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RESEARCH ARTICLE

Regional enlarged controllability of a fractional derivative of an output linear system

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ABSTRACT

This new research aims to extend the topic of the enlarged controllability of a fractional output linear system. Thus, we characterize the optimal control by two methods, ensuring that the Riemann-Liouville fractional derivative of the final state of the considered system lies between two given functions on a subregion of the evolution domain. Firstly, we transform the considered problem into the saddle point using the Lagrangian multiplier approach. Then, in the second one, we provide the technique of the subdifferential, which allows us to present the cost-explicit formula of the minimum energy control. Moreover, we construct an algorithm of Uzawa type to illustrate the theoretical results obtained through numerical simulations.

1. Introduction

The concept of fractional calculus has attracted increasing attention from many researchers, and it was introduced in the 19th century by Riemann, Liouville, and Letnikov. Their objective was to extend classic differentiation and integration using non-integer orders, they have been used in mechanics since the 1930s and later in electrochemistry in the 1960s (see [1]). In addition, the integer derivative of a function φ at a point x_0 remains a local property. However, the fractional-order differentiation of a function φ at x_0 depends on all values of φ, including those that are not in the neighborhood of x_0.

The regional controllability is a crucial and modern topic in advancing control theory and engineering. It is a qualitative property of controlled systems and has an exceptional property in control theory. The last notion is the basis of a mathematical description of a dynamical system, which is also related to the realization theory of quadratic optimality in linear time-invariant controlled systems.

The problem of regional controllability involves determining whether it is possible to find a control that can bring the state of a system from its initial state to the desired state exclusively within a subregion ω at a finite moment. The concept of regional controllability for distributed systems was introduced in the 1990s by Professors El Jai and Zerrik (see [2–4]), in which it was possible to study the idea only on a subregion ω of the domain Ω.

This topic has admitted many applications and has led to crucial results such as the possibility of reaching a state of the system only on an internal subregion ω of Ω or on a subregion of the boundary $\partial\Omega$ of Ω (see [5], [6]). Also, the problem of driving a system to a state between two known functions is well detailed in [7]. Furthermore, in [8], they have investigated and developed the problem of the regional controllability of the gradient state. This problem involves directing the state gradient of the considered system towards a specified function that is only defined in the domain subset $\omega \subset \Omega$. Furthermore, authors

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have examined a problem of regional gradient controllability, which is emphasized by concentrating on a control that would realize a given final gradient on ω with minimum energy (see [9]). Finally, in [10], the authors have proved the fractional controllability of linear hyperbolic systems by using an extension of HUM.

As the optimal linear filter and estimator, the Kalman Filter design for linear infinite-dimensional systems has been widely employed for state estimation and prediction in the realm of lumped parameter systems (see [11]). Besides that, fractional derivatives have been applied to the modelling of combustion processes, offering unique insights into the dynamics and features of these systems. They allow us to characterize processes that involve under- or superdiffusion, where the diffusion rate does not follow the classical diffusion equations. Our problematic is about studying the regional controllability of the fractional state of the considered system. In particular, if $\omega = \Omega$ and $\alpha = 0$, we obtain global enlarged controllability over the evolution domain. On the other hand, we achieve enlarged regional controllability of the system's state gradient with $\alpha = 1$ in all parts of ω within Ω. Hence, we show that the obtained control allows us to generalize the latter cases using the concept of the fractional derivative of order $\alpha \in [0,\ 1]$. In order to solve this problem, we employ the approaches of subdifferential and Lagrangian to determine the optimal control that steers the fractional derivative of an output of the considered system between two known functions on subregion ω in the interior of Ω as shown in the following figure: (e.g. see Figure 1).

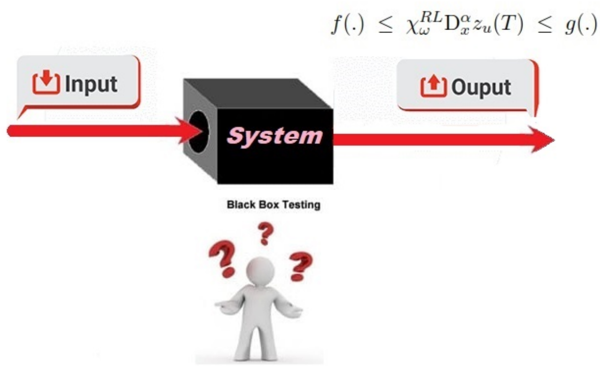

Figure 1. The goal of this research.

The structure of this research is as follows. Section 2 is devoted to recalling some definitions and the statement of the considered problem. In Section 3, we use two procedures, one based on subdifferential tools and the other on the Lagrangian approach, which allows us to determine the explicit formula of optimal control. Finally, the theoretical results achieved are illustrated through numerical simulations by applying an algorithm to the one-dimensional diffusion equation.

2. Problem statement

Let Ω be an open bounded subset of $\mathbb{R}^n$ with a boundary $\partial\Omega$. For $T > 0$ we denote $Q = \Omega \times]0, T[$.
Let's consider the linear system with internal control described by:

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = Az(x,t) + Bu(t) & Q \\ z(\eta,t) = 0 & \partial\Omega \times]0,T[\\ z(x,0) = z_0(x) & \Omega \end{cases} \tag{1}$$

where A generates a C_0-semigroup S(t), $t \geq 0$ in $H_0^1(\Omega)$ and B $\in \mathcal{L}(\mathbb{R}^p, H_0^1(\Omega))$, u $\in \mathbb{U} = L^2(0,T;\mathbb{R}^p)$ and $z_0 \in H_0^1(\Omega)$.

- The problem (1) admits a unique solution $z_u(.)$ such that $z_u(T) \in H_0^1(\Omega)$ and given by the variation of constants formula (see [12], page 106)

$$z_u(t) = S(t)z_0 + \int_0^t S(t-r)Bu(r)dr.$$

- The operator of controllability L_T is defined by:

$$\begin{aligned} L_T : \mathbb{U} &\rightarrow H_0^1(\Omega) \\ u &\mapsto \int_0^T S(T-t)Bu(t)dt \end{aligned}$$

and its adjoint $L_T^* z = B^* S^*(T-.)z$.
- Let ${}^{RL}\mathrm{D}_x^\alpha : H_0^1(\Omega) \rightarrow L^2(\Omega)$ the fractional Riemann-Liouville operator of order α and $({}^{RL}\mathrm{D}_x^\alpha)^*$ its adjoint (see [13]).
- Consider ω as a subregion of Ω. Let $\chi_\omega : L^2(\Omega) \rightarrow L^2(\omega)$ be the restriction operator to ω. The adjoint operator of χ_ω is denoted by χ_ω^* and is given by

$$(\chi_\omega^* z)(x) = \begin{cases} z(x), & x \in \omega, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1. *(see [1] and [14]) Let $\Re(\alpha) > 0$ and $\psi : [a,b) \rightarrow \mathbb{R}$ be continuous and integrable. For $x > a$, we call*

$$\mathrm{I}_a^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\psi(t)dt. \tag{2}$$

the Riemann-Liouville fractional integral of ψ of order α

Definition 2. *(see [1] and [14]) Let α such that $0 \leq \alpha < 1$.*

The fractional derivative of Riemann-Liouville of order α of a function ψ is given by:

$$\begin{aligned} {}^{RL}\mathrm{D}_x^{\alpha}\psi(x) &= \frac{d}{dx}\mathrm{I}_a^{1-\alpha}\psi(x) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}\psi(t)dt. \end{aligned} \tag{3}$$

• Let $f,\ g \in L^2(\omega)$ with $f(.) \le g(.)$ a.e in ω. In all the following we set:

$$[f(.),\ g(.)] = \left\{ \begin{array}{l} \chi_\omega^{RL}\mathrm{D}_x^{\alpha} z \in L^2(\omega)\ / \\ f(.) \le\ \chi_\omega^{RL}\mathrm{D}_x^{\alpha} z \le g(.)\ a.e.\ on\ \omega \end{array} \right\}$$

Definition 3. *(General definition of old one [8]) System (1) is said to be $[f(.),\ g(.)]$-controllable on ω, if there exists $u \in \mathbb{U}$ such that*

$$f(.) \le \chi_\omega^{RL}\mathrm{D}_x^{\alpha} z_u(T) \le g(.) \quad a.e\ on\ \omega.$$

Definition 4. *(General definition of old one [8]) We say that the actuator $(D,\ h)$ is $[f(.),\ g(.)]$-strategic on ω, if the excited system is $[f(.),\ g(.)]$-controllable on ω.*

3. Minimization problem

In this section, we exploit two methods to find a control with minimum energy that allows driving the system (1) from z_0 to the fractional output between $f(.)$ and $g(.)$ on ω. Later, let's consider the following minimization problem:

$$\left\{ \begin{array}{l} \min \frac{1}{2}\|u\|^2 \\ u \in \mathcal{U}_{ad} \end{array} \right. \tag{4}$$

where the set of admissible controls is given by

$$\mathcal{U}_{ad} = \left\{ \begin{array}{l} u \in \mathbb{U}\ / \\ f(.) \le\ {}^{RL}\mathrm{D}_x^{\alpha} z_u(T) \le g(.)\ a.e.\ on\ \omega \end{array} \right\}.$$

Proposition 1. *Problem (4) has a unique solution if the system (1) is $[f(.),\ g(.)]$-controllable on ω.*

Proof. By hypothesis, system (1) is $[f(.),\ g(.)]$-controllable on ω, then $\mathcal{U}_{ad} \neq \emptyset$. Moreover, $u \to \frac{1}{2}\|u\|^2$ is strictly convex and lower semi-continuous in $\mathbb{U}$. As result, it suffices to verify that $\mathcal{U}_{ad}$ is a closed convex set of $\mathbb{U}$.

We can deduce the convexity of $\mathcal{U}_{ad}$ from the linearity of the map $u \to \chi_\omega^{RL}\mathrm{D}_x^{\alpha} z_u(T)$.

Now, we show that $\mathcal{U}_{ad}$ is closed. Let $(u_n)_n$ in $\mathcal{U}_{ad}$ such that $u_n \to u$ strongly in $\mathbb{U}$. Since that $\chi_\omega^{RL}\mathrm{D}_x^{\alpha} L_T$ is continuous, then $\chi_\omega^{RL}\mathrm{D}_x^{\alpha} L_T u_n \to \chi_\omega^{RL}\mathrm{D}_x^{\alpha} L_T u$ strongly in $L^2(\omega)$, we know that $\chi_\omega^{RL}\mathrm{D}_x^{\alpha} z_{u_n} \in [f(.),\ g(.)]$ which is closed, then $\chi_\omega^{RL}\mathrm{D}_x^{\alpha} z_u \in [f(.),\ g(.)]$. We deduce that $u \in \mathcal{U}_{ad}$.

Consequently, $\mathcal{U}_{ad}$ is closed.

Therefore, problem (4) admits a unique solution. □

We will provide two methods to characterize the optimal control solution of (4) in the later subsections.

3.1. First method: Subdifferential method

In this subsection, we provide an expression that characterizes the solution to the problem (4) using the subdifferential approach.

Problem (4) is equivalent to solve the following problem without fractional constraints:

$$\left\{ \begin{array}{l} min\left(\frac{1}{2}\|u\|^2 +\ \Psi_{\mathcal{U}_{ad}}(u)\right) \\ u \in \mathbb{U} \end{array} \right. \tag{5}$$

where, for a nonempty subset F of $\mathbb{U}$, we have

$$\Psi_F(u) = \left\{ \begin{array}{ll} 0\ if & u \in F \\ +\infty & otherwise, \end{array} \right. \tag{6}$$

the indicator function of F.

• Let

$$\Sigma(\mathbb{U}) = \left\{ \begin{array}{l} \sigma : \mathbb{U} \to]-\infty,\ +\infty],\ convex\ proper \\ and\ lower\ semi-continuous\ on\ \mathbb{U} \end{array} \right\}.$$

• Let $\sigma \in \Sigma(\mathbb{U})$, $\mathrm{dom}(\sigma) = \{u \in \mathbb{U}\ /\ \sigma(u) < \infty\}$ and σ^* is the polar function of σ defined by:

$$\sigma^*(v^*) = \sup_{u \in dom(\sigma)} \{\langle v^*,\ u\rangle - \sigma(u)\} \quad \forall v^* \in \mathbb{U}.$$

Definition 5. *(see [15]) The set of subgradients of σ at $u_0 \in \mathbb{U}$ is called the subdifferential of σ at u_0. We denote it as follows:*

$$\partial\sigma(u_0) = \left\{ \begin{array}{l} v^* \in \mathbb{U}\ / \\ \sigma(u) \ge \sigma(u_0) + \langle v^*,\ u - u_0\rangle\ \forall\ u \in \mathbb{U} \end{array} \right\}.$$

The following result characterizes the solution to the problem (5):

Proposition 2. *Assume that system (1) is $[f(.),\ g(.)]$-controllable on ω, then $u^\star$ is the solution of Equation (5) if and only if*

$$u^\star \in \mathcal{U}_{ad} \quad and \quad \Psi^*_{\mathcal{U}_{ad}}(-u^\star) = -\|u^\star\|^2. \tag{7}$$

Proof. By the properties of the subdifferential, we deduce that $u^\star$ is a solution of (5) if and only if $0 \in \partial(\sigma + \Psi_{\mathcal{U}_{ad}})(u^\star)$.

Therefore, $\sigma(u) = \frac{1}{2}\|u\|^2 \in \Sigma(\mathbb{U})$, and $\mathcal{U}_{ad}$ is closed, convex not empty, then $\Psi_{\mathcal{U}_{ad}} \in \Sigma(\mathbb{U})$. In addition, system (1) is $[f(.),\ g(.)]$-controllable on ω and $dom(\sigma) \cap dom(\Psi_{\mathcal{U}_{ad}}) \neq \emptyset$. However, σ is continuous, where

$$\partial\left(\sigma + \Psi_{\mathcal{U}_{ad}}\right)(u^\star) = \partial\sigma(u^\star) + \partial\Psi_{\mathcal{U}_{ad}}(u^\star).$$

Consequently, $u^\star$ is the solution of Equation (5) if and only if $0 \in \partial\sigma(u^\star) + \partial\Psi_{\mathcal{U}_{ad}}(u^\star)$.

On the other hand, we know that σ is Freshet-differentiable, then $\partial\sigma(u^\star) = \{\nabla\sigma(u^\star)\} = \{u^\star\}$. We conclude that $u^\star$ is the solution of (5) if and only if $-u^\star \in \partial\Psi_{\mathcal{U}_{ad}}(u^\star)$, one has

$$\begin{aligned}\Psi_{\mathcal{U}_{ad}}(u) &\geq \Psi_{\mathcal{U}_{ad}}(u^\star) + \langle u^\star,\ u - (-u^\star)\rangle \\ \Leftrightarrow 0 &\geq \Psi_{\mathcal{U}_{ad}}(u^\star) + \langle u^\star,\ u - (-u^\star)\rangle - \Psi_{\mathcal{U}_{ad}}(u) \\ 0 &= \Psi_{\mathcal{U}_{ad}}(u^\star) + \|u^\star\|^2 + \sup_{u\in\mathcal{U}_{ad}} \{\langle u^\star,\ u\rangle \\ &\quad - \Psi_{\mathcal{U}_{ad}}(u)\}.\end{aligned}$$

Then, $u^\star \in \mathcal{U}_{ad}$ and $\Psi_{\mathcal{U}_{ad}}(u^\star) + \Psi^*_{\mathcal{U}_{ad}}(-u^\star) = -\|u^\star\|^2$. We know that $u^\star \in \mathcal{U}_{ad}$, so $\Psi_{\mathcal{U}_{ad}}(u^\star) = 0$. Finally, we obtain that $u^\star \in \mathcal{U}_{ad}$ and $\Psi^*_{\mathcal{U}_{ad}}(-u^\star) = -\|u^\star\|^2$. $\square$

We put $\alpha(.) = f(.) - \chi_\omega^{RL}\mathrm{D}_x^\alpha S(T)z_0$ and $\beta(.) = g(.) - \chi_\omega^{RL}\mathrm{D}_x^\alpha S(T)z_0$, then

$$\mathcal{U}_{ad} = \left\{u \in \mathbb{U}\ /\ \chi_\omega^{RL}\mathrm{D}_x^\alpha L_T u \in [\alpha(.),\ \beta(.)]\right\}.$$

As a result, we get the following:

Proposition 3. *$u^\star$ is the solution of Equation (5) if and only if*

$$\begin{aligned}&\min\left\{\begin{array}{l}\left\langle(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\alpha(.),\ u^\star\right\rangle, \\ \left\langle(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\beta(.),\ u^\star\right\rangle\end{array}\right\} \\ &= \|u^\star\|^2,\end{aligned} \tag{8}$$

where the pseudo-inverse operator of $\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T$ is given by (see [16]):

$$\begin{aligned}&(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger \\ &= (\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^*\left((\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^*\right)^{-1}.\end{aligned}$$

Proof. We have

$$\mathcal{U}_{ad} = (\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\left([\alpha(.),\ \beta(.)]\right).$$

Applying the proposition 2, we get $u^\star$ which is the solution of (5) if and only if $u^\star \in \mathcal{U}_{ad}$ and $\Psi^*_{\mathcal{U}_{ad}}(-u^\star) = -\|u^\star\|^2$.

In addition, for all $u^\star \in \mathbb{U}$, we have

$$\begin{aligned}\Psi^*_{\mathcal{U}_{ad}}(-u^\star) &= \sup_{v\in\mathbb{U}}\left\{\langle -u^\star,\ v\rangle - \Psi_{\mathcal{U}_{ad}}(v)\right\}, \\ &= \sup_{v\in\mathcal{U}_{ad}}\langle -u^\star,\ v\rangle = -\inf_{v\in\mathcal{U}_{ad}}\langle u^\star,\ v\rangle, \\ &= -\inf_{v\in(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger([\alpha(.),\ \beta(.)])}\langle u^\star,\ v\rangle \\ &= -\inf_{z\in[\alpha(.),\ \beta(.)]}\left\langle u^\star,\ (\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger z\right\rangle \\ &= -\inf_{\lambda\in[0,1]}\left\langle\left((\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\right)^* u^\star, \lambda\alpha(.) + (1-\lambda)\beta(.)\right\rangle.\end{aligned}$$

The mapping

$$L : [0,\ 1] \ \rightarrow\ \mathbb{R}$$

$$L(\lambda) = \left\langle\left((\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\right)^* u^\star,\ \lambda\alpha(.) + (1-\lambda)\beta(.)\right\rangle,$$

is convex and continuous, using the Krein-Milman Theorem (see [17], page 362), we obtain

$$\begin{aligned}&\Psi^*_{\mathcal{U}_{ad}}(-u^\star) = \\ &-\inf_{\lambda\in\{0,1\}}\left\langle\left((\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\right)^* u^\star, \lambda\alpha(.) + (1-\lambda)\beta(.)\right\rangle,\end{aligned}$$

from (7), we conclude that

$$\begin{aligned}&\Psi^*_{\mathcal{U}_{ad}}(-u^\star) = \\ &= -\min\left\{\begin{array}{l}\left\langle u^\star,\ (\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\alpha(.)\right\rangle, \\ \left\langle u^\star,\ (\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^\dagger\beta(.)\right\rangle\end{array}\right\} \\ &= -\|u^\star\|^2.\end{aligned}$$

$\square$

3.2. Second method: Lagrangian multiplier method

Problem (4) is equivalent to solve the coming problem:

$$\left\{\begin{array}{l}\min \dfrac{1}{2}\|u\|^2 \\ (u,\ y) \in \mathcal{V}\end{array}\right. \tag{9}$$

where

$$\mathcal{V} = \left\{\begin{array}{l}(u,\ y) \in \mathbb{U}\times[f(.),\ g(.)]\ / \\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_u(T) - y = 0\end{array}\right\}.$$

We define the assistant variable $y \in [f(.),\ g(.)]$ related to u by equation $\chi_\omega^{RL}\mathrm{D}_x^\alpha z_u(T) - y = 0$. We transform problem (9) into a saddle point problem using the Lagrange multiplier.

Definition 6. *(see [8]) We call the Lagrangian associated with problem (9) the function $\mathcal{L}$ defined by: $\forall\ (u,\ y,\ \mu) \in \mathbb{U}\times[f(.),\ g(.)]\times L^2(\omega)$,*

$$\mathcal{L}(u,\ y,\ \mu) = \frac{1}{2}\|u\|^2 + \langle\mu,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_u(T) - y\rangle_{L^2(\omega)}.$$

Definition 7. *(see [8]) We say that $(u^\star,\ y^\star,\ \mu^\star)$ is a saddle point of $\mathcal{L}$ if*

$$\begin{aligned}\max_{\mu\in L^2(\omega)}\mathcal{L}(u^\star,\ y^\star,\ \mu) &= \mathcal{L}(u^\star,\ y^\star,\ \mu^\star) \\ &= \min_{u\in\mathbb{U},\ y\in[f(.),\ g(.)]}\mathcal{L}(u,\ y,\ \mu^\star).\end{aligned}$$

Suppose that system (1) is excited by a zone actuator $(D,\ h)$. Then, we consider the problem (4) and can characterize its solution by the following result:

Proposition 4. *If the actuator $(D,\ h)$ is $[f(.),\ g(.)]$-strategic on ω, then the solution of (4) is characterized by*

$$u^\star = -(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^*\mu^\star \tag{10}$$

whither $\mu^\star$ verifies

$$\left\{\begin{array}{l}G_{\alpha,\omega}\mu^\star + y^\star = 0 \\ y^\star = \mathcal{P}_{[f(.),\ g(.)]}(r\mu^\star + y^\star)\end{array}\right. \tag{11}$$

where $\mathcal{P}_{[f(.),\ g(.)]} : L^2(\Omega) \to [f(.),\ g(.)]$ *designates the projection operator,* $G_{\alpha,\omega} = \left(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T\right)\left(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T\right)^*$ *and* $r > 0$.

Proof. Suppose that the actuator $(D,\ h)$ is $[f(.),\ g(.)]$-strategic on ω, then $\mathcal{U}_{ad} \neq \emptyset$ and (4) has a unique solution.

It's clear that $\mathbb{U} \times [f(.),\ g(.)]$ is nonempty and closed convex. Moreover, we know that the function $\mu \to \mathcal{L}(u,\ y,\ \mu)$ is differentiable, concave, and upper semi-continuous. Likewise the function $(u,\ y) \to \mathcal{L}(u,\ y,\ \mu)$ is differentiable, convex and lower semi-continuous.

We deduce that there exists $\mu_0 \in L^2(\omega)$ and $(u_0,\ y_0) \in \mathbb{U} \times [f(.),\ g(.)]$ such that

$$\lim_{\|(u,\ y)\| \to +\infty} \mathcal{L}(u,\ y,\ \mu_0) = +\infty, \tag{12}$$

and

$$\lim_{\|\mu\| \to +\infty} \mathcal{L}(u_0,\ y_0,\ \mu) = -\infty. \tag{13}$$

As a result, $\mathcal{L}$ possesses a saddle point.
In the following, assume that $(u^\star,\ y^\star,\ \mu^\star)$ is a saddle point of $\mathcal{L}$ and prove that $u^\star$ is a solution of (4).
Now, for all $(u,\ y,\ \mu) \in \mathbb{U} \times [f(.),\ g(.)] \times L^2(\omega)$, we have

$$\mathcal{L}(u^\star,\ y^\star,\ \mu) \ \leq\ \mathcal{L}(u^\star,\ y^\star,\ \mu^\star) \ \leq\ \mathcal{L}(u,\ y,\ \mu^\star).$$

The inequality one gives

$$\langle \mu,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_{u^\star}(T) - y^\star\rangle \leq \langle \mu^\star,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_{u^\star}(T) - y^\star\rangle, \quad \forall\ \mu \in L^2(\omega),$$

means that $\chi_\omega^{RL}\mathrm{D}_x^\alpha z_{u^\star}(T) = y^\star$. Consequently, $\chi_\omega^{RL}\mathrm{D}_x^\alpha z_{u^\star}(T) \in [f(.),\ g(.)]$.
Using the second inequality, we obtain

$$\begin{aligned} \frac{1}{2}\|u^\star\|^2 + \langle \mu^\star,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_{u^\star}(T) - y^\star\rangle \\ \leq \frac{1}{2}\|u\|^2 + \langle \mu^\star,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_u(T) - y\rangle, \\ \forall (u,\ y) \in \mathbb{U} \times [f(.),\ g(.)]\,. \end{aligned}$$

Since $\chi_\omega^{RL}\mathrm{D}_x^\alpha z_{u^\star}(T) = y^\star$, we will have

$$\begin{aligned} \frac{1}{2}\|u^\star\|^2 \ \leq\ \frac{1}{2}\|u\|^2 + \langle \mu^\star,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_u(T) - y\rangle, \\ \forall\ (u,\ y) \in \mathbb{U} \times [f(.),\ g(.)]\,. \end{aligned}$$

For $\chi_\omega^{RL}\mathrm{D}_x^\alpha z_u(T) = y$, we get $\frac{1}{2}\|u^\star\|^2 \ \leq\ \frac{1}{2}\|u\|^2$.
Therefore, $u^\star$ is of minimum energy.
On the other hand, if $(u^\star,\ y^\star,\ \mu^\star)$ is a saddle point of $\mathcal{L}$, then the following assumptions are satisfied:

$$\langle u^\star,\ u - u^\star\rangle + \langle \mu^\star,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha L_T(u - u^\star)\rangle = 0, \quad \forall u \in \mathbb{U} \tag{14}$$

$$-\langle \mu^\star,\ y - y^\star\rangle \geq 0, \quad \forall y \in [f(.),\ g(.)] \tag{15}$$

$$\langle \mu - \mu^\star,\ \chi_\omega^{RL}\mathrm{D}_x^\alpha z_{u^\star}(T) - y^\star\rangle = 0, \quad \forall \mu \in L^2(\omega). \tag{16}$$

From the equation (14) gives (10).
Then, using (16), we get $\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T(u^\star) = y^\star$.
Hence, with (10,) we deduce (11).
Applying inequality (15), we get

$$\langle (r\mu^\star + y^\star) - y^\star,\ y - y^\star\rangle \ \leq\ 0, \quad \forall y \in [f(.),\ g(.)]$$

and $r > 0$, that is equivalent to

$$y^\star = \mathcal{P}_{[f(.),\ g(.)]}(r\mu^\star + y^\star).$$

□

Corollary 1. *If system (1) is* $[f(.),\ g(.)]$*-controllable on* ω*, then* $(y^\star,\ \mu^\star)$ *is a unique solution of system (11), where* $r > 0$ *is suitably chosen.*

Proof. Assume that system (1) is $[f(.),\ g(.)]$-controllable, implies that $(\chi_\omega^{RL}\mathrm{D}_x^\alpha L_T)^*$ and $G_{\alpha,\omega}$ are one to one. In addition, if $(u^\star,\ y^\star,\ \mu^\star)$ is a saddle point of $\mathcal{L}$, we deduce then system (11) is equivalent to

$$\begin{cases} \mu^\star = -G_{\alpha,\omega}^{-1} y^\star \\ y^\star = \mathcal{P}_{[f(.),\ g(.)]}(-rG_{\alpha,\omega}^{-1} y^\star + y^\star). \end{cases} \tag{17}$$

Therefore, $y^\star$ is a fixed point of

$$\begin{array}{rccl} N_r : & [f(.),\ g(.)] & \to & [f(.),\ g(.)] \\ & x & \mapsto & \mathcal{P}_{[f(.),\ g(.)]}(-rG_{\alpha,\omega}^{-1} x + x), \end{array}$$

since that the operator $G_{\alpha,\omega}^{-1}$ is coercive, which means

$$\exists\ k \geq 0 \quad such \quad that \quad \langle G_{\alpha,\omega}^{-1} x,\ x\rangle \geq k\|x\|^2.$$

Hence,

$$\begin{aligned} &\|N_r(x) - N_r(y)\|^2 \\ &= \|\mathcal{P}_{[f(.),\ g(.)]}(-rG_{\alpha,\omega}^{-1} x + x) \\ &- \mathcal{P}_{[f(.),\ g(.)]}(-rG_{\alpha,\omega}^{-1} y + y)\|^2 \\ &= \ \|(-rG_{\alpha,\omega}^{-1} x + x) - (-rG_{\alpha,\omega}^{-1} y + y)\|^2 \\ &= \ \|(-rG_{\alpha,\omega}^{-1}(x - y)) + (x - y)\|^2 \\ &= |\ \left\langle -rG_{\alpha,\omega}^{-1}(x - y),\ -rG_{\alpha,\omega}^{-1}(x - y)\right\rangle \\ &- 2r\left\langle G_{\alpha,\omega}^{-1}(x - y),\ x - y\right\rangle + \langle x - y,\ x - y\rangle\ | \\ &\leq\ (1 + r^2\|G_{\alpha,\omega}^{-1}\|^2 - 2rk)\|x - y\|^2, \\ &\forall x,\ y \in [f(.),\ g(.)]\,. \end{aligned}$$

If we chose $r < \dfrac{2k}{\|G_{\alpha,\omega}^{-1}\|^2}$, we conclude that N_r is a contraction, which implies that $y^\star$ and $\mu^\star$ are unique. □

4. Applications and simulations

In this section, we solve the equations (10) and (11) numerically and propose an Uzawa-type algorithm to evaluate the effectiveness of the Lagrangian method (see [18], page 3).

4.1. Algorithm

step 1: Initial data: Ω, zone of action D, subregion ω, precision threshold ε is sufficiently small and a fractional order α.
step 2: Initiate two functions $(y_0,\ \mu_1) \in [f(.),\ g(.)] \times L^2(\omega)$.
step 3: $(y_{n-1},\ \mu_n)$ is known, we determine u_n and y_n by the equations

$$u_n(t) = -\sum_{k=1}^{\infty} e^{\lambda_k(T-t)} \left(\int_D \varphi_k(x)dx\right) \times \left(\int_\Omega \chi_\omega^{RL}\mathrm{D}_x^\alpha \varphi_k(x)\mu_n(x)dx\right), \tag{18}$$

$$y_n(x) = \begin{cases} f(x) & if\ r\mu_n(x) + y_{n-1}(x) \leq f(x) \\ r\mu_n(x) + y_{n-1}(x) & if\ f(x) \leq r\mu_n(x) + y_{n-1}(x) \leq g(x) \\ g(x) & if\ r\mu_n(x) + y_{n-1}(x) \geq g(x). \end{cases} \tag{19}$$

step 4: While $\|y_n - y_{n-1}\|_{L^2(\omega)} > \varepsilon$,

$$\mu_{n+1}(x) = \mu_n(x) + \sum_{k=1}^{\infty} \left(\int_D \varphi_k(x)dx\right) \chi_\omega^{RL}\mathrm{D}_x^\alpha \varphi_k(x) \times \int_0^T e^{\lambda_k(T-t)} u_n(t)dt - y_n(x), \tag{20}$$

and return to step 3.
Where $(\varphi_n)_{n\in\mathbb{N}}$ is a complete basis of eigenfunctions of A in $H^1(\Omega)$ associated with the eigenvalues λ_n.

4.2. Simulations

This part aims to test the effectiveness of the Lagrangian approach through numerical simulations.
Example 1:
Let $\Omega =]0,\ 1[$ and consider the ensuing system:

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \dfrac{\partial^4 z}{\partial x^4}(t,x) + \mathcal{X}_D u(t), & \Omega \times]0,\ T[, \\ z(0,x) = 0, & x \in \Omega, \\ z(t,0) = z(t,1) = 0, & t \in]0,\ T[, \\ \dfrac{\partial^2 z}{\partial x^2}(t,0) = \dfrac{\partial^2 z}{\partial x^2}(t,1) = 0, & t \in]0,\ T[, \end{cases} \tag{21}$$

taking T $= 2$ and the actuator is located at D.
Let $f(x) = \dfrac{1}{2}x^2(1-x)$ and $g(x) = 4x^2(1-x^3)$.
The operator $Az = \dfrac{\partial^4 z}{\partial x^4}$ admits a complete set of eigenfunctions

$$\varphi_n(x) = \sqrt{2}\sin(n\pi x)$$

and the associated eigenvalues $\lambda_n = -n^4\pi^4$. Applying the above Algorithm, the simulations give the following results.

4.3. First case: $\omega =]0.3,\ 0.9[$

▷ Zone of action $D =]0.4,\ 0.8[$.

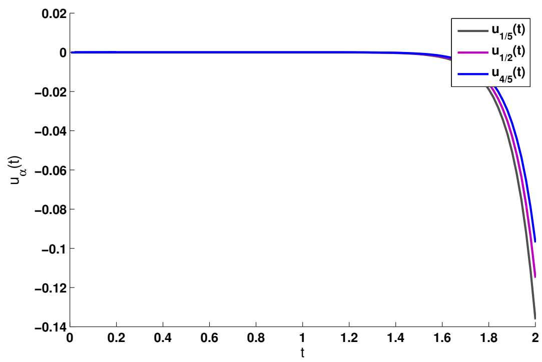

Figure 2. Control Function.

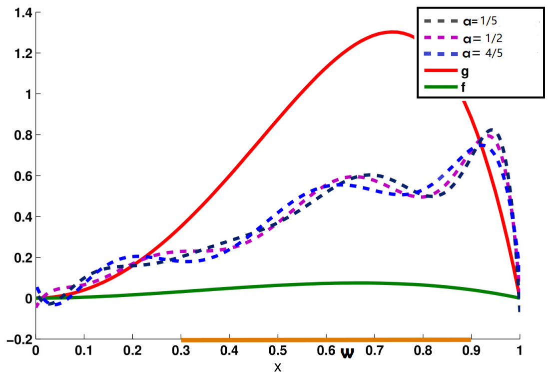

Figure 3. Final state between $[f(.),\ g(.)]$

Figure 2 displays the evolution of the control function over $[0,\ T = 2]$. Figure 3 shows that the fractional final state with different values of α is between $f(.)$ and $g(.)$ on ω. Therefore, the $[f(.),\ g(.)]$-controllability on ω is obtained with transfer cost $\|u^\star_{\frac{1}{5}}\|^2 = 0.258$, $\|u^\star_{\frac{1}{2}}\|^2 = 0.164$ and $\|u^\star_{\frac{4}{5}}\|^2 = 0.0912$.

Example 2:
Let $\Omega =]0,\ 1[$ and consider the following system:

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \dfrac{\partial^2 z}{\partial x^2}(t,x) + \mathcal{X}_D u(t), & \Omega \times]0,\ T[, \\ z(0,x) = 0, & x \in \Omega, \\ z(t,0) = z(t,1) = 0, & t \in]0,\ T[, \end{cases} \tag{22}$$

taking T $= 2$ and the actuator is located at D.
Let $f(x) = \dfrac{1}{2}x^2(1-x^2)$ and $g(x) = 4x(1-x)$.

The operator $Az = \dfrac{\partial^2 z}{\partial x^2}$ admits a complete set of eigenfunctions

$$\varphi_n(x) = \sqrt{2}\sin(n\pi x)$$

and the associated eigenvalues $\lambda_n = -n^2\pi^2$. The simulations provide the following outcomes after applying the aforesaid Algorithm.

4.4. Second case: $\omega =]0.25,\ 0.6[$

$\triangleright$ Zone of action $D =]0.1,\ 0.4[$

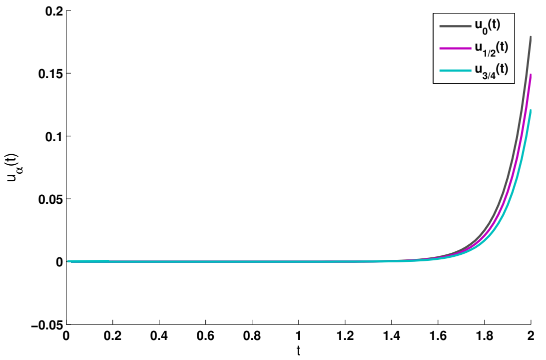

Figure 4. Control function

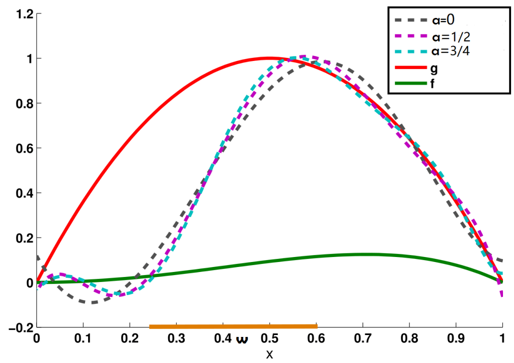

Figure 5. Final state between $[f(.),\ g(.)]$

Figure 4 displays the evolution of the control function over $[0,\ T = 2]$. Figure 5 shows that the fractional final state for various values of α is between $f(.)$ and $g(.)$ on ω. Therefore, the $[f(.),\ g(.)]$-controllability on ω is obtained with transfer cost $\|u_0^\star\|^2 = 0.0054$, $\|u_{\frac{1}{2}}^\star\|^2 = 0.0183$ and $\|u_{\frac{3}{4}}^\star\|^2 = 0.0038$.

Remark 1.

1) *The simulation results show the effectiveness of the proposed control approach in achieving the desired goal of maintaining the fractional state between two given functions over the subregion. Overall, Figures 2 and 4 provide a clear visual representation of the simulation results of the proposed control approach. The different plots in Figures 3 and 5 depict the behavior of the system's fractional state with varying values of the fractional order and constraint.*
2) *The relationship study between the monotonicity of the cost function and the order of the fractional derivative α is not obvious. However, the question still remains open.*

5. Conclusion

We studied the concept of regional controllability, which realizes a situation in which the fractional output of the system lies on between two known functions in a subregion of the evolution domain. Hence, we used two methods to characterize the optimal control. Additionally, we explored the numerical simulations to check the implementation of the theoretical part with different values of α and subregion ω.

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