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RESEARCH ARTICLE

On the regional boundary observability of semilinear time-fractional systems with Caputo derivative

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ABSTRACT

This paper considers the regional boundary observability problem for semilinear time-fractional systems. The main objective is to reconstruct the initial state on a subregion of the boundary of the evolution domain of the considered fractional system using the output equation. We proceed by providing a link between the regional boundary observability of the considered semilinear system on the desired boundary subregion, and the regional observability of its linear part, in a well chosen subregion of the evolution domain. By adding some assumptions on the nonlinear term appearing in the considered system, we give the main theorem that allows us to reconstruct the initial state in the well-chosen subregion using the Hilbert uniqueness method (HUM). From it, we recover the initial state on the boundary subregion. Finally, we provide a numerical example that backs up the theoretical results presented in this paper with a satisfying reconstruction error.

1. Introduction

The analysis of distributed parameter systems leads to the introduction of many useful concepts such as controllability, stability, detectability, and observability [1, 2]. These notions permit researchers to understand those systems and their behaviors, which enable us to manipulate them. In the nineties, the concept of regional analysis was brought to life in [3, 4], bringing with it many tools for investigating real-world problems [5]. In particular, the concept of regional observability, which consists of finding and reconstructing the initial state in a desired subregion of the evolution domain, has great importance in the domain of control theory [3, 6–8].

Fractional calculus (FC) is a field of mathematics that investigates the notions of integration and differentiation of arbitrary or non-integer order. By fractional systems, we mean systems in which a fractional derivative appears. FC is growing in a fast manner nowadays, and this is because fractional operators present a powerful tool for modeling real-world phenomena [9–11]. For example, in [12], authors have generalized the linear prediction (LP) to fractional linear prediction (FLP) and described it with applications to one-dimensional (1D) and two-dimensional (2D) signals. They presented some numerical simulations where, for the 1D case, authors considered standard test signals, namely the square wave, sine wave, sawtooth wave, and real data signals such as speech and electrocardiogram. As for the 2D case, they choose grayscale images. The authors stated that, for the 1D case, the proposed FLP has the same construction as the LP, i.e. it uses linear combinations of non-integer derivatives with non-identical orders of derivatives. As for the 2D case, the FLP model uses a linear combination of fractional derivatives in horizontal and vertical directions. After comparing the performance of LP and FLP, the authors concluded that FLP could be used in processing 1D and 2D signals due to

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the comparable or better performance, using the same or even smaller number of parameters.

Recently, FC started to penetrate the domain of control theory [10, 13, 14]; in particular, it is used to investigate the notion of regional observability; see [15–18] for linear fractional systems and [19, 20] for semilinear ones. In this paper, we investigate the notion of regional boundary observability, which is basically regional observability where the desired subregion is a part of the boundary of the evolution domain [21, 22]. The principal goal is to reconstruct the initial state of the considered system, on a desired boundary subregion B, of the evolution domain Ω. Our contribution can be summarized in the following: Firstly, we define a new internal subregion $\omega_p \subset \Omega$, such that $B \subset \partial\omega_p$, which enables us to give a link between regional boundary observability of the considered semilinear system on B, and the regional observability of its linear part in ω_p. Secondly, we develop a method, which is based on the Hilbert uniqueness method (HUM), in order to reconstruct the initial state in ω_p, and from it we extract the value of the initial state on B.

The proposed method can be applied to real-world situations; for instance, we can use it to determine the initial population for a certain species at the frontiers of some geographical place. The diffusive logistic population growth model is given in general by,

$$D^\alpha y(x,t) - \Delta y(x,t) = my(x,t)\left(1 - \frac{y(x,t)}{b}\right),$$

where x is the spacial variable, t is time and D^α is some type of a fractional derivative. The above system is given with some boundary conditions and an unknown initial state. The quantities m and b are positive constants that are given depending on the species under investigation.

This manuscript is organized as follows: In section (2), we lay out the considered system and its properties, we also give some basic definitions and recalls covering both the field of control theory and fractional calculus. Section (3) is reserved for showing the link between the regional boundary observability of the considered semilinear system and the regional observability of its linear part throughout the subregion ω_p. In section (4), we use an extension of the Hilbert uniqueness method to reconstruct the considered system's initial state in ω_p, which led us to give an algorithm that was implemented numerically and gave us some satisfying numerical results.

2. Considered system and problematic

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with smooth enough boundary $\partial\Omega$, let $[0,T]$ be a time interval and α an element of $]0,1]$. From now on, we denote $Q := \Omega\times]0,T[$ and $\Sigma := \partial\Omega\times]0,T[$. Let $X = H^1(\Omega)$ be the state space and $\mathcal{O}$ a Hilbert space called the observation space, we consider the following fractional system,

$$\begin{cases} {}^C D^\alpha_{0+} y(x,t) = \mathcal{A}y(x,t) + Fy(x,t) & \textit{in } Q, \\ \dfrac{\partial y}{\partial \nu_{\mathcal{A}}}(\xi,t) = 0 & \textit{on } \Sigma, \\ y(x,0) = y_0(x) & \textit{in } \Omega, \end{cases} \tag{1}$$

augmented with the output equation,

$$z(t) = Cy(.,t), \quad 0 \leq t \leq T, \tag{2}$$

where :

- $\mathcal{A}$ is a second order linear differential operator which generates a C_0-semigroup $\{R(t)\}_{t\geq 0}$ on X.
- F is a nonlinear, globally Lipschitz and continuous operator.
- $C : X \longrightarrow \mathcal{O}$ is the observation operator, considered to be bounded.

- ${}^C D^\alpha_{0+} y(x,t) := \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_a^t (t-s)^{-\alpha} \frac{\partial}{\partial s} y(x,s)ds$, is the left sided time-fractional derivative, of order α, of y in the sense of Caputo and $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} t^{\alpha-1}e^{-t}dt$ is the Euler gamma function.
- $\dfrac{\partial y}{\partial \nu_{\mathcal{A}}}$ is the co-normal derivative of y with respect to $\mathcal{A}$, see [23].
- y_0 is the initial state in X, supposedly unknown.

Definition 1. *[20] A function $y \in C(0,T;X)$, is called a mild solution of (1), if it satisfies*

$$y(.,t) = (R_\alpha(t)y_0)(.) + \int_0^t (t-\tau)^{\alpha-1}\mathcal{W}_\alpha(t-\tau)Fy(.,\tau)d\tau, \tag{3}$$

in $[0,T]$, where $R_\alpha(t) = \int_0^\infty \varpi_\alpha(\theta)R(t^\alpha\theta)d\theta$ and $\mathcal{W}_\alpha(t) = \alpha\int_0^\infty \theta\varpi_\alpha(\theta)R(t^\alpha\theta)d\theta$. In addition,

$$\varpi_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{\Gamma(n)\Gamma(1-\alpha n)}, \quad \theta \geq 0, \tag{4}$$

is the Mainardi function.

Proposition 1. *[24] The operators R_α and $\mathcal{W}_\alpha$ are strongly continuous. Furthermore,*

$$\exists M > 0, \textit{ such that } \|R_\alpha(t)\|_{\mathcal{L}(X)} \leq M. \tag{5}$$

For the sake of simplicity, we define the operator $K : L^2(0,T;X) \longrightarrow L^2(0,T;X)$ by

$$(Ky)(t) = \int_0^t (t-\tau)^{\alpha-1}\mathcal{W}_\alpha(t-\tau)y(.,\tau)d\tau,$$

$\forall y \in L^2(0,T;X)$, $\forall t \in [0,T]$.
For the rest of this paper and without any loss of generality, we denote $y(t) := y(.,t)$ and for every operator A we denote its adjoint by A^*.

Let B be a non empty subset of the boundary $\partial\Omega$ with positive Lebesgue measure. We recall the following operators,

- $\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$, the trace operator of order zero, from Ω, on $\partial\Omega$. It is defined by $\gamma_0 v = v_{|\partial\Omega}$.
- $\chi_B : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{\frac{1}{2}}(B)$, the restriction operator, from $\partial\Omega$, on B. It is defined by $\chi_B v = v_{|B}$.
- $H_\alpha : X \longrightarrow L^2(0,T;\mathcal{O})$, the observability operator which is defined as follows $(H_\alpha x)(t) = CR_\alpha(t)x$.

This manuscript aims to study the regional boundary observability of the system (1). In other words, we are looking to reconstruct the initial state of system (1) on the boundary subregion B; this is equivalent to recover the value of y_0 on B, which we denote by y_0^1. One can see that $y_0^1 = \chi_B\gamma_0 y_0$. Then, we give the following definition.

Definition 2. *We say that system* (1)*, augmented with* (2)*, is $\mathcal{B}$-observable on B ($\mathcal{B}$ stands for boundary), if it is possible to reconstruct y_0^1 using the output equation* (2)*.*

Remark 1. *An alternative way to define the regional boundary observability on B is that for two different measurements, $z_1(.)$ and $z_2(.)$, we obtain two different values of y_0^1 on B.*

We associate to the considered system (1) the following linear system,

$$\begin{cases} {}^C D^\alpha_{0^+} y(x,t) = \mathcal{A}y(x,t) & in\ Q, \\ \dfrac{\partial y}{\partial \nu_{\mathcal{A}}}(\xi,t) = 0 & on\ \Sigma, \\ y(x,0) = y_0(x) & in\ \Omega, \end{cases} \tag{6}$$

which plays an important role in achieving the goal of this paper. We formulate the problem of this work as follows.

Problem: Given any system (1) with the output equation (2), can we reconstruct y_0^1?

3. Link between boundary and internal observability

In this section, we design a method for linking the regional boundary observability on B and the regional internal observability in a well-chosen subregion $\omega \subset \Omega$, such that $B \subset \partial\omega$. After reconstructing y_0 in ω, we obtain y_0^1 by taking the restriction on B of the trace of the reconstructed initial state on $\partial\omega$.
For a sufficiently small number $p > 0$, we define

$$U_p = \bigcup_{\xi\in B} \overline{B(\xi,p)} \quad \text{and} \quad w_p = U_p \bigcap \Omega,$$

where $\overline{B(\xi,p)}$ is the closed ball of center ξ and radius p.

Remark 2. *Notice that $\omega_p \subset \Omega$ and $B \subset \partial\Omega \cap \partial\omega_p$.*

As we did for Ω, we recall, for ω_p, the following operators:

- $\chi_{\omega_p} : H^1(\Omega) \longrightarrow H^1(\omega_p)$, the restriction operator in ω_p, which is defined by $\chi_{\omega_p} v = v_{|\omega_p}$.
- $\tilde{\gamma}_0 : H^1(\omega_p) \longrightarrow H^{\frac{1}{2}}(\partial\omega_p)$, the trace operator of order zero, from ω_p, on $\partial\omega_p$. It is defined by $\tilde{\gamma}_0 v = v_{|\partial\omega_p}$.
- $\tilde{\chi}_B : H^{\frac{1}{2}}(\partial\omega_p) \longrightarrow H^{\frac{1}{2}}(B)$, the restriction operator, from $\partial\omega_p$, on B. It is defined by $\tilde{\chi}_B v = v_{|B}$.

Remark 3. *One can see that $y_0^1 = \chi_B\gamma_0 y_0 = \tilde{\chi}_B\tilde{\gamma}_0\chi_{\omega_p} y_0$.*

Remark 4. *The adjoint of χ_{ω_p} is given by*

$$\chi^*_{\omega_p} g = \begin{cases} g & in\ \ \omega_p. \\ 0 & in\ \ \Omega\setminus\omega_p. \end{cases}, \forall g \in H^1(\omega_p).$$

Definition 3. *[25] We say that the linear system* (6)*, augmented with* (2)*, is approximately ω_p-observable if, and only if,*

$$\mathcal{K}er\left(H_\alpha\chi^*_{\omega_p}\right) = \{0\}.$$

Remark (3) allows us to deduce that in order to reconstruct y_0^1, it is sufficient to reconstruct $\chi_{\omega_p}y_0$, which is the initial state in ω_p, after that, we take the restriction on B, of its trace on $\partial\omega_p$. In order to illustrate this, we have the following theorem.

Theorem 1. *If the linear system* (6)*, augmented with* (2)*, is approximately ω_p-observable, then the semilinear system* (1)*, augmented with* (2)*, is $\mathcal{B}-$observable on B, and y_0^1 is the restriction on B of the trace on $\partial\omega_p$ of the restriction in ω_p of a fixed point of the function ϕ at $t = 0$, where*

$\phi : L^2(0,T;X) \longrightarrow L^2(0,T;X)$ is defined, for every $(t,y) \in [0,T] \times L^2(0,T;X)$, as follows:

$$\phi(y)(t) = R_\alpha(t)\overline{y}_0 + (KFy)(t) + R_\alpha(t)\chi^*_{\omega_p}\left[H_\alpha\chi^*_{\omega_p}\right]^\dagger\Big(z(.) - (H_\alpha\overline{y}_0)(.) - C(KFy)(.)\Big), \tag{7}$$

with

$$\left[H_\alpha\chi^*_{\omega_p}\right]^\dagger := \left[\left(H_\alpha\chi^*_{\omega_p}\right)^*\left(H_\alpha\chi^*_{\omega_p}\right)\right]^{-1}\left(H_\alpha\chi^*_{\omega_p}\right)^*,$$

is the pseudo (generalized) inverse of $H_\alpha\chi^*_{\omega_p}$. Moreover, $\overline{y}_0$ has the value of y_0 in $\Omega\setminus\omega_p$ and zero in ω_p.

Proof. Taking into account remark (4), we see that equation (3) can be written as follows:

$$y(t) = R_\alpha(t)\chi^*_{\omega_p}\chi_{\omega_p}y_0 + R_\alpha(t)\overline{y}_0 + (KFy)(t), \tag{8}$$

Using equations (2) and (8), we have,

$$(H_\alpha\chi^*_{\omega_p}\chi_{\omega_p}y_0)(.) = z(.) - (H_\alpha\overline{y}_0)(.) - C\,(KFy)\,(.), \tag{9}$$

and since (6) is approximately ω_p-observable, then, by the same arguments in [2], the operator $H_\alpha\chi^*_{\omega_p}$ has a generalized inverse, denoted $\left[H_\alpha\chi^*_{\omega_p}\right]^\dagger$, hence:

$$\chi_{\omega_p}y_0 = \left[H_\alpha\chi^*_{\omega_p}\right]^\dagger\Big(z(.) - (H_\alpha\overline{y}_0)(.) - C\,(KFy)\,(.)\Big). \tag{10}$$

So, by substituting (10) in (8), we get that:

$$y(t) = R_\alpha(t)\overline{y}_0 + (KFy)(t) = \phi(y)(t) + R_\alpha(t)\chi^*_{\omega_p}\left[H_\alpha\chi^*_{\omega_p}\right]^\dagger\Big(z(.) - (H_\alpha\overline{y}_0)(.) - C\,(KFy)\,(.)\Big), \tag{11}$$

hence, y is a fixed point of ϕ and $y(0)_{|\omega_p} = \chi_{\omega_p}y_0$. Thus $y_0^1 = \tilde{\chi}_B\tilde{\gamma}_0 y(0)_{|\omega_p} = \tilde{\chi}_B\tilde{\gamma}_0\chi_{\omega_p}y_0$. □

4. Reconstruction method

In consequence of theorem (1) and the discussion in section (3), we shall reconstruct the initial state in ω_p. For that we use an extension of the Hilbert uniqueness method for fractional systems. Let's start by introducing the following set,

$$\mathcal{E} = \left\{h \in H^1(\Omega) \;\middle|\; h = 0 \quad in \quad \Omega\setminus\omega_p\right\},$$

in which we define the following semi-norm,

$$\begin{aligned}\|h\|_{\mathcal{E}} &= \sqrt{\int_0^T \|CR_\alpha(t)h\|^2_{\mathcal{O}}dt}, \\ &= \sqrt{\int_0^T \|(H_\alpha h)(t)\|^2_{\mathcal{O}}dt}.\end{aligned}$$

Remark 5. *If g is in $\mathcal{E}$, then $\chi^*_{\omega_p}\chi_{\omega_p}g = g$.*

For every Θ_0 in $\mathcal{E}$, we consider the system,

$$\begin{cases} {}^C D^\alpha_{0+}\Theta(x,t) = \mathcal{A}\Theta(x,t) + F\Theta(x,t) & in\ Q, \\ \dfrac{\partial\Theta}{\partial\nu_{\mathcal{A}}}(\xi,t) = 0 & on\ \Sigma, \\ \Theta(x,0) = \Theta_0(x) & in\ \Omega, \end{cases} \tag{12}$$

which has a unique mild solution, see [26], written as follows,

$$\Theta(t) = R_\alpha(t)\Theta_0 + (KF\Theta)(t), \quad in \quad [0,T], \tag{13}$$

which we decompose as follows $\Theta = \Theta_1 + \Theta_2$, where Θ_1 and Θ_2 are given by the two systems:

$$\begin{cases} {}^C D^\alpha_{0+}\Theta_1(x,t) = \mathcal{A}\Theta_1(x,t) & in\ Q, \\ \dfrac{\partial\Theta_1}{\partial\nu_{\mathcal{A}}}(\xi,t) = 0 & on\ \Sigma, \\ \Theta_1(x,0) = \Theta_0(x) & in\ \Omega, \end{cases}. \tag{14}$$

and

$$\begin{cases} {}^C D^\alpha_{0+}\Theta_2(x,t) = \mathcal{A}\Theta_2(x,t) & in\ Q, \\ \qquad\qquad +F\left(\Theta_1(x,t) + \Theta_2(x,t)\right) & \\ \dfrac{\partial\Theta_2}{\partial\nu_{\mathcal{A}}}(\xi,t) = 0 & on\ \Sigma, \\ \Theta_2(x,0) = 0 & in\ \Omega, \end{cases} \tag{15}$$

with solutions,

$$\Theta_1(t) = R_\alpha(t)\Theta_0, \quad in \quad [0,T], \tag{16}$$

and

$$\Theta_2(t) = R_\alpha(t)\Theta_0 + (KF[\Theta_1+\Theta_2])(t), \quad in \quad [0,T]. \tag{17}$$

<u>*Assumption*</u> : We assume, for the rest of this manuscript, that system (14), augmented with (2), is approximately ω_p-observable.

Proposition 2. *[18] If the above Assumption is satisfied, then the semi-norm $\|.\|_{\mathcal{E}}$ becomes a norm on $\mathcal{E}$.*

We introduce the following auxiliary system

$$\begin{cases} {}^{RL} D^\alpha_{T-}\Xi(x,t) = \mathcal{A}^*\Xi(x,t) & in\ Q, \\ \qquad\qquad -F\Xi(x,t) - C^*C\Theta_1(t) & \\ \dfrac{\partial\Xi}{\partial\nu_{\mathcal{A}^*}}(\xi,t) = 0 & on\ \Sigma, \\ \lim\limits_{t\to T^-}\mathcal{I}^{1-\alpha}_{T^-}\Xi(x,t) = 0 & in\ \Omega, \end{cases} \tag{18}$$

where

$$\mathcal{I}^\alpha_{T^-}y(x,t) := -\frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}y(x,s)ds,$$

is the right sided Riemann-Liouville time-fractional integral of order α, and

$${}^{RL} D^\alpha_{T-}y(x,t) := -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T (s-t)^{-\alpha}y(x,s)ds,$$

is the right sided Riemann-Liouville time-fractional derivative, of order α.

If Θ_0 is chosen in $\mathcal{E}$ such that $CR_\alpha\Theta_0(.) = z(.)$, then (18) is considered to be the adjoint system of (12).

System (18) has a unique mild solution, given by:

$$\begin{aligned}\Xi(x,t) &= \int_t^T (s-t)^{\alpha-1}\mathcal{W}_\alpha^*(s-t)\,[-F\Xi(s)\\ &\qquad -C^*C\Theta_1(s)]\,ds,\end{aligned} \tag{19}$$

which we also decompose into $\Xi = \Xi_1 + \Xi_2$, where Ξ_1 and Ξ_2 are solutions of

$$\begin{cases} {}^{RL}D^\alpha_{T^-}\Xi_1(x,t) = \mathcal{A}^*\Xi_1(x,t) & in\ Q,\\ \qquad -C^*C\Theta_1(t) & \\ \dfrac{\partial \Xi_1}{\partial\nu_{\mathcal{A}^*}}(\xi,t) = 0 & on\ \Sigma,\\ \lim\limits_{t\to T^-}\mathcal{I}^{1-\alpha}_{T^-}\Xi_1(x,t) = 0 & in\ \Omega,\end{cases} \tag{20}$$

and

$$\begin{cases} {}^{RL}D^\alpha_{T^-}\Xi_2(x,t) = \mathcal{A}^*\Xi_2(x,t) & in\ Q,\\ \qquad -F[\Xi_1(x,t)+\Xi_2(x,t)] & \\ \dfrac{\partial \Xi_2}{\partial\nu_{\mathcal{A}^*}}(\xi,t) = 0 & on\ \Sigma,\\ \lim\limits_{t\to T^-}\mathcal{I}^{1-\alpha}_{T^-}\Xi_2(x,t) = 0 & in\ \Omega.\end{cases} \tag{21}$$

Furthermore, they are written as follows,

$$\Xi_1(x,t) = -\int_t^T (s-t)^{\alpha-1}\mathcal{W}_\alpha^*(s-t)C^*C\Theta_1(s)ds, \tag{22}$$

and

$$\Xi_2(x,t) = -\int_t^T (s-t)^{\alpha-1}\mathcal{W}_\alpha^*(s-t)F\left[\Xi_1(s)+\Xi_2(s)\right]ds. \tag{23}$$

Let's denote by $P_{\omega_p} := \chi^*_{\omega_p}\chi_{\omega_p}$ the projection operator in $\mathcal{E}$, we have:

$$\begin{aligned} P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi(0)\right) &= \Lambda\Theta_0 + L\Theta_0,\\ &:= P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi_1(0)\right) + P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi_2(0)\right),\end{aligned}$$

where:

$$\begin{aligned}\Lambda\ :\ \mathcal{E} &\longrightarrow \mathcal{E},\\ \Theta_0 &\longmapsto P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi_1(0)\right),\end{aligned}$$

and

$$\begin{aligned}L\ :\ \mathcal{E} &\longrightarrow \mathcal{E},\\ \Theta_0 &\longmapsto P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi_2(0)\right).\end{aligned}$$

Thus,

$$\Lambda\Theta_0 = P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi(0)\right) - L\Theta_0,$$

and, as proven in [18], since (14) is approximately ω_p-observable, then Λ is an isomorphism. Therefore,

$$\begin{aligned}\Theta_0 &= \Lambda^{-1}P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi(0)\right) - \Lambda^{-1}L\Theta_0,\\ &:= \mathcal{N}\Theta_0.\end{aligned} \tag{24}$$

Hence, in order to reconstruct the initial state in ω_p, it is sufficient to solve the fixed point problem (24). For that, we give the following theorem.

Theorem 2. *Under the following assumptions:*

- H_1 *- System* (14)*, augmented with* (2)*, is approximately ω_p-observable.*
- H_2 *-* $\exists c > 0$*, such that:*

$$\|Fu(t)\|_X \le c\|\mathcal{I}^{1-\alpha}_{T^-}u(t)\|_X,\ \ \forall u\in L^2(0,T;X).$$

The operator $\mathcal{N}$ has a unique fixed point which corresponds with the initial state in ω_p.

Before proving this last theorem, let us give the following proposition.

Proposition 3. *[18] Let α be in $]0,1]$, t in $[0,T]$ and f in $L^2(0,T;X)$, we have:*

$$\begin{aligned}&\mathcal{I}^{1-\alpha}_{T^-}\int_t^T (s-t)^{\alpha-1}\mathcal{W}_\alpha^*(s-t)f(s)ds\\ &\qquad = \int_t^T R_\alpha^*(s-t)f(s)ds.\end{aligned} \tag{25}$$

Proof. : of theorem (2)
We use Schauder's fixed point theorem in our proof. In other words, we need to show that $\mathcal{N}$ is compact and $\mathcal{N}(B(0,s)) \subseteq B(0,s)$ for some $s > 0$, where $B(0,s)$ is the open ball of center zero and radius s.
Remark that $\mathcal{N}$ is compact if, and only if, L is compact. The operator L is compact if,

$$L\left(B(0,r)\right) = \left\{L\Theta_0 = P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi_2(0)\right),\ \Theta_0\in B(0,r)\right\},$$

is relatively compact, for every $r > 0$, and since

$$L\left(B(0,r)\right) \subset \mathcal{J}_p,$$

with

$$\mathcal{J}_p := \left\{P_{\omega_p}\left(\mathcal{I}^{1-\alpha}_{T^-}\Xi_2(t)\right),\ \Theta_0\in B(0,r),\ t\in[0,T]\right\},$$

hence it is sufficient to prove that $\mathcal{J}_p$ is relatively compact.

Step 1: We show that $\mathcal{J}_p$ is uniformly bounded. From proposition (3) and (21), we have:

$$\mathcal{I}^{1-\alpha}_{T^-}\Xi_2(t) = -\int_t^T R_\alpha^*(s-t)F\left[\Xi_1(s)+\Xi_2(s)\right]ds,$$

which gives, by using the property (5) and H_2,

$$\|\mathcal{I}^{1-\alpha}_{T^-}\Xi_2(t)\|_X \le Mc\int_0^T \|\mathcal{I}^{1-\alpha}_{T^-}\Xi_1(s)\|_X ds$$

$$+Mc\int_0^T \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(s)\|_X ds.$$

Furthermore, from (22) and proposition (3), we have:

$$\mathcal{I}_{T^-}^{1-\alpha}\Xi_1(t) = -\int_t^T R_\alpha^*(s-t)\left[C^*C\Theta_1\right]ds,$$

hence, by using Cauchy-Schwartz,

$$\begin{aligned}\|\mathcal{I}_{T^-}^{1-\alpha}\Xi_1(t)\|_X &\leq M\|C\|_{\mathcal{L}(X,\mathcal{O})}\int_0^T \|C\Theta_1\|_{\mathcal{O}}ds,\\ &\leq M\|C\|_{\mathcal{L}(X,\mathcal{O})}T^{\frac{1}{2}}\|\Theta_0\|_{\mathcal{E}},\end{aligned} \tag{26}$$

thus,

$$\|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t)\|_X \leq M^2c\|C\|_{\mathcal{L}(X,\mathcal{O})}T^{\frac{3}{2}}\|\Theta_0\|_{\mathcal{E}} +Mc\int_0^T \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(s)\|_X ds.$$

By Gronwall's inequality, we obtain,

$$\|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t)\|_X \leq M^2c\|C\|_{\mathcal{L}(X,\mathcal{O})}T^{\frac{3}{2}}\|\Theta_0\|_{\mathcal{E}}e^{McT}. \tag{27}$$

Therefore, the set $\mathcal{J}_p$ is uniformly bounded.

Step 2: We show that $\mathcal{J}_p$ is equicontinuous. Let's consider $\varepsilon > 0$, for t_1 and t_2 in $[0,T]$, such that $t_2 > t_1$, we have:

$$\begin{gathered}\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_1) - \mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_2)\\ =\\ \int_{t_2}^T R_\alpha^*(s-t_2)F\left[\Xi_1(s)+\Xi_2(s)\right]ds\\ -\int_{t_1}^T R_\alpha^*(s-t_1)F\left[\Xi_1(s)+\Xi_2(s)\right]ds\\ =\\ \underbrace{\int_{t_2}^T \left(R_\alpha^*(s-t_2)-R_\alpha^*(s-t_1)\right)F\left[\Xi_1(s)+\Xi_2(s)\right]ds}_{:=\mathcal{R}_1}\\ \underbrace{-\int_{t_1}^{t_2} R_\alpha^*(s-t_1)F\left[\Xi_1(s)+\Xi_2(s)\right]ds}_{:=\mathcal{R}_2},\end{gathered}$$

thus,

$$\|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_1) - \mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_2)\|_X \leq \|\mathcal{R}_1\|_X + \|\mathcal{R}_2\|_X.$$

Since the operator R_α is strongly continuous, then for every $\varepsilon_1 > 0$, $\exists\sigma > 0$, such that,

$$|t_1-t_2| < \sigma \implies \|R_\alpha^*(s-t_2)-R_\alpha^*(s-t_1)\|_{\mathcal{L}(X)} \leq \varepsilon_1,$$

hence, by using (26) and (27), we get,

$$\begin{aligned}\|\mathcal{R}_1\|_X &\leq \varepsilon_1 c\int_0^T \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(s)\|_X + \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(s)\|_X ds,\\ &\leq \varepsilon_1 \times \underbrace{McT^{\frac{1}{2}}\|C\|_{\mathcal{L}(X,\mathcal{O})}\|\Theta_0\|_{\mathcal{E}}\left[1+McTe^{McT}\right]}_{:=Z_1},\end{aligned} \tag{28}$$

and

$$\begin{aligned}\|\mathcal{R}_2\|_X &\leq Mc\int_{t_1}^{t_2} \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(s)\|_X + \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(s)\|_X ds,\\ &\leq \sigma \times \underbrace{M^2cT^{\frac{1}{2}}\|C\|_{\mathcal{L}(X,\mathcal{O})}\|\Theta_0\|_{\mathcal{E}}\left[1+McTe^{McT}\right]}_{:=Z_2}.\end{aligned} \tag{29}$$

Since P_{ω_p} is a projection operator, then, from (28) and (29), we have

$$\begin{aligned}&\|P_{\omega_p}\left(\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_1)\right) - P_{\omega_p}\left(\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_2)\right)\|_X\\ &\leq \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_1) - \mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_2)\|_X,\\ &\leq \varepsilon_1 Z_1 + \sigma Z_2,\end{aligned}$$

therefore, by taking $\varepsilon_1 \leq \dfrac{\varepsilon}{2Z_1}$ and $\sigma \leq \dfrac{\varepsilon}{2Z_2}$, we conclude that:

$$\begin{aligned}\|P_{\omega_p}\left(\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_1)\right) - P_{\omega_p}\left(\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(t_2)\right)\|_X &\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2},\\ &\leq \varepsilon.\end{aligned}$$

Thus, $\mathcal{J}_p$ is equicontinuous.

From step 1 and 2, we get that L is compact hence so does $\mathcal{N}$.

Step 3: We show that $\mathcal{N}\left(B(0,s)\right) \subseteq B(0,s)$ for some $s > 0$.

we have that

$$\|\mathcal{N}\Theta_0\|_X \leq \|\Lambda^{-1}\|\left(\|\mathcal{I}_{T^-}^{1-\alpha}\Xi(0)\|_X + \|\mathcal{I}_{T^-}^{1-\alpha}\Xi_2(0)\|_X\right).$$

We know that Ξ and Ξ_2 are in $C(0,T;X)$, then so does $\mathcal{I}_{T^-}^{1-\alpha}\Xi$ and $\mathcal{I}_{T^-}^{1-\alpha}\Xi_2$, which means that they are in $L^\infty(0,T;X)$. Thus, $\exists\beta_1,\beta_2 > 0$ such that,

$$\|\mathcal{N}\Theta_0\|_X \leq \|\Lambda^{-1}\|\left(\beta_1+\beta_2\right).$$

In other words, if we take $s > \|\Lambda^{-1}\|\left(\beta_1+\beta_2\right)$, we get that $\mathcal{N}\left(B(0,s)\right) \subseteq B(0,s)$.

By Schauder's fixed point theorem, $\mathcal{N}$ admits a fixed point.

Step 4: We show that the fixed point is unique.

Let $\tilde{\Theta}_0$ and $\overline{\Theta}_0$ be two fixed points of $\mathcal{N}$. Then, as discussed in the paragraph before equation (19), they satisfy

$$CR_\alpha(.)\overline{\Theta}_0 = CR_\alpha(.)\tilde{\Theta}_0 = z(.),$$

hence, using remark 5, we have:

$$CR_\alpha(.)\left(\tilde{\Theta}_0 - \overline{\Theta}_0\right) = CR_\alpha(.)\chi_{\omega_p}^*\chi_{\omega_p}\left(\tilde{\Theta}_0 - \overline{\Theta}_0\right) = 0,$$

and since (14) is approximately ω_p-observable, we obtain that:

$$\chi_{\omega_p}\left(\tilde{\Theta}_0 - \overline{\Theta}_0\right) = 0,$$

since $\tilde{\Theta}_0$ and $\overline{\Theta}_0$ are in $\mathcal{E}$, then:

$$\tilde{\Theta}_0 = \overline{\Theta}_0.$$

Finally, $\mathcal{N}$ has a unique fixed point. $\square$

Now that we recovered the initial state in ω_p, we can apply, to the recovered function, the trace operator $\tilde{\gamma}_0$ and the restriction operator $\tilde{\chi}_B$ to obtain the initial state on B.

5. Algorithm and numerical Simulation

This section is reserved to give an algorithm that allows us to reconstruct the initial state in ω_p and back up our theoretical results by presenting a successful numerical simulation. Following the steps of the above method, we obtain the following algorithm.

5.1. Algorithm

1 - Initialization of : α, ω_p, $\varepsilon = 10^{-6}$, Θ_0.
2 - Solve (14) and get Θ_1.
3 - Solve (20) and get Ξ_1.
4 - Solve (21) and get Ξ_2.
5 - Do $\Xi = \Xi_1 + \Xi_2$.
7 - If $\|\Theta_0 - \mathcal{N}\Theta_0\| > \varepsilon$, then:
  - $\Theta_0 = \mathcal{N}\Theta_0$.
  - go back to step 2.
else
  - Stop.

The reconstructed initial state in ω_p is $\chi_{\omega_p}\Theta_0$. Therefore, $y_0^1 = \tilde{\chi}_B\tilde{\gamma}_0\chi_{\omega_p}\Theta_0$ is the reconstructed initial state on B.

5.2. Numerical simulation

Let us take for this example $\Omega = [0,\pi]\times[0,1]$, $T = 2$, $\alpha = 0.5$, and $B = \{0\}\times[0,1]$. The dynamic of the system, $\mathcal{A}$, is considered to be $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, which has a complete set of eigenfunctions,

$$\left\{\varphi_{ij}(x_1,x_2) = \frac{2}{\sqrt{\pi(1-\lambda_{ij})}}\cos(ix_1)\cos(j\pi x_2)\right\}_{i,j\geq 0},$$

which forms an orthonormal basis of X, associated with the set of eigenvalues $\left\{\lambda_{ij} = -\left(\frac{i^2}{\pi^2}+j^2\right)\pi^2\right\}_{i,j\geq 0}$. The nonlinear operator F is defined as follows :

$$Fy(x_1,x_2,t) = \sum_{i,j\geq 0}^{\infty}\langle \mathcal{I}_{T^-}^{1-\alpha}y(t),\varphi_{ij}\rangle_X^2\varphi_{ij}(x_1,x_2).$$

After specifying all the needed parameters, we consider now the semilinear system,

$$\begin{cases} {}^C D_{0^+}^{\alpha} y(x_1,x_2,t) = \Delta y(x_1,x_2,t) & \text{in } Q, \\ \qquad\qquad +Fy(x_1,x_2,t) & \\ \dfrac{\partial y}{\partial \nu_\Delta}(\xi_1,\xi_2,t) = 0 & \text{on } \Sigma, \\ y(x_1,x_2,0) = y_0(x_1,x_2) & \text{in } \Omega. \end{cases} \tag{30}$$

The output equation is given by a zonal sensor (D,f), where $D\subset\Omega$ is called the geometric support (location) of the sensor and $f\in L^2(D)$ is its spatial distribution. Note that $\mathcal{O} = \mathbb{R}$ and (2) takes the form:

$$z(t) = \langle y(t), f\rangle_{L^2(D)}, \quad 0\leq t\leq T.$$

We set $f\equiv 1$, $D = [1.2\ ,\ 2.4]\times[0.1\ ,\ 0.9]$, $B = \{0\}\times[0,1]$, $\omega_p = [0\ ,\ 0.09]\times[0,1]$, and

$$y_0(x_1,x_2) = \left(\left(\frac{x_1}{\pi}+1\right)\ln\left(\frac{x_1}{\pi}+1\right) - \frac{x_1}{\pi} - \ln\left(\frac{x_1}{\pi}+1\right)^2\right).\left((x_2+1)\ln(x_2+1) - x_2 - \ln(x_2+1)^2\right),$$

which we suppose to be unknown on B.

In order to solve the systems (14), (20), and (21), we use a combination of two methods. The first is the spectral method [27], where instead of solving a fractional partial differential equation, we solve multiple fractional ordinary differential equations. The second method, which we use to solve the fractional ordinary differential equations derived from the first method, is the predictor-corrector method presented in [28].

By applying the proposed algorithm, and after eight iterations, we obtained the Figures (1), (2), (3) and (4).

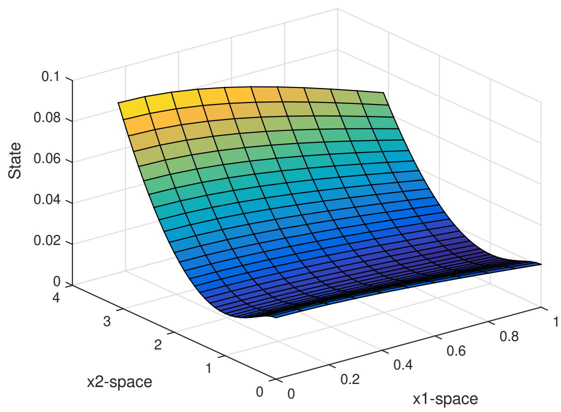

Figure 1. Initial state in Ω.

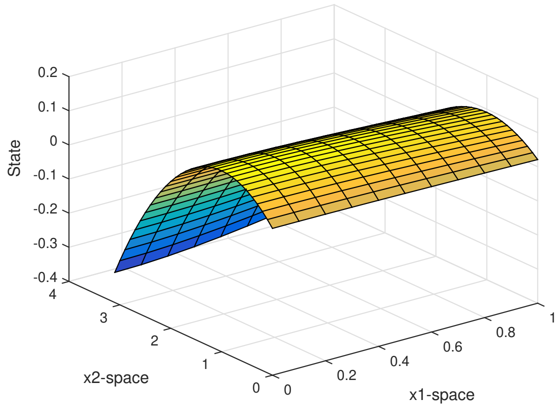

Figure 2. Reconstructed initial state in Ω.

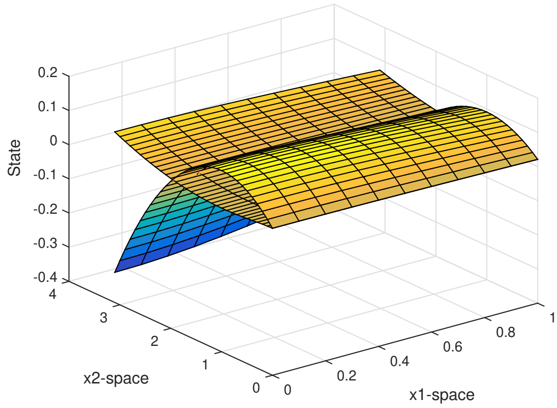

Figure 3. Initial state and the reconstructed one in Ω.

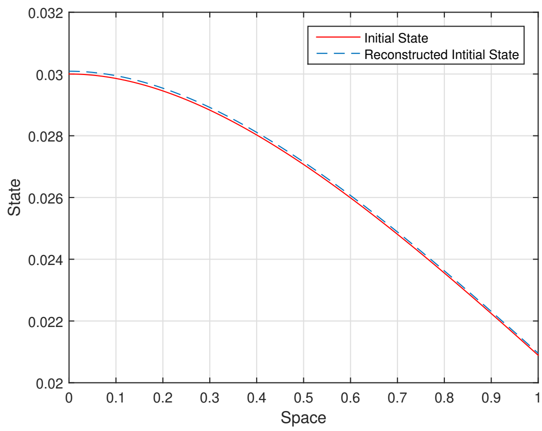

Figure 4. Initial state and the reconstructed one on in B.

Figures 1, 2, and 3 represent, respectively, the real initial state, the reconstructed initial state, and both of them in Ω. By taking a vertical cut in Figure 3 at $x_2 = 0$, we obtain Figure 4 where we can see the values of the two initial states on the boundary subregion B. It is clear, in Figure 4, that the initial state (y_0) is very close to the estimated initial one (Θ_0) on B. Furthermore, the reconstruction error is:

$$\|y_0 - \Theta_0\|^2_{L^2(B)} = 6.41 \times 10^{-9}.$$

In Figure 3, we remark that the two plots present very different behaviors unless in the desired boundary subregion, where they appear to be coinciding, which means that the proposed algorithm does not take into consideration other regions different than the desired one. This means that the cost and time needed to observe the system and reconstruct the initial state regionally is less than if we do it globally.

The efficiency of the proposed method is shown in Figure 4, where we can see that the plots of the initial state and the reconstructed one coincide. This is also backed up by the value of the reconstruction error, which is small.

Table 1 shows how the reconstruction error changes in the function of the sensor's location. We remark that the reconstruction error gets smaller as the area of B gets smaller. This proportionality proves that observing the initial state in a subregion is less expansive than observing it in the whole domain.

Table 1. Evolution of the reconstruction error with respect to the subregion B area.

Subregion B	Error $\|y_0 - \Theta_0\|^2_{L^2(B)}$
$\{0\} \times [0.00\ ,\ 1.00]$	6.41×10^{-9}
$\{0\} \times [0.05\ ,\ 0.95]$	5.80×10^{-9}
$\{0\} \times [0.10\ ,\ 0.90]$	5.18×10^{-9}
$\{0\} \times [0.15\ ,\ 0.85]$	4.55×10^{-9}
$\{0\} \times [0.20\ ,\ 0.80]$	3.92×10^{-9}
$\{0\} \times [0.25\ ,\ 0.75]$	3.27×10^{-9}
$\{0\} \times [0.30\ ,\ 0.70]$	2.63×10^{-9}
$\{0\} \times [0.35\ ,\ 0.65]$	1.67×10^{-9}
$\{0\} \times [0.40\ ,\ 0.60]$	1.32×10^{-9}
$\{0\} \times [0.45\ ,\ 0.55]$	6.59×10^{-10}

6. Conclusion

The present paper studied the regional boundary observability problem for time-fractional systems. We succeeded in reconstructing the initial state of the considered system in the desired boundary subregion by passing through an internal subregion and using the HUM approach. The method used in this work is very effective for regional boundary reconstruction problems. This is shown in the numerical simulation, where we obtained the initial state of a two-dimensional time-fractional diffusion system on the desired boundary subregion with a satisfying value of the reconstruction error. All along this paper, we worked with a bounded observation operator, but we opt to see what happens if we take an unbounded one for future works. We are also investigating the concept of regional gradient observability for fractional systems, where the goal is to reconstruct the gradient or flux of the initial state.

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