An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.13, No.1, pp.68-80 (2023)
http://doi.org/10.11121/ijocta.2023.1218

RESEARCH ARTICLE

A predator-prey model for the optimal control of fish harvesting through the imposition of a tax

Anal Chatterjee [a*], Samares Pal [b]

[a] *Department of Mathematics, Barrackpore Rastraguru Surendranath College, Barrackpore, Kolkata, India*
[b] *Department of Mathematics, University of Kalyani, Kalyani, India*
chatterjeeanal172@gmail.com, samaresp@yahoo.co.in

ARTICLE INFO

Article History:
Received 24 January 2022
Accepted 1 December 2022
Available 26 January 2023

Keywords:
Harvesting efforts
Equilibria
Global stability
Optimal taxation policy

AMS Classification 2010:
92B05; 92D40

ABSTRACT

This paper is devoted to the study of ecosystem based fisheries management. The model represents the interaction between prey and predator population with Holling II functional response consisting of different carrying capacities and constant intrinsic growth rates. We have considered the continuous harvesting of predator only. It is observed that if the intrinsic growth rate of predator population crosses a certain critical value, the system enters into Hopf bifurcation. Our observations indicate that tax, the management object in fisheries system play huge impacts on this system. The optimal harvesting policy is disposed by imposing a tax per unit of predator biomass. The optimal harvest strategy is determined using Pontryagin's maximum principle, which is subject to state equations and control limitations. The implications of tax are also examined. We have derived different bifurcations and global stability of the system. Finally, numerical simulations are used to back up the analytical results.

1. Introduction

Ecosystems are made up of live creatures, plants, and non-living things that coexist and 'interact' with one another. Fish are part of the marine ecosystem since they do not live in isolation. They have a strong connection to their physical, chemical, and biological environments. They rely on the environment to supply the necessary conditions for their growth, reproduction, and survival. They also serve as a food supply for other species such as seabirds and marine mammals, making them an essential component of the marine food chain.

Fishing activity has an impact not just on the fish populations, but also on the habitat in which the fish dwell. Fishing has both direct and indirect effects on the ecosystem. As a result of fishing, other species are caught and/or discarded, and fishing gear affects the seabed. Fishing can have an indirect influence on the ecosystem by harvesting fish from the marine food chain, for example. The ecosystem approach to fisheries recognises that fisheries must be managed as part of its ecosystem and that their environmental impact should be kept to a minimum. It is well known that interspecies competition between multi-fish species is vary much complicated. Competition between two fish species with the combined harvesting have been discussed in [1]. The authors in [2] have studied the combined harvesting of two-species predator-prey model with discrete time delay in fishery system. However, restricting to harvest fishes above a certain age or size only can help the fishery and prevent its extinction. Harvesting of two competing fish species in the presence of toxicity has been discussed in [3]. The authors studied bionomic equilibrium and optimal harvesting policy with help of control theory. The author in [4] have analyzed a coral

*Corresponding Author

reef ecosystem to explore the effects of changes in economic, biological, and social parameters in a multiple-species coral-reef ecosystem with adaptive harvesters. Recently, the authors in [5] and [6] have developed a bio-economic model that combines a model of competition and a model of prey-predator of multi-fish populations. They have calculated the fishing effort which maximizes the income of the fishing fleet. In ecology, a stochastic differential fishery game for a two species fish population has been studied in [7]. Recently, several aspects of the optimal harvest of a stage structured model of a fishery have been discussed in [8]. The authors also looked at how changes in costs and harvesting technologies will affect whether the optimal harvesting strategy is to target one age group, the other age group, or both age groups. The authors in [9] have emphasized the importance of age-structured modelling in practical fishery economics. In [10], a prey-predator type fishery model with solely prey harvesting was investigated. Also many papers on prey-predator model with harvesting have been studied in [11–22]. In [23] and [24], have explored deterministic chaos vs stochastic oscillation in a prey-predator model and global stability of a three-species food chain model with diffusion, respectively. Recently, in [25], local and global stability have been examined in a fractional prey-predator model in presence of harvesting rate.

Motivated by the above theoretical and experimental literatures, the dynamics of such system in which implications of tax on harvesting of predator is studied. It should be noted from the aforementioned literature review that no attempt has been made to research prey-predator fishery harvesting with taxation as a control device. The present paper investigates a dynamic reaction model in the context of a prey-predator type fishery system in which only the predator species are harvested. The imposition of a tax serves as a deterrent to fishermen while also protecting the predator from over-exploitation. The main goal of this paper is to determine the proper taxation strategy that will benefit the community as much as possible through harvesting while preventing the extinction of the predator.

The main target in present manuscript is to investigate the subsequent biological topics:

- How does carrying capacity of both population influence the prey-predator dynamics.
- Can imposing tax influence to stabilize the fishery system.
- How does constant price per unit biomass of predator influence the prey-predator dynamics.

In this paper, we present a new deterministic prey-predator model. It incorporates a feature that appears for the first time in this situation, the emergence of a Holling type II response function that has only been suggested by [26] in presence of harvesting of predator with taxation as a control instrument. The use of a Holling type II functional response is however in contrast to other models, such as [27], where the predation term of the model exhibits ratio-dependent type. In contrast to other current models [26], we account for an alternative food source for the predator, which helps in stabilizing the system. In the present article, predator do not only depend on prey but also grow logistically. In this plankton-fish interaction model, two logistic growth rates of the phytoplankton and zooplankton populations are incorporated in [28].

In this paper, a prey-predator interaction model in presence of harvesting is described in fishery management. The stability of equilibrium point is analyzed. Conditions for globally asymptotically stable of coexistence equilibrium have been studied. We also found the requirements for system instability near the coexiestence equilibrium and Hopf bifurcation. We analyze the optimal tax policy by using Pontryagin's maximum principle. To back up our analytical result, we ran numerical simulations with a set of parametric variables. The paper comes to a close with a brief conclusion.

2. The mathematical model

Let $x(t)$ be the concentration of the prey population at time t with carrying capacity K_1 and constant intrinsic growth rate r_1. Let $y(t)$ denotes the predator population at time t with carrying capacity K_2 and constant intrinsic growth rate r_2. Let α_1 be the maximal prey's ingestion rate and β_1 be the maximal conversion rate for the growth of predator population respectively ($\beta_1 \leq \alpha_1$). Let d_1, d_2 be the mortality rates of the prey and predator population respectively. To characterise the grazing phenomenon, we use the Holling type II functional form with a_1 as half saturation constant.

The predator's catchability coefficient is constant c_1 in this case and E is harvesting effort. In this paper, we consider E to be a dynamic (i.e. time-dependent) variable regulated by the equation

$$E(t) = \mu_1 Q(t),\ 0 \leq \mu_1 \leq 1, \tag{1}$$

$$\frac{dQ}{dt} = I(t) - \gamma_1 Q(t),\ Q(0) = Q_0, \quad (2)$$

where $I(t)$ is the gross investment rate at time t. The amount of capital invested fishery at time t is $Q(t)$ and constant rate of depreciation of capital is γ_1. A regulatory agency controls exploitation of the fishery by imposing a tax $\tau_1(>0)$ per unit biomass of the predator. Here $\tau_1(<0)$ be the subsidy given to the fisherman. The net economic revenue to the fisherman is $[c_1(p_1-\tau_1)y - C]\,E$, where p_1 is the constant price per unit biomass of predator species and C is the constant cost per unit of harvesting effort. Moreover, we assume that the gross rate of investment of capital is proportional to the net economic revenue to the fisherman. Therefore we can write

$$I(t) = \mu_2 \left[c_1(p_1-\tau_1)y - C\right] E(t),\ 0 \le \mu_2 \le 1. \quad (3)$$

Equation (3) shows that the maximum investment rate at any time equals the net economic revenue (for $\mu_2 = 1$) at that time. By virtue of (2) and (3) yield the result

$$\frac{dE}{dt} = \{\mu_1\mu_2 \left[c_1(p_1-\tau_1)y - C\right] - \gamma_1\}E. \quad (4)$$

In this paper, we consider tax as the management objective when discussing the impact of harvesting in the fishery system and assume $p_1 - \tau_1 > 0$. Let $m = \mu_1\mu_2$; as a result, the following system of equation is given by :

$$\frac{dx}{dt} = r_1 x(1-\frac{x}{K_1}) - \frac{\alpha_1 xy}{a_1+x} - d_1 x \equiv G_1,$$
$$\frac{dy}{dt} = r_2 y(1-\frac{y}{K_2}) + \frac{\beta_1 xy}{a_1+x} - d_2 y - c_1 yE \equiv G_2,$$
$$\frac{dE}{dt} = \{m\left[c_1(p_1-\tau_1)y - C\right] - \gamma_1\}E \equiv G_3, \quad (5)$$

where $G_i = G_i(x, y, E)$, i=1,2,3. The system (5) will be analyzed with the following initial conditions,

$$x(0) = X_1 \ge 0,\ y(0) = X_2 \ge 0,\ E(0) = X_3 \ge 0. \quad (6)$$

3. Some preliminary results

3.1. Positive invariance

By setting $X = (x,\ y,\ E)^T \in \mathbf{R}^3$ and $G(X) = [G_1(X),\ G_2(X),\ G_3(X)]^T$, with $G : \mathbf{R}_+{}^3 \to \mathbf{R}^3$ and $G \in C^\infty(\mathbf{R}^3)$, equation (5) becomes

$$\dot{X} = G(X), \quad (7)$$

together with $X(0) \in \mathbf{R}_+{}^3$. It is easy to check that whenever $X(0) \in \mathbf{R}_+{}^3$ with $X_i \ge 0$, for i=1, 2, 3, then $G_i(X)\ |_{X_i=0} \ge 0$. Then any solution of equation (7) with $X_0 \in \mathbf{R}_+{}^3$, say $X(t) = X(t; X_0)$, is such that $X(t) \in \mathbf{R}_+{}^3$ for all $t > 0$.

Lemma 1. *All the non negative solutions of the system (5) are ultimately bounded.*

Proof. From the first equation of the system (5) we have $\frac{dx}{dt} \le r_1 x\left(1-\frac{x}{K_1}\right)$, which gives $x(t) \to K_1$ as $t \to \infty$.
Therefore, corresponding to $\epsilon_1 > 0$, there exists $t_{\epsilon_1} > 0$ such that $x(t) \le K + \epsilon_1$ for all $t \ge t_{\epsilon_1}$.
For all $t \ge t_{\epsilon_1}$, from the second equation of (5), we have $\frac{dy}{dt} \le y\left[r_2(1-\frac{y}{K_2}) + \frac{\beta_1(K_1+\epsilon_1)}{a_1+K_1+\epsilon_1}\right]$ and so, corresponding to $\epsilon_2 > 0$ there exists $t_{\epsilon_2} > 0$ such that $y(t) \le K_2 + \frac{\beta_1(K_1+\epsilon_1)K_2}{r_2(a_1+K_1+\epsilon_1)}$ for all $t \ge \max\{t_{\epsilon_1}, t_{\epsilon_2}\}$.
This gives, $\lim_{t\to\infty}\{x(t) + y(t)\} \le K_1 + K_2 + \frac{\beta_1 K_1 K_2}{r_2(a_1+K_1)}$. Using the previous conditions in the third equation of system (5) we can easily to verify that E is bounded and less than some positive constant when $t \to \infty$.
Now we consider, $w(t) = x(t) + y(t) + \frac{1}{m(\rho_1-\tau_1)}E$.
The time derivative of w along the solutions of is $\frac{dw}{dt} \le r_1 x(1-\frac{x}{K_1}) + r_2 y(1-\frac{y}{K_2}) - d_1 x - d_2 y - \frac{(mc+\gamma_1)}{(\rho_1-\tau_1)m}E$,
$\frac{dw}{dt} \le -D_0 w + r_1 x(1-\frac{x}{K_1}) + r_2 y(1-\frac{y}{K_2})$ where $D_o = Min\{d_1, d_2, (mc+\gamma_1)\}$,
$\frac{dw}{dt} + D_0 w \le \frac{r_1K_1}{4} + \frac{r_2K_2}{4}$.
Integrating the above inequality and using initial condition we get, $0 < w(t) \le w(0)e^{-D_0 t} + (\frac{r_1K_1+r_2K_2}{4})(1-e^{-D_0 t})$.
As $t \to \infty$, the above inequality simplifies to
$0 < w(t) \le (\frac{r_1K_1+r_2K_2}{4})$. Hence, all the solutions of the system is uniformly bounded.

□

3.2. Equilibria

The system (5) possesses the following equilibria:
(i) The prey-predator equilibrium $S_0 = (0, 0, 0)$.
(ii) The predator free equilibrium $S_1 = (x_1, 0, 0) = (\frac{K_1(r_1-d_1)}{r_1}, 0, 0)$, which exists if $r_1 > d_1$.
(iii) The prey free equilibrium in absence of harvesting effort $S_2 = (0, y_2, 0) = (0, \frac{K_2(r_2-d_2)}{r_2}, 0)$, which exists if $r_2 > d_2$.
(iv) The prey free equilibrium in presence of harvesting effort $S_3(0, y_3, E_3)$ with $y_3 = \frac{\gamma_1+Cm}{mc_1(p_1-\tau_1)}$, $E_3 = \frac{K_2 mc_1(p_1-\tau_1)(r_2-d_2) - r_2(\gamma_1+Cm)}{K_2 mc_1(p_1-\tau_1)}$, which exists if $p_1 > Max\{\tau_1, \tau_1 + \frac{r_2(\gamma_1+Cm)}{K_2 mc_1(r_2-d_2)}\}$.
(v) The harvesting effort free equilibrium $S_4(x_4, y_4, 0)$ with $x_4 = \frac{a_1(d_2-r_2+\frac{r_2y_4}{K_2})}{\beta_1-(d_2-r_2+\frac{r_2y_4}{K_2})}$

and $y_4 = \frac{[r_1(1-\frac{x_4}{K_1})-d_1][a_1+x_4]}{\alpha_1}$, which exists if $x_4 < \frac{r_1-d_1}{K_1}$ and $\frac{(r_2-d_2)K_2}{r_2} < y_4 < \frac{K_2(\beta_1-d_2+r_2)}{r_2}$.
(vi) The coexistence equilibrium $S^* = (x^*, y^*, E^*)$ with $y^* = \frac{\gamma_1+Cm}{mc_1(p_1-\tau_1)}$,
$E^* = \frac{r_2(1-\frac{y^*}{K_2})-\frac{\beta_1 x^*}{a_1+x^*}}{c_1} - \frac{d_2}{c_1}$ and x^* satisfies $x^{*2} + \{\frac{K_1 d_1}{r_1} + (a_1 - K_1)\}x^* + \frac{K_1\alpha(\gamma_1+Cm)}{\gamma_1 mc_1(p_1-\tau_1)} + \frac{a_1K_1d_1}{r_1} - a_1K_1 = 0$.
Let x_1 and x_2 be the roots of above equation. We only consider that x_1, x_2 have only one positive root then $x_1x_2 = \frac{K_1\alpha_1(\gamma_1+Cm)}{\gamma_1 mc_1(p_1-\tau_1)} + \frac{a_1K_1d_1}{r_1} - a_1K_1 < 0 \implies \tau_1 < \frac{a_1K_1(r_1-d_1)mc_1p_1 - K_1\alpha_1(\gamma_1+Cm)}{aK_1(r_1-d_1)mc_1}$, and $\Delta = \{\frac{K_1d_1}{r_1} + (a_1 - K_1)\}^2 + 4\{a_1K_1 - \frac{a_1K_1d_1}{r_1} - \frac{K_1\alpha_1(\gamma_1+Cm)}{\gamma_1 mc_1(p_1-\tau_1)}\} > 0$.
Hence, x^* exists as a positive root: $x^* = \frac{1}{2}[\frac{K_1d_1}{r_1} + (a_1 - K_1) + \sqrt{\Delta}]$. Thus, the coexistence equilibrium exists if $x^* > 0,\ y^* > 0$ *and* $E^* > 0$ i.e. $\tau_1 < Min\{\frac{a_1K_1(r_1-d_1)mc_1p_1 - K_1\alpha_1(\gamma_1+Cm)}{aK_1(r_1-d_1)mc_1}, p_1\}$.

3.3. Stability analysis of the system (5)

In this section, local stability analysis of the system around the biologically feasible equilibria is performed. Let $\overline{S} = (\overline{x}, \overline{y}, \overline{E})$ be any arbitrary equilibrium. Then the Jacobian matrix about $\overline{S}$ is given by

$$\overline{V} = \begin{bmatrix} \overline{v}_{11} & -\frac{\alpha_1\overline{x}}{a_1+\overline{x}} & 0 \\ \frac{a_1\beta_1\overline{y}}{(a_1+\overline{x})^2} & \overline{v}_{22} & -c_1\overline{y} \\ 0 & mc_1(p_1-\tau_1)\overline{E} & \overline{v}_{33} \end{bmatrix},$$

where $\overline{v}_{11} = r_1(1 - \frac{2\overline{x}}{K_1}) - \frac{a_1\alpha_1\overline{y}}{(a_1+\overline{x})^2} - d_1$, $\overline{v}_{22} = r_2(1 - \frac{2\overline{y}}{K_2}) + \frac{\beta_1\overline{x}}{a_1+\overline{x}} - d_2 - c_1\overline{E}$ and $\overline{v}_{33} = m[c_1(p_1 - \tau_1)\overline{y} - C] - \gamma_1$.

By calculating the Jacobian matrix for the equilibrium S_0 of the system (5). We see that the eigenvalues of the variational matrix V_0 are $\lambda_1 = r_1 - d_1 > 0$, $\lambda_2 = r_2 - d_2 > 0$, $\lambda_3 = -mC - \gamma_1 < 0$. Clearly S_0 is always unstable. It is clear that $S_0(0, 0, 0)$ is always unstable.

Lemma 2. *If* $R_0 = \frac{\beta_1K_1(r_1-d_1)}{(d_2-r_2)[a_1r_1+K_1(r_1-d_1)]} > 1$ *then the predator free steady state* S_1 *of the system (5) is unstable.*

Proof. Now again computing the Jacobian matrix for the equilibrium S_1 of the system (5) we find that the eigenvalues of the Jacobian matrix V_1 are $\lambda_{11} = -(mC+\gamma_1) < 0$, $\lambda_{12} = -r_1+d_1 < 0$ and $\lambda_{13} = r_2 + \frac{\beta_1K_1(r_1-d_1)}{a_1r_1+K_1(r_1-d_1)} - d_2$. It is clear that $\lambda_{13} < 0$ if $\frac{\beta_1K_1(r_1-d_1)}{(d_2-r_2)[a_1r_1+K_1(r_1-d_1)]} < 1$ i.e. $R_0 < 1$ where $R_0 = \frac{\beta_1K_1(r_1-d_1)}{(d_2-r_2)[a_1r_1+K_1(r_1-d_1)]}$.
So, S_1 is asymptotically stable if and only if $R_0 < 1$. Clearly if $R_0 > 1$, then predator free steady state S_1 is unstable which indicates the proof of lemma 1. □

Lemma 3. *There exists a feasible prey free steady state* S_2 *in absence of harvesting effort of predator of the system (5) which is unstable if*

$$R_1 = \frac{a_1r_2(r_1 - d_1)}{\alpha_1K_2(r_2 - d_2)} > 1. \quad (8)$$

Proof. Now again computing the Jacobian matrix for the equilibrium S_2 of the system (5) we find that the eigenvalues of the Jacobian matrix V_2 are $\lambda^1_{11} = -(mC+\gamma_1) < 0$, $\lambda'_{12} = -r_2+d_2 < 0$ and $\lambda'_{13} = r_1 - d_1 - \frac{\alpha_1K_2(r_2-d_2)}{a_1r_2}$. It is clear that $\lambda'_{13} < 0$ if $\frac{a_1r_2(r_1-d_1)}{\alpha_1K_2(r_2-d_2)} < 1$ i.e. $R_1 < 1$ where $R_1 = \frac{a_1r_2(r_1-d_1)}{\alpha_1K_2(r_2-d_2)}$.
So, S_1 is asymptotically stable if and only if $R_1 < 1$. Clearly, if $R_1 > 1$, then prey free without harvesting effort steady state S_2 is unstable which indicates the proof of lemma 2. □

Lemma 4. *There exists a prey free steady state* S_3 *of the system (5) which is unstable if*

$$R_2 = \frac{a_1mc_1(r_1 - d_1)(p_1 - \tau_1)}{\alpha_1(\gamma_1 + Cm)} > 1. \quad (9)$$

Proof. Further the eigenvalues of the Jacobian matrix V_3 around the equilibrium S_3 of the system (5) are θ_1, θ_2 which are the roots of the equation $\theta^2 + \frac{r_2y_3}{K_2}\theta + mc_1^2(p_1 - \tau_1)y_3E_3 = 0$ and $\theta_3 = r_1 - d_1 - \frac{\alpha_1y_3}{a_1}$. Clearly, θ_1 and θ have negative real parts for equilibrium point $S_3(0, y_3, E_3)$. So, prey free equilibrium S_3 is asymptotically stable if $\theta_3 < 0$ i.e. $\frac{a_1mc_1(r_1-d_1)(p_1-\tau_1)}{\alpha_1(\gamma_1+Cm)} < 1$, i.e. $R_2 < 1$ where $R_2 = \frac{a_1mc_1(r_1-d_1)(p_1-\tau_1)}{\alpha_1(\gamma_1+Cm)}$. Therefore, S_3 is unstable if condition (9) i.e $R_2 > 1$ is satisfied. □

Lemma 5. *The harvesting effort free equilibrium of the (5) is locally asymptotically stable if* $B_i > 0$ *where i=1,2,3 and* $B_1B_2 - B_3 > 0$.

Proof. The Jacobian matrix of system (5) around the harvesting effort free equilibrium $S_4 = (x_4, y_4, 0)$ is

$$V^* = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & n_{33} \end{bmatrix},$$

where $m_{11} = r_1 - \frac{2r_1x_4}{K_1} - \frac{a_1\alpha_1y_4}{(a_1+x_4)^2} - d_1 = -\frac{r_1x_4}{K_1} + \frac{\alpha_1x_4y_4}{(a_1+x_4)^2}$, $m_{12} = -\frac{\alpha_1x_4}{a_1+x_4} < 0, m_{21} = \frac{a_1\beta_1y_4}{(a_1+x_4)^2} > 0$, $m_{22} = -\frac{r_2y_4}{K_2} < 0$, $m_{23} = -c_1y_4 < 0$, $m_{33} = m[c_1(p_1 - \tau_1)y_4 - C] - \gamma_1 > 0$.
The characteristic equation is given by

$$Q_1^3 + B_1Q_1^2 + B_2Q_1 + B_3 = 0,$$

where $B_1 = -(m_{11}+m_{22}+m_{33})$, $B_2 = m_{11}m_{22}+m_{11}m_{33}+m_{22}m_{33}-m_{12}m_{21}$, $B_3 = m_{33}m_{12}m_{21}-m_{33}m_{11}m_{22}$.
Case 1: If $m_{11} < 0$, which shows that $B_3 < 0$. Then S_4 is unstable.
Case 2: If $m_{11} > 0$, Then $B_1 = -(m_{11}+m_{22}+m_{33}) > 0$ if $m_{22} < m_{11}+m_{33})$. Also $B_2 = m_{11}m_{22}+m_{11}m_{33}+m_{22}m_{33}-m_{12}m_{21} > 0$ if $m_{11}m_{33}-m_{12}m_{21} > -(m_{11}m_{22}+m_{22}m_{33})$ since $m_{11}m_{22} < 0$, $m_{11}m_{33} > 0$, $m_{22}m_{33} < 0$ and $m_{12}m_{21} < 0$. Clearly $B_3 = m_{33}m_{12}m_{21}-m_{33}m_{11}m_{22} > 0$ if $m_{33}m_{12}m_{21} > m_{33}m_{11}m_{22}$ since $m_{33}m_{12}m_{21} < 0$ and $m_{33}m_{11}m_{22} < 0$. Now $B_1B_2 - B_3 > 0$ if $B_1B_2 > B_3$. Therefore, according the Routh-Hurwitz criteria, all roots of above equation have negative real parts. Thus S_4 is locally asymptotically stable. □

Lemma 6. *The coexistence equilibrium of the system (5) is locally asymptotically stable if $\Theta_i > 0$ where i=1,2,3 and $\Theta_1\Theta_2 - \Theta_3 > 0$.*

Proof. The Jacobian matrix of system (5) around the positive interior equilibrium $S^* = (x^*, y^*, E^*)$ is

$$V^* = \begin{bmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & n_{23} \\ 0 & n_{32} & 0 \end{bmatrix},$$

where $n_{11} = r_1 - \frac{2r_1x^*}{K_1} - \frac{a_1\alpha_1y^*}{(a_1+x^*)^2} - d_1 = -\frac{r_1x^*}{K_1} + \frac{\alpha_1x^*y^*}{(a_1+x^*)^2}$, $n_{12} = -\frac{\alpha_1x^*}{a_1+x^*} < 0$, $n_{21} = \frac{a_1\beta_1y^*}{(a_1+x^*)^2} > 0$, $n_{22} = -\frac{r_2y^*}{K_2} < 0$, $n_{23} = -c_1y^* < 0$, $n_{32} = mc_1(p_1-\tau_1)E^* > 0$.
The characteristic equation is

$$Q^3 + \Theta_1Q^2 + \Theta_2Q + \Theta_3 = 0,$$

where $\Theta_1 = -(n_{11}+n_{22})$, $\Theta_2 = n_{11}n_{22}-n_{32}n_{23}-n_{12}n_{21}$, $\Theta_3 = n_{11}n_{32}n_{23}$.
Case 1: If $n_{11} > 0$, which shows that $\Theta_3 < 0$. Then S^* is unstable.
Case 2: If $n_{11} < 0$, Then $\Theta_1 = -(n_{11}+n_{22}) > 0$ Also, $\Theta_2 = n_{11}n_{22} - n_{32}n_{23} - n_{12}n_{21} > 0$ since $n_{11}n_{22} > 0$, $n_{32}n_{23} < 0$ and $n_{12}n_{21} < 0$. Clearly, $\Theta_3 = n_{11}n_{32}n_{23} > 0$. Now $\Theta_1\Theta_2 - \Theta_3 > 0$ if $\Theta_1\Theta_2 > \Theta_3$. Therefore according the Routh-Hurwitz criteria, all roots of above equation have negative real parts. Thus S^* is locally asymptotically stable. □

The analytical results are summarized in the Table 1.

Theorem 1. *When the intrinsic growth rate of predator r_2 crosses a critical value, say r_2^*, the system (5) enters into a Hopf-bifurcation around the coexistence equilibrium, which induces oscillations of the populations.*

Proof. If the Hopf-bifurcation exists for $r_2 = r_2^*$, the following are the necessary and sufficient conditions: (i) $\Theta_i(r_2^*) > 0$, $i = 1,2,3$ (ii)$\Theta_1(r_2^*)\Theta_2(r_2^*) - \Theta_3(r_2^*) = 0$ and (iii) the eigenvalues of above characteristic equation should be of the form $\lambda_i = u_i + \mathbf{i}v_i$, and $\frac{du_i}{dr_2} \neq 0$, $i = 1,2,3$. The Hopf-bifurcation condition (iii) will now be tested by putting $\lambda = u + \mathbf{i}v$ in the above equation, we get

$$(u+\mathbf{i}v)^3+\Theta_1(u+\mathbf{i}v)^2+\Theta_2(u+\mathbf{i}v)+\Theta_3 = 0. \quad (10)$$

On distinguishing the real and imaginary parts and removing v, we get

$$8u^3+8\Theta_1u^2+2u(\Theta_1^2+\Theta_2)+\Theta_1\Theta_2-\Theta_3 = 0. \quad (11)$$

From the foregoing, it is apparent that $u(r_2^*) = 0$ iff $\Theta_1(r_2^*)\Theta_2(r_2^*)-\Theta_3(r_2^*) = 0$. Further, at $r_2 = r_2^*$, $u(r_2^*)$ is the only root, since the discriminant $8u^2+8\Theta_1u+2(\Theta_1^2+\Theta_2) = 0$ if $64\Theta_1^2-64(\Theta_1^2+\Theta_2) < 0$. Further, differentiating (11) with respect to r_2, we have
$24u^2\frac{du}{dr_2}+16\Theta_1u\frac{du}{dr_2}+2(\Theta_1^2+\Theta_2)\frac{du}{dr_2}+2u[2\Theta_1\frac{d\Theta_1}{dr_2}+\frac{d\Theta_2}{dr_2}] + \frac{dS}{dr_2} = 0$ where $S = \Theta_1\Theta_2 - \Theta_3$.
Since at $r_2 = r_2^*$, $u(r_2^*) = 0$ we get $\left[\frac{du}{dr_2}\right]_{r_2=r_2^*} = \frac{-\frac{dS}{dr_2}}{2(\Theta_1^2+\Theta_2)} \neq 0$.
This ensures that the above system has a Hopf-bifurcation around the coexistence equilibrium E^*.

□

Theorem 2. *If the coexistence equilibrium S^* exists, then (x^*, y^*, E^*) is globally asymptotically stable in the $x-y-E$ plane.*

Proof. Let's start by defining a Lyapunov function

$$W(x,y,E) = \int_{x^*}^{x} \frac{\xi - x^*}{\xi}d\xi + D_1\int_{y^*}^{y} \frac{\eta - y^*}{\eta}d\eta + D_2\int_{E^*}^{E} \frac{\rho - E^*}{\rho}d\rho, \quad (12)$$

where D_1 and D_2 are positive constants.
It is easy to examine that $W(x, y, E)$ is zero at the equilibrium point and the positive for all other positive values of $W(x, y, E)$.
The time derivative of W along the trajectories of the subsystem is

$$\frac{dW}{dt} = \frac{dx}{dt}\left[\frac{x-x^*}{x}\right] + D_1\left[\frac{dy}{dt}\right]\left[\frac{y-y^*}{y}\right] + D_2\left[\frac{dE}{dt}\right]\left[\frac{E-E^*}{E}\right]$$

Table 1. The table depicting thresholds and stability of steady states.

Thresholds (R_0, R_1, R_2)	$S_0(0,0,0)$	$S_1(x_1,0,0)$	$S_2(0,y_2,0)$	$S_3(0,y_3,E_3)$	$S_4(x_4,y_4,0)$	$S^*(x^*,y^*,E^*)$
$R_0 < 1$	Unstable	Asymptotically stable	Not feasible	Not feasible	Not feasible	Not feasible
$R_0 > 1,\ R_1 < 1$	Unstable	Unstable	Asymptotically stable	Not feasible	Not feasible	Not feasible
$R_1 > 1,\ R_2 < 1$	Unstable	Unstable	Unstable	Asymptotically stable	Not feasible	Not feasible
$R_2 > 1$	Unstable	Unstable	Unstable	Unstable	Asymptotically stable	Not feasible
$S^* > 0, \Theta_i > 0, i = 1,2,3., \Theta_1\Theta_2 - \Theta_3 > 0.$	Unstable	Unstable	Unstable	Unstable	Unstable	Asymptotically stable

$$= [x - x^*]\left[r_1(1 - \frac{x}{K_1}) - \frac{\alpha_1 y}{a_1 + x} - d_1\right] + D_1\left[r_2(1 - \frac{y}{K_2}) + \frac{\beta_1 x}{a_1 + x} - d_2 - c_1E\right] [y - y^*] + D_2\left[mc_1(p_1 - \tau_1)y - mC\right][E - E^*]$$

$$= [x - x^*][-\frac{r_1}{K_1}(x - x^*) - \frac{\alpha_1}{(a_1 + x)(a_1 + x^*)}[(a_1 + x^*)(y - y^*) - y^*(x - x^*)]] + D_1\left[r_2(1 - \frac{y}{K_2}) + \frac{\beta_1 x}{a_1 + x} - d_2 - c_1E\right][y - y^*] + D_2\left[mc_1(p_1 - \tau_1)y - mC\right][E - E^*]$$

$$= \left[-\frac{r_1}{K_1} + \frac{\alpha_1 y^*}{(a_1 + x)(a_1 + x^*)}\right](x - x^*)^2 - \frac{\alpha_1(x - x^*)(y - y^*)}{a_1 + x} - \frac{r_2 D_1}{K_2}(y - y^*)^2 + D_1(\frac{\beta_1 x}{a_1 + x} - \frac{\beta_1 x^*}{a_1 + x^*})(y - y^*) - D_1c_1(E - E^*)(y - y^*) + D_2mc_1(p_1 - \tau_1)(E - E^*)(y - y^*).$$

Now here we choose arbitrary constants D_1 and D_2 such as $D_1 = \frac{\alpha_1(a_1+x^*)}{a_1\beta_1}$, $D_2 = \frac{\alpha_1(a_1+x^*)}{a_1 m\beta_1\beta(p_1-\tau_1)}$. Then

$$\frac{dW(x,y,E)}{dt} = \left[-\frac{r_1}{K_1} + \frac{\alpha_1 y^*}{(a_1 + x)(a_1 + x^*)}\right] \times (x - x^*)^2 - \frac{r_2}{K_2}\frac{\alpha_1(a_1 + x^*)}{a_1\beta_1}(y - y^*)^2.$$

Clearly, the second term is negative. Now after some calculation in first term we see that if $x^* > (K_1 - a_1) - \frac{D_1K_1}{r_1}$ then $\frac{dW(x,y,E)}{dt} \leq 0$. Clearly, $\frac{dW(x,y,E)}{dt} = 0$ if and only if $x = x^*$ and $y = y^*$ which yields $E = E^*$. Hence, $\frac{dW(x,y,E)}{dt} = 0$ if and only if $x = x^*$, $y = y^*$ and $E = E^*$. So from Lasalle invariant principle we say that S^* is globally asymptotically stable. □

4. Optimal Taxation policy

Biologically, we care more about the coexistence equilibrium in the presence of harvesting in order to ensure the existence of both species. Our main goal is to save each species while also maximising the monetary and social benefits. The profits (revenues) received by the fisherman and regulatory agency are saved to the society through the fishery in a large society. The entire economic revenue is

$$(c_1p_1y - C)E = [c_1(p_1 - \tau_1)y - C]E + \tau_1c_1yE. \quad (13)$$

It is equal to the sum of the entire fisherman's economic revenue and the regulating agency's economic revenue. It is obvious that

$$\pi(x,y,E) = (c_1p_1y - C)E. \quad (14)$$

In order to maximise the present value J of a continuous time steam of revenues, we analyse optimal harvest policy

$$J = \int_0^\infty e^{-\delta t}(c_1p_1y - C)Edt, \quad (15)$$

where δ be the instantaneous annual rate of discount [29–31].

Now we want to discover the path tracked by $(x(t); y(t); E(t))$ with the tax policy τ_1 so that if fish populations and harvesting effort stay on this path, the regulatory authority will be assumed to have accomplished its goal.

Our goal is to use Pontryagin's maximal principle [32] to determine a tax policy $\tau_1 = \tau_1(t)$ that maximises J under the state equation (5). The control variable $\tau_1(t)$ is subjected to the constraints $min\ \tau_1 \leq \tau_1 \leq max\ \tau_1$. When $min\ \tau_1 < 0$, we can explore subsidies, which in this case would have the effect of increasing the rate of expansion of the fishery.

Hamiltonian function is defined as follows:

$$H = e^{-\delta t}(c_1p_1y - C)E + \lambda_1\frac{dx}{dt} + \lambda_2\frac{dy}{dt} + \lambda_3\frac{dE}{dt}. \quad (16)$$

where λ_1, λ_2 and λ_3 are the adjoint variables. Hamiltonian (16) must be maximized when $\tau_1 \in [min\ \tau_1, max\ \tau_1]$. We assume that the optimal solution does not occur at $\tau_1 = min\ \tau_1\ or\ max\ \tau_2$ which imply that constraints are not binding. Therefore, singular control is represented by

$$\frac{\delta H}{\delta\tau_1} = -\lambda_3 mc_1yE \Longrightarrow \lambda_3 = 0. \quad (17)$$

The adjoint equations, according to Pontryagin's maximal principle, are

$$\frac{d\lambda_1}{dt} = \frac{\delta H}{\delta x}, \ \frac{d\lambda_2}{dt} = \frac{\delta H}{\delta y}, \ \frac{d\lambda_3}{dt} = \frac{\delta H}{\delta E}. \quad (18)$$

As a result of the substitution and simplification, we arrive at

$$\frac{d\lambda_1}{dt} = -\frac{\delta H}{\delta x} = -\lambda_1\left[r_1(1-\frac{2x}{K_1}) - \frac{a_1\alpha_1 y}{(a_1+x)^2} - d_1\right] - \lambda_2\frac{a_1\beta_1 y}{(a_1+x)^2}, \quad (19)$$

$$\frac{d\lambda_2}{dt} = -\frac{\delta H}{\delta y} = -e^{-\delta t}c_1p_1E + \lambda_1\frac{\alpha_1 x}{a_1+x} -\lambda_2\left[(r_2 - \frac{2r_2y}{k_2}) + \frac{\beta_1 x}{a_1+x} - d_2 - c_1E\right], \quad (20)$$

$$\frac{d\lambda_3}{dt} = -\frac{\delta H}{\delta E} = -(c_1p_1y - C)e^{-\delta t} + \lambda_2 c_1 y = 0. \quad (21)$$

The solution of (21) is described in the following in order to obtain an optimal equilibrium solution by considering the coexisting equilibrium as follows:

$$\lambda_2 = e^{-\delta t}(p_1 - \frac{C}{c_1y^*}). \quad (22)$$

Let

$$A_1 = -\left[r_1 - \frac{2r_1x^*}{K_1} - \frac{a_1\alpha_1y^*}{(a_1+x^*)^2} - d_1\right],$$
$$A_2 = \frac{a_1\beta_1y^*}{(a_1+x^*)^2}(p_1 - \frac{C}{c_1y^*})e^{-\delta t},$$
$$A_3 = c_1p_1E^* - \frac{A_2}{A_1+\delta}\frac{\alpha_1x^*}{a_1+X^*} - \frac{r_2y^*}{\alpha_2}(p_1 - \frac{C}{c_1y^*}). \quad (23)$$

Now the equations (19) and (20) can be written as

$$\frac{d\lambda_1}{dt} = A_1\lambda_1 - A_2e^{-\delta t}\frac{d\lambda_2}{dt} = -A_3e^{-\delta t}. \quad (24)$$

Solving the above linear differential equation we get

$$\lambda_1 = \frac{A_2}{A_1+\delta}e^{-\delta t}, \ \lambda_2 = -\frac{A_3}{\delta}e^{-\delta t}. \quad (25)$$

Substituting the value of λ_2 from (22) into (25), we get

$$(p_1 - \frac{C}{c_1y^*}) = \frac{A_3}{\delta}. \quad (26)$$

Now putting the value of x^*, y^* and E^* into (26), we get an equation for τ_1; let τ_1^* be a solution of this equation. We get the optimal equilibrium solutions $x = x(\tau_1^*)$, $y = y(\tau_1^*)$ and $E = E(\tau_1^*)$ by using the value of $\tau_1 = \tau_1^*$. As a result, we have established the existence of an optimal equilibrium solution that satisfies the necessary conditions of the maximum principle. From the above analysis carried out in this section, we observe the following.

From (21), we get

$$\lambda_2 c_1 y = (c_1p_1y - C)e^{-\delta t} = e^{-\delta t}\frac{\delta\pi}{\delta E}. \quad (27)$$

Putting the value of $\lambda_2(t)$ into (27), we get

$$c_1p_1y - \frac{A_3}{\delta}c_1y = C. \quad (28)$$

When $\delta \longrightarrow \infty$, (28) leads to the results $c_1p_1y = C$, which implies that the economic rent is completely dissipated.

(ii) By (26) we get the optimal equilibrium populations $x = x(\tau_1^*), y = y(\tau_1^*), E = E(\tau_1^*)$, hence, we have

$$\pi = (c_1p_1y - C)E = \frac{A_3}{\delta}c_1yE. \quad (29)$$

Thus π is a decreasing function of δ we, therefore, conclude that π leads to maximization when δ leads to 0.

5. Numerical simulations

In this section, some numerical simulations are performed with the help of used RK4 schemes to discuss the dynamical behavior of system (5) and to verify analytical results. To examine the dynamic of fishery system, we start with a set of parametric values (Ref. [26])

$$r_1 = 7, \ r_2 = 1 \ K_1 = 3, \ K_2 = 20, \ \alpha_1 = 1.5,$$
$$\beta_1 = 0.8, \ d_1 = 0.01, \ d_2 = 0.01, \ m = 0.8, \ c_1 = 1.2,$$
$$p_1 = 0.7, \ \tau_1 = 0.08, \ C = 0.49, \ a_1 = 0.05, \ \gamma_1 = 0.1. \quad (30)$$

Considering the parametric values, we find the coexistence equilibrium point $S^* = (2.81, 0.83, 1.44)$ which is locally asymptotically stable (cf. Figure 1a (black line)). Taking $K_2 = 200$, the system (5) exhibits oscillation around S^* (cf. Figure 1a (red line)). In case of $K_2 = 0.4$, then Figure 1a (blue line) shows the system switches to harvesting effort free equilibrium S_4.

Further from Figure 1b (black line), it follows tax per unit biomass of the predator, $\tau_1 = 0.52$, the equilibrium S_3 is locally asymptotically stable. Increasing the value of $\tau_1 = 0.7$, the system switches to prey free equilibrium S_2 in absence of harvesting effort (cf. Figure 1b (blue line)). It is observe that the system switches to stable to oscillatory behavior around S^* due to low value of $r_2 = 0.2$ (cf. Figure 1c). But in both case of $r_1 = 1$ and $K_1 = 0.3$ the system switches to prey free equilibrium simultaneously (cf. Figure 1d) . Figures 3a,3b and 3c illustrate the different steady state behaviour of each species in the system (5) for the parameter τ_1. We note that Hopf point (red star (H)) situated

Table 2. Natures of equilibrium points.

Parameters	Values	Eigenvalues	Equilibrium points
τ_1	0.436074	$(4.23719, \pm 0.900417i)$	Hopf (H)
	0.557934	$(0, -0.0901945 \pm 0.92798i)$	Limit Point (LP)
p_1	0.343926	$(4.23719, \pm .900417i)$	Hopf (H)
	0.222066	$(0, -0.0901945 \pm 0.92798i)$	Limit Point (LP)
	2.279571	$(0, -0.005825 \pm 0.693768)$	Branch Point (BP)
	4.942273	$(0, 1.52281, -10.147,)$	Branch Point (BP)
(τ_1, r_1)	$(0.046608, 2.014876)$	$(0.8937, \pm 0.934174i)$	Generalized Hopf (GH)
(p_1, r_1)	$(0.733392, 2.014876)$	$(0.8937, \pm 0.934174i)$	Generalized Hopf (GH)
	$(1.010732, 1.443315)$	$(0.655091, \pm 0.936168i)$	Generalized Hopf (GH)
(p_1, K_2)	$(1.590494, 0.669121)$	$(1.64826, \pm 0.688981i)$	Generalized Hopf (GH)

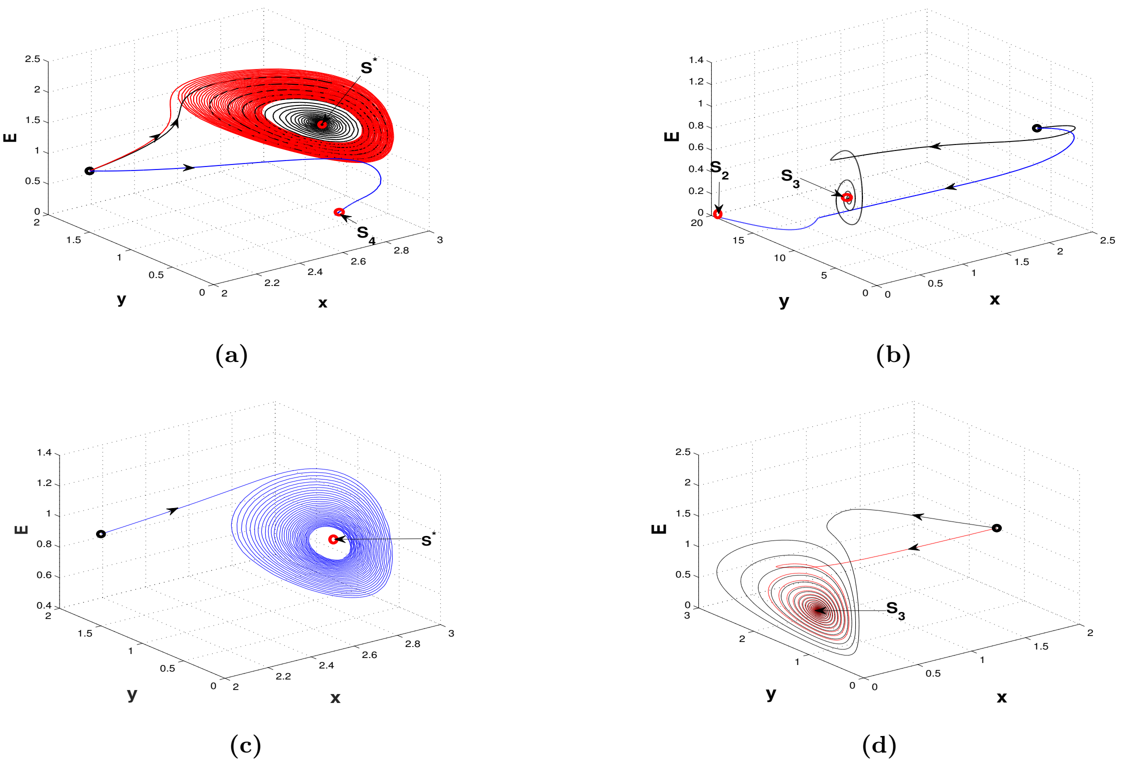

Figure 1. **(a)** Phase plane diagram showing local stability of S^* for $K_2 = 20$ (black solid line), oscillatory behavior around S^* for $K_2 = 200$ (red solid line) and local stability at S_4 for $K_2 = 0.4$ (blue solid line). **(b)** The solution of trajectory approaches to two different equilibrium points S_3 and S_2 for $\tau_1 = 0.52$ (black line) and 0.7 (blue line) respectively. **(c)** The system switches to oscillatory behavior for $r_2 = 0.2$. **(d)** Phase plane diagram indicating the local stability of S_3 for $r_1 = 1$ (red solid line) and $K_1 = 0.3$ (black line) respectively.

at 0.436074 with two complex parts of eigenvalues ≈ 0. We also observe that at that particular point, the value of first Lyapunov coefficient is positive 0.01112548 which indicates unstable limit cycle bifurcates from Hopf point. To proceed further, we have a limit point (LP) at $\tau_1 = 0.557934$ with eigenvalues $0, 0.0901945 \pm 0.92798i$. From Figures 3d, it is evident that at $\tau_1 = .4360747$ and 0.4578429 we have two Limit point cycle (LPC) and Branch Point cycle (BPC).

Figures 4a, 4b and 4c depict different behavior of each species when p_1 is a free parameter. Here we observe that a Hopf points (H), two Branch points (BP) and one Limit point (LP). In this case, Hopf point is situated at 0.343929 with first Lyapunov coefficients a .01112584 indicating subcritical bifurcation. Further, it is observed that one LP and two Branch points are located at 0.222066, 2.279571 and 4.942273 respectively. Starting from Hopf point and proceed further, a family of unstable limit cycle is generated (cf. Figure 4d).

To demonstrate the clear picture of changes in dynamical system when K_2 and r_2 be the free parameters, we plot two bifurcation diagrams (cf. Figure 2a, 2b) respectively. Finally, we draw two parameter bifurcation diagrams for $\tau_1 - r_1$, $p_1 - r_1$

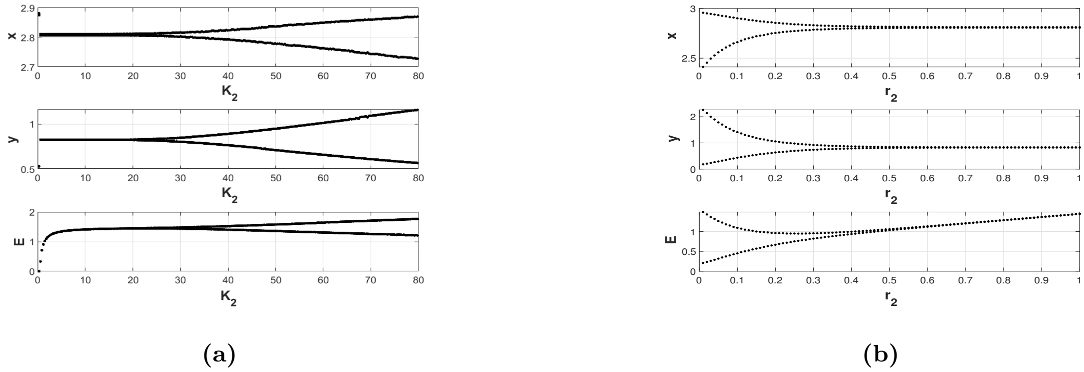

Figure 2. **(a)** Bifurcation diagram for K_2. **(b)** Bifurcation diagram for r_2.

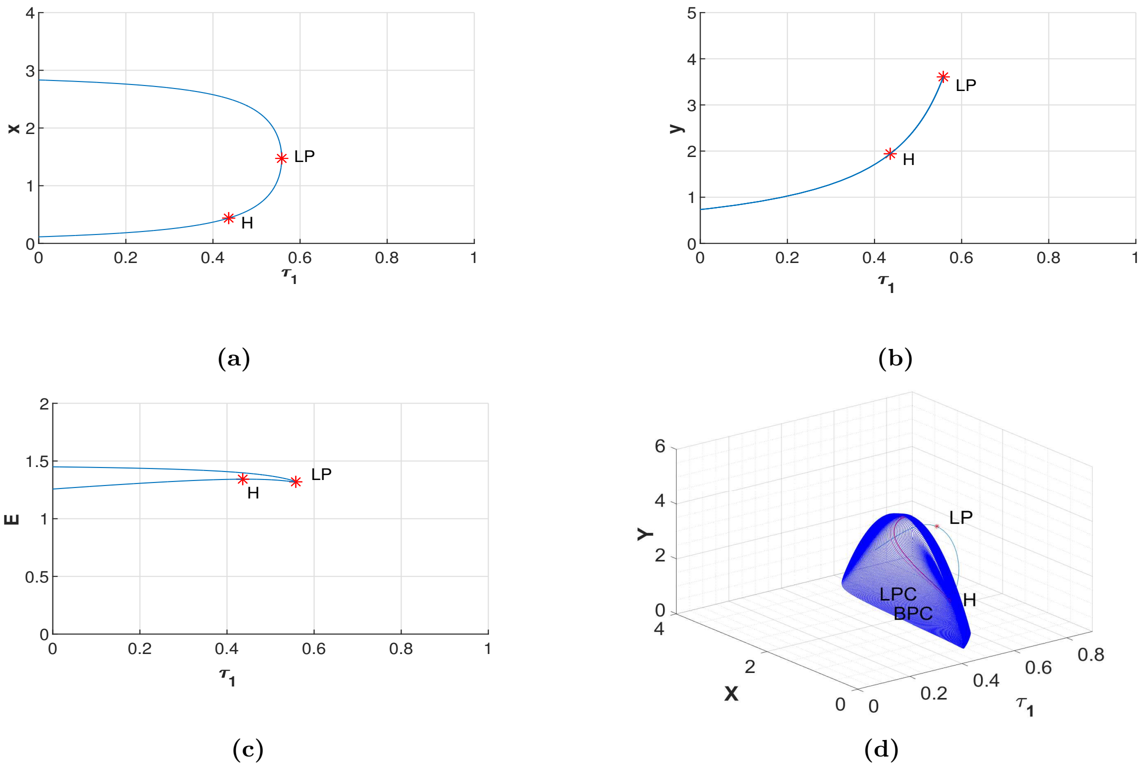

Figure 3. **(a)** Different steady state behavior of prey for the effect of τ_1. **(b)** Different steady state behavior of predator for the effect of τ_1. **(c)** Different steady state behavior of harvesting effort for the effect of τ_1. **(d)**The family of limit cycles bifurcating from the Hopf point H for τ_1.

and $p_1 - K_2$ (cf. Figure 5a, 5b and 5c). In each cases, we have generalized Hopf (GH) point. Actually, two branches of sub and supercritical Andronov-Hopf bifurcations split at GH point.

6. Conclusion

We investigate the interspecies competition of prey and predator in a fishery system in this work. We assumed that predator undergo exploitation due to consume of prey. This work has a dual goal, namely economic and ecological. The economic goal is to maximize monetary benefit to society while also preventing the predator from extinction. Here we implement a tax to regulate the harvesting effort in order to preserve the ecological balance.

As a result, one of the most important features of this approach is the harvesting effort and net economic revenue to the fisherman. The first step is to perform analytical conditions for the existence and stability of various steady states. We also look into the global stability of coexistence equilibrium while the tax remains certain threshold value. The outcome of global stability shows that when a tax provides a sustainable threshold value, predator are not from a body of water at a rate greater than that the species can replenish

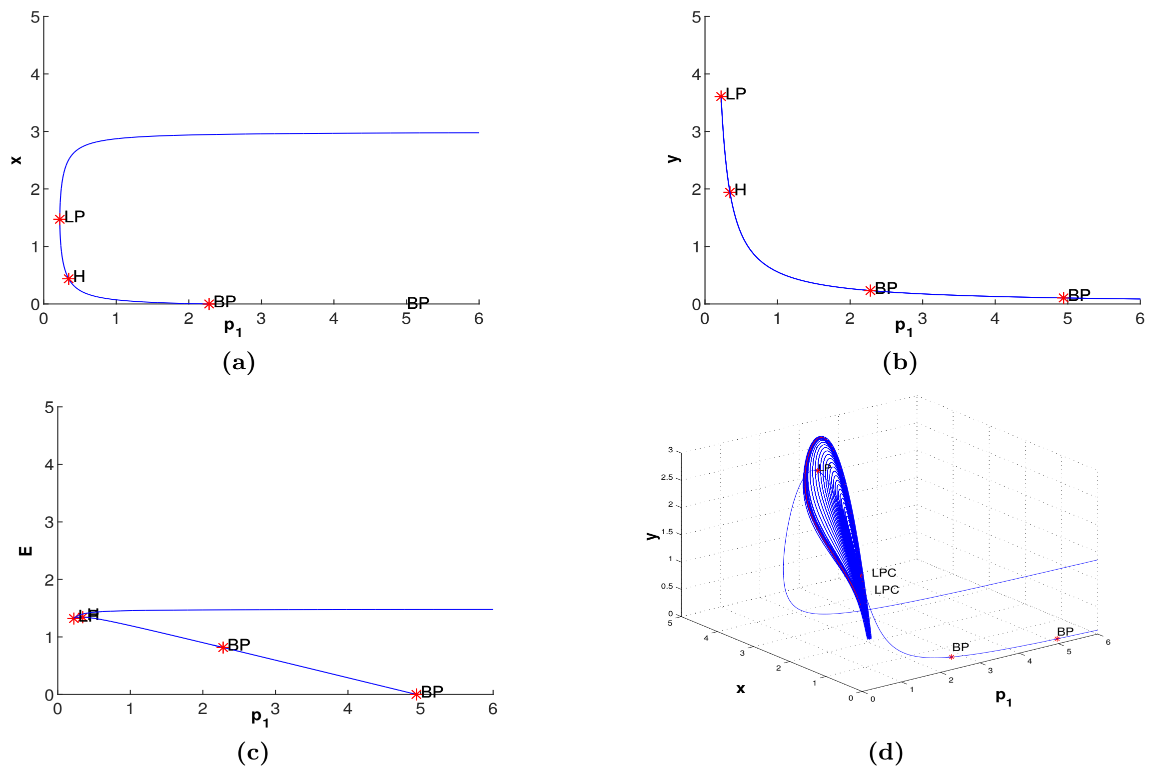

Figure 4. **(a)** Different steady state behavior of prey population for the effect of p_1. **(b)** Different steady state behavior of predator population for the effect of p_1. **(c)** Different steady state behavior of harvesting effort for the effect of p_1. **(d)** The family of limit cycles bifurcating from the Hopf point H for p_1.

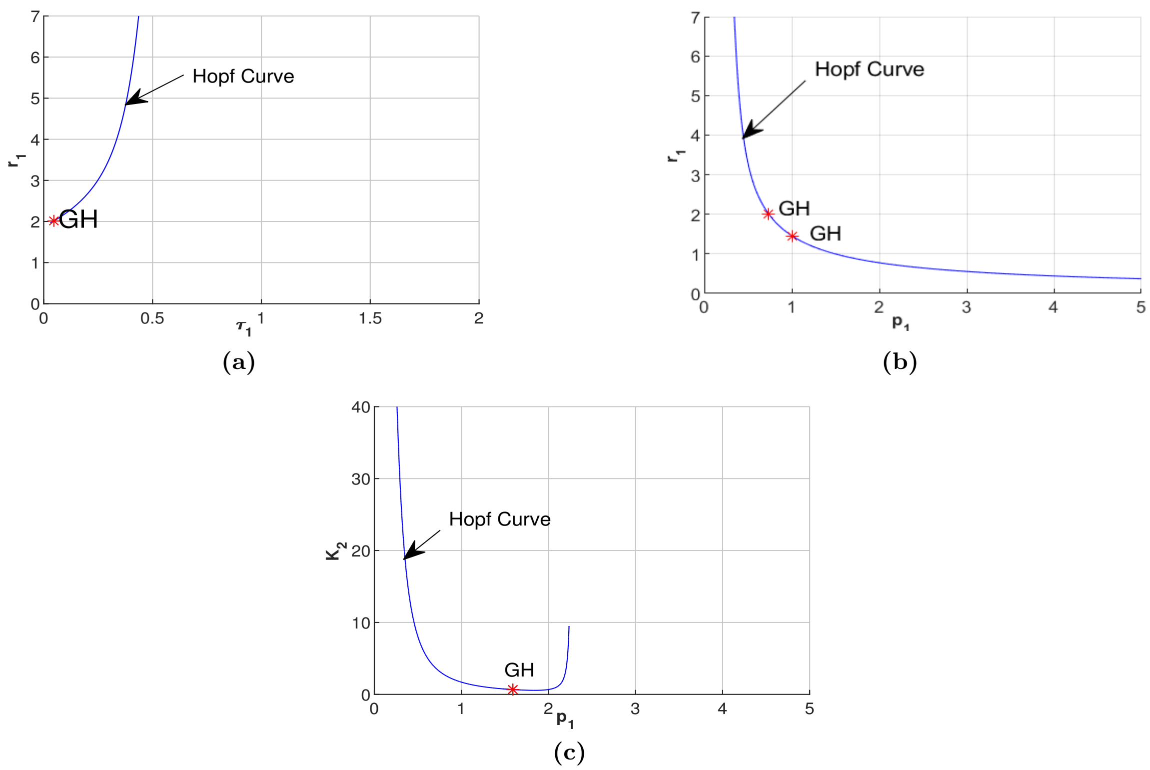

Figure 5. **(a)** The two parameters bifurcation diagram for $\tau_1 - r_1$. **(b)** The two parameters bifurcation diagram for $p_1 - r_1$. **(c)** The two parameters bifurcation diagram for $p_1 - K_2$.

its population naturally. We note the following observations by numerical simulation:

Two different scenarios are shown when changing the value of carrying capacity of predator. Here we observe that each fish species are present in the system in absence of harvesting efforts for low values of carrying capacity of predator. On the other hand, because to the high value of predators' carrying capacity, all species become unstable. Due to high tax levels, coexistence equilibrium switches to different boundary equilibrium, which is related to transcritical bifurcations. The system becomes oscillate due to low values of intrinsic growth rate of predator. Our research also shows that maintaining carrying capacity and imposing a tax on harvesting of predator are critical factors in keeping predator exploitation under control. In addition, we impose a tax to study the the optimal harvesting policy for harvesting predator. When the monetary social benefit is subject to maximisation, it is demonstrated by utilising Pontryagin's maximal principle. We established the optimal equilibrium solution by using optimal tax $\tau_1 = \tau_1^*$. It has been demonstrated that zero discounting maximises economic revenue and that an infinite discount rate causes economic rent to dissipate completely. It should be noted that in this paper, several crucial factors are disregarded, including ecological fluctuations, refuge, allee effect etc. Therefore, further study is required to meet the demands in this area.

References

[1] Chaudhuri, K. (1988). Dynamic optimization of combined harvesting of a two-species fishery. *Ecological Modelling*, 41(1-2), 17-25.

[2] Mukhopadhyay, A. Chattopadhyay, J. & Tapaswi, P.K. (1997). Selective harvesting in a two species fishery model. *Ecological Modelling*, 94, 243-253.

[3] Kar, T.K. & Chaudhuri, K.S. (2003). On non-selective harvesting of two competing fish species in the presence of toxicity. *Ecological Modelling*, 161(1-2), 125-137.

[4] Kramer, D. B. (2008). Adaptive harvesting in a multiple-species coral-reef food web. *Ecology and Society*, 13(1).

[5] El Foutayeni, Y., & Khaladi, M. (2012(a)). A bio-economic model of fishery where prices depend on harvest. *Journal of Advanced Modeling and Optimization*, 14(3), 543-555.

[6] Elfoutayeni, Y & Khaladi, M. (2012(b)). A generalized bio-economic model for competing multiple-fish populations where prices depend on harvest. *Journal of Advanced Modeling and Optimization*, 14(3), 531-542.

[7] Wang, W. K. & Ewald, C.O. (2010). A stochastic differential fishery game for a two species fish population with ecological interaction. *Journal of Economic Dynamics and Control*, 34(5), 844-857.

[8] Skonhoft, A., Vestergaard, N. & Quaas, M. (2012). Optimal harvest in an age structured model with different fishing selectivity. *Environmental and Resource Economics*, 51(4), 525-544.

[9] Diekert, F.K., Hjermann, D., Nævdal, E. & Stenseth, N.C. (2010). Spare the young fish: optimal harvesting policies for north-east arctic cod. *Environmental and Resource Economics*, 47, 455-475.

[10] Kar, T.K. & Chakraborty, K.S. (2010). Effort dynamics in a prey-predator model with harvesting. *International Journal of Information and Systems Sciences*, 6(3), 318-332.

[11] Chen, Q. Mao, J. & Li, W. (2006). Stability analysis of harvesting strategies in a cellular automata based predator-prey model. *Cellular Automata Lecture Notes in Computer Science*, 4173, 268-276.

[12] Pal, D., Mahaptra, G.S. & Samanta, G.P. (2013). Optimal harvesting of prey–predator system with interval biological parameters: A bioeconomic model. *Mathematical Biosciences*, 241, 181-187.

[13] Srinivas, M.N., Srinivas, M.A.S., Das, K. & Gazi, N.H. (2011). Prey predator fishery model with stage structure in two patchy marine aquatic environment. *Applied Mathematics*, 2, 1405-1416.

[14] Wuhaib, S.A., & Hasan, Y.A. (2013). Predator-prey interactions with harvesting of predator with prey in refuge. *Communications in Mathematical Biology and Neuroscience*, 2013.

[15] Huang, L., Cai, D., & Liu, W. (2021). Optimal tax policy of a one-predator–two-prey system with a marine protected area. *Mathematical Methods in the Applied Sciences*, 44(8), 6876-6895.

[16] Ibrahim, M. (2021). Optimal harvesting of a predator-prey system with marine reserve. *Scientific African*, 14, e01048.

[17] Sharma, A., Gupta, B., Dhar, J., Srivastava, S. K., & Sharma, P. (2021). Stability analysis and optimal impulsive harvesting for a delayed stage-structured self dependent two compartment commercial fishery model. *International Journal of Dynamics and Control*, 1-11.

[18] Meng, X.Y., & Meng, F.L. (2021). Bifurcation analysis of a special delayed predator-prey model with herd behavior and prey harvesting. *AIMS Mathematics*, 6(6), 5695-5719.

[19] Meng, X.Y., & Wu, Y.Q. (2018). Bifurcation and control in a singular phytoplankton-zooplankton-fish model with nonlinear fish harvesting and taxation. *International Journal of Bifurcation and Chaos*, 28(03), 1850042.

[20] Meng, X.Y., Wu, Y.Q., & Li, J. (2020). Bifurcation analysis of a singular nutrient-plankton-fish model with taxation, protected zone and multiple delays. *Numerical Algebra, Control & Optimization*, 10(3), 391.

[21] Juneja, N., & Agnihotri, K. (2018). Conservation of a predator species in SIS prey-predator system using optimal taxation policy. *Chaos, Solitons & Fractals*, 116, 86-94.

[22] Rani, R., Gakkhar, S., & Moussaoui, A. (2019). Dynamics of a fishery system in a patchy environment with nonlinear harvesting. *Mathematical Methods in the Applied Sciences*, 42(18), 7192-7209.

[23] Upadhyay, R.K., Banerjee, M., Parshad, R. & Raw, S.N. (2011). Deterministic chaos versus stochastic oscillation in a prey-predator-top predator model. *Mathematical Modelling and Analysis*, 16(3), 343-364.

[24] Zhang, X., Xu, R. & Li, Z. (2011). Global stability of a three-species food-chain model with diffusion and nonlocal delays. *Mathematical Modelling and Analysis*, 16(3), 376-389.

[25] Yavuz, M., & Sene, N. (2020). Stability analysis and numerical computation of the fractional predator–prey model with the harvesting rate. *Fractal and Fractional*, 4(3), 35.

[26] Wang, Y. & Wang, H. (2014). Stability and selective harvesting of a phytoplankton-zooplankton system. *Journal of Applied Mathematics*, 2014.

[27] Kar, T.K. (2005). Conservation of a fishery through optimal taxation: a dynamic reaction model. *Communications in Nonlinear Science and Numerical Simulation*, 10(2), 121-131.

[28] Sharma, A.K., Sharma, A., & Agnihotri, K. (2015). Dynamical analysis of a harvesting model of phytoplankton-zooplankton interaction. *International Journal of Mathematical and Computational Sciences*, 8(6), 1013-1018.

[29] Dubeya, B. Chandra, P. & Sinha, P. (2003). A model for fishery resource with reserve area. *Nonlinear Analysis: Real World Applications*, 4, 625-637.

[30] Kar, T.K. (2006). Modelling and analysis of a harvested prey–predator system incorporating a prey refuge. *Journal of Computational and Applied Mathematics*, 185, 19-33.

[31] Kar, T.K. & Chattopadhyay, S.K (2009). Bioeconomic modelling: an application to the north-east-atlantic cod fishery. *Journal of Mathematics Research*, 1(2), 164-178.

[32] Pontryagin, L.S. (1987). *Mathematical Theory of Optimal Processes*. CRC Press.

Anal Chatterjee *received his M.Sc degree in Applied Mathematics from University of Kalyani. He has qualified NET (2007) and GATE (2008). He became a Lecturer in Sheikhpara A.R.M. Polytechnic, Sheikhpara, and Murshidabad in 2012 and later joined Barrackpore Rastraguru Surendranath College, 85, Middle Road, 6, River Side Rd, Kolkata as an Assistant Professor in 2017. He obtained his PhD in Mathematical Biology from Kalyani University in 2014. His field of research interests is in mathematical ecology. The current research work is devoted to present several mathematical models on the dynamics of marine plankton ecology with planktivorous fish and eco-epidemiological models with prey refuge. He has already published more than 25 international research articles. He participated and delivered lectures in conferences, workshops in India.*

https://orcid.org/0000-0001-5780-9511

Samares Pal *received his MSc and MPhil degree in Applied Mathematics from University of Calcutta. He obtained his PhD in Mathematical Biology from Jadavpur University in 2004. He became a Lecturer in Ramakrishna Mission Vivekananda Centenary College, Rahara, Kolkata, in 1998 and later joined University of Kalyani as an Associate Professor in 2008 and promoted to professor in 2013. His field of research interests are in mathematical ecology and epidemiology. The current research work is devoted to present several mathematical models on the dynamics of marine plankton ecology, coral bleaching due to invasive predators, nonlinear transmission of a stage structured eco-epidemiological models with prey refuge, non-autonomous prey- predator model and ecosystem modeling including nitrogen stable isotopes in marine environments. He has already published more than 75 international research articles. He received INDO-US Research Fellowship for collaborative research work at Georgia Institute of Technology, Atlanta, Georgia, USA. Obtained fellowship under INSA International bilateral exchange programme for collaborative research work at National Centre for Theoretical Sciences (NCTS), Department of Mathematics, National Tsing Hua University, Taiwan. He participated and delivered invited lectures in conferences, workshops and visited Institutions in India and abroad.*

https://orcid.org/0000-0002-8792-0031

An International Journal of Optimization and Control: Theories & Applications (http://ijocta.balikesir.edu.tr)