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RESEARCH ARTICLE

Genocchi polynomials as a tool for solving a class of fractional optimal control problems

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ABSTRACT

In this research, we use operational matrix based on Genocchi polynomials to obtain approximate solutions for a class of fractional optimal control problems. The approximate solution takes the form of a product consisting of unknown coefficients and the Genocchi polynomials. Our main task is to compute the numerical values of the unknown coefficients. To achieve this goal, we apply the initial condition of the problem, the Tau and Lagrange multiplier methods. We do error analysis as a means to study the behaviour of the approximate solutions.

1. Introduction

In the real world, Optimal control problems are concerned with the effective allocation of limited resources, with the main objective of achieving the set goals. A classic example will be how a firm can use the limited budgets or labour hours of its employees with an overall objective of minimizing costs and maximizing revenue.

Fractional optimal control problems (FOCPs) are an extension of the optimal control problems [1–3]. In an optimal control problem, the restrictions are presented as integer order differential equations. On the other hand, in a FOCP, the restrictions are in the form of fractional differential equations.

Thus, optimal control problems are subsets of FOCPs. Imposing specific conditions on an FOCP yields the optimal control problem. As in the fractional and integer order derivatives scenario, there is a general consensus that models formulated from FOCPs capture real world phenomenon more effectively that the models constructed from Optimal control problems. However, one major drawback with FOCPs models is that they are cumbersome to solve. Generally, we solve FOCPs either indirectly or directly. Methods that solve indirectly convert the original problem into a different state, for example into a boundary value problem in Pontryagin's maximum principle. Directly solving an optimal control problem involves approximating its solution numerically.

Polynomials play a crucial role in most numerical methods for approximating differential equations. Some of them that we frequently make use of are, Legendre polynomials [4, 5], Fibonacci polynomials [6], Bernstein polynomials [7], Laguerre polynomials [8], shifted Chebyshev polynomials [9] and Genocchi polynomials [10].

The application of polynomials has also filtered through to the numerical approximation of the

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FOPCs. In [11], the authors make of the Legendre orthonormal polynomials to numerically solve FOCPs, they apply the Legendre-Gauss quadratic formula and the Lagrange multiplier method to convert the problem to a system of equations. Agrawal makes use of the fractional integration by parts, variational calculus and the Lagrange multiplier to create Euler-Lagrange equations in a quadratic numerical scheme that approximates the solutions of FOCPs [12].

Sweilam and Al-Ajami apply the Legendre polynomials as they use two different approaches to approximate the solutions of FOCPs [13]. In the first approach, they approximate the necessary optimality conditions associated with the Hamiltonian. Then, in the second approach, they employ the trapezoidal rule and the Rayleigh-Ritz to formulate a system of equations.

Rabiei, Ordokhani and Babolian applied the Bernoulli polynomials together with the Newton iterative method to approximate the solutions of one and two dimensional systems of FOCPs [14]. The same authors used the Boubaker polynomials to approximate FOCPs [15].

This research seeks to make a contribution towards the direct solution of FOCPs. In essence, we approximate the solution of the FOCPs through the use of Gennochi polynomials.

We divide our work into sections that address different aspects of the research. Section two deals with definitions and theorems that lay the necessary mathematical foundation. Section three presents our main results, we concisely describe our suggested numerical scheme. We substantiate our proposed scheme by giving practical examples in section four. Detailed explanations that cover our research findings are provided in the last section.

2. Mathematical framework

This section of the paper lays some mathematical background for the next section. We give some definitions and theorems that will be applied in section three.

Definition 1. *The Genocchi polynomial, $\mathcal{P}_q(t)$, of degree q is defined as,*

$$\mathcal{P}_q(t) = \sum_{d=0}^{q} \binom{q}{d} g_{q-d} t^d, \tag{1}$$

$g_q = 2(1-2^q)\beta_q$, g_q and β_q are Genocchi and Bernoulli numbers respectively.

The Genocchi polynomials exhibit some of the following properties,

(i) $\int_0^1 \mathcal{P}_q(t)\mathcal{P}_a(t)dt = \dfrac{2(-1)^q q!a!}{(a+q)!} g_{a+q}, \ q,a \geqslant 1.$

(ii) $\mathcal{P}_q'(t) = q\mathcal{P}_{q-1}(t), \quad q \geqslant 1.$

(iii) $\mathcal{P}_q^{(d)}(t) = \begin{cases} 0 & q \le d, \\ d!\dbinom{q}{d} g_{q-d}, & q > d. \end{cases}$

(iv) $\mathcal{P}_q(1) + \mathcal{P}_q(0) = 0, \quad q > 1.$

Definition 2. *The Caputo fractional derivative of order γ acting on $x(t)$ is defined as,*

$$D_t^\gamma x(t) = \begin{cases} \frac{1}{\Gamma(q-\gamma)}\int_0^t \frac{x(\tau)}{(t-\tau)^{1+\gamma-q}}d\tau, & q-1<\gamma<q \\ x^{(q)}(t), & \gamma = q. \end{cases}$$

Theorem 1. *Let $f(t) \in L^2[0,1]$, $\mathcal{P}(t) = [\mathcal{P}_1(t), \mathcal{P}_2(t), \cdots, \mathcal{P}_N(t)]^T$ and $C = [c_1, c_2, \cdots, c_N]^T$, then we can approximate $f(t)$ in the form,*

$$f(t) \approx \sum_{b=0}^{N} \chi_b \mathcal{P}_b(t) = C^T \mathcal{P}(t), \tag{2}$$

with $C = \langle \mathcal{P}(t), \mathcal{P}(t)\rangle^{-1}\langle f(t), \mathcal{P}(t)\rangle$, where $\langle .,. \rangle$ denotes an inner product over the defined interval.

Theorem 2. *Suppose that $\Omega_r = span\{\mathcal{P}_1(t), \mathcal{P}_2(t), \cdots, \mathcal{P}_N(t)\} \subset H = L^2[0,1]$, and $s(t) \in C^{q+1}[0,1]$. If $s_q(t)$ is the best approximation to $s(t)$ out of Ω_r, then an analytical expression for the error can be expressed as,*

$$\| s(t) - s_q(t) \|_2 \le \frac{\mu}{(q+1)!}\frac{1}{\sqrt{2q+3}}. \tag{3}$$

Where $\mu = max_{t\in[0,1]} \mid s^{(q+1)}(t) \mid$.

Proof. We consider the Taylor series,

$$\begin{aligned} s_1(t) &= s(0) + s'(0)t + s''(0)\frac{t^2}{2!} + \cdots \\ &+ s^{(q)}(0)\frac{t^q}{q!}. \end{aligned} \tag{4}$$

Since $s_q(t)$ is the best approximation of $s(t)$ in Ω_r, we have,

$$
\begin{aligned}
\| s(t)-s_q(t) \|_2 &\leq \| s(t)-s_1(t) \|_2 \\
&= \left(\int_0^1 \mid s(\tau)-s_1(\tau) \mid^2 d\tau\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 \left(\mu \frac{\tau^{q+1}}{(q+1)!}\right)^2 d\tau\right)^{\frac{1}{2}} \\
&\leq \frac{\mu}{(q+1)!}\left(\int_0^1 \tau^{2q+2} d\tau\right)^{\frac{1}{2}} \\
&= \frac{\mu}{(q+1)!}\frac{1}{\sqrt{2q+3}}.
\end{aligned}
$$

Therefore, we have $\lim\limits_{q\longrightarrow\infty} \| s(t)-s_q(t) \|_2 = 0$. This completes the proof. □

Theorem 3. *Given $\mathcal{P}(t)=[\mathcal{P}_1(t),\mathcal{P}_2(t),\cdots,\mathcal{P}_N(t)]^T$, then the $N\times N$ operational matrix of the derivative D,*

$$\mathcal{P}'(t) = D\mathcal{P}(t), \tag{5}$$

is given as,

$$
D=\begin{bmatrix}
0 & 0 & 0 & \dots & 0 & 0 & 0\\
2 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 3 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & 4 & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & N-1 & 0 & 0\\
0 & 0 & 0 & \dots & 0 & N & 0
\end{bmatrix}. \tag{6}
$$

Corollary 1. *The q^{th} order derivative operational matrix is represented as,*

$$\mathcal{P}^{(q)}(t) = D^q\mathcal{P}(t). \tag{7}$$

Lemma 1. *If $f=1,\dots,\lceil\gamma\rceil-1$, then the derivative of order γ acting on Genocchi polynomials is given as,*

$$D^\gamma\mathcal{P}_f(t) = 0, \tag{8}$$

$\lceil\gamma\rceil$ denotes the ceiling of γ, this is the least integer that is greater than γ.

Theorem 4. *The Caputo derivative of $\mathcal{P}(t)$ with order γ can be approximated as,*

$$D^\gamma\mathcal{P}(t) \simeq Z^\gamma\mathcal{P}(t), \tag{9}$$

where Z^γ is called the operational matrix of Caputo derivative based on $\mathcal{P}(t)$.

$$
Z^\gamma=\begin{bmatrix}
0 & 0 & \dots & 0\\
\vdots & \vdots & \dots & \vdots\\
0 & 0 & \dots & 0\\
\sum_{d=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\Omega_{\lceil\gamma\rceil,1,d} & \sum_{d=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\Omega_{\lceil\gamma\rceil,2,d} & \dots & \sum_{d=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\Omega_{\lceil\gamma\rceil,N,d}\\
\vdots & \vdots & \dots & \vdots\\
\sum_{d=\lceil\gamma\rceil}^{f}\Omega_{f,1,d} & \sum_{d=\lceil\gamma\rceil}^{f}\Omega_{f,2,d} & \dots & \sum_{d=\lceil\gamma\rceil}^{f}\Omega_{f,N,d}\\
\vdots & \vdots & \dots & \vdots\\
\sum_{d=\lceil\gamma\rceil}^{N}\Omega_{N,1,d} & \sum_{d=\lceil\gamma\rceil}^{N}\Omega_{N,2,d} & \dots & \sum_{d=\lceil\gamma\rceil}^{N}\Omega_{N,N,d}
\end{bmatrix} \tag{10}
$$

Proof.

$$
\begin{aligned}
D^\gamma\mathcal{P}_f(t) &= \sum_{k=0}^{f}\frac{i!g_{f-d}}{(f-d)!d!}D^\gamma t^d \\
&= \sum_{d=\lceil\gamma\rceil}^{f}\frac{f!g_{f-d}}{(f-d)!\Gamma(d+1-\gamma)}t^{d-\gamma},
\end{aligned} \tag{11}
$$

The term $t^{d-\gamma}$ is approximated in the form of a product that consists of coefficients and Genocchi polynomials,

$$t^{d-\gamma} = \sum_{b=1}^{N}\chi_{b,k}\mathcal{P}_b(t), \qquad d=\lceil\gamma\rceil,\cdots,f \tag{12}$$

Inserting (12) in (11) gives,

$$
\begin{aligned}
& D^\gamma\mathcal{P}_f(t) \\
= & \sum_{b=1}^{N}\left(\sum_{d=\lceil\gamma\rceil}^{f}\frac{f!g_{f-d}}{(f-d)!\Gamma(d+1-\gamma)}\chi_{b,d}\right) \\
\times & \mathcal{P}_f(t) \\
= & \sum_{b=1}^{N}\left(\sum_{d=\lceil\gamma\rceil}^{f}\Omega_{f,b,d}\right)\mathcal{P}_b(t),
\end{aligned} \tag{13}
$$

where,

$$\Omega_{f,b,d} = \frac{f!g_{f-d}}{(f-k)!\Gamma(d+1+\eta)}\chi_{b,d}. \tag{14}$$

Eq. (13) can be evaluated into the form,

$$
\begin{aligned}
& D^\gamma\mathcal{P}_f(t) \\
& = \left[\sum_{d=\lceil\gamma\rceil}^{f}\Omega_{\lceil\gamma\rceil,1,d}\ \sum_{d=\lceil\gamma\rceil}^{f}\Omega_{\lceil\gamma\rceil,2,d}\ \cdots\right. \\
& \left.\sum_{d=\lceil\gamma\rceil}^{f}\Omega_{\lceil\gamma\rceil,N,d}\right]\mathcal{P}(t), \\
& f=\lceil\gamma\rceil,\dots,N.
\end{aligned} \tag{15}
$$

Applying Lemma 1 and Eq. (15) completes the proof. □

Lemma 2. *If H denotes a Hilbert space and η is an arbitrary element of H, let $V = \{\tau_1, \tau_2, \ldots, \tau_n\}$ be a closed subspace of Hilbert space whose τ_0 is the unique best approximation to η out of V. Then*

$$\|\eta - \tau_0\|^2 = \frac{Gram(\eta, \tau_1, \tau_2, \ldots, \tau_q)}{Gram(\tau_1, \tau_2, \ldots, \tau_q)}, \tag{16}$$

with,

$$Gram(\eta, \tau_1, \tau_2, \ldots, \tau_q) = \begin{vmatrix} \langle \eta, \eta \rangle & \langle \eta, \tau_1 \rangle & \ldots & \langle \eta, \tau_q \rangle \\ \langle \tau_1, \eta \rangle & \langle \tau_1, \tau_1 \rangle & \ldots & \langle \tau_1, \tau_q \rangle \\ \vdots & \vdots & & \vdots \\ \langle \tau_q, \eta \rangle & \langle \tau_q, \tau_1 \rangle & \ldots & \langle \tau_q, \tau_q \rangle \end{vmatrix} \tag{17}$$

Proof. Kreyszig, 1978 [16].

Lemma 3. *If s_N is an approximation of $s \in L^2[0,1]$, then [17],*

$$s(t) \simeq s_N(t) = \sum_{q=1}^{N} s_q \mathcal{P}_q(t), \tag{18}$$

we define,

$$L_N(s) = \int_0^1 [s(t) - s_N(t)]^2 dt, \tag{19}$$

such that,

$$\lim_{N \longrightarrow \infty} L_N(s) = 0 \tag{20}$$

Theorem 5. *The fractional differentiation matrix has an error that is bounded as,*

$$\| W_q^\gamma \| \leq \left| \sum_{d=\lceil\gamma\rceil}^{f} \frac{f! g_{f-d}}{(f-d)!\Gamma(d+1-\gamma)} \right| \times \left(\frac{Gram(t^{d-\gamma}, \mathcal{P}_1(t), \ldots, \mathcal{P}_N(t))}{Gram(\mathcal{P}_1(t), \ldots, \mathcal{P}_N(t))} \right)^{\frac{1}{2}} \tag{21}$$

$$W^\gamma = D^\gamma \mathcal{P}(t) - Z^\gamma \mathcal{P}(t), \tag{22}$$

W^γ is composed of individual elements that are represented as,

$$W^\gamma = [W_q^\gamma]_{N\times 1}, \quad q = 1, \cdots, N. \tag{23}$$

Proof. Taking into consideration both Eq. (12) and Lemma 2 yields,

$$\| t^{d-\gamma} - \sum_{b=1}^{N} \chi_{b,d} \mathcal{P}_b(t) \| = \left(\frac{Gram(t^{d-\gamma}, \mathcal{P}_1(t), \ldots, \mathcal{P}_N(t))}{Gram(\mathcal{P}_1(t), \ldots, \mathcal{P}_N(t))} \right)^{\frac{1}{2}}. \tag{24}$$

Therefore, we have

$$\begin{aligned} \| W_q^\gamma \| &= \| D^\gamma \mathcal{P}(t) - Z^\gamma \mathcal{P}(t) \| \\ &\leq \left| \sum_{d=\lceil\gamma\rceil}^{f} \frac{f! g_{f-d}}{(f-d)!\Gamma(k+1-\gamma)} \right| \times \| t^{d-\gamma} - \sum_{b=1}^{N} \chi_{b,d} \mathcal{P}_b(t) \| \\ &\leq \left| \sum_{d=\lceil\gamma\rceil}^{f} \frac{f! g_{f-d}}{(f-d)!\Gamma(d+1-\gamma)} \right| \times \left(\frac{Gram(x^{d-\gamma}, \mathcal{P}_1(t), \ldots, \mathcal{P}_N(t))}{Gram(\mathcal{P}_1(t), \ldots, \mathcal{P}_N(t))} \right)^{\frac{1}{2}}. \end{aligned} \tag{25}$$

□

3. Methodology

This section constitutes the main results of this research, we discuss the algorithm of our numerical technique.

We attempt to approximate the solution of the problem,

$$min \quad \mathcal{K} = \int_0^1 \mathsf{L}(t, m(t), r(t)) dt, \tag{26}$$

with the system dynamics,

$$A\dot{m}(t) + BD_t^\gamma m(t) = f(t, m(t), r(t)), \quad 0 < \gamma \leq 1, \tag{27}$$

and the initial condition,

$$m(0) = m_0, \tag{28}$$

$A, B \neq m_0$ denote fixed real numbers, $m(t)$ and $r(t)$ are state and control variables respectively.

We express both the state and control variables as a product of coefficients to be computed and Genocchi polynomials,

$$m(t) \simeq M^T \mathcal{P}(t), \tag{29}$$
$$r(t) \simeq R^T \mathcal{P}(t). \tag{30}$$

M^T and R^T are expressed as,

$$M^T = [m_0, \cdots, m_N], \tag{31}$$
$$R^T = [r_0, \cdots, r_N]. \tag{32}$$

Introducing the derivative operator (9) in (29) means,

$$D^\gamma m(t) = X^T Z^\gamma \mathcal{P}(t). \tag{33}$$

Substituting (29) and (30) in (26) gives,

$$min \quad \mathcal{K} \simeq \int_0^1 \mathsf{L}(t, M^T \mathcal{P}(t), R^T \mathcal{P}(t))dt. \tag{34}$$

Substituting (29) and (30) in (27), thereafter applying (5) and (33), we get,

$$\begin{aligned} &AM^T D\mathcal{P}(t) + BM^T Z^\gamma \mathcal{P}(t) \\ &\quad = f(t, M^T \mathcal{P}(t), R^T \mathcal{P}(t)), \quad 0 < \gamma \leq 1. \end{aligned} \tag{35}$$

Substituting (29) in the initial condition (28) gives us,

$$M^T \mathcal{P}(0) = m_0. \tag{36}$$

We note that (36) is an equation with unknowns in M^T.

We deduce the residual $\mathcal{R}(t)$ from (35) as,

$$\begin{aligned} \mathcal{R}(t) \;\; = \;\; & AM^T D\mathcal{P}(t) + BM^T Z^\gamma \mathcal{P}(t) \\ & - f(t, M^T \mathcal{P}(t), R^T \mathcal{P}(t)). \end{aligned} \tag{37}$$

We then create a system of $N-1$ equations through the application of the Tau method as,

$$\int_0^1 \mathcal{R}(t)\mathcal{P}_b(t)dt = 0, \quad b = 1, \cdots, N-1. \tag{38}$$

The number of equations from (38) and one equation from (36) are not enough to match the number of unknowns in M^T and R^T. Thus additional equations are required, we will apply the Lagrange multipliers method to come up with more equations.
We state the Lagrange function as,

$$\begin{aligned} \mathcal{K}^*(M, R, \lambda) \;\; = \;\; & \int_0^1 \mathsf{L}(t, M^T \mathcal{P}(t), R^T \mathcal{P}(t))dt \\ & + \sum_{b=1}^{N} \lambda_b \int_0^1 \mathcal{R}(t)\mathcal{P}(t)dt, \end{aligned} \tag{39}$$

where $\lambda = [\lambda_1, \cdots, \lambda_N]^T$ are Lagrange multipliers to be determined. Imposing the extremum conditions on (39) yields,

$$\begin{aligned} &\frac{\partial \mathcal{K}^*}{\partial m_b} = 0, \quad \frac{\partial \mathcal{K}^*}{\partial r_b} = 0, \quad \frac{\partial \mathcal{K}^*}{\partial \lambda_b} = 0, \\ &b = 1, \ldots, N-1. \end{aligned} \tag{40}$$

Combining (36), equations from (38) and equations from (40), we get the sufficient number of equations that match the number of unknowns in M^T and R^T. Thus, we are able to approximate $m(t)$ and $r(t)$ in (29) and (30) respectively.

4. Illustrative examples

We support the concepts discussed in the previous section through the solution of problems and some comparison with established solutions.

Example 1. *Consider the following FOCP [11, 18]:*

$$\min \quad \mathcal{K} = \frac{1}{2} \int_0^1 3m^2(t) + r^2(t)dt, \tag{41}$$

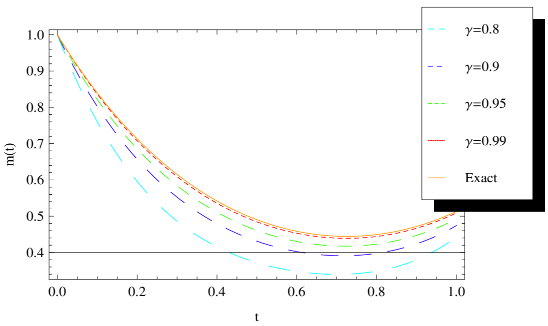
subjected to the dynamical system,

$$\begin{aligned} &\frac{1}{4}\dot{m}(t) + \frac{3}{4} D^\gamma m(t) = m(t) - r(t) \\ &m(0) = 1. \end{aligned} \tag{42}$$

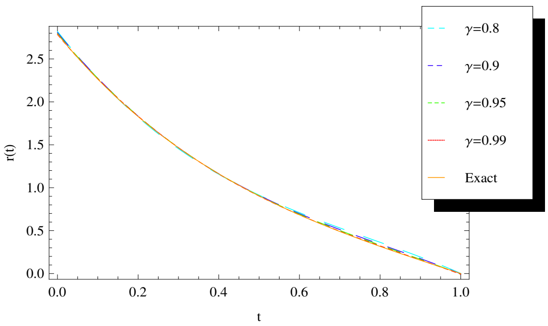
If $\gamma = 1$ the solution of (41) is,

$$\begin{aligned} m(t) &= e^{2t}(3 + e^4)^{-1}(3 + e^{4-4t}) \\ r(t) &= 3e^{2t}(3 + e^4)^{-1}(e^{4-4t} - 1) \end{aligned} \tag{43}$$

Figure 1 compares (43) with our approximate solution, as γ approaches 1, our approximate solution agrees with (43).
Figure 2, Table 1 and Table 2 display absolute errors under different conditions for Example 1. We realise that as the value of N increases, then accuracy of the proposed scheme improves.



(a) Exact and approximation solutions of $m(t)$.



(b) Exact and approximation solutions of $r(t)$.

Figure 1. Exact and approximation solutions of $m(t)$ and $r(t)$ for $N = 5$ in Example 1.

Example 2. *Suppose we have the FOCP [13],*

$$\min \quad \mathcal{K} = \int_0^1 (r(t) - m(t))^2 dt, \tag{44}$$

subjected to the dynamical system and the initial condition,

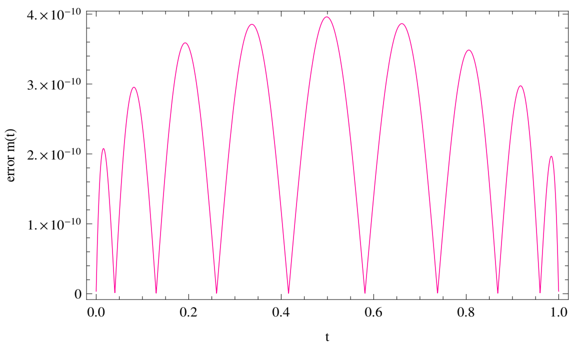
$$\begin{aligned} &\dot{m}(t) + D^\gamma m(t) = r(t) - m(t) + t^3 + \frac{6t^{\gamma+2}}{\Gamma(\gamma+3)} \\ &m(0) = 0, \end{aligned} \tag{45}$$

for $\gamma = 1$,

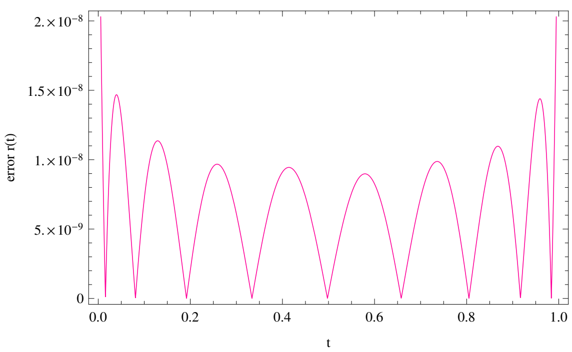
$$m(t) = r(t) = \frac{t^4}{4} \tag{46}$$

Comparison of (46) with our approximate solution is depicted in Fig. 3. Fig. 4, Table 3 and Table 4 demonstrate the behaviour of absolute errors for Example 2 under different conditions.

As we increase the number of polynomials used for approximation, the accuracy improves.

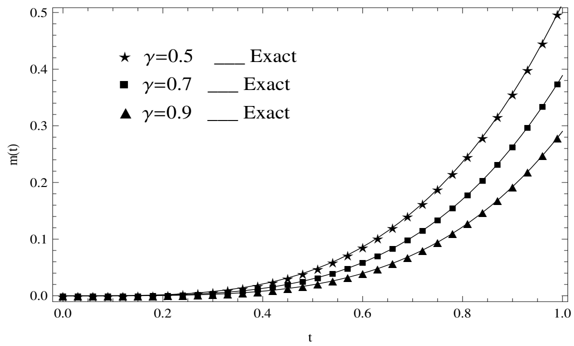


(a) Absolute errors of $m(t)$.

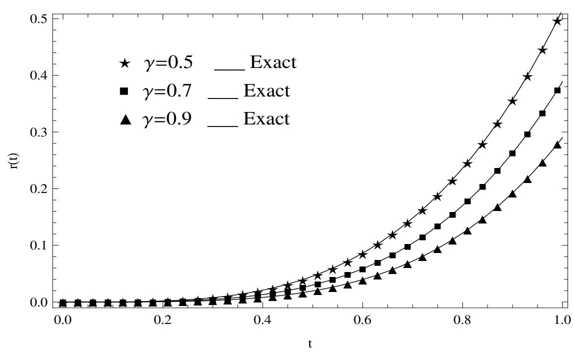


(b) Absolute errors of $r(t)$.

Figure 2. Absolute errors of $m(t)$ and $r(t)$ for $N = 10$ and $\gamma = 1$ for Example 1.



(a) Exact and approximation solutions of $m(t)$.



(b) Exact and approximation solutions of $r(t)$.

Figure 3. Exact and approximation solutions of $m(t)$ and $r(t)$ for $N = 5$ in Example 2.

Table 1. The absolute errors of $m(t)$ and $r(t)$ for $\gamma = 1$ for Example 1.

	m(t)			**r(t)**		
t	$N=5$	$N=8$	$N=10$	$N=5$	$N=8$	$N=10$
0.1	1.09006×10^{-4}	9.57294×10^{-8}	2.39863×10^{-10}	2.66755×10^{-3}	2.06794×10^{-16}	7.17614×10^{-9}
0.2	7.93386×10^{-5}	7.4338×10^{-9}	3.53015×10^{-10}	3.68719×10^{-3}	2.9584×10^{-6}	2.04521×10^{-9}
0.3	1.74739×10^{-4}	1.3894×10^{-7}	2.8304×10^{-10}	4.50409×10^{-4}	3.37535×10^{-7}	6.10702×10^{-9}
0.4	1.03393×10^{-4}	1.43259×10^{-8}	1.18485×10^{-10}	2.58382×10^{-3}	2.63025×10^{-6}	9.10892×10^{-9}
0.5	4.98148×10^{-5}	1.44398×10^{-7}	3.95863×10^{-10}	3.31745×10^{-3}	2.17556×10^{-7}	3.87368×10^{-10}
0.6	1.59367×10^{-4}	5.32508×10^{-9}	1.39608×10^{-10}	1.52384×10^{-3}	2.64159×10^{-6}	8.25309×10^{-9}
0.7	1.42128×10^{-4}	1.37905×10^{-7}	2.71924×10^{-10}	1.56668×10^{-3}	8.12583×10^{-8}	7.24176×10^{-9}
0.8	8.67009×10^{-6}	1.38793×10^{-8}	3.45382×10^{-10}	3.62268×10^{-3}	2.86573×10^{-6}	1.20762×10^{-9}
0.9	1.15034×10^{-4}	9.10652×10^{-8}	2.48459×10^{-10}	1.52344×10^{-3}	2.2886×10^{-6}	6.15546×10^{-9}
1	5.31549×10^{-6}	2.62501×10^{-12}	3.58774×10^{-12}	8.3610×10^{-3}	8.96794×10^{-6}	3.51907×10^{-8}

Table 2. Absolute errors of $\mathcal{K}$ at $\gamma = 1$ for Example 1.

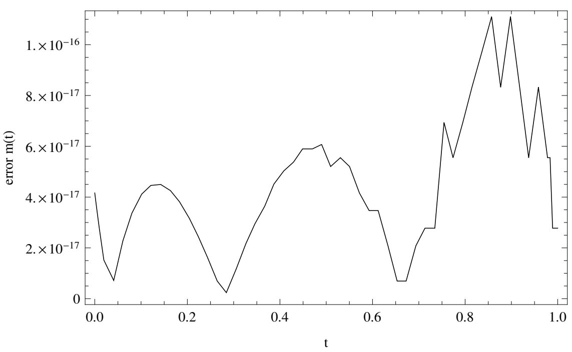
N	$N=5$	$N=8$	$N=10$
$\lvert\mathcal{K} - \mathcal{K}_N\rvert$	3.33043×10^{-7}	1.37542×10^{-10}	1.79479×10^{-12}

Table 3. Absolute errors of J at $N = 5$ for different values of γ in Example 2.

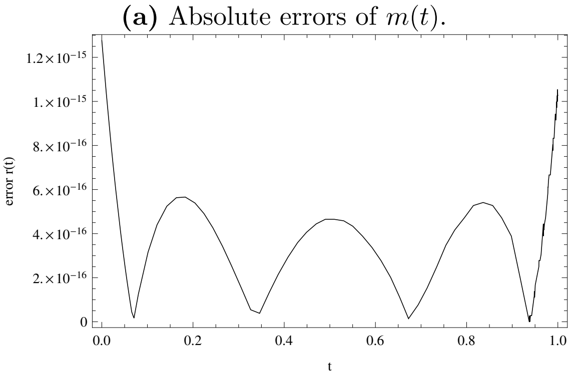
	$\gamma = 0.5$		$\gamma = 0.7$		$\gamma = 0.9$	
t	$N=5$	$N=8$	$N=5$	$N=8$	$N=5$	$N=8$
0.1	8.67932×10^{-5}	1.77534×10^{-6}	6.01988×10^{-5}	8.3122×10^{-7}	2.14671×10^{-5}	1.89079×10^{-7}
0.2	7.09407×10^{-5}	6.42261×10^{-7}	4.44603×10^{-5}	2.73675×10^{-7}	1.36285×10^{-5}	5.31956×10^{-8}
0.3	1.45414×10^{-4}	1.58479×10^{-6}	9.8158×10^{-5}	7.62797×10^{-7}	3.35696×10^{-5}	1.81782×10^{-7}
0.4	8.61693×10^{-5}	8.70422×10^{-7}	6.18471×10^{-5}	3.98654×10^{-7}	2.30861×10^{-5}	8.21489×10^{-8}
0.5	3.89408×10^{-5}	1.54044×10^{-6}	2.20596×10^{-5}	7.4929×10^{-7}	5.01391×10^{-6}	1.77558×10^{-7}
0.6	1.31492×10^{-3}	5.31372×10^{-7}	8.71043×10^{-5}	2.32235×10^{-7}	2.84443×10^{-5}	4.25814×10^{-8}
0.7	1.20904×10^{-4}	1.43896×10^{-6}	8.32356×10^{-5}	6.87549×10^{-7}	2.92973×10^{-5}	1.60014×10^{-7}
0.8	7.05935×10^{-6}	5.49734×10^{-7}	7.62429×10^{-6}	2.54608×10^{-7}	5.06112×10^{-6}	4.81956×10^{-8}
0.9	1.06771×10^{-4}	7.65855×10^{-7}	7.06895×10^{-5}	3.60429×10^{-7}	2.22565×10^{-5}	8.56972×10^{-8}
1	1.2497×10^{-5}	6.52598×10^{-8}	9.58809×10^{-6}	3.49445×10^{-8}	1.64419×10^{-6}	2.96755×10^{-9}

Table 4. Absolute errors of $\mathcal{K}$ at $N = 5$ for different values of γ in Example 2.

N	$\gamma = 0.5$	$\gamma = 0.7$	$\gamma = 0.9$	$\gamma = 1$
$\lvert\mathcal{K} - \mathcal{K}_N\rvert$	2.25994×10^{-31}	4.99374×10^{-30}	9.95394×10^{-32}	1.44241×10^{-31}



(a) Absolute errors of $m(t)$.



(b) Absolute errors of $r(t)$.

Figure 4. Absolute errors of $m(t)$ and $r(t)$ for $N = 5$ and $\gamma = 1$ for Example 2.

Example 3. *We intend to approximate the solution of the FOCP [14],*

$$\min \quad \mathcal{K} = \int_0^1 (m_1(t) - 1 - t^{\frac{3}{2}})^2 + (m_2(t) - 1 - t^{\frac{5}{2}})^2 + (r(t) - 0.75\pi^{\frac{1}{2}} + t^{\frac{5}{2}})^2 dt, \tag{47}$$

restricted to the conditions,

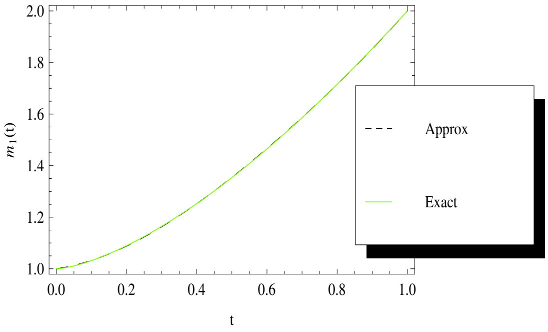
$$\begin{aligned} &\dot{m}_1(t) + D^{0.5} m_1(t) = m_2(t) + r(t) + \frac{3}{2}\sqrt{t}, \quad (48)\\ &\dot{m}_2(t) + D^{0.5} m_2(t) = \frac{5}{2} m_1(t) + \frac{15\sqrt{\pi}}{16} t^2 - \frac{5}{2},\\ &m_1(0) = 0, \quad m_2(0) = 0, \end{aligned}$$

whose exact solution is,

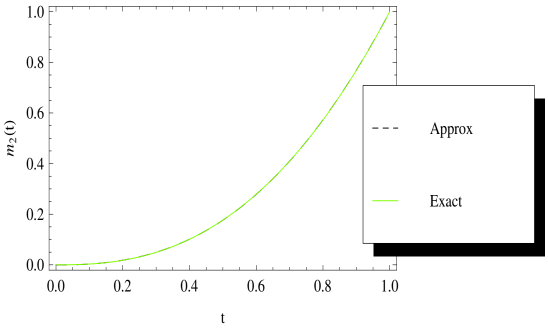
$$\begin{aligned} m_1(t) &= 1 + t^{\frac{3}{2}},\\ m_2(t) &= \sqrt{t^5}, \qquad (49)\\ r(t) &= \frac{15\sqrt{\pi}}{4} t - \sqrt{t^5}. \end{aligned}$$

Graphical comparisons of our approximate solution and (49) is depicted in Fig. 5 and fig. 6.

The absolute errors between (49) and the approximate solution are displayed in Table 5 and Table 6. Generally, the accuracy of the approximate solution improves with increasing N.



(a) Exact and approximation solutions of $m(t)$.



(b) Exact and approximation solutions of $r(t)$.

Figure 5. Exact and approximation solutions of $m_1(t)$ and $m_2(t)$ for $N = 5$ in Example 3.

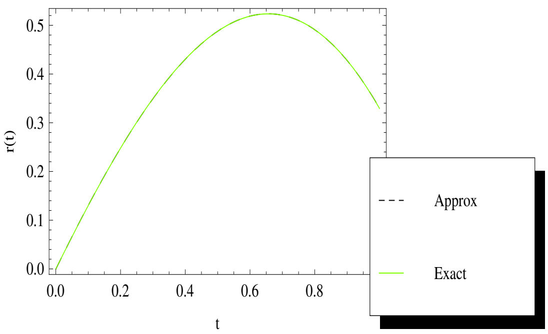


Figure 6. Exact and approximation solutions of $r(t)$ for $N = 5$ in Example 3.

5. Conclusion

We demonstrated how to apply the Genocchi polynomials in the approximation of FOCPs. The developed technique proved to give accurate and consistent results for both the state and control variables. Computed errors between our approximate solutions and the analytical solutions of specific problems were negligible, proving the accuracy of our suggested scheme. Even though the main purpose of this research was on optimal control problems of fractional order, imposing appropriate conditions on our solutions to fit that of integer order gave expected results, confirming the reliability of our approach. In light of the results emanating from this work, we are of the view that it is a worthwhile adventure to pursue the use of other polynomials in optimal control problems.

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Table 5. Absolute errors of J at $N = 5$ for different values of γ in Example 3.

	$\gamma = 0.5$		$\gamma = 0.7$		$\gamma = 0.9$	
t	$N = 5$	$N = 8$	$N = 5$	$N = 8$	$N = 5$	$N = 8$
0.1	6.82149×10^{-4}	1.42924×10^{-4}	3.77116×10^{-5}	1.47412×10^{-6}	2.9760×10^{-4}	1.45935×10^{-5}
0.2	1.3244×10^{-3}	1.83017×10^{-4}	3.27293×10^{-4}	8.71204×10^{-6}	1.31524×10^{-5}	1.67875×10^{-6}
0.3	1.52515×10^{-3}	6.9459×10^{-5}	3.0291×10^{-4}	1.57391×10^{-5}	1.99801×10^{-4}	2.93186×10^{-7}
0.4	4.62929×10^{-4}	1.2989×10^{-4}	2.04439×10^{-5}	9.14772×10^{-6}	1.73188×10^{-4}	8.39681×10^{-6}
0.5	7.85417×10^{-4}	8.84718×10^{-5}	3.58353×10^{-4}	1.12922×10^{-5}	1.98335×10^{-5}	4.46794×10^{-6}
0.6	1.3727×10^{-3}	9.80591×10^{-5}	4.62719×10^{-4}	1.49544×10^{-6}	9.6854×10^{-5}	2.31929×10^{-7}
0.7	9.56508×10^{-4}	9.28018×10^{-5}	2.35876×10^{-4}	2.50712×10^{-5}	7.70024×10^{-5}	1.72475×10^{-6}
0.8	1.72911×10^{-4}	3.6848×10^{-5}	2.02618×10^{-4}	8.3339×10^{-6}	4.89367×10^{-5}	1.97021×10^{-6}
0.9	1.02372×10^{-3}	9.84388×10^{-5}	4.72812×10^{-4}	3.47823×10^{-6}	7.70153×10^{-5}	2.24429×10^{-6}
1	1.38385×10^{-4}	2.55125×10^{-5}	9.54884×10^{-5}	9.76438×10^{-6}	4.00056×10^{-4}	1.04777×10^{-5}

Table 6. Absolute errors of $\mathcal{K}$ for different values of N in Example 3.

N	$N = 5$	$N = 8$	$N = 10$
$\lvert\mathcal{K} - \mathcal{K}_N\rvert$	1.1802×10^{-6}	4.14576×10^{-7}	2.84016×10^{-8}

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