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RESEARCH ARTICLE

The processes with fractional order delay and PI controller design using particle swarm optimization

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ABSTRACT

In this study, the stability analysis of systems with fractional order delay is presented. Besides, PI controller design using particle swarm optimization (PSO) technique for such systems is also presented. The PSO algorithm is used to obtain the controller parameters within the stability region. As it is known that it is not possible to investigate the stability of systems with fractional order delay using analytical methods such as the Routh-Hurwitz criterion. Furthermore, stability analysis of such systems is quite difficult. In this study, for stability testing of such systems, an approximation method previously introduced in the literature by the corresponding author is used. In addition, the unit step responses have been examined to evaluate the systems' performances. It should be noted that examining unit step responses of systems having fractional-order delay is not possible due to the absence of analytical methods. One of the aims of this study is to overcome this deficiency by using the proposed approximation method. Besides, a solution to the question of which controller parameter values should be selected in the stability region, which provides the calculation of all stabilizing PI controllers, is proposed using the PSO algorithm.

1. Introduction

In practice, many dynamic systems cannot be satisfactorily modeled with ordinary differential equations. Actually, in many systems, the future behaviors of state variables depend on both their current values and their past values [1]. Such systems are called time delay systems. Time delay systems can occur in practice for many reasons. So, many processes contain dead time in their inner dynamics. Due to the increasing demands of dynamic performances, we need models behaving more like the real process. Therefore, the notion of time delay keeps on growing attraction for many scientific disciplines such as control engineering.

Analysis and control of systems having time delay are more complicated than integer order ones [2]. However, describing systems without using time delay component may lead to incorrect conclusions in terms of evaluating and obtaining desired control aims. Thus, analysis and control of time delay systems are very important, and considerable attention and effort have been given to the stability of linear time delay systems during the past 30 years [1, 3] (and references therein). It is still an active research area in the literature. However, to the best knowledge of the authors of this article, studies related to the stability of control systems having fractional order time delay are not extensive. Only particular cases such as a special class of distributed parameter systems have been given in [4–7].

Distributed parameters and/or delay elements described by partial differential equations can be seen in many industrial systems such as fluid lines, transmission lines, nuclear rocket engines,

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diffusion processes and chemical processes etc. [5–8]. The distributed parameter approach provides more accurate design results than lumped parameter approach [6]. Transfer functions of distributed parameter systems contain $\sqrt{s}$ or e^{-hs}, $e^{-(ks^{0.5})}$ functions, where, real $h \geq 0$ and $k \geq 0$ stand for time delay and distributed lag, respectively [6,8]. Calculating the inverse Laplace transforms of these functions is an extremely complex matter [8]. Distributed parameter systems can be considered as a special form of fractional order systems when $\sqrt{s}$ term is used. The stability analysis of systems with distributed parameters, i.e., fractional order systems has some challenges since their mathematical descriptions have irrational functions of "s" [5, 6]. The analytic methods to evaluate time response of irrational transfer functions are inadequate. Furthermore, to solve this problem, used graphical methods are inaccurate [8]. These are important shortcomings of the works have been done on this topic. In the literature, there are some important studies. In [9], necessary and sufficient conditions for the B.I.B.O. (Bounded Input Bounded Output) stability and the asymptotic stability of systems whose transfer functions are functions of s, $\sqrt{s}$, and $e^{\sqrt{Ts}}$ are established, where T is a positive constant. An algorithm for the inverse Laplace transform to obtain time response of irrational transfer functions is developed in [8] by using the Fast Fourier transform. The Hurwitz stability test is extended to lumped-distributed RC networks in [10]. An algebraic test procedure such as Routh algorithm and Hurwitz determinant is improved to a certain class of distributed parameter systems with multiple delays in [4, 5]. Two-dimensional stability criterion to a special class of distributed parameter systems has been studied in [6]. It has been showed that the conditions of stability for such systems are independent of time delay and distributed lag [6]. Similarly, a stability test independent of distributed lag and another stability test to find the intervals of distributed lag are proposed for a special class of distributed parameter systems in [7]. First order plus fractional diffusive delay is studied in [11]. However, the works in this area are mostly focused on systems containing terms e^{-hs} or $\sqrt{s}$. The current studies on $e^{-\sqrt{s}}$ consist of complex stability test procedures. What is more, the time-response analysis of such systems is quite a few [12] One of the motivations of this study is to examine the unit step responses of systems having fractional order delay. Various methods have been used in the literature to determine the parameters of PI controllers for time-delay systems [12–16]. One of these methods is the weighted geometrical center method, and the other is the centroid of the convex stability region method. In this study, a method based on the centroid of the convex stability region is presented. In this new method, the most optimum PI controller parameters are obtained by creating a triangular area under the stability curve of the system and searching this area with PSO algorithm. The first application of this method can be found in [17].

This paper is organised as follows. A brief introduction of fractional order calculus and fractional order systems are given in section 2. Fractional order systems with fractional order delay are also introduced in section 2. Brief information about particle swarm optimization is given in section 3. PI controller design for systems with fractional order delay and a stability test procedure for such systems are given in section 4. Finally, numerical examples are given in section 5.

2. Fractional order calculus and fractional order systems

Fractional order calculus, namely, non-integer order calculus of control systems is gaining more and more attention from many science disciplines. The notion of non-integer order calculus which is related to the development of regular calculus has been known for 300 years [18]. But it has mostly remained as a subject studied by prominent mathematicians owing to its complex structure. There are studies in various fields related to fractional expressions, especially in mathematics [11, 19–22]. Since it requires advanced mathematical analysis techniques, engineers and other science disciplines could not use it effectively till the development of analyzing and solution methods [18]. After obtaining important achievements in fractional calculus recently, the real order of dynamic systems can be investigated. In fractional calculus, the order of derivatives or integrals can be real or complex number [11]. Thus, the order of the fractional integrals and derivatives can be considered as a function of time or some other variable [23]. Using the fractional order models to describe real-world systems has some advantages in terms of both having more degrees of freedom in the model and having an unlimited memory which is very important to predict and influence present and future behaviors. Fractional order systems and their applications are one of the most popular research topics of today. Recently, it has been reported that fractional order representation is more accurate to describe real world systems than those of integer order models since the real-world processes are generally and/or most likely fractional order [24]. It is known that using integer order model to define a system can lead to distinction between mathematical model and actual system [24].

A fractional order system (FOS) has transfer function consisting of fractional order derivatives s^{α}, where $\alpha \in \mathbb{R}$. In the literature, many studies conducted on FOS use integer order approximations due to the lack of analytical solution methods. There are some integer order approximation methods of FOS such as continued fraction expansion, Oustaloup etc. (Details can be found in [25]). Stability analysis of such systems is one of the most challenging problems. To the best knowledge of the authors, there are no analytical stability test procedures such as Routh that can be applied to such systems, directly. Although using integer order approximations provide the stability analysis of such systems, time domain analysis has remained the most challenging and important problem. However, some important studies to obtain inverse Laplace transform and time response of FOS have been studied in [26, 27], recently.

2.1. Fractional order systems with fractional order delay

A system represented by a differential equation where the orders of derivatives can take any real number not necessarily integer number can be considered FOS. Thus, FOS can be defined by the fractional-order transfer function with fractional order time delay. The transfer function of the system non-integer order time delay is defined by

$$\begin{aligned} G_p(s) &= G(s)e^{-\sqrt{\tau s}} = \frac{N(s)}{D(s)}e^{-\sqrt{\tau s}} \\ &= \frac{b_m s^{\beta m} + b_{m-1}s^{\beta m-1} + \ldots + b_0 s^{\beta_0}}{a_n s^{\gamma n} + a_{n-1}s^{\gamma n-1} + \ldots + a_0 s^{\gamma_0}} e^{-\sqrt{\tau s}} \end{aligned} \tag{1}$$

or in general form, it can be described as follows.

$$\begin{aligned} G_p(s) &= G(s)e^{-(\tau s)^{\alpha}} = \frac{N(s)}{D(s)}e^{-(\tau s)^{\alpha}} \\ &= \frac{b_m s^{\beta m} + b_{m-1}s^{\beta m-1} + \ldots + b_0 s^{\beta_0}}{a_n s^{\gamma n} + a_{n-1}s^{\gamma n-1} + \ldots + a_0 s^{\gamma_0}} e^{-(\tau s)^{\beta}} \end{aligned} \tag{2}$$

where τ is fractional order time delay, a_k ($k = 0, ..., n$) and b_k ($k = 0, ..., m$) are constants, γ_k ($k = 0, ..., n$) and β_k ($k = 0, ..., m$) are arbitrary real numbers. And, also $\beta_m > \beta_{m-1} > .... > \beta_0$, $\gamma_n > \gamma_{n-1} > .... > \gamma_0$ without loss of generality. As stated before, the studies related to systems with fractional order time delay are not extensive. Thus, new studies need to be done on this research topic.

3. Particle swarm pptimization

Particle Swarm Optimization (PSO) is a powerful metaheuristic optimization technique based on the movement and intelligence of swarms. PSO algorithm is inspired by flocks of birds and schools of fish in nature. For instance, when birds flying and searching randomly for food, they help each other in the flock to find the best food place. In 1995, Dr. Kennedy and Dr. Eberhart have discovered PSO algorithm by examining the behavior of bird flocks [28].

The PSO algorithm can consider like a flock of birds. Particles come together to form a swarm. PSO algorithm can find problems' minimum or maximum value. In other words, it is finding the optimum value of the problem. PSO algorithm has individuals also referred to as particles. These particles are solution sets of a problem. In PSO, particles generate randomly between problems boundaries.

In the PSO algorithm, it is necessary to evaluate whether the particles are suitable for the result according to a certain criterion. This is done by the fitness function. The fitness function tests the fitness of particles. In some previous studies, performance indexes such as "ISE", "IAE", "ITSE", "ITAE" were used as fitness functions [29–31]. These fitness functions are chosen according to the problem. If it is desired to reach the minimum point in the problem, the best particle of the solution set, which gives the minimum value of the fitness function, is selected. However, if it is desired to reach the maximum point in the problem, the maximum value of the fitness function should be selected. PSO algorithm is an iterative algorithm, so it needs to be updated some parameters about the problems. There are two updates in the PSO algorithm. The first one is velocity update and the second one is position update. The velocity update formula is given in Eq. (3) [28].

$$v_{ij} = \epsilon * v_{ij} + c_1 * r_1 * (x_{ij}^{Pb} - x_{ij}) + c_2 * r_2 * (x_{ij}^{Sb} - x_{ij}) \tag{3}$$

ϵ: Coefficient of inertia
c_1: Cognitive coefficient
c_2: Social coefficient
r_1, r_2: Random coefficient(0-1)
x_{ij}: Position of particle
x_{ij}^{Pb}: Position of the best particle
x_{ij}^{Sb}: Position of the best of the swarm

The velocity equation can be handled in three different parts. In the first part, every particle has inertia and wants to maintain its motion. The expression $\epsilon * v_{ij}$ is used to express this situation. In the second part, the particle wants to reach its best position. The expression $c_1 * r_1 * (x_{ij}^{Pb} - x_{ij})$ is used to express this situation. In the last part, the particle seeks to reach the best position of the swarm. The expression $c_2 * r_2 * (x_{ij}^{Sb} - x_{ij})$ is used to express this. The combination of all these components gives us a new velocity. This process is done for all particles. After the velocity update, the position update is done.

$$\Delta v = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{1} \Rightarrow \Delta v = \Delta x \tag{4}$$

Position update formula is given in Eq. (4). In this equation, one unit time change in the PSO algorithm is equal to one iteration cycle. Thus Eq. (5) is used when updating the position [28].

$$x_{ij} = x_{ij} + v_{ij} \tag{5}$$

After the position update, the PSO algorithm tests the new positions of the swarm to avoid leaving the determined search space. For points outside the search space, a correction is made so that they fall back into the search space. Otherwise, the algorithm may give incorrect results when it leaves the search space. When the iteration is complete or the PSO algorithm satisfies the stopping condition, the outputs of the PSO algorithm are the best solutions for a problem.

4. PI controller design for systems with fractional order delay and stability analysis

PID controllers are the most common controller type in practical systems due to their simple structure. And, they have been applied to many complex systems. They are widely used in practice even today despite significant development in control theory [13]. A large number of studies have been carried out to determine appropriate parameters for these popular controllers and some methods have been developed in [32–34]. In general, the studies to obtain optimum controller parameters are still in progress and the concept of the best approach is not yet available. Thus, it is still a research topic for control engineering. In this section, PI controller design is presented for the systems having fractional order time delay. To obtain stabilizing controller parameters, the PSO algorithm has been combined with the centroid of the convex stability region concept based on the stability boundary locus method [33]. To explain PI controller design procedure, first, we need to obtain some equations. Consider the single input single output (S.I.S.O.) control system as shown in Figure 1

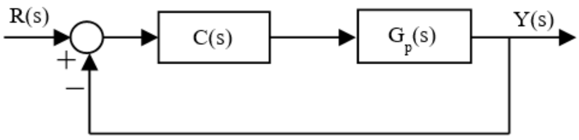

Figure 1. A S.I.S.O. control system.

where

$$G_p(s) = G(s)e^{-(\tau s)^\alpha} = \frac{N(s)}{D(s)}e^{-(\tau s)^\alpha} \tag{6}$$

is the plant to be controlled and C(s) is a PI controller of the form

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s} \tag{7}$$

The closed loop characteristic polynomial $\Delta(s)$ of the system of Figure 1, i.e. the numerator of $1 + C(s)G_p(s)$ can be written as

$$\Delta(s) = sD(s) + (k_p s + k_i)N(s)e^{-(\tau s)^\alpha} \tag{8}$$

Separating the numerator and the denominator polynomials of $G(s)$ in Eq. (6) into even and odd parts, and substituting $s = j\omega$ in the equation provides the following

$$G(j\omega) = \frac{N_e(-\omega^2) + j\omega N_o(-\omega^2)}{D_e(-\omega^2) + j\omega D_o(-\omega^2)} \tag{9}$$

In the rest of the paper, $(-\omega^2)$ notation will not be used in the following equations for the simplicity. Using Eq. (10), Eq. (12) can be obtained instead of Eq. (11).

$$(j\omega)^\alpha = \omega^\alpha(\cos\frac{\pi}{2}\alpha + j\sin\frac{\pi}{2}\alpha) \tag{10}$$

$$e^{-(s\tau)^\alpha} = e^{-(j\omega)^\alpha \tau^\alpha} \tag{11}$$

$$\begin{aligned} &e^{-\left[(\cos\frac{\pi}{2}\alpha + j\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha\right]} \\ &= e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha - j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \\ &= e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha}.e^{-j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \end{aligned} \tag{12}$$

Where the first term which is a constant, and the second term can be written as follows, respectively.

$$e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \tag{13}$$

$$e^{-j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} \tag{14}$$

The second term can be rearranged as in Eq. (15).

$$e^{-j(\sin\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} = \cos[(\sin\tfrac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha] - j\sin[(\sin\tfrac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha] \tag{15}$$

Substituting Eqs. (16) and (17) in Eq. (11) the closed loop characteristic polynomial of Eq. (8) can be written as in Eq. (18)

$$e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha} = e^{-m} \tag{16}$$

$$\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha) = n \tag{17}$$

$$\begin{aligned}\Delta(j\omega) &= -\omega^2 D_o - \omega^2 k_p N_o e^{-m}\cos(n)\\ &+k_i N_e e^{-m}\cos(n) + \omega k_p N_e e^{-m}\sin(n)\\ &+\omega k_i N_o e^{-m}\sin(n)\\ &+j[\omega k_p N_e e^{-m}\cos(n) + \omega k_i N_o e^{-m}\cos(n)\\ &+\omega^2 k_p N_o e^{-m}\sin(n) - k_i N_e e^{-m}\sin(n) + \omega D_e]\end{aligned} \tag{18}$$

Then, equating the real and imaginary parts of $\Delta(j\omega)$ to zero, one obtains

$$\begin{aligned}&k_p[-\omega^2 N_o e^{-m}\cos(n) + \omega N_e e^{-m}\sin(n)]\\ &+k_i[N_e e^{-m}\cos(n) + \omega N_o e^{-m}\sin(n)] = \omega^2 D_o\end{aligned} \tag{19}$$

and

$$\begin{aligned}&k_p[\omega^2 N_o e^{-m}\sin(n) + \omega N_e e^{-m}\cos(n)]\\ &+k_i[\omega N_o e^{-m}\cos(n) - N_e e^{-m}\sin(n)] = -\omega D_e\end{aligned} \tag{20}$$

Eqs. (19) and (20) can be rearranged as follows.

$$k_p X_3(\omega) + k_i X_4(\omega) = X_1(\omega) \tag{21}$$

$$k_p X_5(\omega) + k_i X_6(\omega) = X_2(\omega) \tag{22}$$

Where

$$X_1(\omega) = \omega^2 D_o \tag{23}$$

$$X_2(\omega) = -\omega D_e \tag{24}$$

$$\begin{aligned}X_3(\omega) &= -\omega^2 N_o e^{-m}\cos(n) + \omega N_e e^{-m}\sin(n)\\ &= e^{-m}[-\omega^2 N_o\cos(n) + \omega N_e\sin(n)]\end{aligned} \tag{25}$$

$$\begin{aligned}X_4(\omega) &= N_e e^{-m}\cos(n) + \omega N_o e^{-m}\sin(n)\\ &= e^{-m}[N_e\cos(n) + \omega N_o\sin(n)]\end{aligned} \tag{26}$$

$$\begin{aligned}X_5(\omega) &= \omega^2 N_o e^{-m}\sin(n) + \omega N_e e^{-m}\cos(n)\\ &= e^{-m}[\omega^2 N_o\sin(n) + \omega N_e\cos(n)]\end{aligned} \tag{27}$$

$$\begin{aligned}X_6(\omega) &= \omega N_o e^{-m}\cos(n) - N_e e^{-m}\sin(n)\\ &= e^{-m}[\omega N_o\cos(n) - N_e\sin(n)]\end{aligned} \tag{28}$$

From Eqs. (21) and (22), k_p and k_i can be obtained as in Eqs. (29) and (30).

$$k_p = \frac{X_1(\omega)X_6(\omega) - X_2(\omega)X_4(\omega)}{X_3(\omega)X_6(\omega) - X_5(\omega)X_4(\omega)} \tag{29}$$

$$k_i = \frac{X_2(\omega)X_3(\omega) - X_1(\omega)X_5(\omega)}{X_3(\omega)X_6(\omega) - X_5(\omega)X_4(\omega)} \tag{30}$$

The stability boundary locus represented as $l(k_p, k_i, \omega)$ can be obtained in the (k_p, k_i) plane using Eqs. (29) and (30) when the denominator $X_3(\omega)X_6(\omega) - X_5(\omega)X_4(\omega) \neq 0$. It should be noted that it is necessary to investigate whether stabilizing controllers exist or not since the stability boundary locus $l(k_p, k_i, \omega)$ and the line $k_i = 0$ can divide the (k_p, k_i) plane into sub-regions as stable and unstable [33] (Details can be found in [33,35]).

Using Eqs. (29) and (30), PI controller parameters k_p and k_i are obtained as follows.

$$k_p = \frac{\begin{array}{c}(\omega^2 N_o D_o + N_e D_e)\cos(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha))\\ +\omega(N_o D_e - N_e D_o)\sin(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha))\end{array}}{-e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha}({N_e}^2 + \omega^2 {N_o}^2)} \tag{31}$$

$$k_i = \frac{\begin{array}{c}\omega^2(N_o D_e - N_e D_o)\cos(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha))\\ -\omega(N_e D_e + \omega^2 N_o D_o)\sin(\omega^\alpha\tau^\alpha(\sin\frac{\pi}{2}\alpha))\end{array}}{-e^{-(\cos\frac{\pi}{2}\alpha)\omega^\alpha\tau^\alpha}({N_e}^2 + \omega^2 {N_o}^2)} \tag{32}$$

Exponential functions have an infinite number of isolated roots [36]. The stability analysis of time delay systems is difficult. Moreover, when the system has a fractional order time delay, the stability analysis becomes much more complicated. Recently, an approximation method has been proposed in the literature to analyze the stability and time response of such systems in [12, 37]. Stability analysis of such systems is possible with this method. Besides, using this method time response analysis of these systems can be obtained. The 1^{st}, 2^{nd} and 3^{rd} order approximations of $e^{-(s\tau)^\alpha}$ are given in Eqs. (33), (34) and (35), respectively [12,37].

First order approximation:

$$\frac{\frac{-(s\tau)^\alpha}{2}+1}{\frac{(s\tau)^\alpha}{2}+1} \tag{33}$$

Second order approximation:

$$\frac{\frac{(s\tau)^{2\alpha}}{12}-\frac{(s\tau)^{2\alpha}}{2}+1}{\frac{(s\tau)^{2\alpha}}{12}+\frac{(s\tau)^{2\alpha}}{2}+1} \tag{34}$$

Third order approximation:

$$\frac{-\frac{(s\tau)^{2\alpha}}{120}+\frac{(s\tau)^{2\alpha}}{10}-\frac{(s\tau)^{2\alpha}}{2}+1}{-\frac{(s\tau)^{2\alpha}}{120}+\frac{(s\tau)^{2\alpha}}{10}-\frac{(s\tau)^{2\alpha}}{2}+1} \tag{35}$$

The value of α is in the range of $0 \leq \alpha \leq 1$. In Figure 2, in order to see the efficiency of this approximation method, the stability regions are drawn by taking $\alpha = 0.999$ and $\alpha = 1$ in the transfer function of the system given by $(1/(s+1))e^{-(s)^\alpha}$. Here, the second order approximation for the fractional order time delay is used. The higher the approximation degree, the more the system's approximation model will resemble the real system model. However, the higher the degree of approximation, the more difficult the system model will be to analyze. We used the second-order approximation for the fractional order time delay in Examples 1 and 2. With the help of this approximation method, analysis related to a system with a fractional order time delay can be made easily. More detailed information can be found in [12, 37].

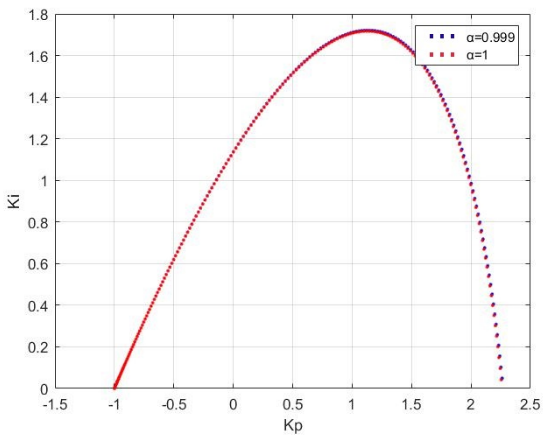

Figure 2. Stability regions for $\alpha = 0.999$ and $\alpha = 1$ using second order approximation.

5. Numerical examples

5.1. Example 1

Consider the control system of Figure 1 with the transfer function of Eq. (36)

$$G_p(s) = \frac{1}{s+1}e^{-\sqrt{s}} \tag{36}$$

The characteristic equation of the system without using PI controller is obtained as follows.

$$\Delta(s) = s + 1 + e^{-\sqrt{s}} \tag{37}$$

This equation can be rearranged as

$$\Delta(s) = (\sqrt{s})^2 + 1 + e^{-\sqrt{s}} \tag{38}$$

For the stability test of the system in Eq.(36), if the second order approximation given by Eq.(34) is substituted for the fractional order time delay in Eq.(38), the new characteristic equation will be as in Eq.(39).

$$\Delta(s) = (\sqrt{s})^2 + 1 + \frac{\frac{s}{12}-\frac{s}{2}+1}{\frac{s}{12}+\frac{s}{2}+1} \tag{39}$$

If Eq.(39) is set to zero, Eq.(40) is obtained.

$$s^2 + 6s\sqrt{s} + 14s + 24 = 0 \tag{40}$$

In Eq.(40), the $q = \sqrt{s}$ transform is performed to find the roots of the characteristic equation. Thus, the following equation is obtained.

$$q^4 + 6q^3 + 14q^2 + 24 = 0 \tag{41}$$

The roots of Eq.(41) are obtained as follows.
$q_{1,2} = -3.2937 \pm 2.3575i = 4.0504\angle \pm 144.406$
$q_{3,4} = 0.2937 \pm 1.1733i = 1.2095\angle \pm 75.9465$
The roots of the characteristic equation are shown in Figure 3. As seen from the figure, the system is stable. (Details about the stability can be found in [37]).

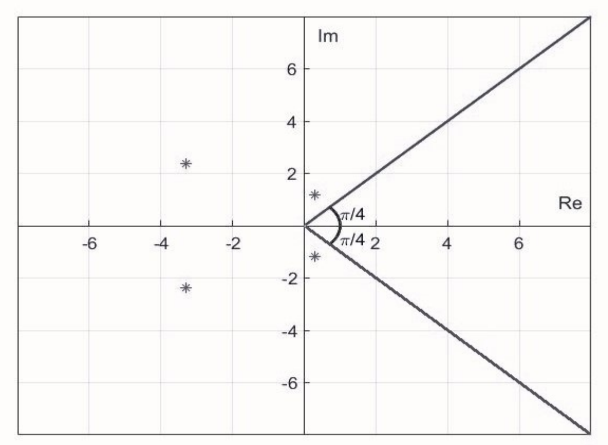

Figure 3. Roots of the characteristic equation in $q = \sqrt{s}$ plane.

As stated before, conventional stability test methods such as Routh-Hurwitz cannot be applied to the analysis of distributed systems whose transfer functions are irrational in s. However, to investigate the stability of fractional order systems, geometric techniques based on the principle of argument can be applied. These techniques provide information about the number of singularities of the function by observing the development of the function's argument. The argument principle (Nyquist diagram) is a curve surrounding the right half plane of the Riemann main sheet [38], the stability of the system can be obtained by determining the number of cycles of this curve around the origin.

The Nyquist curve of the system has been shown in Figure 4. As seen from Figure 4, the system is stable because it does not include critical point (-1, j0). The Nyquist curves for different values of α, and constant value of $\tau = 1$ are presented in Figure 5. As seen from Figure 5, while the value of α increases for constant τ, i.e., it gets closer to 1, a curve similar to the time delay (e^{-s}) in the classical calculation is obtained. The Nyquist curves for different values of τ, and constant value of $\alpha = 0.9$ are given in Figure 6. It can be seen from Figure 6, when τ increases for the constant value of α, the Nyquist curve approaches to the critical point (-1, j0).

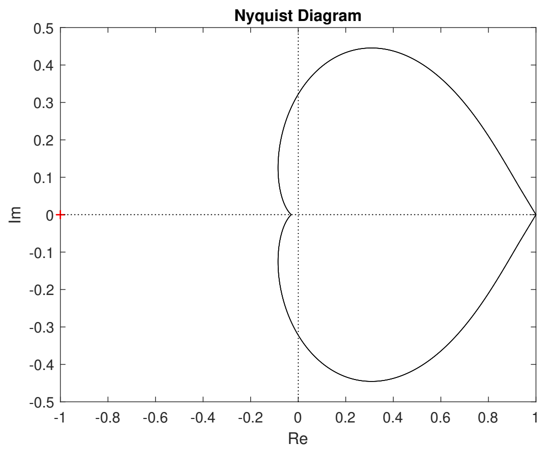

Figure 4. Nyquist diagram of Example 1.

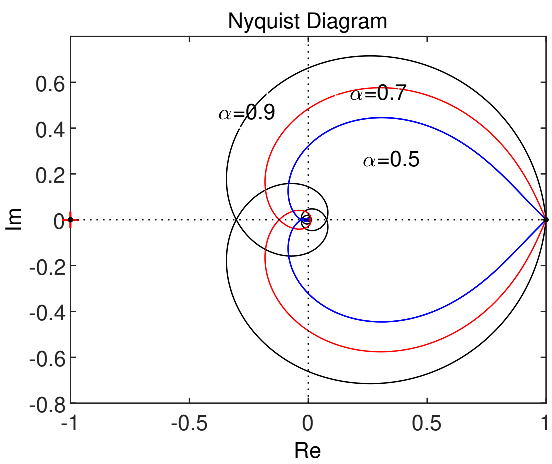

Figure 5. Nyquist diagram of Example 1 for different values of α, and fixed $\tau = 1$.

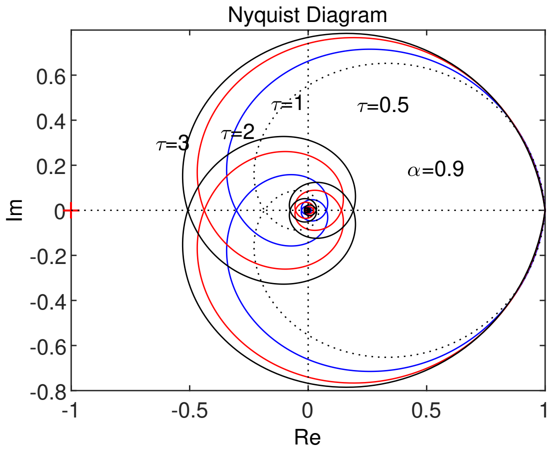

Figure 6. Nyquist diagram of Example 1 for different values of τ, and fixed $\alpha = 0.9$.

To compute all stabilizing PI controllers for the system, k_p and k_i are obtained as follows.

$$k_p = \frac{\cos(0.707\omega^{0.5}) - \omega\sin(0.707\omega^{0.5})}{-e^{-0.707\omega^{0.5}}} \tag{42}$$

$$k_i = \frac{-\omega^2\cos(0.707\omega^{0.5}) - \omega\sin(0.707\omega^{0.5})}{-e^{-0.707\omega^{0.5}}} \tag{43}$$

Stability region and unit step responses for the system can be seen in Figures 7 and 8, respectively. Any point selected within the stability region guarantees system stability. However, in order to ensure a good result in terms of system performance, it is necessary to determine new criteria to choose controller parameters from the stability region. For this purpose, a tuning method presented in [17] is used. Thus, a triangular region which is known as convex stability region has been determined in the stability curve as shown in Figure 7. In this region, the optimum point search is made with the PSO algorithm. In the PSO algorithm, the number of swarms is taken as 100 and the number of iterations is taken as 300. The number of swarms and the number of iterations are obtained by trial and error method according to the ITAE performance index.

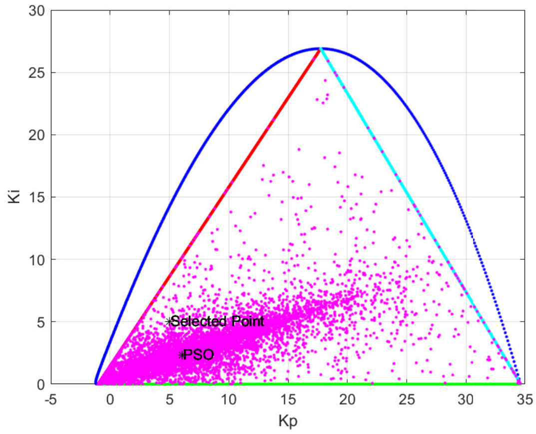

Figure 7. Stability Region and convex stability region of Example 1

Step responses of Example 1 are shown in Figure 8. As seen in Figure 8, the PSO algorithm provides a very good result.

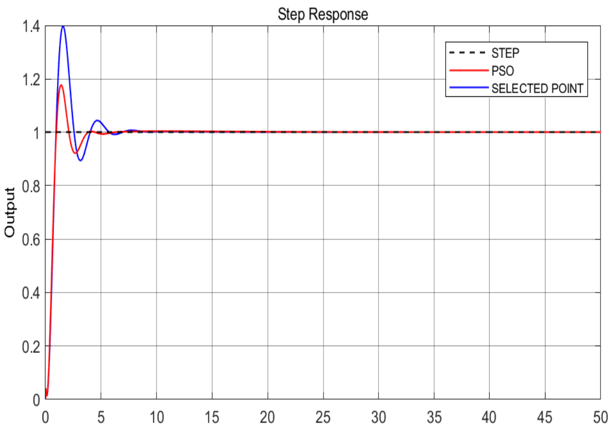

Figure 8. Unit step responses of Example 1 for $k_p = k_i = 5$ and PSO parameters $k_p = 6.0417, k_i = 2.3146$.

Using a PI controller in Example 1, the characteristic equation is given by Eq.(44) when $k_p = k_i = 5$.

$$s^3 + 6s^2\sqrt{s} + 18s^2 - 24s\sqrt{s} + 77s - 30\sqrt{s} + 60 = 0 \quad (44)$$

Substituting $q = \sqrt{s}$ in Eq.(44), Eq.(45) is found.

$$q^6 + 6q^5 + 18q^4 - 24q^3 + 77q^2 - 30q + 60 = 0 \quad (45)$$

The roots are obtained as follows.

$q_{1,2} = -3.8401 \pm 3.6049i = 5.2670\angle \pm 136.8094$
$q_{3,4} = 0.8401 \pm 1.2071i = 1.4707\angle \pm 55.1634$
$q_{5,6} = 0 \pm i = 1\angle \pm 90$

The roots of the system are shown in Figure 9. As seen in Figure 9, the system is stable.

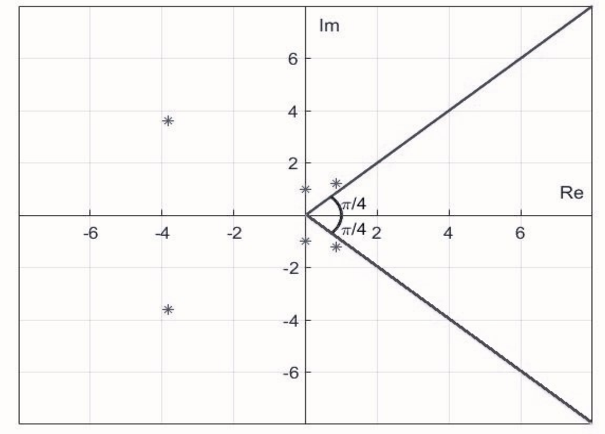

Figure 9. The roots of the PI controlled system.

5.2. Example 2

Consider the fractional order control system of Figure 1 with the transfer function of Eq.(46).

$$G_p(s) = \frac{1}{s^{1.5} + 1} e^{-\sqrt{s}} \quad (46)$$

The characteristic equation of the system without controller is obtained as

$$\Delta(s) = s^{1.5} + 1 + e^{-\sqrt{s}} \quad (47)$$

This equation can be rearranged as follows.

$$\Delta(s) = (\sqrt{s})^3 + 1 + e^{-\sqrt{s}} \quad (48)$$

The Nyquist plot of Example 2 is shown in Figure 10. As seen from Figure 10, the system is stable since it does not include critical point (-1, j0). The stability region is shown in Figure 11. The triangular region under the stability curve is also obtained for Example 2. As shown in Figure 11, the PSO algorithm has been searched for the optimum point within the triangular region. The swarm number of the PSO algorithm was taken as 100 and the number of iterations was taken as 300. ITSE performance index was used as a fitness function. The unit step changes of the system for $k_p = 1, k_i = 0.7$ and PSO parameters $k_p = 0.229, k_i = 0.4267$ are given in Figure 12. As seen from this figure, the PSO algorithm provides a better result than the selected point.

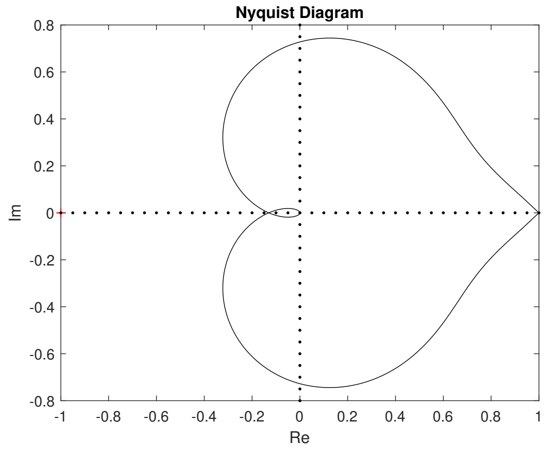

Figure 10. Nyquist Diagram for Example 2

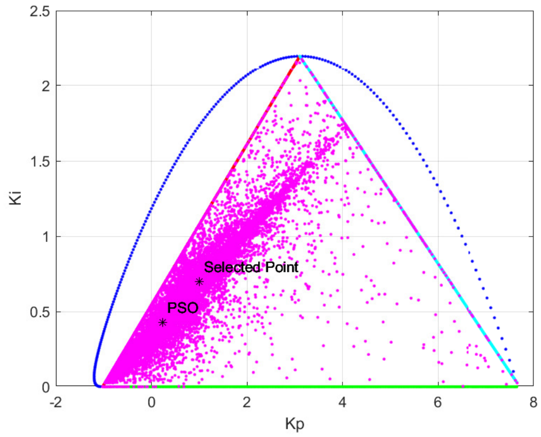

Figure 11. Stability region for Example 2

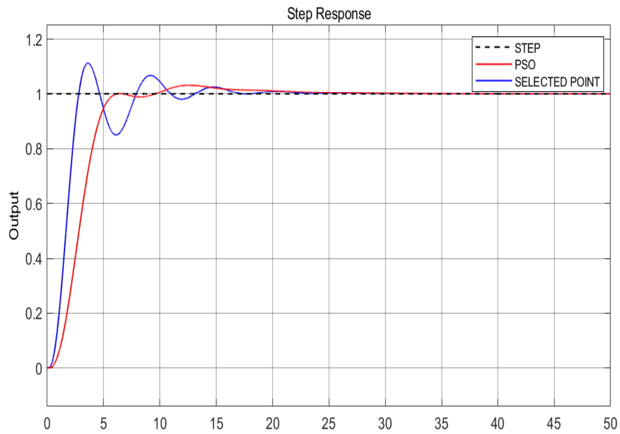

Figure 12. Unit step responses of Example 2 for $k_p = 1, k_i = 0.7$ and PSO parameters $k_p = 0.229, k_i = 0.4267$

6. Conclusion

In this paper, stability analysis and PI controller design for the systems with fractional order time delay are presented. it is known that analysis of the stability and time response of systems having fractional order delay is not possible using classical methods. To overcome this difficulty, an approximation method to investigate the stability of such systems is used. Using this approximation method, time response analysis can also be made for these systems. As for PI controller design part, a new tuning algorithm is aimed. This tuning method uses the PSO algorithm under the stability region. Thus, it has been shown that optimum PI controller parameters can be obtained with the PSO algorithm. This tuning method provides very good results as seen from the numerical examples. For future works, tuning of different controller types such as fractional order PI, PD and PID can be investigated for systems having fractional order delay. Since the reported studies are very restricted for such systems, these investigations would be very important.

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