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RESEARCH ARTICLE

Certain Saigo type fractional integral inequalities and their q-analogues

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ABSTRACT

The main purpose of the present article is to introduce certain new Saigo fractional integral inequalities and their q-extensions. We also studied some special cases of these inequalities involving Riemann- Liouville and Erdélyi-Kober fractional integral operators.

1. Introduction

Fractional calculus has great importance in mathematical analysis, also in consideration of its numerous applications in modeling and in applications. Recently there have been several significant contributions to the theory of fractional operators [1].

From last decades various types of integral inequalities have attracted the attention of many mathematicians [2–6] and also fractional integral inequalities have been found many interesting applications in the fields of engineering and physics.

Very recently in 2019, Ekinci and Ozdemir [2] have studied Hermite-Hadamard type inequalities involving intermediate values of $| f' |$ by using Riemann-Liouville fractional operator and Butt et al. [3] established some new integral inequalities involving Caputo fractional derivatives for exponential s-convex functions. In this sequence Kizil and Ardiç [4] have introduced inequalities for strongly convex functions via Atangana-Baleanu integral operators. Later in 2022, Kalsoom *et.al.* [5] proposed few new inequalities of Ostrowski type by means of newly derived identity and considered some special cases. Our present work is fully motivated by the mentioned work.

Saigo fractional integral operator is one of the important operators of fractional calculus theory due to involving the Gauss hypergeometric function ${}_2F_1(.)$. This operator has already found various applications in solving problems in the theory of special functions, integral transforms and theory of inequalities.

Before setting out the main findings of our article, it is useful to briefly review the contributions on which it is based.

We say that, fixed $v \in \mathbb{R}$ a real valued function $g(x)$ defined for $x > 0$ belongs to the function space C_v, if there exists a real number $q > v$ such

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that $g(x) = x^q\Phi(x)$, where $\Phi(x) \in C(0,1)$. Moreover, for $m \in \mathbb{R}a$ we say that $g(x)$ belongs to the function space C_v^m , if $g^m \in C_v$.

For $a > 0$, Riemann-Liouville fractional integral operator of a function g such that $g \in C_v(, v \geq -1)$ defined as follows [7]:

$$R_{0,y}^a\{g(y)\} = \frac{1}{\Gamma(a)}\int_0^y (y-t)^{a-1}g(t)dt, \tag{1}$$

here, $y > 0$.

For $a > 0$, Erdélyi-Kober fractional integral operator of a function g such that $g \in C_v(v \geq -1)$ defined as follows [8]:

$$K_{0,y}^{a,b}\{g(y)\} = \frac{y^{-a-b}}{\Gamma(a)}\int_0^y t^b(y-t)^{a-1}g(t)dt, \tag{2}$$

here, $b \in \mathbb{R}$.

For $a > 0$, Saigo fractional integral operator of a function g, such that $g \in C_v(v \geq -1)$ defined as follows [9]:

$$I_{0,y}^{a,a',b}\{g(y)\} = \frac{y^{-a-a'}}{\Gamma(a)}\times \int_0^y (y-t)^{a-1}{}_2F_1\left(a+a', -b, a, 1-\frac{t}{y}\right)g(t)dt, \tag{3}$$

here, $a', b \in \mathbb{R}$, and ${}_2F_1(r_1, r_2, r_3; z)$ is classical Gauss hypergeometric function defined in [10].

$${}_2F_1(r_1; r_2; r_3; z) = \sum_{n=0}^{\infty}\frac{(r_1)_n(r_2)_n}{(r_3)_n}\frac{z^n}{n!}, \tag{4}$$

where $(s)_k$ denotes the Pochhammer symbol (shifted factorial) defined as follows [10,11]:

$$(s)_k := \frac{\Gamma(s+k)}{\Gamma(s)} = \begin{cases} 1 & (k=0; s \in \mathbb{C}\setminus\{0\}) \\ s(s+1)\cdots(s+k-1) & (k \in \mathbb{N}; s \in \mathbb{C}) \end{cases} \tag{5}$$

By observing, we note that Saigo fractional integral operators contains both Reimann-Liouville fractional integral operators as well as Erdélyi-Kober fractional integral operators.

Remark 1. *(i)Taking $a' = -a$ in the equations* (3) *then, we get Riemann-Liouville fractional integral operator* (1).

$$R_{0,y}^a\{g(y)\} = I_{0,y}^{a,-a,b}\{g(y)\} \tag{6}$$

Remark 2. *(ii)Taking $a' = 0$ in the equations* (3) *then, we get Erdélyi-Kober fractional integral operator* (2).

$$K_{0,y}^{a,b}\{g(y)\} = I_{0,x}^{a,0,b}\{g(y\} \tag{7}$$

Fractional integral inequalities are an important tool to prove the key result, the uniqueness of solutions of fractional partial differential equations and fractional boundary value problems. Also, they give information about the boundness of the solutions of partial differential equations and fractional boundary value problems. These features have led many researchers in the area of integral inequalities to analyze some more extensions and generalizations by involving fractional calculus integral operators.

Introduce the following functional:

$$\begin{aligned} T(k,l,m,n) = &\int_c^d n(t)dt\int_c^d m(t)k(t)l(t)dt \\ &+ \int_c^d m(t)dt\int_c^d n(t)k(t)l(t)dt \\ &-\left(\int_c^d n(t)k(t)dt\right)\left(\int_c^d m(t)l(t)dt\right) \\ &-\left(\int_c^d m(t)k(t)dt\right)\left(\int_c^d n(t)l(t)dt\right) \end{aligned} \tag{8}$$

here $k, l : [c,d] \to \mathbb{R}$ are two integrable functions defined on the interval $[c,d]$ and $m(t)$ and $n(t)$ are positive integrable functions defined on $[c,d]$.

Consider two functions Φ and Ψ defined on $[c,d]$, then they are synchronous on $[c,d]$, if they satisfies the following inequality:

$$(\Phi(t) - \Phi(s))(\Psi(t) - \Psi(s)) \geq 0, \tag{9}$$

for arbitrary $t, s \in$[c,d], then from [12,13], we observe that

$$T(\Phi, \Psi, m, n) \geq 0, \tag{10}$$

If the inequality defined in (9) is reversed, then functions Φ and Ψ are called asynchronous on [c,d] and satisfies the following inequality:

. $$(\Phi(t) - \Phi(s))(\Psi(t) - \Psi(s)) \leq 0, \tag{11}$$

for any $t, s \in$[c,d].

From [14], we have the Chebyshev inequality for the special case when $m(t) = \Psi(t)$, for any t,s $\in [c,d]$, .

The functional $T(k,l,m,n)$ defined in (8) has drawn many researchers' attentions due to the wide range of applications in engineering mathematics, statistical probability, transform theory, and probability and numerical quadrature. From all these applications, the functional $T(k,l,m,n)$ (8) has also been used to produce many integral inequalities (see, e.g., [15–22]; for more details regarding very recent work, we can refer [23])

In the last few years, many researchers have more attention to the q-calculus and fractional q-differential equations due to many applications of the q-calculus in physics, statistics and mathematics. The q-calculus is also called the quantum calculus can be dated back to 1908, Jackson's work [24] and fractional q-calculus is the q-analogous of the ordinary fractional calculus. Recently, q-calculus operators have been applied in various fields like optimal control problems, ordinary fractional calculus, solutions of the q-difference (differential),q-transform analysis and q-integral equations, and many more such areas.

In 1966, Al-Salam gives the idea of fractional q-calculus by introducing the q-analogue of Cauchy's formula ([25–27]). Then, in 1969 Agrawal [28] studied some fractional q-integral operators and q-derivatives and their basic properties. Then later in 2007, Rajkovic et al. [29] extended the notion of the left fractional q-integral operators and fractional q-derivatives by introducing variable lower limit and proved the semi-group properties. In the sequence, Isogawa et al. [30] studied various basic properties of fractional q-derivatives.

For $0 <| q |< 1$ the q-shifted factorial is defined as [31]:

$$(b;q)_k = \begin{cases} 1 & (k=0) \\ \prod_{s=0}^{k-1}(1-bq^s) & (k \in \mathbb{N}), \end{cases} \tag{12}$$

here, $b, q \in \mathbb{C}$ and $b \neq q^{-l} (l \in N_0)$.

For $k \in N_0$, q-shifted factorial with negative subscript is defined as follows:

$$(b;q)_k = \frac{1}{(1-bq^{-1})(1-bq^{-2})(1-bq^{-3})...(1-bq^{-k})}. \tag{13}$$

From (12) and (13), we can conclude that:

$$(b;q)_\infty = \prod_{s=0}^{\infty}(1-bq^s), \tag{14}$$

here, $b, q \in \mathbb{C}$

By using the equations (12), (13) and (14), we observe that:

$$(b;q)_\infty = \frac{(b;q)_\infty}{(bq^k;q)_\infty}, \tag{15}$$

here, $k \in \mathbb{N}$.

Then from above equations, for any complex number β,

$$(b;q)_\beta = \frac{(b;q)_\infty}{(bq^\beta;q)_\infty}, \tag{16}$$

here, only the principal value of q^β is valid for the above equation.

For the power function $(c-d)^m$, we can define its q-analogy as follows:

$$\begin{aligned} (c-d)_q^m &= c^m \left(\frac{d}{c};q\right)_m \quad (m \in \mathbb{N}) \\ &= c^m \frac{\left(\frac{d}{c};q\right)_\infty}{\left(\frac{d}{c}q^m;q\right)_\infty} \quad (c \neq 0) \\ &= c^m \prod_{l=0}^{\infty}\left[\frac{1-\left(\frac{d}{d}\right)q^l}{1-\left(\frac{d}{c}\right)q^{l+m}}\right]. \end{aligned} \tag{17}$$

From above (17), we conclude that:

$$(c-d)_q^m = \begin{cases} 1 & (m=0) \\ (c-d)(c-dq)\cdots(c-dq^{m-1}) & (m \in \mathbb{N}). \end{cases} \tag{18}$$

In 1910, Jackson was the first researcher who introduced q-derivative and q-integral in systematic way.

The q-derivative of a function $g(x)$ is defined as [31]:

$$D_q\{g(x)\} = \frac{d_q}{d_q x}\{g(x)\} = \frac{g(qx)-g(x)}{qx-x}. \tag{19}$$

From above, we observe and notice that

$$\lim_{q \to 1} D_q\{g(x)\} = \frac{d}{dx}\{g(x)\}, \tag{20}$$

if, given function $g(x)$ is differentiable.

The q-integral of a function $g(x)$ is defined as [31]:

$$\int_0^t g(x) d_q x = t(1-q)\sum_{l=0}^{\infty} q^l g(tq^l), \tag{21}$$

$$\int_t^\infty g(x) d_q x = t(1-q)\sum_{l=0}^{\infty} q^{-l} g(tq^{-l}), \tag{22}$$

$$\int_0^\infty g(x) d_q x = t(1-q)\sum_{l=-\infty}^{\infty} q^l g(q^l). \tag{23}$$

For $0 < q < 1$, q-gamma function is given by [31]:

$$\Gamma_q(b) = \frac{(q;q)_\infty}{(q^b;q)_\infty}(1-q)^{(1-b)}. \tag{24}$$

For $b > 0$, q-analogue of Riemann-Liouville fractional integral operator of a function $g(x)$ defined as [28]:

$$R_q^b\{g(x)\} = \frac{x^{b-1}}{\Gamma_q(b)}\int_0^x \left(\frac{qt}{x};q\right)_{b-1} g(t) d_q t, \tag{25}$$

where,$(b,q)_\beta$ is given in the equation (16).

For, $a > 0$ and $b \in R$ and $0 < q < 1$, q-analogue of the Erdélyi-Kober fractional operator is defined as [28]:

$$K_q^{a,b}\{f(x)\} = \frac{x^{b-1}}{\Gamma_q(a)} \int_0^x \big(\frac{qt}{x};q\big)_{a-1} t^b f(t) d_q t. \tag{26}$$

For $a > 0$, a' and $b \in \mathbb{R}$, q-analogue of Saigo's fractional integral is defined as [32]:

$$I_q^{a,a',b}\{f(x)\} = \frac{x^{-a'-1}}{\Gamma_q(a)} \int_0^x \big(\frac{qt}{x};q\big)_{a-1} \sum_{k=0}^{\infty} \frac{(q^{a+a'};q)_k (q^{-b};q)_k}{(q^{-a};q)_k (q,q)_k} q^{(b-a')k}(-1)^k q^{-\binom{k}{2}} \big(\frac{t}{x}-1\big)_q^k f(t) d_q t \tag{27}$$

2. Certain inequalities involving Saigo type fractional integral operator

In this section, we introduce some inequalities involving the Saigo type fractional integral operator and their special cases.

Theorem 1. *Assume u and v are two positive integrable and synchronous mapping on $[0,\infty]$. Suppose $\exists$ four positive integrable mappings m_1, m_2, n_1 and n_2 such that:*

$$\begin{aligned} &0 < m_1(t) \le u(t) \le m_2(t), \\ &0 < n_1(t) \le v(t) \le n_2(t), \\ &(t \in [0,x], x > 0). \end{aligned} \tag{28}$$

Then the following inequality holds true:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1 n_2 u^2\}(x) \times I_{0,x}^{a,a',b}\{m_1 m_2 v^2\}(x) \\ &\quad \le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{(m_1 n_1 + m_2 n_2)uv\}(x)\Big)^2 \end{aligned} \tag{29}$$

Proof. By using the relations that are given in (28), for $t \in [0,x], \forall x > 0$, we can easily have:

$$\left(\frac{m_2(t)}{n_1(t)} - \frac{u(t)}{v(t)}\right) \ge 0 \tag{30}$$

$$\left(\frac{u(t)}{v(t)} - \frac{m_1(t)}{n_2(t)}\right) \ge 0 \tag{31}$$

If we product (30) and (31) side by side , we can write

$$\left(\frac{m_2(t)}{n_1(t)} - \frac{u(t)}{v(t)}\right)\left(\frac{u(t)}{v(t)} - \frac{m_1(t)}{n_2(t)}\right) \ge 0$$

Then we have:

$$\begin{aligned} &\big(m_1(t)n_1(t) + m_2(t)n_2(t)\big)u(t)v(t) \\ &\ge n_1(t)n_2(t)u^2(t) + m_1(t)m_2(t)v^2(t). \end{aligned} \tag{32}$$

Consider the following function $F(x,t)$ defined by:

$$\begin{aligned} F(x,t) =& \frac{x^{-a-a'}(x-t)^{a-1}}{\Gamma(a)} \\ &\times {}_2F_1\left(a+a', -b, a, 1-\frac{t}{x}\right), \\ &(t \in (0,x); x > 0). \end{aligned} \tag{33}$$

Then multiplying both sides of (32), by $F(x,t)$ defined by (33) and integrating the resulting inequality with respect to t from 0 to x and using the definition (3), we have:

.

$$\begin{aligned} &I_{0,x}^{a,a',b}\{(m_1 n_1 + m_2 n_2)uv\}(x) \\ &\ge I_{0,x}^{a,a',b}\{(n_1 n_2)u^2\}(x) + I_{0,x}^{a,a',b}\{(m_1 m_2)v^2\}(x) \end{aligned} \tag{34}$$

Let us recall the A.M -G.M inequality, i.e $(a+b) \ge 2\sqrt{ab}$, $a, b \in \mathbb{R}^+$. By applying this classical inequality to (34), we obtain:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{(m_1 n_1 + m_2 n_2)uv\}(x) \\ &\ge 2\sqrt{I_{0,x}^{a,a',b}\{(n_1 n_2)u^2\}(x) \times I_{0,x}^{a,a',b}\{(m_1 m_2)v^2\}(x)} \end{aligned} \tag{35}$$

By making use of some necessary operations, we deduce that:

$$\begin{aligned} &I_{0,x}^{a,a',b}\{n_1 n_2 u^2\}(x) \times I_{0,x}^{a,a',b}\{m_1 m_2 v^2\}(x) \\ &\quad \le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{(m_1 n_1 + m_2 n_2)uv\}(x)\Big)^2 \end{aligned} \tag{36}$$

This complete the proof of Theorem 1. □

If we substitute $a' = -a$ and $a' = 0$ in above results we get following special cases of the inequalities respectively.

Corollary 1. *For Riemann-Liouville fractional integral operator the following inequality holds true:*

$$\begin{aligned} &R_{0,x}^{a}\{n_1 n_2 u^2\}(x) \times R_{0,x}^{a}\{m_1 m_2 v^2\}(x) \\ &\quad \le \frac{1}{4}\Big(R_{0,x}^{a}\{(m_1 n_1 + m_2 n_2)uv\}(x)\Big)^2 \end{aligned} \tag{37}$$

Corollary 2. *For Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$\begin{aligned} &K_{0,x}^{a,b}\{n_1 n_2 u^2\}(x) \times K_{0,x}^{a,b}\{m_1 m_2 v^2\}(x) \\ &\quad \le \frac{1}{4}\Big(K_{0,x}^{a,b}\{(m_1 n_1 + m_2 n_2)uv\}(x)\Big)^2. \end{aligned} \tag{38}$$

Theorem 2. *Consider u and v are two positive integrable and synchronous mapping on $[0,\infty]$. Assume $\exists$ four positive integrable mapping m_1, m_2, n_1 and n_2 such that:*

$$\begin{aligned}&0 < m_1(t) \le u(t) \le m_2(t),\\ &0 < n_1(t) \le v(t) \le n_2(t),\\ &(t \in [0,x], x > 0).\end{aligned} \tag{39}$$

Then the following inequality holds true:

$$\begin{aligned}&I_{0,x}^{a,a',b}\{n_1n_2\}(x)I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\quad\times I_{0,x}^{a,a',b}\{m_1m_2\}(x)I_{0,x}^{a,a',b}\{v^2\}(x)\\ &\quad\le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{n_1v\}(x)I_{0,x}^{a,a',b}\{m_1u\}(x)\\ &\quad+I_{0,x}^{a,a',b}\{n_2v\}(x)I_{0,x}^{a,a',b}\{m_2u\}(x)\Big)^2.\end{aligned} \tag{40}$$

Proof. With similar steps to the proof of the previous Theorem, if we consider the inequalities are given in (39), we have

$$\left(\frac{m_2(t)}{n_1(s)} - \frac{u(t)}{v(s)}\right) \ge 0, \tag{41}$$

$$\left(\frac{u(t)}{v(s)} - \frac{m_1(t)}{n_2(s)}\right) \ge 0. \tag{42}$$

Then we can write above inequality as the following form:

$$\left(\frac{m_1(t)}{n_2(s)} + \frac{m_2(t)}{n_1(s)}\right)\frac{u(t)}{v(s)} \ge \frac{u^2(t)}{v^2(s)} + \frac{m_1(t)m_2(t)}{n_1(s)n_2(s)}. \tag{43}$$

If we multiply both sides of (43), by $n_1(s)n_2(s)v^2(s)$, we get

$$\begin{aligned}&m_1(t)u(t)n_1(s)v(s) + m_2(t)u(t)n_2(s)v(s)\\ &\quad\ge n_1(s)n_2(s)u^2(t) + m_1(t)m_2(t)v^2(s).\end{aligned} \tag{44}$$

Then on multiplying both sides of the equation (44), by $F(x,t)$ defined in (33) and integrating with respect to t from 0 to x, and using the definition (3), we have

$$\begin{aligned}&n_1(s)v(s)I_{0,x}^{a,a',b}\{m_1u\}(x)\\ &\quad+n_2(s)v(s)I_{0,x}^{a,a',b}\{m_2u\}(x)\\ &\quad\ge n_1(s)n_2(s)I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\quad\quad+v^2(s)I_{0,x}^{a,a',b}\{m_1m_2\}(x).\end{aligned} \tag{45}$$

Again multiplying both sides of the equation (45), by $F(x,s)$ defined in (33), a nd integrating with respect to s from 0 to x and using the definition (3), we have

$$\begin{aligned}&I_{0,x}^{a,a',b}\{n_1v\}(x)I_{0,x}^{a,a',b}\{m_1u\}(x)\\ &+I_{0,x}^{a,a',b}\{n_2v\}(x)I_{0,x}^{a,a',b}\{m_2u\}(x)\\ &\ge I_{0,x}^{a,a',b}\{n_1n_2\}(x)I_{0,x}^{a,a',b}\{u^2\}(x)\\ &+I_{0,x}^{a,a',b}\{v^2\}(x)I_{0,x}^{a,a',b}\{m_1m_2\}(x).\end{aligned} \tag{46}$$

Now, using the AM-GM inequality, we have:

$$\begin{aligned}&I_{0,x}^{a,a',b}\{n_1v\}(x)I_{0,x}^{a,a',b}\{m_1u\}(x)\\ &+I_{0,x}^{a,a',b}\{n_2v\}(x)I_{0,x}^{a,a',b}\{m_2u\}(x)\\ &\qquad\ge 2\Big\{I_{0,x}^{a,a',b}\{n_1n_2\}(x)I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\qquad\quad\times I_{0,x}^{a,a',b}\{v^2\}(x)I_{0,x}^{a,a',b}\{m_1m_2\}(x)\Big\}^{\frac{1}{2}}.\end{aligned} \tag{47}$$

By making use of some necessary operations, we deduce that:

$$\begin{aligned}&I_{0,x}^{a,a',b}\{n_1n_2\}(x)I_{0,x}^{a,a',b}\{u^2\}(x)\\ &\times I_{0,x}^{a,a',b}\{m_1m_2\}(x)I_{0,x}^{a,a',b}\{v^2\}(x)\\ &\le \frac{1}{4}\Big(I_{0,x}^{a,a',b}\{n_1v\}(x)I_{0,x}^{a,a',b}\{m_1u\}(x)\\ &+I_{0,x}^{a,a',b}\{n_2v\}(x)I_{0,x}^{a,a',b}\{m_2u\}(x)\Big)^2.\end{aligned} \tag{48}$$

This proofs the Theorem (2). □

On putting $a' = -a$ and $a' = 0$ in above results we get following special cases of the inequalities respectively.

Corollary 3. *For Riemann-Liouville fractional integral operator the following inequality holds true:*

$$\begin{aligned}&R_{0,x}^{a}\{n_1n_2\}(x)R_{0,x}^{a}\{u^2\}(x)\\ &\times R_{0,x}^{a}\{m_1m_2\}(x)R_{0,x}^{a}\{v^2\}(x)\\ &\le \frac{1}{4}\Big(R_{0,x}^{a}\{n_1v\}(x)R_{0,x}^{a}\{m_1u\}(x)\\ &+R_{0,x}^{a}\{n_2v\}(x)R_{0,x}^{a}\{m_2u\}(x)\Big)^2\end{aligned} \tag{49}$$

Corollary 4. *For Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$
\begin{aligned}
&K_{0,x}^{a,b}\{n_1n_2\}(x)K_{0,x}^{a,b}\{u^2\}(x)\\
&\times K_{0,x}^{a,b}\{m_1m_2\}(x)K_{0,x}^{a,b}\{v^2\}(x)\\
&\leq \frac{1}{4}\bigg(K_{0,x}^{a,b}\{n_1v\}(x)K_{0,x}^{a,b}\{m_1u\}(x)\\
&+K_{0,x}^{a,b}\{n_2v\}(x)K_{0,x}^{a,b}\{m_2u\}(x)\bigg)^2
\end{aligned}
\tag{50}
$$

3. Saigo type fractional q-integral inequalities

Here, we established some q-integral inequalities involving q-Saigo type fractional integral operator which are the q-analogues of the Theorems proved in the previous section.

Theorem 3. *Consider $0 < q < 1$, let u and v are two positive integrable and synchronous mapping on $[0,\infty]$. Assume $\exists$ four positive integrable mapping m_1, m_2, n_1 and n_2 such that:*

$$
\begin{aligned}
&0 < m_1(t) \leq u(t) \leq m_2(t),\\
&0 < n_1(t) \leq v(t) \leq n_2(t),\\
&(t \in [0,x], x > 0).
\end{aligned}
\tag{51}
$$

Then the following inequality holds true:

$$
\begin{aligned}
&I_q^{a,a',b}\{(m_1n_1+m_2n_2)uv\}(x)\\
&\geq I_q^{a,a',b}\{(n_1n_2)u^2\}(x)\\
&+I_q^{a,a',b}\{(m_1m_2)v^2\}(x).
\end{aligned}
\tag{52}
$$

Proof. To prove our result we need to recall function with their conditions defined by Choi [33],

$$
\begin{aligned}
H(t,x,u(x);a,a',b;q) &= \frac{x^{-a'-1}}{\Gamma_q(a)}\big(\frac{qt}{x},q\big)_{a-1}\\
&\sum_{k=0}^{\infty}\frac{(q^{a+a'},q)_k(q^{-b},q)_k}{(q^{-a},q)_k(q,q)_k}\\
&\times q^{(b-a')k}(-1)^k q^{-\binom{k}{2}}\big(\frac{t}{x}-1\big)_q^k u(t)
\end{aligned}
\tag{53}
$$

where $x > 0, 0 \leq t \leq x$; $a > 0$, $a', b \in \mathbb{R}$ with $a + a' > 0$ and $b < 0$, $0 < q < 1$, $u : [0,\infty) \to [0,\infty)$ it is seen that

$$
H(t,x,u(x);a,a',b;q) \geq 0. \tag{54}
$$

Then from (44), we have

$$
\begin{aligned}
&\big(m_1(t)n_1(t)+m_2(t)n_2(t)\big)u(t)v(t)\\
&\geq n_1(t)n_2(t)u^2(t)+m_1(t)m_2(t)v^2(t).
\end{aligned}
\tag{55}
$$

Now multiplying both sides of (55) by $H(t,x,1;a,a',b;q)$ given in (53) together with (54) and taking q-integration with respect to t from 0 to x with aid of (27), we get our desired result.

$$
\begin{aligned}
&I_q^{a,a',b}\{(m_1n_1+m_2n_2)uv\}(x)\\
&\geq I_q^{a,a',b}\{(n_1n_2)u^2\}(x)+I_q^{a,a',b}\{(m_1m_2)v^2\}(x)
\end{aligned}
\tag{56}
$$

□

If we substitute $a' = -a$ and $a' = 0$ in above results we get following special cases of the inequalities respectively.

Corollary 5. *For q-analogue of Riemann-Liouville fractional integral operator the following inequality holds true:*

$$
\begin{aligned}
&R_q^{a}\{(m_1n_1+m_2n_2)uv\}(x)\\
&\geq R_q^{a}\{(n_1n_2)u^2\}(x)+R_q^{a}\{(m_1m_2)v^2\}(x)
\end{aligned}
\tag{57}
$$

Corollary 6. *For q-analogue of Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$
\begin{aligned}
&K_q^{a,b}\{(m_1n_1+m_2n_2)uv\}(x)\\
&\geq K_q^{a,b}\{(n_1n_2)u^2\}(x)+K_q^{a,b}\{(m_1m_2)v^2\}(x)
\end{aligned}
\tag{58}
$$

Theorem 4. *Let $0 < q < 1$, consider u and v are two positive integrable and synchronous mapping on $[0,\infty]$. Assume $\exists$ four positive integrable mapping m_1, m_2, n_1 and n_2 such that:*

$$
\begin{aligned}
&0 < m_1(t) \leq u(t) \leq m_2(t),\\
&0 < n_1(t) \leq v(t) \leq n_2(t),\\
&(t \in [0,x], x > 0).
\end{aligned}
\tag{59}
$$

Then the following inequality holds true:

$$
\begin{aligned}
&I_q^{a,a',b}\{n_1v\}(x)I_q^{a,a',b}\{m_1u\}(x)\\
&+I_q^{a,a',b}\{n_2v\}(x)I_q^{a,a',b}\{m_2u\}(x)\\
&\geq I_q^{a,a',b}\{n_1n_2\}(x)I_q^{a,a',b}\{u^2\}(x)\\
&+I_q^{a,a',b}\{v^2\}(x)I_q^{a,a',b}\{m_1m_2\}(x).
\end{aligned}
\tag{60}
$$

Proof. From (44), we have

$$
\begin{aligned}
&m_1(t)u(t)n_1(s)v(s)+m_2(t)u(t)n_2(s)v(s)\\
&\geq n_1(s)n_2(s)u^2(t)+m_1(t)m_2(t)v^2(s).
\end{aligned}
\tag{61}
$$

Then on multiplying both sides of the equation (61), by $H(t,x,1;a,a',b;q)$ defined in (53) together with (54) and taking q-integration with respect to t from 0 to x with aid of (27)

$$\begin{aligned} & n_1(s)v(s)I_q^{a,a',b}\{m_1u\}(x) \\ & \quad + n_2(s)v(s)I_q^{a,a',b}\{m_2u\}(x) \\ & \quad \geq n_1(s)n_2(s)I_q^{a,a',b}\{u^2\}(x) \\ & \quad\quad + v^2(s)I_q^{a,a',b}\{m_1m_2\}(x) \end{aligned} \tag{62}$$

Again multiplying both sides of the equation (62), by $H(t,x,1;a,a',b;q)$ defined in (53) together with (54) and taking q-integration with respect to s from 0 to x with aid of (27), we get our desired result.

$$\begin{aligned} & I_q^{a,a',b}\{n_1v\}(x)I_q^{a,a',b}\{m_1u\}(x) \\ & +I_q^{a,a',b}\{n_2v\}(x)I_q^{a,a',b}\{m_2u\}(x) \\ & \geq I_q^{a,a',b}\{n_1n_2\}(x)I_q^{a,a',b}\{u^2\}(x) \\ & \quad\quad +I_q^{a,a',b}\{v^2\}(x)I_q^{a,a',b}\{m_1m_2\}(x) \end{aligned} \tag{63}$$

□

By setting $a' = -a$ and $a' = 0$ in above results we get following special cases of the inequalities respectively.

Corollary 7. *For q-analogue of Riemann-Liouville fractional integral operator the following inequality holds true:*

$$\begin{aligned} & R_q^a\{n_1v\}(x)R_q^a\{m_1u\}(x) \\ & +R_q^a\{n_2v\}(x)R_q^a\{m_2u\}(x) \\ & \geq R_q^a\{n_1n_2\}(x)R_q^a\{u^2\}(x) \\ & \quad\quad +R_q^a\{v^2\}(x)R_q^a\{m_1m_2\}(x) \end{aligned} \tag{64}$$

Corollary 8. *For q-analogue of Erdélyi-Kober fractional integral operator the following inequality holds true:*

$$\begin{aligned} & K_q^{a,b}\{n_1v\}(x)K_q^{a,b}\{m_1u\}(x) \\ & +K_q^{a,b}\{n_2v\}(x)K_q^{a,b}\{m_2u\}(x) \\ & \geq K_q^{a,b}\{n_1n_2\}(x)K_q^{a,b}\{u^2\}(x) \\ & \quad\quad +K_q^{a,b}\{v^2\}(x)K_q^{a,b}\{m_1m_2\}(x) \end{aligned} \tag{65}$$

4. Concluding remark

We summarize our research work by mentioning that all the results derived in this paper are novel and important. Firstly, we have established certain inequalities involving Saigo type fractional integral operator and derived some special cases of it. Then we have derived q-analogues of the inequalities involving Saigo type fractional integral operator that means certain q-integral inequalities. Some special cases of q-integral inequalities are also derived. We also notice that when q approaches to 1 then the resulting inequalities presented in Section 3, are become those demonstrated in Section 2.

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