An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.14, No.1, pp.41-49 (2024)
http://doi.org/10.11121/ijocta.1392

RESEARCH ARTICLE

On the upper bounds of Hankel determinants for some subclasses of univalent functions associated with sine functions

Muhammet Kamali*, Alina Riskulava,

Department of Mathematics, Faculty of Sciences, Kyrgyz-Türkish Manas University, Chyngz Aitmatov Avenue Bishkek, Kyrgyz Recuplic
muhammet.kamali@manas.edu.kg, alinka.ryskulova97@gmail.com

ARTICLE INFO

Article History:
Received 13 April 2023
Accepted 15 July 2023
Available Online 12 December 2023

Keywords:
Analytic functions
Coefficient estimates
Subordination
Hankel determinant
Starlike functions
Convex function
Sine function

AMS Classification 2010:
30C45; 30C50

ABSTRACT

Let a normalized analytic function be given on the open unit disk. In this paper, we define and consider some familiar subsets of analytic functions associated with sine functions in the region of unit disk on the complex plane. For these classes, we aim to find the upper bounds of the modules of the Hankel determinants obtained from the coefficients of the functions belonging to some classes defined by subordination.

1. Introduction

Let A be the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E \tag{1}$$

which are analytic in the open unit disk $E = \{z \in C : |z| < 1\}$ and let S denote the subclass of A consisting of all univalent functions in E. With a view to recalling the principal of subordination between analytic functions, let $f(z)$ and $g(z)$ be analytic functions in E. Then we say that the function $f(z)$ is subordinate to $g(z)$ in E, if there exits a Schwarz function $w(z)$, analytic in E with

$$w(0) = 0 \text{ and } |w(z)| < 1, (z \in E)$$

such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z).$$

If g is a univalent function in E, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

The famous coefficient conjucture Beiberbach conjucture for the functions $f \in S$ of the form (1) was first presented by Beiberbach [1] in 1916 and proven by de-Branges [2] in 1985. In between the years 1916 and 1985, many mathematicians worked to prove Beiberbach's conjucture. Consequently, they defined several subclasses of S connected with different image domains. Among these, the families S^*, C and K of starlike functions, convex functions, and close-to-convex functions, respectively, are the most fundamental subclasses of S and have a

*Corresponding Author

nice geometric interpretation. These families are defined as follows:

$$S^{*}=\left\{f\in S:\frac{zf^{'}(z)}{f(z)}\prec\frac{1+z}{1-z},(z\in E)\right\}\quad(2)$$

$$C=\left\{f\in S:1+\frac{zf^{''}(z)}{f^{'}(z)}\prec\frac{1+z}{1-z},(z\in E)\right\}\quad(3)$$

$$K=\left\{f\in S:\frac{zf^{'}(z)}{g(z)}\prec\frac{1+z}{1-z},g(z)\in S^{*},(z\in E)\right\}.\quad(4)$$

We recall here which are connected with trigonometric functions and are defined as folows:

$$S_{\sin}^{*}=\left\{f\in S:\frac{zf^{'}(z)}{f(z)}\prec1+\sin(z),(z\in E)\right\}\quad(5)$$

$$C_{\sin}=\left\{f\in S:1+\frac{zf^{''}(z)}{f^{'}(z)}\prec1+\sin(z),(z\in E)\right\}\quad(6)$$

$$R_{\sin}=\left\{f\in S:f^{'}(z)\prec1+\sin(z),(z\in E)\right\}\quad(7)$$

The class $S_{\sin}^{*}$ of analytic function defined in (5) was introduced by Cho et al. [3].

In the 1960s Pommerenke [4], [5] defined the Hankel determinant $H_{q,n}(f)$for a given f of the form (1) f as follows

$$H_{q,n}(f)=\begin{vmatrix}a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix},\quad(8)$$

where $q,n\in N=\{1,2,3,...\}$. In particular,

$$H_{2,1}(f)=\begin{vmatrix}1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-a_2^2,$$

$$H_{2,2}(f)=\begin{vmatrix}a_2 & a_3\\ a_3 & a_4\end{vmatrix}=a_2a_4-a_3^2$$

and

$$H_{3,1}(f)=\begin{vmatrix}1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}$$

$$=a_3\left(a_2a_4-a_3^2\right)+a_4\left(a_2a_3-a_4\right)+a_5\left(a_3-a_2^2\right).$$

The studies on Hankel determinants are concentrated on estimating $H_{2,2}(f)$ and $H_{3,1}(f)$ for different subclasses of S.The absolute sharp bounds of the functional $H_{2,2}(f)$ were found in [6], [7] for each of the families S^{*},C, and R, where the family$R$ contains functions of bounded turning. In [7] , Janteng et al. proved that $|H_{2,2}(f)|\leq1$ for $S^{*}$and $|H_{2,2}(f)|\leq\frac{1}{8}$ for K,where $S^{*}$and $K$are very well known classes of starlike and convex functions. The estimation of the determinant $|H_{3,1}(f)|$ is very hard as compared to deriving the bound of $|H_{2,2}(f)|$. The paper on $|H_{3,1}(f)|$ was given in 2010 by Babalola [8], in which he obtained the upper bound of $H_{3,1}(f)$ for the families of S^{*},C, and R. Later on, many authors published their work regarding $|H_{3,1}(f)|$ for different subclasses of univalent functions; see [9–16]. In 2017, Zaprawa [17] improved the results of Babalola. In 2018, Kowalczyk et al. [18] and Lecko et al. [19] obtained the sharp inequalities:

$$|H_{3,1}(f)|\leq\frac{4}{35}\text{ and }|H_{3,1}(f)|\leq\frac{1}{9}$$

for the recognizable families K and $S^{*}\left(\frac{1}{2}\right)$, respectively , where the symbol $S^{*}\left(\frac{1}{2}\right)$ stands for the family of starlike functions of order $\frac{1}{2}$. Arif M. et al. [20] obtained the upper bound of $|H_{3,1}(f)|$for the subclasses $S_{\sin}^{*}$,$C_{\sin}$ and $R_{\sin}$ in 2019. In 2019, Shi et al. [21] investigated the estimate of $|H_{3,1}(f)|$for the subclasses S_{car}^{*},C_{car} and R_{car}in of analytic functions connected with the cardioid domain. In 2019, Zaprawa [22] studied the Hankel Determinant for Univalent Functions Related to the Exponential Function.

For $f\in A$,$n\in N=\{0,1,2,,3,...\}$, the operator D^nf is defined by $D^n:A\to A$ [23]

$$D^0f(z)=f(z)$$
$$D^{n+1}f(z)=z\left[D^nf(z)\right]^{'},z\in E.$$

If $f\in A$, $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, then $D^nf(z)=z+\sum_{k=2}^{\infty}k^na_kz^k$, $z\in E$.

Let $n\in N=\{0,1,2,,3,...\}$ and $\lambda\geq0$. We let D_{λ}^{n} denote [24] the operator defined by

$$D_{\lambda}^{n}:A\to A,D_{\lambda}^{0}f(z)=f(z),$$
$$D_{\lambda}^{1}f(z)=(1-\lambda)D_{\lambda}^{0}f(z)+\lambda z\left(D_{\lambda}^{0}f(z)\right)^{'}$$
$$=(1-\lambda)f(z)+\lambda zf^{'}(z),$$
$$\cdots$$
$$D_{\lambda}^{n+1}f(z)=(1-\lambda)D_{\lambda}^{n}f(z)+\lambda z\left(D_{\lambda}^{n}f(z)\right)^{'}.$$

We observe that D_{λ}^{n} is a linear operator and for $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, we have [25]

$$D_{\lambda}^{n}f(z)=z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}a_{k}z^{k}.$$

Now, we define a subclass of analytic functions as follows:

Definition 1. *Let $\lambda \geq 0$, $n \in N=\{0,1,2,3,...\}$ and $f(z)$ is defined by (1). We define the classes of $S^{*}_{(\lambda,n)}$ and $C_{(\lambda,n)}$ in the following way*

$$S^{*}_{(\lambda,n)}=\left\{f\in S:\frac{z\left(D_{\lambda}^{n}f(z)\right)'}{D_{\lambda}^{n}f(z)}\prec 1+\sin(z)\right\} \tag{9}$$

and

$$C_{(\lambda,n)}=\left\{f\in S:1+\frac{z\left(D_{\lambda}^{n}f(z)\right)''}{\left(D_{\lambda}^{n}f(z)\right)'}\prec 1+\sin(z)\right\}. \tag{10}$$

In this present article, our aim is to investigate the estimate of $|H_{2,2}(f)|$ and $|H_{3,1}(f)|$ for the subclasses $S^{*}_{(\lambda,n)}$ and $C_{(\lambda,n)}$ of analytic functions related with sine function.

2. Auxiliary lemmas

Let P denote the family of all functions p which are analytic in E with $Re\ p(z)>0$ and has the following series representation

$$p(z)=1+p_{1}z+p_{2}z^{2}+p_{3}z^{3}+... \tag{11}$$
$$=1+\sum_{n=1}^{\infty}p_{n}z^{n}\ (z\in E).$$

Here $p(z)$ is called the Caratheodory function [26].

Lemma 1. *([27]) Let $p(z)\in P$. Then $|p_n|\leq 2$, $n=1,2,...$.*

Lemma 2. *([28], [29]) Let the function $p(z)\in P$ be given by (11), then*

$$2p_{2}=p_{1}^{2}+x\left(4-p_{1}^{2}\right) \tag{12}$$

for some x, $|x|\leq 1$, and

$$4p_{3}=p_{1}^{3}+2p_{1}\left(4-p_{1}^{2}\right)x-p_{1}\left(4-p_{1}^{2}\right)x^{2}+2\left(4-p_{1}^{2}\right)\left(1-|x|^{2}\right)\eta \tag{13}$$

for some complex value η, $|\eta|\leq 1$.

Lemma 3. *([20], [30]) Let $p(z)\in P$ and has the form (11) then*

$$\left|p_{2}-\frac{p_{1}^{2}}{2}\right|\leq 2-\frac{|p_{1}|^{2}}{2} \tag{14}$$

$$\left|p_{n+2k}-\mu p_{n}p_{k}^{2}\right|\leq 2(1+2\mu)\ \textit{for}\ \mu\in R, \tag{15}$$

$$|p_{n+k}-\eta p_{n}p_{k}|<2,\quad \text{for } 0\leq\eta\leq 1, \tag{16}$$

$$|p_{m}p_{n}-p_{k}p_{l}|\leq 4\quad \text{for } m+n=k+l, \tag{17}$$

and for complex number λ, we have

$$\left|p_{2}-\lambda p_{1}^{2}\right|\leq\max\{2,2|\lambda-1|\}. \tag{18}$$

For the results in (14),(15),(16),(17) see ([31], [32]) for the inequality (18).

Lemma 4. *([20]) Let $p(z)\in P$ and has the form (11),then*

$$\left|Jp_{1}^{3}-Kp_{1}p_{2}+Lp_{3}\right| \tag{19}$$
$$\leq 2|J|+2|K-2J|+2|J-K+L|.$$

3. Main results

3.1. The upper bound of the modules of Hankel's determinants for the coefficients of functions belonging to class $S^{*}_{(\lambda,n)}$

Theorem 1. *If the function $f(z)\in S^{*}_{(\lambda,n)}$ and of the form (1), then*

$$|a_{2}|\leq\frac{1}{(1+\lambda)^{n}},|a_{3}|\leq\frac{1}{2(1+2\lambda)^{n}},$$
$$|a_{4}|\leq\frac{13}{36(1+3\lambda)^{n}},|a_{5}|\leq\frac{7}{24(1+4\lambda)^{n}}.$$

Proof. From the definition of the class $S^{*}_{(\lambda,n)}$, we have

$$\frac{z\left(D_{\lambda}^{n}f(z)\right)'}{D_{\lambda}^{n}f(z)}=1+\sin(w(z)) \tag{20}$$

where w is analytic in E with $w(0)=0$ and $|w(z)|<1, z\in E$. Consider a function p such that

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+p_{1}z+p_{2}z^{2}+...$$

then $p\in P$. This implies that

$$w(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_{1}z+p_{2}z^{2}+p_{3}z^{3}+...}{2+p_{1}z+p_{2}z^{2}+p_{3}z^{3}+...}.$$

From (1), we can write

$$\frac{z\left(D_{\lambda}^{n}f(z)\right)'}{D_{\lambda}^{n}f(z)} = 1+(1+\lambda)^{n}a_2 z + \left[2(1+2\lambda)^{n}a_3-(1+\lambda)^{2n}a_2^2\right]z^2 + [3(1+3\lambda)^{n}a_4- 3(1+2\lambda)^{n}(1+\lambda)^{n}a_2a_3+(1+3\lambda)^{3n}a_2^3\Big]z^3 + [4(1+4\lambda)^{n}a_5-4(1+3\lambda)^{n}(1+\lambda)^{n}a_2a_4 + 4(1+2\lambda)^{n}(1+\lambda)^{2n}a_2^2a_3-2(1+2\lambda)^{2n}a_3^2 -(1+\lambda)^{4n}a_2^4\Big]z^4+\ldots \tag{21}$$

After some simple calculations, we obtain

$$1+\sin(w(z)) = 1+w(z)-\frac{(w(z))^3}{3!}+\frac{(w(z))^5}{5!}-\frac{(w(z))^7}{7!}+\ldots$$
$$= 1+\tfrac{1}{2}p_1z+\left(\tfrac{p_2}{2}-\tfrac{p_1^2}{4}\right)z^2+\left(\tfrac{5}{48}p_1^3+\tfrac{p_3}{2}-\tfrac{p_1p_2}{2}\right)z^3+ \left(\frac{p_4}{2}+\frac{5}{16}p_1^2p_2-\frac{p_1^4}{32}-\frac{p_1p_3}{2}-\frac{p_2^2}{4}\right)z^4+\ldots \tag{22}$$

From (20), (21) and (22), it follows that

$$a_2=\frac{p_1}{2(1+\lambda)^{n}} \tag{23}$$

$$a_3=\frac{p_2}{4(1+2\lambda)^{n}} \tag{24}$$

$$a_4=\frac{1}{6(1+3\lambda)^{n}}\left(p_3-\frac{p_1p_2}{4}-\frac{p_1^3}{24}\right) \tag{25}$$

$$a_5=\frac{1}{8(1+4\lambda)^{n}}\left(p_4-\frac{p_1p_3}{3}+\frac{5p_1^4}{144}-\frac{p_1^2p_2}{24}-\frac{p_2^2}{4}\right). \tag{26}$$

Applying Lemma 1 in (23) and (24), we obtain

$$|a_2|=\left|\frac{p_1}{2(1+\lambda)^{n}}\right|\le\frac{1}{(1+\lambda)^{n}} \text{ and}$$
$$|a_3|=\left|\frac{p_2}{4(1+2\lambda)^{n}}\right|\le\frac{1}{2(1+2\lambda)^{n}}.$$

If the expression (25) is calculated by taking $J=-\frac{1}{24},\ K=\frac{1}{4},\ L=1$ in the inequality (19), we obtain

$$|a_4|=\tfrac{1}{6(1+3\lambda)^{n}}\left|-\tfrac{1}{24}p_1^3-\tfrac{1}{4}p_1p_2+p_3\right| \tag{27}$$
$$\le\frac{1}{6(1+3\lambda)^{n}}\left\{2\left|-\frac{1}{24}\right|+2\left|\frac{1}{4}+\frac{1}{12}\right|+2\left|-\frac{1}{24}-\frac{1}{4}+1\right|\right\}$$
$$\le\frac{1}{6(1+3\lambda)^{n}}\left\{\frac{1}{12}+\frac{2}{3}+\frac{17}{12}\right\}\le\frac{13}{36(1+3\lambda)^{n}}.$$

If the expression (26) is calculated using the Lemma 1 and (16) inequality, we have

$$|a_5|\le\frac{1}{8(1+4\lambda)^{n}} \left\{\frac{1}{2}\left|p_4-\frac{2}{3}p_1p_3\right|+\frac{1}{2}\left|p_4-\frac{1}{2}p_2^2\right|+\frac{|p_1|^2}{24}\left|p_2-\frac{5}{6}p_1^2\right|\right\} <\frac{7}{24(1+4\lambda)^{n}}. \tag{28}$$

Taking $\lambda=0, n=0$ in Theorem 1, we obtain the following Corollary 1. □

Corollary 1. *If the function $f(z)\in S^{*}_{(0,0)}=S^{*}$ and of the form (1), then $|a_2|\le 1\ |a_3|\le\frac{1}{2}, |a_4|\le\frac{13}{36}, |a_5|<\frac{7}{24}$.*

Theorem 2. *If the function $f(z)\in S^{*}_{(\lambda,n)}$ and of the form (1), then*

$$\left|a_3-a_2^2\right|\le\frac{1}{2(1+2\lambda)^{n}} \tag{29}$$

Proof. To obtain the (29) inequality, we will use the expression (18). If equations (23) and (24) are used,we can write

$$\left|a_3-a_2^2\right| = \left|\frac{p_2}{4(1+2\lambda)^{n}}-\left[\frac{p_1}{2(1+\lambda)^{n}}\right]^2\right|$$
$$=\left|\frac{p_2}{4(1+2\lambda)^{n}}-\frac{p_1^2}{4(1+\lambda)^{2n}}\right|$$
$$=\frac{1}{4(1+2\lambda)^{n}}\left|p_2-\frac{(1+2\lambda)^{n}}{(1+\lambda)^{2n}}p_1^2\right|$$
$$\le \frac{1}{4(1+2\lambda)^{n}}\max\left\{2,2\left|\frac{(1+2\lambda)^{n}}{(1+\lambda)^{2n}}-1\right|\right\}.$$

Here, considering that

$$\frac{(1+2\lambda)^{n}}{(1+\lambda)^{2n}}\le 1\Rightarrow(1+2\lambda)^{n}\le(1+\lambda)^{2n}$$
$$\Rightarrow 1+2\lambda\le 1+2\lambda+\lambda^2\Rightarrow\lambda^2\ge 0,$$
$$\left|a_3-a_2^2\right|\le\frac{1}{2(1+2\lambda)^{n}}$$

is written. □

Theorem 3. *If the function $f(z)\in S^{*}_{(\lambda,n)}$ and of the form (1), then*

$$|a_2a_3-a_4| \tag{30}$$

$$\leq \begin{cases} \frac{1}{2(1+\lambda)^n(1+2\lambda)^n} - \frac{1}{9(1+3\lambda)^n}, & 0 \leq \lambda \leq \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \\ \frac{13}{36(1+3\lambda)^n}, & \lambda \geq \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \end{cases}.$$

Here, it is defined as $\nu = \left(\frac{18}{17}\right)^{\frac{1}{n}} - 1$.

Proof. From (23), (24) and (25), it follows that

$$|a_2a_3 - a_4| = \left| \begin{array}{c} \frac{p_1^3}{144(1+3\lambda)^n} + \\ \left(\frac{1}{8(1+\lambda)^n(1+2\lambda)^n} + \frac{1}{24(1+3\lambda)^n}\right) p_1p_2 \\ -\frac{p_3}{6(1+3\lambda)^n} \end{array} \right|. \tag{31}$$

If the expression (31) is calculated by taking

$$\begin{aligned} J &= \frac{1}{144(1+3\lambda)^n}, \\ K &= -\left(\frac{1}{8(1+\lambda)^n(1+2\lambda)^n} + \frac{1}{24(1+3\lambda)^n}\right), \\ L &= -\frac{1}{6(1+3\lambda)^n} \end{aligned}$$

in the inequality (19),we obtain

$$\begin{aligned} |a_2a_3 - a_4| \leq & \frac{1}{72(1+3\lambda)^n} \\ & + 2\left|\frac{1}{8(1+\lambda)^n(1+2\lambda)^n} + \frac{1}{18(1+3\lambda)^n}\right| \\ & + 2\left|\frac{1}{8(1+\lambda)^n(1+2\lambda)^n} - \frac{17}{144(1+3\lambda)^n}\right| \end{aligned}$$

Now, Let's Look at the sign of $\frac{1}{8(1+\lambda)^n(1+2\lambda)^n} - \frac{17}{144(1+3\lambda)^n}$ expression. Let's assume that

$$\frac{1}{8(1+\lambda)^n(1+2\lambda)^n} - \frac{17}{144(1+3\lambda)^n} \leq 0.$$

With a simple calculation, we write

$$\begin{aligned} & \frac{1}{8(1+\lambda)^n(1+2\lambda)^n} \leq \frac{17}{144(1+3\lambda)^n} \\ & \Rightarrow 144(1+3\lambda)^n \leq 136\left[(1+\lambda)(1+2\lambda)\right]^n \Rightarrow \\ & 18(1+3\lambda)^n \leq 17\left(1+3\lambda+2\lambda^2\right)^n \\ & \Rightarrow \left(\tfrac{18}{17}\right)^{\frac{1}{n}} \leq 1 + \tfrac{2\lambda^2}{1+3\lambda} \Rightarrow \left(\tfrac{18}{17}\right)^{\frac{1}{n}} - 1 \leq \tfrac{2\lambda^2}{1+3\lambda}. \end{aligned}$$

If $\left(\frac{18}{17}\right)^{\frac{1}{n}} - 1 = \nu$, then $\left(\frac{18}{17}\right)^{\frac{1}{n}} - 1 \geq 0$. Thus, we obtain $\nu \leq \frac{2\lambda^2}{1+3\lambda} \Rightarrow 2\lambda^2 - 3\nu\lambda - \nu \geq 0$.

If this inequality is solved, we find

$$\lambda_1 = \frac{3\nu - \sqrt{9\nu^2+8v}}{4} \text{ and } \lambda_2 = \frac{3\nu + \sqrt{9\nu^2+8v}}{4}.$$

Thus, since $\lambda \geq 0$ must be, we get $|a_2a_3 - a_4| \leq \begin{cases} \frac{1}{2(1+\lambda)^n(1+2\lambda)^n} - \frac{1}{9(1+3\lambda)^n}, & 0 \leq \lambda \leq \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \\ \frac{13}{36(1+3\lambda)^n}, & \lambda \geq \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \end{cases}$. □

Theorem 4. *If the function* $f(z) \in S^*_{(\lambda,n)}$ *and of the form (1), then*

$$\begin{aligned} |H_{2,2}(f)| &= \left|a_2a_4 - a_3^2\right| \\ &\leq \frac{13}{36(1+\lambda)^n(1+3\lambda)^n} + \frac{1}{4(1+2\lambda)^{2n}}. \end{aligned} \tag{32}$$

Proof. From (23), (24) and (25), it follows that

$$\begin{aligned} & \left|a_2a_4 - a_3^2\right| \\ = & \left| \begin{array}{c} \frac{p_1p_3}{12(1+\lambda)^n(1+3\lambda)^n} - \frac{p_1^2p_2}{48(1+\lambda)^n(1+3\lambda)^n} \\ -\frac{p_1^4}{288(1+\lambda)^n(1+3\lambda)^n} - \frac{p_2^2}{16(1+2\lambda)^{2n}} \end{array} \right|. \end{aligned}$$

If the triangle inequality is applied to this last equation, we obtain

$$\begin{aligned} & \left|a_2a_4 - a_3^2\right| \\ \leq & \left\{ \begin{array}{c} \frac{|p_1|}{12(1+\lambda)^n(1+3\lambda)^n}\left|p_3 - \frac{p_1p_2}{4} - \frac{p_1^3}{24}\right| \\ + \left|\frac{p_2^2}{16(1+2\lambda)^{2n}}\right| \end{array} \right\}. \end{aligned}$$

(19) according to inequality, we can write

$$\begin{aligned} & \left|p_3 - \frac{p_1p_2}{4} - \frac{p_1^3}{24}\right| \\ \leq & \left\{ \begin{array}{c} 2\left|-\frac{1}{24}\right| + 2\left|\frac{1}{4} + \frac{1}{12}\right| \\ +2\left|-\frac{1}{24} - \frac{1}{4} + 1\right| \end{array} \right\} \\ \leq & \frac{13}{6}. \end{aligned}$$

Thus , we obtain $\left|a_2a_4 - a_3^2\right| \leq \left\{\frac{2}{12(1+\lambda)^n(1+3\lambda)^n} \cdot \frac{13}{6} + \frac{1}{4(1+2\lambda)^{2n}}\right\}$

$\leq \frac{13}{36(1+\lambda)^n(1+3\lambda)^n} + \frac{1}{4(1+2\lambda)^{2n}}$. □

Theorem 5. *If the function* $f(z) \in S^*_{(\lambda,n)}$ *and of the form (1), then*

$$|H_3(1)| < \begin{cases} \frac{13}{36(1+3\lambda)^n(1+\lambda)^n(1+2\lambda)^n} \\ +\frac{1}{8(1+2\lambda)^{3n}} \\ -\frac{13}{324(1+3\lambda)^{2n}} \\ +\frac{7}{48(1+4\lambda)^n(1+2\lambda)^n}; 0 \le \lambda \le \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \\ \frac{13}{72(1+3\lambda)^n(1+\lambda)^n(1+2\lambda)^n} \\ +\frac{1}{8(1+2\lambda)^{3n}} \\ +\frac{169}{1296(1+3\lambda)^{2n}} \\ +\frac{7}{48(1+4\lambda)^n(1+2\lambda)^n}; \lambda \ge \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \end{cases} \quad (33)$$

Here, it is defined as $\nu = \left(\frac{18}{17}\right)^{\frac{1}{n}} - 1$.

Proof. If the absolute value of both sides of the expression

$$H_3(1) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right)$$

is taken and the triangle inequality is applied, we can write

$$|H_3(1)| \le |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\left|a_4 - a_2a_3\right| + |a_5|\left|a_3 - a_2^2\right|.$$

In this last inequality, if the upper bound expressions discussed in Theorem 1, Theorem 2, Theorem 3, and Theorem 4, are written instead of and necessary operations are done, we obtain

$$|H_3(1)| < \begin{cases} \frac{13}{36(1+3\lambda)^n(1+\lambda)^n(1+2\lambda)^n} \\ +\frac{1}{8(1+2\lambda)^{3n}} \\ -\frac{13}{324(1+3\lambda)^{2n}} \\ +\frac{7}{48(1+4\lambda)^n(1+2\lambda)^n}; 0 \le \lambda \le \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \\ \frac{13}{72(1+3\lambda)^n(1+\lambda)^n(1+2\lambda)^n} \\ +\frac{1}{8(1+2\lambda)^{3n}} \\ +\frac{169}{1296(1+3\lambda)^{2n}} \\ +\frac{7}{48(1+4\lambda)^n(1+2\lambda)^n}; \lambda \ge \frac{3\nu+\sqrt{9\nu^2+8v}}{4} \end{cases}.$$

Here, it is defined as $\nu = \left(\frac{18}{17}\right)^{\frac{1}{n}} - 1$. ☐

3.2. The upper bound of the modules of Hankel's determinants for the coefficients of functions belonging to class $C_{(\lambda,n)}$.

Theorem 6. *If the function* $f(z) \in C_{(\lambda,n)}$ *and of the form (1), then* $|a_2| \le \frac{1}{2(1+\lambda)^n}, |a_3| \le \frac{1}{6(1+2\lambda)^n}, |a_4| \le \frac{13}{144(1+3\lambda)^n}, |a_5| \le \frac{7}{120(1+4\lambda)^n}$.

Proof. From the definition of the class $C_{(\lambda,n)}$, we have

$$1 + \frac{z\left(D_\lambda^n f(z)\right)''}{\left(D_\lambda^n f(z)\right)'} = 1 + \sin(w(z)) \quad (34)$$

where w is analytic in E with $w(0) = 0$ and $|w(z)| < 1, z \in E$.

From (1), we can write

$$\begin{aligned} 1 + \frac{z\left(D_\lambda^n f(z)\right)''}{\left(D_\lambda^n f(z)\right)'} &= 1 + 2(1+\lambda)^n a_2 z \\ &+ \left[6(1+2\lambda)^n a_3 - 4(1+\lambda)^{2n} a_2^2\right] z^2 \\ &+ \left[12(1+3\lambda)^n a_4 \right. \\ &-18(1+\lambda)^n(1+2\lambda)^n a_2a_3 \\ &\left. +8(1+\lambda)^{3n} a_2^3\right] z^3 + \left[20(1+4\lambda)^n a_5 \right. \\ &-32(1+\lambda)^n(1+3\lambda)^n a_2a_4 \\ &-18(1+2\lambda)^{2n} a_3^2 \\ &+48(1+\lambda)^{2n}(1+2\lambda)^n a_2^2a_3 \\ &\left. -16(1+\lambda)^{4n} a_2^4\right] z^4 + ... \end{aligned} \quad (35)$$

From (23) and (35) , it follows that

$$a_2 = \frac{p_1}{4(1+\lambda)^n} \quad (36)$$

$$a_3 = \frac{p_2}{12(1+2\lambda)^n} \quad (37)$$

$$a_4 = \frac{1}{24(1+3\lambda)^n}\left(p_3 - \frac{p_1p_2}{4} - \frac{p_1^3}{24}\right) \quad (38)$$

$$a_5 = \frac{1}{40(1+4\lambda)^n}\left(p_4 - \frac{p_1p_3}{3} + \frac{5p_1^4}{144} - \frac{p_1^2p_2}{24} - \frac{p_2^2}{4}\right). \quad (39)$$

Applying Lemma 1 in (36) and (37), we obtain

$$|a_2| \le \left|\frac{p_1}{4(1+\lambda)^n}\right| \le \frac{1}{2(1+\lambda)^n} \quad (40)$$

and

$$|a_3| \le \left|\frac{p_2}{12(1+2\lambda)^n}\right| \le \frac{1}{6(1+2\lambda)^n}. \quad (41)$$

If the expression (38) is calculated by taking $J = -\frac{1}{24}$, $K = \frac{1}{4}$, $L = 1$ in the inequality (19), we obtain

$$\begin{aligned} |a_4| &\le \frac{1}{24(1+3\lambda)^n}\left|-\frac{1}{24}p_1^3 - \frac{1}{4}p_1p_2 + p_3\right| \\ &\le \frac{13}{144(1+3\lambda)^n}. \end{aligned}$$

If the expression (39) is calculated using the Lemma 1 and (16) inequality, we have

$$|a_5| \leq \frac{1}{40(1+4\lambda)^n} \left\{ \begin{array}{c} \frac{1}{2}\left|p_4 - \frac{2}{3}p_1p_3\right| \\ +\frac{1}{2}\left|p_4 - \frac{1}{2}p_2^2\right| + \frac{|p_1|^2}{24}\left|p_2 - \frac{5}{6}p_1^2\right| \end{array} \right\} = \frac{7}{120(1+4\lambda)^n}. \tag{42}$$

□

Theorem 7. *If the function $f(z) \in C_{(\lambda,n)}$ and of the form (1), then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6(1+2\lambda)^n}. \tag{43}$$

Proof. To obtain the inequality (43) we will use the expression (18). If equations (36) and (37) are used,we can write

$$\begin{aligned} &\left|a_3 - a_2^2\right| \\ = &\left|\frac{p_2}{12(1+2\lambda)^n} - \left[\frac{p_1}{4(1+\lambda)^n}\right]^2\right| \\ \leq &\frac{1}{12(1+2\lambda)^n}\max\left\{2, 2\left|\frac{3(1+2\lambda)^n}{4(1+\lambda)^{2n}} - 1\right|\right\}. \end{aligned}$$

Here , $\frac{3(1+2\lambda)^n}{4(1+\lambda)^{2n}} \leq 1$. Thus, since $\max\left\{2, 2\left|\frac{3(1+2\lambda)^n}{4(1+\lambda)^{2n}} - 1\right|\right\} = 2$, the desired result is obtained in the form of $\left|a_3 - a_2^2\right| \leq \frac{1}{6(1+2\lambda)^n}$. □

Theorem 8. *If the function $f(z) \in C_{(\lambda,n)}$ and of the form (1), then*

$$|a_2a_3 - a_4| \leq \frac{1}{12(1+\lambda)^n(1+2\lambda)^n} + \frac{13}{144(1+3\lambda)^n}. \tag{44}$$

Proof. From (36), (37) and (38), it follows that

$$\begin{aligned} &|a_2a_3 - a_4| \\ = &\left|\begin{array}{c} \frac{p_1p_2}{48(1+\lambda)^n(1+2\lambda)^n} \\ +\frac{1}{24(1+3\lambda)^n}\left(\frac{p_1^3}{24} + \frac{p_1p_2}{4} - p_3\right) \end{array}\right|. \end{aligned}$$

If the triangle inequality is applied to this last equation, we write

$$\begin{aligned} &|a_2a_3 - a_4| \\ \leq &\frac{|p_1||p_2|}{48(1+\lambda)^n(1+2\lambda)^n} \\ &+\frac{1}{24(1+3\lambda)^n}\left|\frac{p_1^3}{24} + \frac{p_1p_2}{4} - p_3\right|. \end{aligned}$$

According to the (19) inequality, it is thought that

$$\begin{aligned} &\left|\frac{p_1^3}{24} + \frac{p_1p_2}{4} - p_3\right| \\ \leq &\left\{ \begin{array}{c} 2\left|\frac{1}{24}\right| + 2\left|-\frac{1}{4} - \frac{1}{12}\right| \\ +2\left|\frac{1}{24} + \frac{1}{4} - 1\right| \end{array} \right\} \\ \leq &\frac{13}{6} \end{aligned}$$

can be written and if Lemma 1 is taken into account, we find

$$\begin{aligned} &|a_2a_3 - a_4| \\ \leq &\frac{|p_1||p_2|}{48(1+\lambda)^n(1+2\lambda)^n} + \frac{1}{24(1+3\lambda)^n}\left|\frac{p_1^3}{24} + \frac{p_1p_2}{4} - p_3\right| \\ \leq &\frac{1}{12(1+\lambda)^n(1+2\lambda)^n} + \frac{13}{144(1+3\lambda)^n}. \end{aligned}$$

□

Theorem 9. *If the function $f(z) \in C_{(\lambda,n)}$ and of the form (1), then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{13}{288(1+\lambda)^n(1+3\lambda)^n} + \frac{1}{36(1+2\lambda)^{2n}}. \tag{45}$$

Proof. From (36), (37) and (38), it follows that

$$\begin{aligned} &\left|a_2a_4 - a_3^2\right| \\ = &\left|\begin{array}{c} \frac{p_1p_3}{96(1+\lambda)^n(1+3\lambda)^n} - \frac{p_1^2p_2}{384(1+\lambda)^n(1+3\lambda)^n} \\ -\frac{p_1^4}{2304(1+\lambda)^n(1+3\lambda)^n} - \frac{p_2^2}{144(1+2\lambda)^{2n}} \end{array}\right|. \end{aligned}$$

If the triangle inequality is applied to this last equation, we write

$$\begin{aligned} &\left|a_2a_4 - a_3^2\right| \\ \leq &\left\{ \begin{array}{c} \frac{|p_1|}{96(1+\lambda)^n(1+3\lambda)^n} \\ \left|p_3 - \frac{p_1p_2}{4} - \frac{p_1^3}{24}\right| + \left|\frac{p_2^2}{144(1+2\lambda)^{2n}}\right| \end{array} \right\}. \end{aligned}$$

According to the (19) inequality, it is thought that

$$\begin{aligned} &\left|p_3 - \frac{p_1p_2}{4} - \frac{p_1^3}{24}\right| \\ \leq &\left\{2\left|-\frac{1}{24}\right| + 2\left|\frac{1}{4} + \frac{1}{12}\right| + 2\left|-\frac{1}{24} - \frac{1}{4} + 1\right|\right\} \\ \leq &\frac{13}{6} \end{aligned}$$

can be written and if Lemma 1. is taken into account, we find

$$\begin{aligned} &\left|a_2a_4 - a_3^2\right| \\ \leq &\left\{\frac{2}{96(1+\lambda)^n(1+3\lambda)^n} \cdot \frac{13}{6} + \frac{1}{36(1+2\lambda)^{2n}}\right\} \\ \leq &\frac{13}{288(1+\lambda)^n(1+3\lambda)^n} + \frac{1}{36(1+2\lambda)^{2n}}. \end{aligned}$$

□

Theorem 10. *If the function $f(z) \in C_{(\lambda,n)}$ and of the form (1), then*

$$|H_3(1)| \leq \frac{13}{864(1+3\lambda)^n(1+\lambda)^n(1+2\lambda)^n} + \frac{1}{216(1+2\lambda)^{3n}} + \frac{169}{20736(1+3\lambda)^{2n}} + \frac{7}{720(1+4\lambda)^n(1+2\lambda)^n}.$$

Proof. The proof of this theorem is similar to the one in Theorem 5. □

References

[1] Bieberbach, L. (1916). Über die Koeffizienten derjenigen Pottenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. Sitzungsberichte Preussische Akademie der Wissenschaften, 138, pp. 940-955.

[2] De-Branges, L. (1985). A proof of the Beiberbach conjecture. *Acta Mathematica*, 154, 137-152.

[3] Cho, N.E., Kumar, V., Kumar, S.S. & Ravichandran, V. (2019). Radius problems for starlike functions associated with the sine function. *Bulletin of the Iranian Mathematical Society*, 45, 213–232.

[4] Pommerenke, C. (1975). On the coefficients and Hankel determinants of univalent functions. *Journal of the London Mathematical Society*, 41, 111–122.

[5] Pommerenke, C. (1967). On the Hankel determinants of univalent functions. *Mathematika*, 14, 108–112.

[6] Janteng, A., Halim, S.A. & Darus, M. (2006). Coefficient inequality for a function whose derivative has a positive real part. *Journal of Inequalities in Pure and Applied Mathematics*, 7, 1-5.

[7] Janteng, A., Halim, S.A. & Darus, M. (2007). Hankel determinant for starlike and convex functions. *International Journal of Mathematical Analysis*, 1, 619-625.

[8] Babalola, K.O. (2010). On Hankel determinants for some classes of univalent functions. *Inequality Theory and Applications*, 6, 1-7.

[9] Altınkaya, S. & Yalçın, S. (2016). Third Hankel determinant for Bazilevic functions. *Advances in Mathematics*, 5, 91-96.

[10] Bansal, D., Maharana, S. & Prajapat, J.K. (2015). Third order Hankel determinant for certain univalent functions. *Journal of Korean Mathematical Society*, 52, 1139–1148.

[11] Krishna, D.V., Venkateswarlu, B. & RamReddy, T. (2015). Third Hankel determinant for bounded turning functions of order alpha. *Journal of the Nigerian Mathematical Society*, 34, 121–127.

[12] Raza, M. & Malik, S.N. (2013). Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. *Journal of Inequalities and Applications*, 412.

[13] Shanmugam, G., Stephen, B.A. & Babalola, K.O. (2014). Third Hankel determinant for α- starlike functions. *Gulf Journal of Mathematics*, 2, 107–113,

[14] Zhang, H.Y., Tang, H. & Ma, L.N. (2017). Upper bound of third Hankel determinant for a class of analytic functions. *Pure and Applied Mathematics*, 33(2), 211- 220.

[15] Breaz, D., Cataş, A. & Cotırla, L. (2022). On the upper bound of the third Hankel determinant for certain class of analytic functions related with exponential function. *Analele Stiintifice ale Universitatii ''Ovidius Constanta. Seria Mathematica*, 30(1), 75-89.

[16] Orhan, H., Çağlar, M. & Cotırla, L. (2023). Third Hankel determinant for a subfamily of holomorphic functions related with lemniscate of Bernoulli. *Mathematics*, 11, 1147.

[17] Zaprawa, P. (2019). Third Hankel determinants for subclasses of univalent functions. *Mediterranean Journal of Mathematics*, 14, 19.

[18] Kowalczyk, B., Lecko, A. & Sim,Y.J. (2018). The sharp bound of the Hankel determinant of the third kind for convex functions. *Bulletin of the Australian Mathematical Society*, 97, 435–445.

[19] Lecko,A., Sim, Y.J. & Smiarowska, B. (2018).The sharp bound of the Hankel determinant of the third kind for starlike functions of order. *Complex Analysis and Operator Theory*, 1–8.

[20] Arif, M., Raza, M., Tang, H., Hussain, S. & Khan, H. (2019). Hankel determinant of order three for familiar subsets of analytic functions related with sine function. *Open Mathematics*, 17, 1615-1630.

[21] Shi, L., Ali, I., Arif ,M., Cho ,N.E., Hussain ,S. & Khan, H. (2019). A Study of third Hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain, *Mathematics*, 7, 418.

[22] Zaprawa, P. (2019). Hankel Determinant for univalent functions related to the exponential function. *Symmetry*, 11, 1211.

[23] Salagean, G. (1983). *Subclasses of Univalent Functions*. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1013, 362-372.

[24] Al-Oboudi, F.M. (2004). On univalent functions defined by a generalized Salagean operator. *Indian Journal of Mathematics and Mathematical Sciences*, 25-28, 1429-1436.

[25] Oros, I.O. & Oros, G. (2008). On a class of univalent functions defined by a generalized Salagean operator. *Complex Variables and Elliptic Equations*, 53(9), 869-877.

[26] Caratheodory, C. (1911). Uber den variabilitatsbereich der Fourier'schen Konstanten Von Positiven harmonischen. *Rendiconti del Circolo Mathematico di Palermo*, 32, 193–217.

[27] Duren, P.L. (1983). Univalent functions, (Springer-Verlag), 114–115. *Mathematicheskii Sbornik*, 37 (79), 471–476.

[28] Libera, J. & Zlotkiewicz, E.J. (1982). Early coefficients of the inverse of a regular convex Function. *Proceedings of American Mathematical Society*, 85(2), 225–230.

[29] Libera, J. & Zlotkiewicz, E.J. (1983). Coefficient bounds for the inverse of a function with derivative in P. *Proceedings of American Mathematical Society*, 87(2), 251–257,

[30] Ayınla, R.O.B. & Risikat, A. (2021). Toeplitz determinants for a subclass of analytic functions. *Journal of Progressive Research in Mathematics*, 18(1), 99-106.

[31] Keough, F. & Merkes, E. (1996). A coefficient inequality for certain subclasses of analytic functions. *Proceedings of American Mathematical Society*, 20, 8–12.

[32] Pommerenke, C. (1975). Univalent Functions, *Studia Mathematica Mathematische Lehrbucher*, Vandenhoeck & Ruprecht, Gottingen, vol. 25.

Muhammet Kamali *received his doctoral degree from Atatürk University. Currently, he is serving as a professor in the Department of Mathematics at Kyrgyz-Turkish Manas University. His area of interest is complex analysis.*
https://orcid.org/0000-0002-6230-6836

Alina Riskulava *received his master's degree from Kyrgyz-Turkish Manas University. His area of interest is complex analysis.*
https://orcid.org/0009-0000-1311-892X