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RESEARCH ARTICLE

The solvability of the optimal control problem for a nonlinear Schrödinger equation

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ABSTRACT

In this paper, we analyze the solvability of the optimal control problem for a nonlinear Schrödinger equation. A Lions-type functional is considered as the objective functional. First, it is shown that the optimal control problem has at least one solution. Later, the Frechet differentiability of the objective functional is proved and a formula is obtained for its gradient. Finally, a necessary optimality condition is derived.

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1. Introduction

The nonlinear Schrödinger equation (NLSE) describes the behavior of wave packets in weakly nonlinear media. It is an adaptable model to many disciplines in applied sciences such as dynamical systems, materials science, nonlinear optics, fluid dynamics, astrophysics, particle physics, and nonlinear transmission networks. NLSE represents the evolution of optical waves in a nonlinear fiber, various biological systems, and the price of options in economics [1].

In the present paper, we consider a specific case of the following Schrödinger equation

$$\varepsilon\frac{\partial u}{\partial \tau}+R_2(\varsigma,\tau,u)\frac{\partial^2 u}{\partial \varsigma^2}+$$
$$R_1(\varsigma,\tau,u)\frac{\partial u}{\partial \varsigma}+R_0(\varsigma,\tau,u)u=0, \qquad (1)$$

where $\varepsilon = const.$, $u(\varsigma,\tau)$ is the wave's complex amplitude. The coefficients $R_j(\varsigma,\tau,u)$ for $j=0,1,2$ describe the variation of the medium. If the functions R_j depend on $u(\varsigma,\tau)$, it shows that the medium has the nonlinear properties [2]. Linear and nonlinear Schrödinger equations are obtained from equation (1) with respect to the characteristics of the coefficients $R_j(\varsigma,\tau,u)$ for $j=0,1,2$ and $\varepsilon=i$.

Optimal control problems (OCPs) arise in many branches of science. They have numerous applications in optics, medical imaging, geophysics, system identification, communication theory, astronomy, medicine [3–11].

As it is known, in the OCPs, there is an objective functional, a controlled system, and a set of admissible controls. The objective functionals can be diversely chosen with regard to our purpose such as final, boundary or Lions-type functional [12]. In the studies [13–22], the objective functional is considered as a final functional and the controlled system is generally stated by the Schrödinger equation. In [23–27], the OCPs with Lions functional has been studied and the controlled system is stated by linear or nonlinear Schrödinger equations. Also, in [28–30], the

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OCPs for systems whose state is expressed by the Schrödinger equation with the boundary functional has been studied.

In this paper, we consider an OCP with Lions functional for NLSE derived from (1). It is proved that the OCP has a unique solution and the objective functional is Frechet differentiable. Also, by proving the continuity of the gradient of the objective functional, a necessary optimality condition is obtained.

Differently from the previous studies, in this paper, we analyze the solution of the OCP for NLSE derived from (1) for $R_2 = R_2(\varsigma, \tau)$, $R_1 = R_1(\varsigma, \tau)$, $R_0(\varsigma, \tau, u) = R_0(\varsigma, \tau, u)$ with different coefficients which are in the larger space than previous works.

2. The statement of optimal control problem

The OCP is the problem of finding the minimum of the objective functional

$$J_\alpha(p) = ||u_1 - u_2||^2_{L_2(\Omega)} + \alpha||p - w||^2_{L_2(I)} \quad (2)$$

subject to

$$i\frac{\partial u_1}{\partial \tau} + a_0\frac{\partial^2 u_1}{\partial \varsigma^2} + ia_1(\varsigma)\frac{\partial u_1}{\partial \varsigma} - a_2(\varsigma)u_1 + p(\varsigma)u_1 + ia_3|u_1|^2u_1 = f_1, \quad (3)$$

$$\left.\begin{array}{c} u_1(\varsigma, 0) = \vartheta_1(\varsigma),\ \varsigma \in I, \\ u_1(0, \tau) = u_1(l, \tau) = 0, \tau \in (0, T) \end{array}\right\} \quad (4)$$

and

$$i\frac{\partial u_2}{\partial \tau} + a_0\frac{\partial^2 u_2}{\partial \varsigma^2} + ia_1(\varsigma)\frac{\partial u_2}{\partial \varsigma} - a_2(\varsigma)u_2 + p(\varsigma)u_2 + ia_3|u_2|^2u_2 = f_2, \quad (5)$$

$$\left.\begin{array}{c} u_2(\varsigma, 0) = \vartheta_2(\varsigma),\ \varsigma \in I, \\ \frac{\partial u_2}{\partial \varsigma}(0, \tau) = \frac{\partial u_2}{\partial \varsigma}(l, \tau) = 0, \tau \in (0, T) \end{array}\right\} \quad (6)$$

on admissible controls set

$$P \equiv \{p \in L_2(I) :\ |p(\varsigma)| \leq b_0 \ for\ almost\ all\ \varsigma \in I\},$$

where $\varsigma \in I = (0, l)$, $\tau \in Q = [0, T]$, $i = \sqrt{-1}$. Let $\Omega = I \times (0, T)$, $\Omega_\tau = I \times (0, \tau)$, $\widetilde{\Omega}_\tau = I \times (\tau, T)$ and $a_0, a_3, b_0 > 0$ are given real numbers, $a_1(\varsigma)$, $a_2(\varsigma)$, ϑ_1, ϑ_2, f_1, f_2 are functions which satisfy the conditions, respectively

$$\left.\begin{array}{c} |a_1(\varsigma)| \leq \mu_1,\ \left|\frac{da_1(\varsigma)}{d\varsigma}\right| \leq \mu_2 \ for\ almost\ all\ \varsigma \in I, \\ a_1(0) = a_1(l) = 0,\ \ \mu_1, \mu_2 = const. > 0, \end{array}\right\} \quad (7)$$

$$0 < \mu_3 \leq a_2(\varsigma) \leq \mu_4 \ for\ almost\ all\ \varsigma \in I, \quad \mu_3, \mu_4 = const. > 0, \quad (8)$$

$$\left.\begin{array}{c} \vartheta_1 \in \mathring{W}^2_2(I),\ \vartheta_2 \in W^2_2(I),\ \frac{\partial \vartheta_2(0)}{\partial \varsigma} = \frac{\partial \vartheta_2(l)}{\partial \varsigma} = 0, \\ f_r \in W^{0,1}_2(\Omega) \text{ for } r = 1, 2, \end{array}\right\} \quad (9)$$

where $W^m_s(I), W^{m,n}_s(\Omega), \mathring{W}^m_s(I)$ for $m \geq 0$, $s \geq 1$ are Sobolev spaces. These Sobolev spaces are in detail explained in [31]. Also, $\alpha \geq 0$ is a Tikhonov regularization parameter [32] and $w \in L_2(I)$ is a given element.

Since the solutions of (3)-(4) and (5)-(6) evidently depend on p, we denote $u_r = u_r(\varsigma, \tau) \equiv u_r(\varsigma, \tau; p)$ for $r = 1, 2$.We are interested in solutions of problems (3)-(4) and (5)-(6) in the following sense:

Definition 1. *A function* $u_1 \in U_1 \equiv C^0(Q, \mathring{W}^2_2(I)) \cap C^1(Q, L_2(I))$ *is said to be a solution of problem (3)-(4), if it holds (3) for almost all* $\varsigma \in I$ *and any* $\tau \in Q$, *(4) for almost all* $\varsigma \in I$ *and for almost all* $\tau \in (0, T)$, *respectively.*

Definition 2. *A function* $u_2 \in U_2 \equiv C^0(Q, W^2_2(I)) \cap C^1(Q, L_2(I))$ *is said to be a solution of problem (5)-(6), if it holds (5) for almost all* $\varsigma \in I$ *and any* $\tau \in Q$, *(6) for almost all* $\varsigma \in I$ *and for almost all* $\tau \in (0, T)$, *respectively.*

In the definitions above, for any nonnegative integer k, $C^k(Q, B)$ is the Banach space of all $B-$valued, k times continuously differentiable functions on Q with the norm

$$||u||_{C^k(Q,B)} = \sum_{m=0}^{k} \max_{0 \leq t \leq T} ||\frac{d^m u(t)}{dt^m}||_B$$

for $u \in C^k(Q, B)$.

By the methodology in [33], we can readily prove the theorem below:

Theorem 1. *Assume that* a_1, a_2, ϑ_r, f_r *for* $r = 1, 2$ *satisfy the conditions (7), (8) and (9), respectively. Then, problems (3)-(4) and (5)-(6) for each* $p \in P$ *have unique solutions* $u_1 \in U_1$, $u_2 \in U_2$, *respectively, and the functions* u_1, u_2 *satisfy the estimates*

$$||u_1(., \tau)||^2_{\mathring{W}^2_2(I)} + \left\|\frac{\partial u_1}{\partial \tau}\right\|^2_{L_2(I)} \leq \quad (10)$$
$$c_1\left(||\vartheta_1||^2_{\mathring{W}^2_2(I)} + ||f_1||^2_{W^{0,1}_2(\Omega)} + ||\vartheta_1||^6_{\mathring{W}^1_2(I)}\right),$$

$$\|u_2(.,\tau)\|^2_{W_2^2(I)} + \left\|\frac{\partial u_2}{\partial \tau}\right\|^2_{L_2(I)} \leq c_2\left(\|\vartheta_2\|^2_{W_2^2(I)} + \|f_2\|^2_{W_2^{0,1}(\Omega)} + \|\vartheta_2\|^6_{W_2^1(I)}\right), \tag{11}$$

for any $\tau \in Q$, where the constants c_1, $c_2 > 0$ are independent from $\vartheta_1, f_1, \vartheta_2, f_2$ and τ.

For simplicity, let's rewrite problems (3)-(4) and (5)-(6) in the form

$$i\frac{\partial u_r}{\partial \tau} + a_0\frac{\partial^2 u_r}{\partial \varsigma^2} + ia_1(\varsigma)\frac{\partial u_r}{\partial \varsigma} - a_2(\varsigma)u_r + p(\varsigma)u_r + ia_3|u_r|^2 u_r = f_r \quad for\ r = 1,2, \tag{12}$$

$$u_r(\varsigma, 0) = \vartheta_r(\varsigma), \varsigma \in I \ for\ r = 1,2, \tag{13}$$

$$\left.\begin{array}{l} u_1(0,\tau) = u_1(l,\tau) = 0,\ \tau \in (0,T), \\ \frac{\partial u_2(0,\tau)}{\partial \varsigma} = \frac{\partial u_2(l,\tau)}{\partial \varsigma} = 0,\ \tau \in (0,T). \end{array}\right\} \tag{14}$$

Thus, the OCP is to minimize the objective functional (2) on P under conditions (12)-(14).

3. The solvability of optimal control problem

In this section, we show that the OCP has a unique solution on a dense subset of $L_2(I)$ and it has at least one solution on $L_2(I)$.

Lemma 1. *The functional $J_0(p) = \|u_1 - u_2\|^2_{L_2(\Omega)}$ is continuous on P.*

Proof. Suppose $u_r = u_r(\varsigma, \tau; p)$ and $u_{r\delta} = u_r(\varsigma, \tau; p + \delta p)$ for $r = 1,2$ are solutions of problem (12)-(14) corresponding to $p \in P$, $p + \delta p \in P$, respectively, where $\delta p \in L_\infty(I)$ is an increment of any $p \in P$. Then, for $r = 1,2$, the functions $\delta u_r \equiv u_r(\varsigma, \tau; p + \delta p) - u_r(\varsigma, \tau; p)$ hold the boundary value problem

$$i\frac{\partial \delta u_r}{\partial \tau} + a_0\frac{\partial^2 \delta u_r}{\partial \varsigma^2} + ia_1(\varsigma)\frac{\partial \delta u_r}{\partial \varsigma} - a_2(\varsigma)\delta u_r + (p(\varsigma) + \delta p(\varsigma))\delta u_r + ia_3\left[\left(|u_{r\delta}|^2 + |u_r|^2\right)\delta u_r\right] + ia_3 u_{r\delta}u_r\left(\delta\overline{u}_r\right) = -\delta p(\varsigma)u_r, \tag{15}$$

$$\delta u_r(\varsigma, 0) = 0,\ \varsigma \in I,\ r = 1,2, \tag{16}$$

$$\left.\begin{array}{l} \delta u_1(0,\tau) = \delta u_1(l,\tau) = 0, \tau \in (0,T) \\ \frac{\partial \delta u_2(0,\tau)}{\partial \varsigma} = \frac{\partial \delta u_2(l,\tau)}{\partial \varsigma} = 0,\ \tau \in (0,T). \end{array}\right\} \tag{17}$$

Now we multiply both sides of equation (15) by $\delta\overline{u}_r$ for $r = 1,2$, and integrate over Ω_τ. If we subtract their complex conjugates from equalities obtained with the help of integration by parts and use condition (16), we get

$$\|\delta u_r(.,\tau)\|^2_{L_2(I)} + \int_{\Omega_\tau} \frac{\partial}{\partial \varsigma}\left(a_1(\varsigma)|\delta u_r|^2\right) d\varsigma dt + 2a_3\int_{\Omega_\tau} |\delta u_r|^2\left(|u_{r\delta}|^2 + |u_r|^2\right) d\varsigma dt + 2a_3\int_{\Omega_\tau} Re\left(u_{r\delta}u_r(\delta\overline{u}_r)^2\right) d\varsigma dt = -2\int_{\Omega_\tau} Im\left(\delta p u_r \delta\overline{u}_r\right) d\varsigma dt + \int_{\Omega_\tau} \frac{\partial a_1(\varsigma)}{\partial \varsigma}|\delta u_r|^2 d\varsigma dt. \tag{18}$$

Using conditions (7), (17) and Young's inequality in (18),we obtain

$$\|\delta u_r(.,\tau)\|^2_{L_2(I)} + a_3\int_{\Omega_\tau} |\delta u_r|^2\left(|u_{r\delta}|^2 + |u_r|^2\right) d\varsigma dt \leq (1 + \mu_2)\int_{\Omega_\tau} |\delta u_r|^2 d\varsigma dt + \int_{\Omega_\tau} |\delta p|^2 |u_r|^2 d\varsigma dt. \tag{19}$$

Since $a_3 > 0$ and $\int_{\Omega_\tau} |\delta p|^2 |u_r|^2 d\varsigma dt \leq \|\delta p\|^2_{L_\infty(I)}\left(\int_0^T \|u_r(.,t)\|^2_{L_2(I)} dt\right)$, from (19) by virtue of estimates (10) and (11) we get

$$\|\delta u_r(.,\tau)\|^2_{L_2(I)} + a_3\int_{\Omega_\tau} |\delta u_r|^2\left(|u_{r\delta}|^2 + |u_r|^2\right) d\varsigma dt \leq c_3\|\delta p\|^2_{L_\infty(I)},\ r = 1,2 \tag{20}$$

for any $\tau \in Q$, where the positive constant c_3 does not depend on δp and τ.

Using formula (2) for $\alpha = 0$, we obtain

$$\delta J_0(p) = J_0(p + \delta p) - J_0(p) = 2\int_\Omega Re\left[(u_1 - u_2)\left(\delta\overline{u}_1 - \delta\overline{u}_2\right)\right] dxdt + \|\delta u_1\|^2_{L_2(\Omega)} + \|\delta u_2\|^2_{L_2(\Omega)} - 2\int_\Omega Re\left(\delta u_1 \delta\overline{u}_2\right) dxdt \tag{21}$$

which implies that

$$\begin{aligned}|\delta J_0(p)| \;\leq\;& 2\left\|u_1\right\|_{L_2(\Omega)}\left\|\delta u_1\right\|_{L_2(\Omega)}+\\ & 2\left\|u_1\right\|_{L_2(\Omega)}\left\|\delta u_2\right\|_{L_2(\Omega)}+\\ & 2\left\|u_2\right\|_{L_2(\Omega)}\left\|\delta u_1\right\|_{L_2(\Omega)}+\\ & 2\left\|u_2\right\|_{L_2(\Omega)}\left\|\delta u_2\right\|_{L_2(\Omega)}+\\ & 2\left\|\delta u_1\right\|_{L_2(\Omega)}^2+2\left\|\delta u_2\right\|_{L_2(\Omega)}^2 .\end{aligned}$$

If we use estimates (10), (11), (20) in the inequality above, we get the inequality

$$\begin{aligned}&|J_0(p+\delta p)-J_0(p)| \leq\\ &c_4\left(\|\delta p\|_{L_\infty(I)}+\|\delta p\|_{L_\infty(I)}^2\right)\end{aligned}$$

for any $p \in P$, where c_4 is a positive constant independent from δp. Thus, we obtain that $|\delta J_0(p)| \rightarrow 0$ as $\|\delta p\|_{L_\infty(I)} \rightarrow 0$ for *any* $p \in P$, which concludes the proof. □

Theorem 2. *Let Theorem 1 be satisfied and $w \in L_2(I)$. Then, there exists a dense subset $V \subset L_2(I)$ such that OCP has a unique solution for any $w \in V$ and $\alpha > 0$.*

Proof. From Lemma 1, $J_0(p)$ is a lower semicontinuous functional. Also, it is clear that $J_0(p)$ is lower bounded. As known, $L_2(I)$ is a uniformly convex Banach space. Furthermore, P is a closed, bounded subset of $L_2(I)$. Therefore, based on Theorem 4 in [34] we can say that the OCP has a unique solution on a dense subset $V \subset L_2(I)$. This completes the proof. □

Theorem 3. *Let $w \in L_2(I)$ be a given function and $\alpha \geq 0$. Also, assume that Theorem 1 is satisfied. Then, the OCP has at least one solution.*

Proof. The proof of Theorem 3 is carried out as in [22]. □

4. The gradient of functional and a necessary optimality condition

In this section, we introduce the adjoint problem to investigate the differentiability of the objective functional and get a formula for its gradient. Finally, a necessary optimality condition for the OCP is derived.

By using Lagrange multiplier functions, we obtain the adjoint problem as follows:

$$\begin{aligned}&i\frac{\partial\eta_r}{\partial\tau}+a_0\frac{\partial^2\eta_r}{\partial\varsigma^2}+i\frac{\partial}{\partial\varsigma}\left(a_1(\varsigma)\eta_r\right)-\\ &a_2(\varsigma)\eta_r+p(\varsigma)\eta_r-2ia_3|u_r|^2\eta_r+ \qquad (22)\\ &ia_3u_r^2\bar{\eta}_r=2(-1)^r\left(u_1-u_2\right)\ for\ r=1,2,\end{aligned}$$

$$\eta_r(\varsigma,T)=0\ for\ r=1,2,\ \varsigma\in I, \qquad (23)$$

$$\left.\begin{aligned}&\eta_1(0,\tau)=\eta_1(l,\tau)=0, \tau\in(0,T),\\ &\frac{\partial\eta_2}{\partial\varsigma}(0,\tau)=\frac{\partial\eta_2}{\partial\varsigma}(l,\tau)=0,\ \tau\in(0,T),\end{aligned}\right\} \qquad (24)$$

where the functions $u_r = u_r(\varsigma,\tau)$ are solutions of problem (12)-(14) for any $p \in P$. It can be seen that the adjoint problem (22)-(24) includes the two boundary value problems. One of them is a Dirichlet problem with respect to η_1 and the other is a Neumann problem with respect to η_2. If we use transform $t = T-\tau$ to the adjoint problem, we come to the conclusion that the adjoint problem is in the form of problem (12)-(14). As a solution of (22)-(24), we consider two functions $\eta_1(\varsigma,\tau) \in U_1$, $\eta_2(\varsigma,\tau) \in U_2$ satisfying equation (22) *for almost all* $\varsigma \in I$ and *any* $\tau \in Q$, the condition (23) *for almost all* $\varsigma \in I$ and the conditions (24) *for almost all* $\tau \in (0,T)$, respectively. Hence, we can state the validity of the following theorem for the solution of the adjoint problem (22)-(24):

Theorem 4. *Let the assumptions of Theorem 1 be fulfilled. Then adjoint problem (22)-(24) has a unique solution $\eta_1 \in U_1$, $\eta_2 \in U_2$ for any $p \in P$ and the following estimates hold*

$$\begin{aligned}&||\eta_1(.,\tau)||_{\mathring{W}_2^2(I)}^2+\left\|\frac{\partial\eta_1}{\partial\tau}\right\|_{L_2(I)}^2\leq\\ &c_5\left\|u_1-u_2\right\|_{W_2^{0,1}(\Omega)}, \qquad (25)\end{aligned}$$

$$\begin{aligned}&||\eta_2(.,\tau)||_{W_2^2(I)}^2+\left\|\frac{\partial\eta_2}{\partial\tau}\right\|_{L_2(I)}^2\leq\\ &c_6\left\|u_1-u_2\right\|_{W_2^{0,1}(\Omega)} \qquad (26)\end{aligned}$$

for any $\tau \in Q$, where the positive constants c_5, c_6 do not depend on τ.

This theorem can be easily proved by the Galerkin's method similarly to the proof of Theorem 1.

Now, let's get the enhancement $\delta J_\alpha(p) = J_\alpha(p+\delta p) - J_\alpha(p)$ of $J_\alpha(p)$ for any $p \in P$, where $\delta p \in L_\infty(I)$ is an increment given to any $p \in P$ such that $p+\delta p \in P$. If we use formula (2), we achieve

$$\begin{aligned}\delta J_\alpha(p) \;=\;& \int_\Omega \delta p(\varsigma)Re(u_1\bar{\eta}_1)d\varsigma d\tau+\\ &\int_\Omega \delta p(\varsigma)Re(u_2\bar{\eta}_2)d\varsigma d\tau+ \qquad (27)\\ &2\alpha\int_0^l(p-w)\,\delta p d\varsigma+R,\end{aligned}$$

where

$$
\begin{aligned}
R = & \int_{\Omega} \delta p(\varsigma) Re(\delta u_1 \overline{\eta}_1) d\varsigma d\tau + \\
& \int_{\Omega} \delta p(\varsigma) Re(\delta u_2 \overline{\eta}_2(\varsigma,\tau)) d\varsigma d\tau + \\
& \|\delta u_1\|^2_{L_2(\Omega)} + \|\delta u_2\|^2_{L_2(\Omega)} - \\
& a_3 \int_{\Omega} (|u_{1\delta}|^2 - |u_1|^2) Im(\delta u_1 \bar{\eta}_1) d\varsigma d\tau - \\
& a_3 \int_{\Omega} (|u_{2\delta}|^2 - |u_2|^2) Im(\delta u_2 \bar{\eta}_2) d\varsigma d\tau - \\
& a_3 \int_{\Omega} |\delta u_1|^2 Im\,(u_1 \bar{\eta}_1)\, d\varsigma d\tau - \\
& a_3 \int_{\Omega} |\delta u_2|^2 Im\,(u_2 \bar{\eta}_2)\, d\varsigma d\tau + \alpha \|\delta p\|^2_{L_2(I)}
\end{aligned}
$$

and $\delta u_r \equiv u_r(\varsigma,\tau;p+\delta p) - u_r(\varsigma,\tau;p)$ for $r=1,2$ hold problem (15) for any $p \in P$. By Young's inequality for the term R, we get

$$
\begin{aligned}
|R| \leq & \frac{5}{2}\|\delta u_1\|^2_{L_2(\Omega)} + \frac{5}{2}\|\delta u_2\|^2_{L_2(\Omega)} + \\
& \alpha \|\delta p\|^2_{L_2(I)} + \\
& \frac{T}{2}\left(\max_{0\leq\tau\leq T}\|\eta_1(.,\tau)\|^2_{L_\infty(I)}\right)\|\delta p\|^2_{L_2(I)} + \\
& \frac{T}{2}\left(\max_{0\leq\tau\leq T}\|\eta_2(.,\tau)\|^2_{L_\infty(I)}\right)\|\delta p\|^2_{L_2(I)} + \\
& a_3 \int_{\Omega} \left(|u_{1\delta}|^2 + |u_1|^2\right)|\delta u_1|^2\, d\varsigma d\tau + \\
& a_3 \int_{\Omega} \left(|u_{2\delta}|^2 + |u_2|^2\right)|\delta u_2|^2\, d\varsigma d\tau + \\
& a_3 \int_0^T \|\eta_1(.,\tau)\|^2_{L_\infty(I)} \|\delta u_1(.,\tau)\|^2_{L_2(I)}\, d\tau + \\
& a_3 \int_0^T \|\eta_2(.,\tau)\|^2_{L_\infty(I)} \|\delta u_2(.,\tau)\|^2_{L_2(I)}\, d\tau + \\
& \frac{1}{2}a_3 \int_0^T \|u_1(.,\tau)\|^2_{L_\infty(I)} \|\delta u_1(.,\tau)\|^2_{L_2(0,l)}\, d\tau + \\
& \frac{1}{2}a_3 \int_0^T \|u_2(.,\tau)\|^2_{L_\infty(I)} \|\delta u_2(.,\tau)\|^2_{L_2(I)}\, d\tau.
\end{aligned}
$$

In the inequality above, if we use estimates (10), (11), (20), (25), (26) and the well known inequality in [31]

$$
\begin{aligned}
& \|u(.,\tau)\|^2_{L_\infty(I)} \leq \\
& \beta_2 \left\|\frac{\partial u(.,\tau)}{\partial\varsigma}\right\|_{L_2(I)} \|u(.,\tau)\|_{L_2(I)}, \\
& \beta_2 = const. > 0
\end{aligned} \quad (28)
$$

for any $\tau \in Q$, we achive

$$
|R| \leq c_7 \|\delta p\|^2_{L_2(I)} \leq c_8 \|\delta p\|^2_{L_\infty(I)}
$$

which shows that $R = o\left(||\delta p||_{L_\infty(I)}\right)$, that is, $\underset{||\delta p||_{L_\infty(I)}\to 0}{lim} \frac{R}{||\delta p||_{L_\infty(I)}} = 0$, where the constants $c_7, c_8 > 0$ are independent from δp and τ. So, from (27), we can write

$$
\begin{aligned}
\delta J_\alpha(p) = & \int_0^l \left(\int_0^T Re(u_1\overline{\eta}_1 + u_2\overline{\eta}_2)d\tau\right)\delta p(\varsigma)d\varsigma + \\
& \int_0^l 2\alpha\,(p-w)\,\delta p(\varsigma)d\varsigma + o\left(||\delta p||_{L_\infty(I)}\right)
\end{aligned}
$$

which implies that

$$
J'_\alpha(p) = \int_0^T Re(u_1\overline{\eta}_1 + u_2\overline{\eta}_2)d\tau + 2\alpha\,(p-w)\,. \quad (29)
$$

Consequently, the differentiability of $J_\alpha(p)$ in the meaning of Frechet is shown and the next theorem is proved:

Theorem 5. *Let $w \in L_2(I)$ be a given function. Assume that the conditions of Theorem 4 are satisfied. Then, $J_\alpha(p)$ is a differentiable functional on P and moreover, its gradient is given by formula (29).*

Lemma 2. *The functional $J'_\alpha(p)$ is continuous on P.*

Proof. Let's prove that $|J'_\alpha(p+\delta p) - J'_\alpha(p)| \longrightarrow 0$ as $\|\delta p\|_{L_\infty(I)} \longrightarrow 0$ on the set P. Using formula (29), we get

$$J'_\alpha(p+\delta p)-J'_\alpha(p)= \int_0^T Re\left(u_{1\delta}\delta\overline{\eta}_1+u_{2\delta}\delta\overline{\eta}_2\right)d\tau+ \int_0^T Re\left(\delta u_1\overline{\eta}_1+\delta u_2\overline{\eta}_2\right)d\tau+2\alpha\delta p(\varsigma), \tag{30}$$

where the functions $\delta\eta_r=\delta\eta_r(\varsigma,\tau)\equiv\eta_r(\varsigma,\tau;p+\delta p)-\eta_r(\varsigma,\tau;p)$ for $r=1,2$ satisfy the problem

$$\begin{aligned}
&i\frac{\partial\delta\eta_r}{\partial\tau}+a_0\frac{\partial^2\delta\eta_r}{\partial\varsigma^2}+i\frac{\partial\left(a_1(\varsigma)\delta\eta_r\right)}{\partial\varsigma}-\\
&a_2(\varsigma)\delta\eta_r+(p(\varsigma)+\delta p(\varsigma))\delta\eta_r=-\delta p\eta_r-\\
&ia_3\left(2|u_{r\delta}|^2\eta_{r\delta}-u_{r\delta}^2\overline{\eta}_{r\delta}\right)+\\
&ia_3\left(2|u_r|^2\eta_r-u_r^2\overline{\eta}_r\right)+2(-1)^r\left(\delta u_1-\delta u_2\right),\\
&\quad\delta\eta_r(\varsigma,T)=0,\ \varsigma\in I,\ r=1,2,
\end{aligned}$$

$$\begin{aligned}
\delta\eta_1(0,\tau)&=\delta\eta_1(l,\tau)=0,\ \tau\in(0,T)\\
\frac{\partial\delta\eta_2}{\partial\varsigma}(0,\tau)&=\frac{\partial\delta\eta_2}{\partial\varsigma}(l,\tau)=0,\ \tau\in(0,T).
\end{aligned}$$

For this problem, as similar to obtain of inequality (20), we get the estimate

$$\|\delta\eta_r(.,\tau)\|^2_{L_2(I)}+ a_3\int_{\widetilde{\Omega}_\tau}|u_{r\delta}|^2|\delta\eta_r|^2d\varsigma d\tau\le c_9\|\delta p\|^2_{L_\infty(I)} \tag{31}$$

for any $\tau\in Q$ and $r=1,2$. From (30), we get

$$\begin{aligned}
&\left|J'_\alpha(p+\delta p)-J'_\alpha(p)\right|\le\\
&\int_0^T|u_{1\delta}|\,|\delta\eta_1|\,d\tau+\int_0^T|u_{2\delta}|\,|\delta\eta_2|\,d\tau+\\
&\int_0^T|\delta u_1|\,|\eta_1|\,d\tau+\int_0^T|\delta u_2|\,|\eta_2|\,d\tau+\\
&2\alpha\,|\delta p(\varsigma)|
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left\|J'_\alpha(p+\delta p)-J'_\alpha(p)\right\|^2_{L_2(I)}\le\\
&5T\|u_{1\delta}\|^2_{L_\infty(\Omega)}\int_0^T\|\delta\eta_1\|^2_{L_2(I)}d\tau+\\
&5T\|u_{2\delta}\|^2_{L_\infty(\Omega)}\int_0^T\|\delta\eta_2\|^2_{L_2(I)}d\tau+\\
&5T\|\eta_1\|^2_{L_\infty(\Omega)}\int_0^T\|\delta u_1\|^2_{L_2(I)}d\tau+\\
&5T\|\eta_2\|^2_{L_\infty(\Omega)}\int_0^T\|\delta u_2\|^2_{L_2(I)}d\tau+\\
&20\alpha^2\|\delta p\|^2_{L_2(I)}.
\end{aligned}$$

In inequality above, using estimates (10), (11), (20), (25), (26), (31) and inequality (28) we get

$$\begin{aligned}
&\left\|J'_\alpha(p+\delta p)-J'_\alpha(p)\right\|^2_{L_2(I)}\le\\
&c_{10}\|\delta p\|^2_{L_\infty(I)}\ \text{ for any } p\in P
\end{aligned}$$

which implies that

$$\left|J'_\alpha(p+\delta p)-J'_\alpha(p)\right|\longrightarrow0 \text{ as } \|\delta p\|_{L_\infty(I)}\longrightarrow0,$$

where the constants $c_9,c_{10}>0$ are independent from δp and τ. Thus, the proof is completed. □

Theorem 6. *Presume that the Theorem 5 and Lemma 2 hold and let $p^*=p^*(\varsigma)$ be a solution of the OCP. Then, the inequality*

$$\int_0^l\left(\int_0^T Re(u_1^*\overline{\eta}_1^*+u_2^*\overline{\eta}_2^*)d\tau\right)(p-p^*)\,d\varsigma+ \int_0^l\left(2\alpha\left(p^*-w\right)\right)(p-p^*)\,d\varsigma\le0$$

is valid for any $p\in P$, where the functions u_r^* and η_r^*, $r=1,2$ are solutions of (12)-(14) and the adjoint problem corresponding to $p^*\in P$, respectively.

Proof. It is clear that the functional $J_\alpha(p)$ is the sum of the functionals $J_0(p)$ and $\alpha\|p-w\|^2_{L_2(I)}$. Since $\alpha\|p-w\|^2_{L_2(I)}$ is a continuous functional on P, from Lemma 1, we deduce that the functional $J_\alpha(p)$ is continuous on the set P. Also if we take into account Lemma 2, we say that $J_\alpha(p)$ is a continuous differentiable functional on the convex set P. Thus, by virtue of known theorem in [35], if the functional $J_\alpha(p)$ has a minimum value at $p^* \in P$, then

$$\left(J'_\alpha(p^*), p - p^*\right)_{L_2(I)} \geq 0 \text{ for any } p \in P$$

which concludes the proof. □

5. Conclusions

In this study, we examined an optimal control problem for a system whose state is expressed by the nonlinear Schrödinger equation. We regard Lions functional as the objective functional. As it is seen from the definition of P, the admissible controls set contains the measurable bounded functions from $L_2(I)$. We have shown the existence and uniqueness of the solution to the optimal control problem. By means of an adjoint problem, we demonstrated that the objective functional is differentiable in the sense of Frechet. Finally, by proving that the objective functional is a continuously differentiable functional on the set of admissible controls, we derived a necessary optimality condition for the optimal control problem.

As a future direction, we will consider the optimal control problem, in which the set of admissible controls will be chosen from the wider class of functions.

References

[1] Liu, Wu-M., & Kengne, E. (2019). *Schrödinger Equations in Nonlinear Systems,* Springer Nature Singapore Pte. Ltd., Singapore.

[2] Zhuravlev, V. M. (1996). Models of nonlinear wave processes that allow for soliton solutions. *Journal of Experimental and Theoretical Physics,* 83 (6), 1235-1245.

[3] Elhia, M., Balatif, O., Boujalla, L., & Rachik, M. (2021). Optimal control problem for a tuberculosis model with multiple infectious compartments and time delays. *An International Journal of Optimization and Control: Theories & Applications,* 11 (1), 75-91.

[4] Gisser, M., & Sanchez, D. A. (1980). Competition versus optimal control in groundwater pumping. *Water Resources Research,* 16 (4), 638-642.

[5] Imer, O. C., Yüksel, S., & Başar, T. (2006). Optimal control of LTI systems over unreliable communication links. *Automatica,* 42, 1429 – 1439.

[6] Moussouni, N., & Aliane, M. (2021). Optimal control of COVID-19. *An International Journal of Optimization and Control: Theories & Applications,* 11 (1), 114-122.

[7] Trelat, E. (2012). Optimal Control and Applications to Aerospace: Some Results and Challenges. *Journal of Optimization Theory and Applications,* 154 (2012), 713–758.

[8] Van-Reeth, E., Ratiney, H., Lapert, M., Glaser, S. J., & Sugny, D. (2017). Optimal control theory for applications in Magnetic Resonance Imaging, *Pacific Journal of Mathematics for Industry,* 9(9), 1-10.

[9] Hamdache, A., Saadi, S., & Elmouki, I. (2016). Free terminal time optimal control problem for the treatment of HIV infection. *An International Journal of Optimization and Control: Theories & Applications,* 6(1), 33-51.

[10] Ulus, A. Y. (2018). On discrete time infinite horizon optimal growth problem. *An International Journal of Optimization and Control: Theory and Applications,* 8(1), 102-116.

[11] Vorontsov, M. A., & Shmalgauzen, V. I. (1985). *The Principles of Adaptive Optics,* Izdatel'stvo Nauka, Moscow (in Russian).

[12] Lions, J.-L. (1971). *Optimal Control of Systems Described by Partial Differential Equations.* Berlin; New York, Springer-Verlag.

[13] Baudouin, L., & Salomon, J. (2008). Constructive solution of a bilinear optimal control problem for a Schrödinger equation. *Systems and Control Letters,* 57(6), 453-464.

[14] Butkovskiy, A. G., & Samoilenko, Yu.I. (1990). *Control of Quantum-Mechanical Processes and Systems. Mathematics and its Applications* (56). Kluwer Academic Publishers Group, Dordrecht.

[15] Guliyev, H. F., & Gasimov, Y. S. (2014). Optimal control method for solving the Cauchy-Neumann problem for the Poisson equation. *Journal of Mathematical Physics, Analysis, Geometry,* 10(4), 412-421.

[16] Hao, D. N. (1986). Optimal control of quantum systems. *Automat Remote Control,* 47(2), 162–168.

[17] Iskenderov, A. D. (2005). Identification problem for the time dependent Schrödinger type equation. *Proceedings of the Lankaran State University,* Natural Sciences Series, 31-53.

[18] Iskenderov, A. D., & Yagubov, G. Y. (2007). Optimal control problem with unbounded potential for multidimensional, nonlinear and nonstationary Schrödinger equation. *Proceedings of the Lankaran State University,* Natural Sciences Series, 3-56.

[19] Iskenderov, A. D., Yagubov, G. Ya., & Musayeva, M. A. (2012). *The Identification of Quantum Mechanics Potentials.* Çaşıoğlu, Baku.

[20] Keller, D. (2013). Optimal conrol of a linear stochastic Schrödinger equation. *Discrete Continuous Dynamical Systems Supplement,* 437-446.

[21] Yagubov, G. Ya., & Musayeva, M. A. (1997). On the identification problem for nonlinear Schrödinger equation, *Differ. Equ.* 33 (12), 1695-1702.

[22] Yildirim Aksoy, N., Aksoy, E., & Kocak, Y. (2016). An optimal control problem with final

observation for systems governed by nonlinear Schrödinger equation. *Filomat,* 30(3), 649-665.

[23] Iskenderov, A. D. (1984). On variational formulations of multidimensional inverse problems of mathematical physics. *Doklady Akademii Nauk USSR,* 274(3), 531-533.

[24] Iskenderov, A. D., Yagubov, G. Ya., Ibragimov, N. S., & Aksoy, N.Y. (2014). Variation formulation of the inverse problem of determining the complex-coefficient of equation of quasioptics. *Eurasian Journal of Mathematical and Computer Applications,* 2(2), 102-121.

[25] Kücük, G. D., Yagub, G., & Celik, E. (2019). On the existence and uniqueness of the solution of an optimal control problem for Schrödinger equation. *Discrete Continuous Dynamical Systems Series-S,* 12(3), 503-512.

[26] Mahmudov, N. M. (2010). On an optimal control problem for the Schrödinger equation with the real coefficient. *Izv. VUZOV,* 11, 31-40.

[27] Salmanov, V. (2020). Existence and uniqueness of the solution to the optimal control problem with integral criterion over the entire domain for a nonlinear Schrödinger equation with a special gradient term. *Control and Cybernetics*, 49(3), 277-290.

[28] Yagub, G., Ibrahimov, N.S., Musayeva, M.A., & Zengin, M. (2019). Optimal control problem with the boundary functional for a Schrödinger equation with a special gradient term, 34th International Conference Problems of Decision Making under Uncertainties (PDMU); September 23-27; Lviv, Ukraine, pp. 116-117.

[29] Yildirim Aksoy, N., Celik, E., & Zengin, M. (2022). On optimal control of a charged particle in a varying electromagnetic field, *Waves in Random and Complex Media.*

[30] Zengin, M., Ibrahimov, N. S., & Yagub, G. (2021). Existence and uniqueness of the solution of the optimal control problem with boundary functional for nonlinear stationary quasi-optical equation with a special gradient term. *Sci Proc Lankaran State Univ., Math Nat Sci Ser 1.*, 27-42.

[31] Ladyzhenskaya, O. A., Solonnikov, V. A., & Ural'ceva, N. N. (1968). *Linear and Quasilinear Equations of Parabolic Type.* Amer. Math. Soc. (English Trans.), Providence, RI.

[32] Tikhonov, A. N., & Arsenin, V. Y. (1977). *Solutions of Ill-Posed Problems,* V.H. Winston&Sons, Washington DC.

[33] Yagub, G., Ibrahimov, N. S., & Zengin, M. (2018). The solvability of the initial-boundary value problems for a nonlinear Schrödinger equation with a special gradient term, *Journal of Mathematical Physics, Analysis, Geometry,* 14 (2), 214-232.

[34] Goebel, M. (1979). On existence of optimal control, *Mathematische Nachrichten*, 93(1), 67-73.

[35] Vasilyev, F. P. (1981). *Methods of Solving for Extremal Problems,* Nauka, Moskow (in Russian).

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