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RESEARCH ARTICLE

Some stability results on non-linear singular differential systems with random impulsive moments

Arumugam Vinodkumar[a*], Sivakumar Harinie[a], Michal Fečkan [b,c] Jehad Alzabut [d,e]

[a] *Department of Mathematics, Amrita School of Physical Sciences, Coimbatore-641 112, Amrita Vishwa Vidyapeetham, India*
[b] *Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48, Bratislava, Slovakia*
[c] *Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia*
[d] *Department of Mathematics and General Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia*
[e] *Department of Industrial Engineering, OSTİM Technical University, Ankara 06374, Turkey*
a_vinodkumar@cb.amrita.edu, hariniesiva960@gmail.com, Michal.Fecka@fmph.uniba.sk, jalzabut@psu.edu.sa

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ABSTRACT

This paper studies the exponential stability for random impulsive non-linear singular differential systems. We established some new sufficient conditions for the proposed singular differential system by using the Lyapunov function method with random impulsive time points. Further, to validate the theoretical results' effectiveness, we finally gave two numerical examples that study with graphical illustration and an additional example involving matrices with complex entries, proving the results to be true in that case as well.

1. Introduction

Singular systems are widely connected to various applications such as power systems, electrical networks, and robotics. However, it has some exceptional features like regular and impulse free that do not exist in normal state-space systems. These exceptional characteristics may cause some challenges upon studying the singular systems. Further, because of the singularity matrix E, it is not easy to formulate easy-to-check conditions for analysis and synthesis problems. Due to the above justifications, the study of singular systems has been scrutinized more attention over the past decades [1]. The past two decades have spotted an important development on the theory of singular differential systems (SDSs), and many basic and most significant concepts have been favorably examined including stability analysis, stabilization, guaranteed cost control, filtering, observer design, sliding mode control and so on [2, 3]. The main target is to show the latest developments in the analysis and synthesis of SDSs. Since the system is chronicled by algebraic and differential equations, the SDSs may disclose instability behavior and thus poor performance may be raised on the basis of presence of time delay. Hence the investigation of stability character of SDSs becomes compulsory. By applying various methods and ideas, several authors have studied the SDS. In [4], the author studied the delay-dependent stability criteria by using Writinger-based inequality. The delay-dependent robust stability norms for two classes of SDSs with norm-bounded uncertainties are discussed in [5].

*Corresponding Author

In [6] the exponential stability problems of singular impulsive switched systems was investigated. Impulsive stabilization problem for a class of linear singular systems with time-delays can be seen in [7]. The problem of exponential stability analysis for a class of singular systems with interval time-varying discrete and distributed delays is discussed in [8]. The stability problem of singular systems with time-varying delay by first transforming it into a neutral system with time-varying delay and constructing an appropriate Lyapunov-Krasovskii functional, is studied in [9]. In [10] the control problem of switched singular systems was investigated aiming to compress their inconsistent state jumps when switch occurs between two different singular subsystems. In [11] we can see a definition of a transform that reformulates the system with delays into a singular linear system of differential equations whose coefficients are non-square constant matrices where the number of their columns is greater than the number of their rows. Further, in engineering applications, the complexity increases mean accuracy will not be described by linear singular systems. To overcome this type of problem, we need generalized nonlinear singular systems to solve the problem. Very few authors have studied the nonlinear singular system models [6, 12–15] and the references therein. Moreover, The problem of sliding mode control with torpidity of a class of uncertain nonlinear SDSs had been discussed in [16]. Many other valuable results are obtained for stability and stabilization for SDSs, see [7, 17–26] and the references therein.

Stability is a condition in which a slight disturbance in a system does not generate too disrupting effect on that system. The dynamics of SDS are by a mixture of differential- algebraic equations, so the study of $\mathcal{E}$-exponential stability ($\mathcal{E}$-ES) was first introduced by [12]. In [3, 6], the authors analyzed the connection between the exponential stability (ES) and the $\mathcal{E}$-ES for linear and non-linear singular impulsive differential systems and they claimed that the $\mathcal{E}$-ES is nearly equal to its ES. Hence it is essential to speak about the exponential stability of random impulsive nonlinear SDSs. On the other hand, impulsive systems stand up when dynamics generate discontinuous trajectories. Discontinuities arise when movements of states occur over a small interlude that simulates a point-mass measure. There are several works contributed to study the impulses at fixed point (see the monograph [27, 28] and [29–35]). The significant concepts of impulsive control have been disputed with a wide field of uses in analysis and control of complex systems in [36]. Some stability criteria for impulsive differential systems had been discussed in [37]. Global ES for impulsive system with infinite distributed delay based on flexible impulse frequency are discussed in [38]. In [39], impulse control is used to study nonlinear systems with partial unmeasurable states. Very few research have been carried for random impulsive systems. When the reactions of the impulse drawn at random time points, the results follow as a stochastic process. Random impulses are different from fixed-time impulse effects. Recently in [40], the authors studied the exponential stability based on fixed and random time effect of the impulses while they proved the robust mean square stability for random impulsive control systems in [41]. Then, by considering the impulse moments at random time points in [42], the authors proved the stability results for differential systems. Moreover and to the best of authors' knowledge, we like to point out that there is no paper about the investigation of the ES on the random impulsive SDSs. For further information the reader can refer to [43–48].

Inspired by the above discussion, in this paper, we generalized the $\mathcal{E}$-ES result for p^{th} moment and also proved the equivalence to ES for a nonlinear singular system. Further, we address new sufficient conditions to develop the exponential stability criteria ($\mathcal{E}$-ES and ES) for random impulsive nonlinear SDSs. The waiting time between two consecutive impulses is considered to follow an exponential distribution when the effects of the impulses taken at random time points. By employing the effect of impulses and Lyapunov-function approach, we achieve the desired performance. The rest of this paper follows through some definitions and lemmas in Section 2. In Section 3, we prove the $\mathcal{E}$-ES and ES results for random impulsive SDSs by using the Lyapunov-function approach. In Section 4, three numerical examples are discussed, the last of which involves the usage of matrices with complex entries and finally in Section 5 a conclusion is given.

Notations: Let $\Re$ indicate the set of all real numbers, $\Re_+$ the set of all positive real numbers and Z_+ the set of all positive integers. Let $\Re^n$ be the Euclidean space provided with norm $\|\cdot\|$, and $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. We use $\mathcal{PC}([t_0, T], \Re^n)$, to indicate the set of all piecewise right continual real-valued random variables $\varphi : [t_0, T] \to \Re^n$, with the norm is described by $E\|\varphi\|^p = \sup_{\theta\in[t_0,T]} E\|\varphi(\theta)\|^p$. Furthermore, A^T represents the transpose of A where the maximum and minimum eigenvalues of the matrix indicated by $\lambda_{max}(\cdot)$, and $\lambda_{min}(\cdot)$. Then $E[\cdot]$, indicates the expectation operator with respect to the given probability $\mathcal{P}$.

2. Model description and essential preliminaries

Let $\{\chi_m^{'}\}_{m=0}^{\infty}$ be the non-decreasing sequence of random variables and $\{\tau_m^{'}\}_{m=1}^{\infty}$ is a sequence of an independent exponentially distributed random variable with parameter γ defined on sample space Ω. Note that $\chi_0^{'} = t_0$, where $t_0 \geq 0$ is a fixed point and $\chi_m^{'} = \chi_{m-1}^{'} + \tau_m^{'}$ for $m = 1, 2, \cdots$, where $\tau_m^{'}$ define the delay (waiting) time between two consecutive impulses where $\sum\limits_{m=1}^{\infty} \tau_m^{'} = \infty$ with probability 1.

Consider, the random impulsive non-linear SDSs:

$$\begin{cases} \mathcal{E}\dot{x}(t) = Ax(t) + f(x(t),t), \chi_m^{'} < t < \chi_{m+1}^{'}, \\ x(\chi_m^{'+}) = C_m(\tau_m^{'})x(\chi_m^{'-}), m \in Z_+ \\ x_{t_0} = x_0, \end{cases} \tag{1}$$

where $t \geq t_0$, $x(t) \in \Re^n, A \in \Re^{n\times n}$ is system matrix, $C_m(\tau_m^{'})$ is the jump altitude and the matrix $\mathcal{E} \in \Re^{n\times n}$ is singular with rank $\mathcal{E} = k \leq n$. $f(x(t),t) : \Re^n \times \Re^+ \to \Re^{n\times n}$ are piecewise continual vector-valued functions assuring the existence and uniqueness of solutions for systems (1) with $f(0,t) \equiv 0$ and satisfies the Lipschitz condition for all $(x,t), (x^*,t) \in \Re^n \times \Re^+$

$$\|f(x(t),t) - f(x^*(t),t)\| \leq \|F(x(t) - x^*(t))\|, \tag{2}$$

where F is a constant matrix with an appropriate dimension. Consequently, from (2), we have

$$\|f(x(t),t)\| \leq \|Fx(t)\|. \tag{3}$$

Remark 1. *Let $\{\chi_m\}_{m=0}^{\infty}$ be non-decreasing sequence of points, where χ_m are values of the correlated random variables $\chi_m^{'}, \forall\, m = 1, 2, \cdots$, and $\{\tau_m\}_{m=1}^{\infty}$ be a sequence of points, where τ_m are arbitrary values of the random variable $\tau_m^{'}, \forall\, m = 1, 2, \cdots$. For satisfaction, we define $\chi_0 = t_0$ and $\chi_m = \chi_{m-1} + \tau_m$, $\forall\, m = 1, 2, \cdots$, where τ_m represents the value of the delay (waiting) time. Then system (1) becomes*

$$\begin{cases} \mathcal{E}\dot{x}(t) = Ax(t) + f(x(t),t), t \neq \chi_m, t \geq t_0, \\ x(\chi_m^{+}) = C_m(\tau_m)x(\chi_m^{-}), m \in Z_+ \\ x_{t_0} = x_0. \end{cases} \tag{4}$$

The solutions of the system (4) are controlled not only by the initial condition but also by the moments of impulses χ_m, $m = 1, 2, \cdots$. That is, the result depends on the selected arbitrary values τ_m of the random variable $\tau_m^{'}, \forall\, m = 1, 2, \cdots$. We will assume $x(\chi_m) = \lim\limits_{t\to\chi_m - 0} x(t)$.

Moreover, the set of all solutions of system (4), is known as a sample path solution of system (1). Thus, the sample path solution produces a stochastic process. We can assure that it is a solution of the system (1).

Lemma 1. *[41, 42], When there will be exactly m impulses until the time $t, t \geq t_0$, and the waiting time between two consecutive impulses follow an exponential distribution with parameter γ, then the probability*

$$\mathcal{P}(I_{[\chi_m^{'},\chi_{m+1}^{'})}(t)) = \frac{\gamma^m (t-t_0)^m}{m!} e^{-\gamma(t-t_0)},$$

where the events

$$I_{[\chi_m^{'},\chi_{m+1}^{'})}(t) = \{\omega \in \Omega : \chi_m^{'}(\omega) < t < \chi_{m+1}^{'}(\omega)\},$$

$m = 1, 2, \cdots$.

Remark 2. *[41, 42], Let $x(t)$ be the solution of the random impulsive differential equations then the expected value of $x(t)$ satisfies*

$$E[\|x(t)\|^p] = \sum_{m=0}^{\infty} E[\|x(t)\|^p | I_{[\chi_m^{'},\chi_{m+1}^{'})}(t)] \\ \mathcal{P}(I_{[\chi_m^{'},\chi_{m+1}^{'})}(t)),$$

where $\chi_m^{'}$ is the impulse moments.

Definition 1. *[6], The pair $(\mathcal{E}, A)$ is called regular if $det(s\mathcal{E} - A)$ is not identical zero. The pair $(\mathcal{E}, A)$ is called impulse free if $deg(det(s\mathcal{E} - A)) = rank(\mathcal{E})$.*

Definition 2. *[6, 12], System (1) is said to have a Lyapunov-like property if there exists a matrix P such that $\mathcal{E}^T P = P^T \mathcal{E} \geq 0$ and $[Ax(t) + f(x(t),t)]^T Px + x^T P[Ax(t) + f(x(t),t)] < 0$.*

Remark 3. *[6, 12] For a nonlinear system, it is sufficient that the solution exists and is unique on $[0,\infty)$, if there exists a matrix P satisfying definition 2.*

From [6, 36], we have that the pair $(\mathcal{E}, A)$ is regular and impulse free, then we have that there exists matrices $\mathcal{G}_1 \in \Re^{r\times n}, \mathcal{G}_2 \in \Re^{(n-r)\times n}, \mathcal{Q}_1 \in \Re^{n\times r}, \mathcal{Q}_2 \in \Re^{n\times(n-r)}$, such that $\mathcal{G} = col(\mathcal{G}_1, \mathcal{G}_2)$ and $\mathcal{Q} = row(\mathcal{Q}_1, \mathcal{Q}_2) \in \Re^{n\times n}$ are two non-singular matrices and the following standard decomposition holds

$$\mathcal{G}\mathcal{E}\mathcal{Q} = diag\{I_r, 0\}, \mathcal{G}A\mathcal{Q} = diag\{A_1, I_{n-r}\}$$

where $r = Rank(\mathcal{E}), A_1 \in \Re^{r\times r}$. Non-singularity of $\mathcal{G}$ implies that $\mathcal{G}_2$ is full-row and then $\mathcal{G}_2(\mathcal{G}_2)^T$ is positive definite. Without loss of generality, we always assume that $\|\mathcal{G}_2\| \leq 1$.

Lemma 2. *[6], Let $\mathcal{V} \in \Re^{n\times n}$ be a positive-definite matrix, then*

$$\lambda_{min}(\mathcal{V})x^Tx \leq x^T\mathcal{V}x \leq \lambda_{max}(\mathcal{V})x^Tx, \quad \forall\, x \in \Re^n.$$

Lemma 3. *[49] For any constant $\epsilon > 0$, and vectors $x, y \in \Re^n$, then we have*

$$x^Ty + y^Tx \leq \epsilon^{-1}x^Tx + \epsilon y^Ty, holds.$$

Definition 3. *System (1) is said to be p^{th} moment $\mathcal{E}$-ES, if there exist two positive numbers $\lambda > 0, M > 0$ such that, the solution x of system (1) satisfies*

$$E\,\|\mathcal{E}x(t)\|^p \leq ME[\|\mathcal{E}x_0\|^p]e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Definition 4. *System (1) is said to be p^{th} moment ES, if there exist two positive numbers $\lambda > 0, M > 0$ such that, the solution x of the system (1), satisfies*

$$E\,\|x(t)\|^p \leq ME[\|\mathcal{E}x_0\|^p]e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

If $p = 2$, then it is mean square exponential stable.

3. Main results

Theorem 1. *Let $\tau' = \max_{m\in Z_+}\left\{\chi_m' - \chi_{m-1}'\right\} < \infty$. Assume that system (1) satisfies a Lyapunov-like property and there exists an invertible matrix P, and positive constants $\kappa > 0$, $\omega_m > 0$, such that $E[\omega_m] \leq \kappa$, $\zeta < 0$ be a negative real number, $\epsilon > 0$, exponential distribution parameter γ and the following conditions hold,*

$$\begin{gathered}(A^TP + P^TA) + \lambda_{max}(\tfrac{1}{\epsilon}F^TF + \epsilon P^TP)I \\ < \zeta\mathcal{E}^TP, \\ \Gamma = (C_m(\tau_m)^T\mathcal{E}^TPC_m(\tau_m) - \omega_m\mathcal{E}^TP) \\ \leq 0 \\ \zeta + \gamma(\kappa - 1) < 0.\end{gathered} \tag{5}$$

Then, the trivial solution of system (1) is p^{th} moment $\mathcal{E}$-ES.

Proof. Let x be the sample path solution of systems (4). For convenience we take $V(x(t)) = V(t, x(t))$, and consider the Lyapunov function

$$V(x(t)) = x^T(t)\mathcal{E}^TPx(t). \tag{6}$$

Taking the derivative of $V(x(t))$ along the solution of system (4) at the continuous interval $[\chi_{m-1}, \chi_m), m \in Z_+$, then we have

$$\begin{aligned}&\dot{V}(x(t)) \\ &= \dot{x}^T(t)\mathcal{E}^TPx(t) + x^T(t)P^T\mathcal{E}\dot{x}(t), \qquad (7)\\ &= x^T(t)(A^TP + P^TA)x(t) + 2f^T(x(t), t)Px(t).\end{aligned}$$

From condition (5), we have

$$\begin{aligned}&\dot{V}(x(t)) \\ &= x^T(t)(A^TP + P^TA)x(t) + 2f^T(x(t), t)Px(t), \\ &= x^T(t)(A^TP + P^TA)x(t) + 2f^T(x(t), t)Px(t), \\ &\leq x^T(t)(A^TP + P^TA)x(t) \\ &\quad + x^T(t)(\frac{1}{\epsilon}F^TF + \epsilon P^TP)Ix(t) \\ &\leq x^T(t)\zeta\mathcal{E}^TPx(t) \\ &\leq \zeta V(x(t)).\end{aligned}$$

Hence we have,

$$\dot{V}(x(t)) - \zeta V(x(t)) \leq 0, \tag{8}$$

or

$$\dot{V}(x(t)) \leq \zeta V(x(t)),\ t \in [\chi_{m-1}, \chi_m), m \in Z_+. \tag{9}$$

Note that for any $m \in Z_+$, at instant $t = \chi_m$, we have

$$\begin{aligned}&V(\chi_m{}^+) - \omega_mV(\chi_m{}^-) \\ &= x^T(\chi_m{}^+)\mathcal{E}^TPx(\chi_m{}^+) - \omega_mx^T(\chi_m{}^-)\mathcal{E}^TPx(\chi_m{}^-) \\ &= [C_m(\tau_m)x(\chi_m{}^-)]^T\mathcal{E}^TP[C_m(\tau_m)x(\chi_m{}^-)] \\ &\quad -\omega_m[x(\chi_m{}^-)]^T\mathcal{E}^TP[x(\chi_m{}^-)] \qquad (10) \\ &= [x^T(\chi_m)C_m(\tau_m)^T]\mathcal{E}^TP[C_m(\tau_m)x(\chi_m)] \\ &\quad -\omega_m[x^T(\chi_m)]\mathcal{E}^TP[x(\chi_m)] \\ &= x^T(\chi_m)(C_m(\tau_m)^T\mathcal{E}^TPC_m(\tau_m) - \omega_m\mathcal{E}^TP)x(\chi_m) \\ &= x^T(\chi_m)\Gamma x(\chi_m) \\ &\leq 0.\end{aligned}$$

Therefore, from (4) and by using simple induction, from (9) and (10), we have

$$V(x(t)) \leq V(x_0(t))\prod_{i=1}^{m}\omega_ie^{\zeta(t-t_0)}, \forall m \in Z_+. \tag{11}$$

By the Lyapunov-like property, there exists a positive definite symmetric matrix L such that $\mathcal{E}^TP = \mathcal{E}^TL\mathcal{E}$. Then, we have

$$\begin{aligned}&\lambda_{min}(L)\,\|\mathcal{E}x(t)\|^p \\ &\qquad \leq V(x(t)) \\ &\qquad \leq V(x_0(t))\prod_{i=1}^{m}\omega_ie^{\zeta(t-t_0)}.\end{aligned} \tag{12}$$

Hence, we obtain

$$\begin{aligned}\|\mathcal{E}x(t)\|^p &\leq \frac{\lambda_{max}(L)}{\lambda_{min}(L)}\,\|\mathcal{E}x(t_0)\|^p\prod_{i=1}^{m}\omega_ie^{\zeta(t-t_0)}, \\ &\leq \frac{\lambda_{max}(L)}{\lambda_{min}(L)}\,\|\mathcal{E}x_0\|^p\prod_{i=1}^{m}\omega_ie^{\zeta(t-t_0)},\end{aligned}$$

where $t \in [\chi_{m-1}, \chi_m), m \in Z_+$. This equation generates a stochastic process and it is defined by

$$\|\mathcal{E}x(t)\|^p \leq M\|\mathcal{E}x_0\|^p\prod_{i=1}^{m}\omega_ie^{\zeta(t-t_0)}, \chi_{m-1}' < t < \chi_m',$$

where $M = \frac{\lambda_{max}(L)}{\lambda_{min}(L)}$. Taking expectation, by using Lemma 1, and remark 2, we get

$$\begin{aligned}
&E\left[\|\mathcal{E}x(t)\|^p\right] \\
&= \sum_{m=0}^{\infty} E[\|\mathcal{E}x(t)\|^p | I_{[\chi'_m,\chi'_{m+1})}(t)] \\
&\quad \mathcal{P}(I_{[\chi'_{m-1},\chi'_m)}(t)), \\
&\leq ME[\|\mathcal{E}x_0\|^p] \sum_{m=0}^{\infty}\prod_{i=1}^{m} E\left[\omega_i\right] e^{\zeta(t-t_0)} \\
&\quad \mathcal{P}(I_{[\chi'_{m-1},\chi'_m)}(t)) \\
&= ME[\|\mathcal{E}x_0\|^p] \sum_{m=0}^{\infty}\prod_{i=1}^{m} E\left[\omega_i\right] e^{\zeta(t-t_0)} \\
&\quad \frac{\gamma^m(t-t_0)^m}{m!} e^{-\gamma(t-t_0)}, \\
&= ME[\|\mathcal{E}x_0\|^p] e^{\zeta(t-t_0)} \sum_{m=0}^{\infty} \frac{[\gamma\kappa]^m (t-t_0)^m}{m!} \\
&\quad e^{-\gamma(t-t_0)},
\end{aligned}$$

Hence,

$$E\left[\|\mathcal{E}x(t)\|^p\right] \leq ME[\|\mathcal{E}x_0\|^p] e^{[\zeta+\gamma(\kappa-1)](t-t_0)}, \quad (13)$$

where $\zeta + \gamma(\kappa - 1)$ is the convergent rate. This implies that the trivial solution of (1) is $\mathcal{E}$-exponentially stable. □

Corollary 1. *For system (1), its p^{th} moment $\mathcal{E}$-exponentially stability is equivalent to its p^{th} moment exponential stability and its satisfies $1 - 2^{p-1}E\|F\mathcal{Q}_2\|^p > 0$.*

Proof. The pair $(\mathcal{E}, A)$ is regular and impulse free, we introduce the coordinate transformation

$$x(t) = \mathcal{Q}\ col(x_1, x_2). \quad (14)$$

It follows that system (1) is equivalent to

$$\begin{aligned}
\dot{x_1} &= A_1 x_1 + \mathcal{G}_1 f(x(t), t), && (15)\\
&\quad \chi'_m < t < \chi'_{m+1}, t \geq t_0, \\
0 &= x_2 + \mathcal{G}_2 f(x(t), t), && (16)\\
&\quad \chi'_m < t < \chi'_{m+1}, t \geq t_0 \\
x(\chi'^{+}_m) &= C_m x(\chi'^{-}_m), m \in Z_+ && (17)\\
x_{t_0} &= x_0,
\end{aligned}$$

where $x_1 \in \Re^r, x_2 \in \Re^{n-r}$ and $\mathcal{G} = col(\mathcal{G}_1, \mathcal{G}_2), \mathcal{G}_1 \in \Re^{r\times n}, \mathcal{G}_2 \in \Re^{(n-r)\times n}$, $\mathcal{Q} = row(\mathcal{Q}_1, \mathcal{Q}_2) \in \Re^{n\times n}$, $\mathcal{Q}_1 \in \Re^{n\times r}$, $\mathcal{Q}_2 \in \Re^{n\times(n-r)}$. Hence,

$$\begin{aligned}
\mathcal{G}\mathcal{E}x(t) &= \mathcal{G}\mathcal{E}\mathcal{Q}\ col(x_1, x_2) \\
&= diag(I_r, 0) col(x_1, x_2) \\
&= col(x_1, 0) && (18)
\end{aligned}$$

From (13) and (18), we have

$$\begin{aligned}
E\|x_1\|^p &= E\|\mathcal{G}\mathcal{E}x\|^p \\
&\leq \|\mathcal{G}\|^p E\|\mathcal{E}x\|^p \\
&\leq \|\mathcal{G}\|^p ME[\|\mathcal{E}x_0\|^p] e^{[\zeta+\gamma(\kappa-1)](t-t_0)}.
\end{aligned} \quad (19)$$

Here we understood that the solution of the system (1) is p^{th} moment globally exponentially stable.

Now, It is necessary to prove that that x_2 is also exponentially stable. It follows from equation (3) and (17) that

$$\begin{aligned}
\|x_2\| &\leq \|\mathcal{G}_2\|\|f(x(t), t)\| \leq \|f(x(t), t)\| \\
&\leq \|Fx(t)\| = \|F\mathcal{Q}_1\ x_1 + F\mathcal{Q}_2\ x_2\| \\
&\leq \|F\mathcal{Q}_1\ x_1\| + \|F\mathcal{Q}_2\ x_2\| \\
&\leq \|F\mathcal{Q}_1\|\|x_1\| + \|F\mathcal{Q}_2\|\|x_2\|.
\end{aligned}$$

Thus, taking expectation and the p^{th} moment on both sides, we get

$$(1 - 2^{p-1}E\|F\mathcal{Q}_2\|)E\|x_2\|^p \leq 2^{p-1}E\|F\mathcal{Q}_1\|E\|x_1\|^p,$$

where $\mathcal{Q}$ is non singular matrix can be suitably taken to satisfy $1 - 2^{p-1}E\|F\mathcal{Q}_2\|^p > 0$. Therefore from (19),

$$\begin{aligned}
E\|x_2\|^p &\leq \frac{2^{p-1}E\|F\mathcal{Q}_1\|^p}{1 - 2^{p-1}E\|F\mathcal{Q}_2\|^p} E\|x_1\|^p \\
&\leq \frac{2^{p-1}E\|F\mathcal{Q}_1\|^p}{1 - 2^{p-1}E\|F\mathcal{Q}_2\|^p} \|\mathcal{G}\|^p ME[\|\mathcal{E}x_0\|^p] \\
&\quad e^{[\zeta+\gamma(\kappa-1)](t-t_0)}.
\end{aligned}$$

From (19) and the above equation, we conclude that the trivial solution of (1) is p^{th} moment exponentially stable. The proof is completed. □

When $f(x(t), t) = 0$, then the system (1) becomes a linear SDSs with random impulses. In this case, the following corollary can be easily obtained.

Corollary 2. *Let $\tau' = \max_{m\in Z_+}\left\{\chi'_m - \chi'_{m-1}\right\} < \infty$. Assume that system (1) with $f(x(t), t) = 0$ satisfies a Lyapunov-like property and there exists an invertible matrix P, and there exists positive constant $\kappa > 0$, $\omega_m > 0$, such that $E[\omega_m] \leq \kappa$, $\zeta < 0$, exponential distribution parameter γ and the following conditions hold,*

$$\begin{aligned}
&(A^T P + P^T A) < \zeta\mathcal{E}^T P, && (20)\\
&\Gamma = (C_m(\tau_m)^T \mathcal{E}^T P C_m(\tau_m) - \omega_m \mathcal{E}^T P) \leq 0 \\
&\zeta + \gamma(\kappa - 1) < 0.
\end{aligned}$$

Then, the trivial solution of system (1) is p^{th} moment $\mathcal{E}$-ES.

The proof is similar to the proof of Theorem 1 and hence it is omitted.

When $\mathcal{E} = I_n$, then the system (1) becomes a nonlinear state-space system with random impulses.

In this case, the following corollary can be easily obtained.

Corollary 3. *Let* $\tau^{'} = \max_{m \in Z_+} \left\{ \chi^{'}_m - \chi^{'}_{m-1} \right\} < \infty$. *Assume that system* (1) *with* $\mathcal{E} = I_n$ *satisfies a Lyapunov-like property and there exists a positive definite matrix* P, *and there exists positive constant* $\kappa > 0$, $\omega_m > 0$, *such that* $E[\omega_m] \leq \kappa$, $\zeta < 0$, $\epsilon > 0$, *exponential distribution parameter* γ *and the following conditions hold,*

$$(A^T P + P^T A) + \lambda_{max}(\frac{1}{\epsilon} F^T F + \epsilon P^T P) I < \zeta P,$$
$$\Gamma = (C_m(\tau_m))^T P C_m(\tau_m) - \omega_m P) \leq 0 \qquad (21)$$
$$\zeta + \gamma(\kappa - 1) < 0.$$

Then, the trivial solution of system (1) *is* p^{th} *moment ES.*

The proof is similar to the proof of Theorem 1 and hence it is omitted.

Remark 4. *From the condition* (5) *and* $\mathcal{E}^T P = P^T \mathcal{E} \geq 0$, *different matrices* P *can be chosen based on the matrices* $\mathcal{E}$, A *and* F.

Remark 5. *We carried out the following four conditions from the convergent rate* $\zeta + \gamma(\kappa - 1)$, *in Theorem 1,*

(i) *If* $\zeta < 0$ *in the inequality* $\dot{V}(x(t)) \leq \zeta V(x(t))$, *then the singular system* (1) *is stable. In this case, the impulsive strength* $\kappa \in (0, 1)$ *and the arrival rate of impulses do not necessarily satisfy any condition.*

(ii) *If* $\zeta < 0$ *in the inequality* $\dot{V}(x(t)) \leq \zeta V(x(t))$, *then the singular system* (1) *is stable. In this case, the system does not have an arrival rate of impulses when the impulsive strength* $\kappa = 1$.

(iii) *If* $\zeta < 0$ *in the inequality* $\dot{V}(x(t)) \leq \zeta V(x(t))$, *then the singular system* (1) *is stable. In this case, the arrival rate of impulses must be satisfied with this condition* $\gamma < \frac{-\zeta}{\kappa - 1}$, *where the impulsive strength* $\kappa > 1$.

4. Applications

In this section, numerical examples are discussed to support the proposed results. We illustrate the results by graphs to support the results.

Example 1. *Consider system* (1) *where*

$$\mathcal{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -0.3 & 0.1 & 0.1 \\ -1 & -3 & 1 \\ -0.6 & -1.5 & -2.5 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} -0.1 \\ 0.1 \\ 0.2 \end{bmatrix}, f(x(t), t) = \begin{bmatrix} \frac{1}{10\sqrt{3}} \tanh x_1(t) \\ \frac{1}{10\sqrt{3}} \tanh x_2(t) \\ \frac{1}{10\sqrt{3}} \tanh x_3(t), \end{bmatrix},$$

$$C_m(\tau_m) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

It is easy to verify that $\mathcal{E}^T P = P^T \mathcal{E} \geq 0$ *with* $P = I_3$ *and* $f(x(t), t)$ *satisfies the Lipschitz condition with* $F = \frac{1}{10\sqrt{3}} I$.

Here, $\zeta = -3$, $\epsilon = 0.05$ *with impulse arrival rate* $\gamma = 25, \kappa = 0.5$ *and* $\tau^{'} = \max_{m \in Z_+} \left\{ \xi^{'}_m - \xi^{'}_{m-1} \right\} = 0.026$, *then the conditions* (5) *in Theorem 1 are satisfied. Hence system* (1) *is* $\mathcal{E}$*-ES. Figure 1 illustrates the graphical behaviour of the solution. When there are no impulses, then the above system is unstable.*

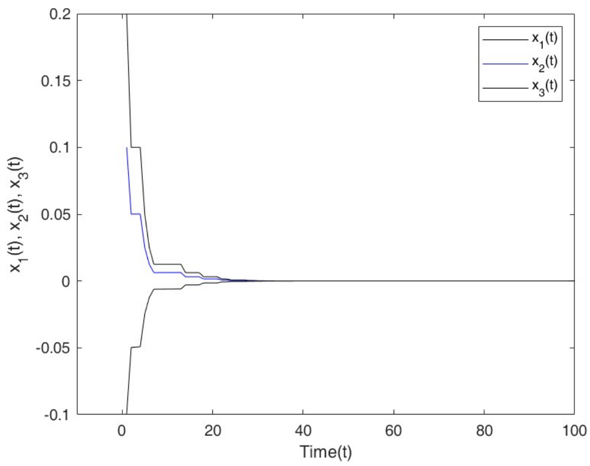

Figure 1. $\mathcal{E}$- Exponential stability.

Example 2. *Consider system* (1) *where*

$$\mathcal{E} = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}, A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},$$
$$C_m(\tau_m) = \begin{bmatrix} -0.7 & 0 \\ 0 & -0.5 \end{bmatrix}, x_0 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix},$$
$$f(x(t), t) = \begin{bmatrix} \frac{\sin x_1(t)}{4\sqrt{3}} \\ \frac{\sin x_2(t)}{4\sqrt{3}} \end{bmatrix}.$$

It is easy to verify that $\mathcal{E}^T P = P^T \mathcal{E} \geq 0$ *with* $P = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$ *and* $f(x(t), t)$ *satisfies the Lipschitz conditions with* $F = \frac{1}{4\sqrt{3}} I$.

Choose $p = 2$, $\mathcal{G} = \begin{bmatrix} 0.2 & 0.1 \\ -0.4 & 0.8 \end{bmatrix}$,
$\mathcal{Q} = \begin{bmatrix} 1 & 0 \\ 0.8 & -0.5 \end{bmatrix}$, *such that* $\mathcal{GEQ} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *and*
$\mathcal{GAQ} = \begin{bmatrix} -0.3 & 0 \\ 0 & 1 \end{bmatrix}$.

Hence, it is easy to verify that $\|\mathcal{G}_2\| \leq 1$ *and* $1 - E\|F\mathcal{Q}_2\|^2 > 0$.

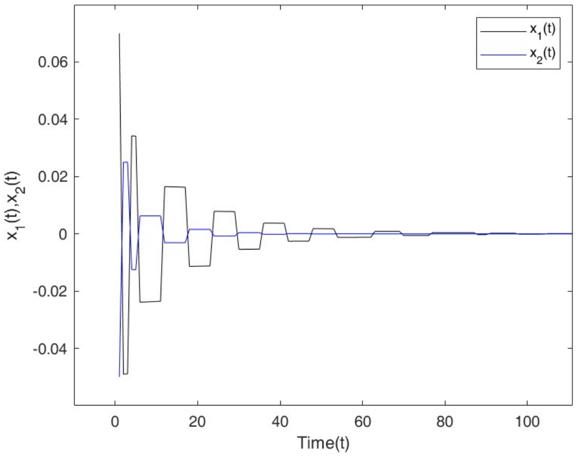

Figure 2. Exponential stability.

Then the singular system (1) *becomes*

$$\begin{aligned} \dot{x}_1(t) &= -0.3x_1(t) + 0.0404\frac{\sin x_1(t)}{4\sqrt{3}} \\ &\quad -0.0072\frac{\sin x_2(t)}{4\sqrt{3}}, \\ &\quad \chi_m^{'} < t < \chi_{m+1}^{'}, t \geq t_0, \end{aligned}$$

and

$$\begin{aligned} 0 &= x_2(t) + 0.0346\frac{\sin x_1(t)}{4\sqrt{3}} - 0.0577\frac{\sin x_2(t)}{4\sqrt{3}}, \\ &\quad \chi_m^{'} < t < \chi_{m+1}^{'}, t \geq t_0, \\ x(\chi_m^+) &= C_m(\tau_m)x(\chi_m^-), m \in Z_+. \end{aligned}$$

Choose $\zeta = -2$, $\gamma = 4$, *and* $\epsilon = 0.05$ *such that* $(A^TP + P^TA) + \lambda_{max}(\frac{1}{\epsilon}F^TF + \epsilon P^TP) - \zeta\mathcal{E}^TP < 0$. *Further, take* $\kappa = 1.5$ *and* $\tau^{'} = \max_{m\in Z_+}\left\{\xi_m^{'} - \xi_{m-1}^{'}\right\} = 0.026$, *then conditions* (5) *in Theorem* 1 *are satisfied. Hence system* (1) *is mean square ES. Figure 2 demonstrates the graphical behaviour of the solution. When there are no impulses, then the above system is unstable.*

Example 3. *Consider system* (1) *where*

$$\mathcal{E} = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 0 \end{bmatrix}, x_0 = \begin{bmatrix} -0.5+0.1i \\ -0.4+0.2i \\ 0.2+i \end{bmatrix},$$

$$A = \begin{bmatrix} -0.5+i & 0.2-0.3i & 0.1+0.3i \\ 0.2+0.5i & -1-0.5i & -0.1-i \\ -1.2+i & -0.4-0.3i & -0.2-0.5i \end{bmatrix},$$

$$f(x(t),t) = \begin{bmatrix} \frac{1}{2}(|x_1(t)+1| - |x_1(t)-1|) \\ \frac{1}{2}(|x_2(t)+1| - |x_2(t)-1|) \\ \frac{1}{2}(|x_3(t)+1| - |x_3(t)-1|) \end{bmatrix},$$

$$C_m(\tau_m) = \begin{bmatrix} 0.25+0.1i & 0 & 0 \\ 0 & 0.25+0.1i & 0 \\ 0 & 0 & 0.25+0.1i \end{bmatrix}.$$

It is easy to verify that $\mathcal{E}^TP = P^T\mathcal{E} \geq 0$ *with* $P = I_3$ *and* $f(x(t),t)$ *satisfies the Lipschitz condition with* $F = \frac{1}{2}I$.

Here, $\zeta = -5$, $\epsilon = 0.01$ *with impulse arrival rate* $\gamma = 10, \kappa = 0.7$ *and* $\tau^{'} = \max_{m\in Z_+}\left\{\xi_m^{'} - \xi_{m-1}^{'}\right\} = 0.005$, *then the conditions* (5) *in Theorem* 1 *are satisfied. Hence system* (1) *is* $\mathcal{E}$*-ES.*

Remark 6. *In the above example, we have proved that the results hold true even when the matrices involved have complex entries. However, the function* f *involved is still a real valued function.*

5. Conclusion

In this paper, we consider the exponential stability of random impulsive nonlinear singular differential system. It is worth mentioning that the system under consideration involves random impulses which may cause some technical difficulties comparing with systems with fixed impulses. Less restrictive conditions are established for the $\mathcal{E}$-ES and ES of the system. To support the theoretical findings, we give two numerical examples along with their graphical representations. We illustrate that the obtained results are consistent with the main theorem. We have additionally proved the truth of the results in case of matrices involving complex entries as well, while the function involved still remains real-valued. Proving the results true for complex valued functions could be considered to be a future problem. Moreover, as done in [11], we can consider analyzing a system with delay by reformulating it into a singular linear system of differential equations, as a future work. We believe that the results of this paper are of great significant for relevant community and can be used for instance to investigate switched singular time delay systems.

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Arumugam Vinodkumar *works as an Associate Professor in the Department of Mathematics, School of Physical Sciences, Amrita Vishwa Vidyapeetham, Coimbatore Campus. He completed his Ph.D. in Random impulsive Differential Systems and Stochastic Differential Systems. Now he is working on qualitative behaviors of the differential systems. He serves as a reviewer for some prestigious journals and societies.*
`https://orcid.org/0000-0002-5314-8768`

Sivakumar Harinie *is currently working as an Amazon and Key Accounts Analyst at Hatley Little Blue House Inc, Montreal, Canada. She completed her Master's Degree, MSc in Mathematics, at the School of Physical Sciences, Amrita Vishwa Vidyapeetham, Coimbatore Campus. Additionally, she has completed a Master's Program in Data Analyst and Data Scientist at Simplilearn, Bangalore.*
`https://orcid.org/0009-0003-3172-3869`

Michal Fečkan *has been a Professor of Mathematics at the Department of Mathematical Analysis and Numerical Mathematics in the Faculty of Mathematics, Physics, and Informatics at Comenius University in Bratislava, Slovakia, since 2003. He received his Master's degree from Comenius University in Bratislava in 1985 and his Ph.D. from the Mathematical Institute of the Slovak Academy of Sciences in Bratislava,*

Slovakia, in 1993. He is interested in nonlinear functional analysis, bifurcation theory, dynamical systems, and fractional calculus with applications to mechanics, vibrations, and economics. He is a Highly Cited Researcher in Mathematics.
https://orcid.org/0000-0002-7385-6737

Jehad Alzabut *is a professor of Applied Mathematics. He received his Ph.D. degree from Middle East Technical University, Turkey. His area of interest is qualitative properties of delay, difference, fractional, and impulsive differential equations. He has particular interest in mathematical models describing biological and medical phenomena. He serves the role of an editor and a reviewer for some prestigious journals and societies.*
https://orcid.org/0000-0002-5262-1138

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