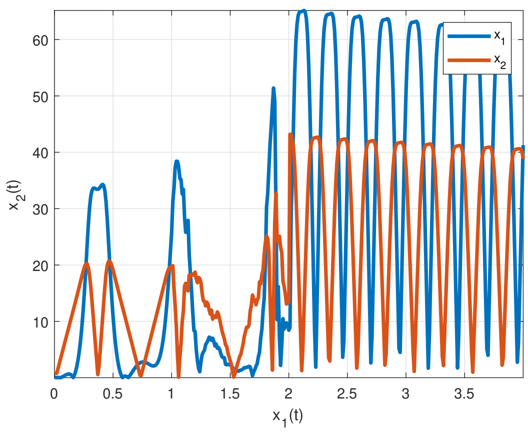

Figure 1. Numerical solutions for $\alpha = 1$.

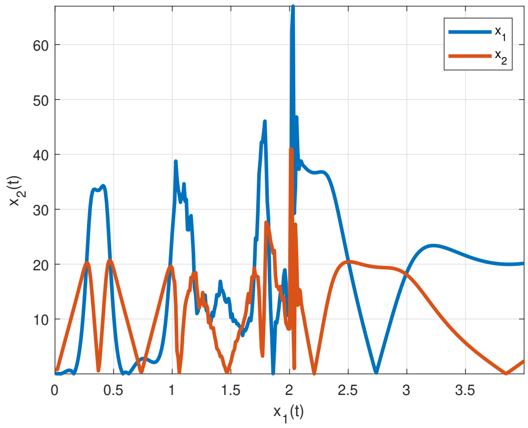

Figure 2. Numerical solutions for $\alpha = 0.97$.

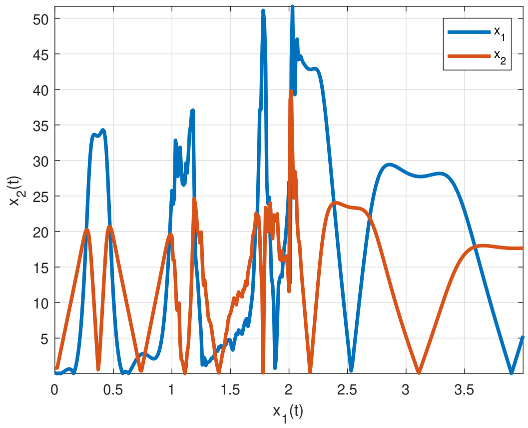

Figure 3. Numerical solutions for $\alpha = 0.98$.

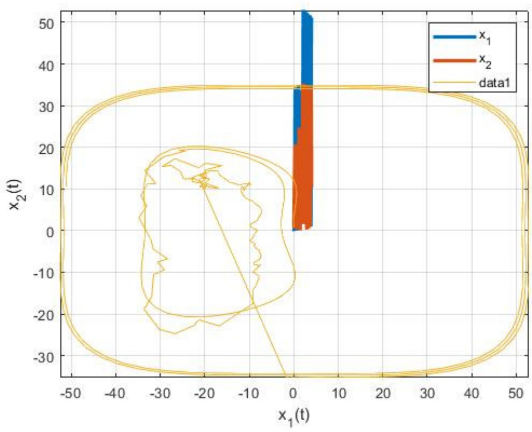

Figure 4. Numerical solutions for $\alpha = 1$.

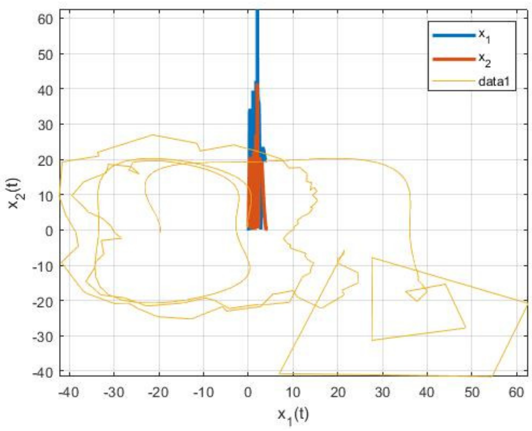

Figure 5. Numerical solutions for $\alpha = 0.97$.

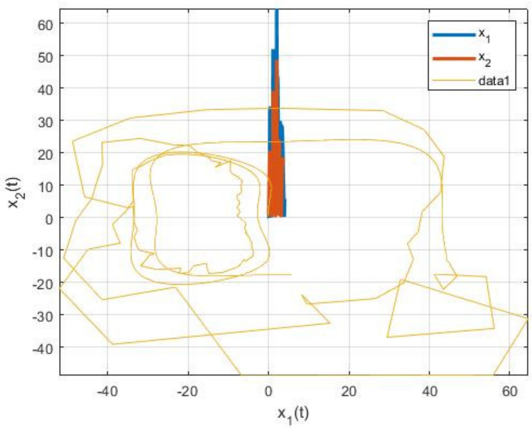

Figure 6. Numerical solutions for $\alpha = 0.98$.

9. Conclusion

In this work, a nonlinear differential equation was taken into consideration, and the Caputo, stochastic process, and piecewise differential operators were used in place of the classical differential operators. Through this work, we have looked into the associated equilibrium points' general approach to stability. We have derived the conditions under which the system admits a singular, unique system of solutions using the linear growth and Lipschitz requirements. To solve this problem numerically in the Caputo, stochastic, and piecewise cases, a numerical approach was adopted.

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