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RESEARCH ARTICLE

Uncertainty-based Gompertz growth model for tumor population and its numerical analysis

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ABSTRACT

For treating cancer, tumor growth models have shown to be a valuable resource, whether they are used to develop therapeutic methods paired with process control or to simulate and evaluate treatment processes. In addition, a fuzzy mathematical model is a tool for monitoring the influences of various elements and creating behavioral assessments. It has been designed to decrease the ambiguity of model parameters to obtain a reliable mathematical tumor development model by employing fuzzy logic.The tumor Gompertz equation is shown in an imprecise environment in this study. It considers the whole cancer cell population to be vague at any given time, with the possibility distribution function determined by the initial tumor cell population, tumor net population rate, and carrying capacity of the tumor. Moreover, this work provides information on the expected tumor cell population in the maximum period. This study examines fuzzy tumor growth modeling insights based on fuzziness to reduce tumor uncertainty and achieve a degree of realism. Finally, numerical simulations are utilized to show the significant conclusions of the proposed study.

1. Introduction

A disease caused by an abnormal proliferation of cells due to unregulated cell proliferation is called cancer. Cancer is becoming increasingly common and has reached alarming levels [1]. To put it another way, cancer is not just an isolated population of mutated cells. Still, it is part of a larger tissue population that actively interacts with and disrupts a varied community of various cellular and micro-environmental interacting components that seek to maintain homeostasis [2]. Mathematical modeling of cellular mechanisms is frequently utilized to improve numeric knowledge of clinical events. This empirical evidence may be used in both experimental and clinical contexts. Tumor research is one significant field where modeling techniques are used [3]. When the system is unclear, fuzzy mathematics provides an outlet for bio-mathematical issues such as cancer to reach a realistic solution and a better and more exciting knowledge of a particular phenomenon [4, 5].

Zadeh introduced fuzzy sets in 1965 to deal with data having non-statistical uncertainty. The actual circumstances are frequently vague or ambiguous in many situations. Due to a lack of understanding, the future status of the network may not be fully known. Probability and statistics have long been used to address this sort of complexity (stochastic nature). Imprecision refers to

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uncertainty rather than a lack of knowledge about the parameter's value. A fuzzy set theory is a structured mathematical framework that allows for the precise and rigorous study of vague cognitive representations.Fuzziness may be noticed in many sectors of daily life, including engineering, medicine, metallurgy, astronomy, and so on. However, it is widespread in all areas where human judgment, evaluation, and decision-making are crucial, such as decision-making, thinking, reasoning, and so on [1, 6, 8, 17].

Uncertainty is prevalent in nearly all scientific disciplines, with a specific focus on biological phenomena, wherein complicated elements such as inheritance, habitat, availability, and so on affect the traits and development. This feature has prompted several recent contributions to develop mathematical models that integrate ambiguity to understand evolutionary processes using differential equations.The traditional perspective of uncertainty has been transformed, and the usual view regards uncertainty as undesirable in research efforts and must be eradicated by all feasible measures. The contemporary viewpoint accepts ambiguity and thinks that science should solve it [9–12]. Possibility theory focuses on ambiguity inherent in natural languages and is "possibilistic" rather than probabilistic. As a result, the term variable is employed in a broader scientific context than in a strictly mathematical context.

As shown in the prolongation, the mathematical structure of fuzzy set theory generates a suitable concept of possibility, giving a characteristic similar to that of measure theory regarding probability theory.An ambiguous restriction is an allocation of possibilities, with its membership function serving as a distribution function of possibilities, and a fuzzy attribute is affiliated with an allocation of possibilities in the same way that a probability distribution is attributed with a random variable suggested by Zadeh in this outlook [13]. Assume that the tumor cell population $n_0$is described in the sense of tumor population dynamics by a fuzzy set n_{01} with membership function$\mu_{n_{01}}(N)$.As a result, for any precise value $N = z_0$, the value $\mu_{n_{01}}(z_0)$represents the rate of probability that the dynamical system'sactual initial state of tumor takes the value z_0 given that the initial cell count of the tumor is precisely n_0. The membership function $\mu_{n_{01}}(N)$ is then the probability distribution associated with the initial state of the tumor. A detailed explanation of the condition of tumor cell number at a specified time $t > 0$ will not be required until we have accurate knowledge about the actual value of the initial tumor status. It is essential to explain a tumor's starting state using a fuzzy constraint since the initial condition is about z_0. In turn, this determines the distribution of possibilities to the outcomes expected by the initial condition.

The extended version of the fuzzy polynomial is the fuzzy equation. Since the ambiguities are explicit undefined parameters of the fuzzy equations, the fuzzy equations are more accessible to apply than regular fuzzy systems. These characteristics, on the other hand, are challenging to get.There are numerous methods for creating fuzzy equations, e.g.,crisp linear systems, extension principle, homotypic analysis, Newton's technique, fixed point methodology, α-level methods, and superimposition of sets. More recently, fuzzy fractional differential and integral equations, which may be used to solve fuzzy numbers, have been intensively investigated [14–19]. Various approaches have been proposed to comprehend the fuzzy uncertainty of differential equations. Some authors employ delay differential equations and Hukuhara–derivatives to create answers to fuzzy uncertainty. To forecast cell proliferation and tumor development models, the idea of fuzzy differential inclusion has been applied. The generalized Seikkala differentiability approach and the Zadeh extension rule are used to determine deterministic solutions in another attempt to solve fuzzy differential equations [20–27].

This study discusses the influence of fuzzy uncertainty in the Gompertz growth equation, which is used in a variety of fields, including statistical mechanics, medicine (tumor growth rate), chemistry (response models), and ecology (population growth) [28, 29]. The Gompertz model, probably second only to the logistic model, is amongst the most widely utilized sigmoid models for fitting growth data and other data. The Gompertz equation was initially designed for actuarial analysis, but it was eventually adopted as a growth curve. According to the literature, the Gompertz model has been used for many phenomena, including plant, bird, fish, and other animal development and tumor and bacterial growth. The Gompertz model is a particular instance of the four-parameter Richards model, and therefore belongs to the Richards family of three-parameter sigmoidal growth models, among famous models like the negative exponential, logistic, and von Bertalanffy. The literature has several parametrizations and re-parameterizations of the Gompertz model. Still, no comprehensive study of them and their characteristics have been done, but the Gompertz model showed the probability density function [30, 31]. Here, we'll gradually clarify the variance in the possibility distribution function, and then we'll decide on the optimal choice of that fuzzy variable, which will result in distinct outcomes.

Uncertainties remain in ecological modeling and must be considered to improve model confidence. For example, it is sometimes impossible to precisely identify the exact cell count or the bearing capability in a specified person in the difficulties of tumor population dynamics. Since the beginning condition is about n_0 or the carrying potential is approximately K_0, verbal statements often obtain information. This may be treated as an uncertain system, with the mark almost ambiguous. These verbal assertions can alternatively be regarded as fuzzy limitations on the values followed by the output characteristic. Since the beginning of cancer research, finding models to predict tumor progression has been critical. Several techniques have been proposed; however, there is no consensus on how tumor cells grow. This is a significant problem since an accurate tumor development model is necessary to evaluate imaging techniques, enhance radiation therapy processes, and provide treatment recommendations for patients [32–34].

The primary purpose of this research is to construct a fuzzy mathematical model based on fuzzy set theory to describe tumor growth and the concept of the possibility distribution function. A similar framework on the initial state of the tumor and parameters utilized in probability theory to address fuzzy uncertainty has been established here. For deterministic outcomes, we may implement the Zadeh extension principle to the early stages of the tumor and its attributes and characterize fuzzy strategies by treating them as fuzzy variables in the possibility distribution function.Because of this, we need to know how many tumor cells are expected to be in each network structure to compute the overall impact of this fuzzy uncertainty.Calculating the desired level to the deterministic solution describes the predicted level of the initial state and parameters. In addition, the Gompertz growth model predicts the absolute growth rate using fuzzy numbers, which are described in the restricted assertions.This work reveals novel ideas and aims that might make the model more practical and suitable.The fuzzy approaches will also aid in the early detection of tumors, which will allow us to customize the medication to the tumor patient. Also, to support the created model,numerical simulation has been presented. The framework of the tumor growth model is depicted in Fig.1.

This paper is organized as follows. In section 2some preliminaries on fuzzy sets, fuzzy variables and their transformation required to determine the solution of the Gompertz model are addressed.Section 3 deals with the Gompertz model of tumor growth with the projected level of the fuzzy variable and its net population rate.Section 4 explains the tumor state at time $t > 0$ in an imprecise environment, and section 5 shows our significant findings with a numerical simulation. The significance of our findings is explored in section 6. The managerial insights are discussed in section 7, and the conclusion is presented in section 8.

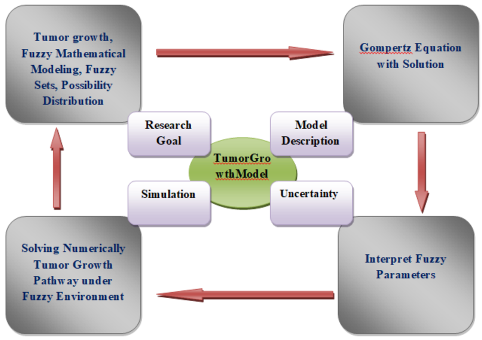

Figure 1. The framework of the tumor growth model.

2. Preliminaries for the proposed work

2.1. Fuzzy theory

Definition 1. *Let$X$ be a non-empty universal set, then a fuzzy set N is defined by a pair $(X,\ \mu_N(X))$ where, $\mu_N(X) : X \to [0,1]$ for each $u \in X$, $\mu_N(X)$ is called the membership grade function of u defined in N.*

A fuzzy set N in X is defined by a membership function$f_N(u)$that correlates any real number in the interval [0, 1] with every point in X, to a value of $f_N(u)$ at u reflecting the membership degree of u in N. Also for any $\alpha \in [0,\ 1]$, the set of points of X where $X_\alpha = \{u \in X \mid \mu_N(u) \geq \alpha\}$ is known as an α-cut set or theα-level set of the fuzzy set $A$and the crisp set that contains all the items of X with nonzero membership grades in N is the support of a fuzzy set N within a universal set X. For $\alpha = 0$, the support of A is the same as the strong α-cut of N.

Now, let us represent the set of fuzzy subsets of $X \subset R$ by $F(X)$, where the α-cuts for each $\alpha \in [0,\ 1]$ are non-void, bounded, closed, and are simply connected. Here's how we may get the distance among two fuzzy sets $a, b \in F(X)$,

$$d_\infty(a,b) = \operatorname*{Sup}_{\alpha\in[0,1]} d_p\left([a]^\alpha,[b]^\alpha\right), \qquad (1)$$

where for compact sets$d_p$is the Hausdorff distance [5, 6, 22, 35, 36].

We also symbolize the characteristic function of the set M by $\chi\{M\}$.

Definition 2. *(Fuzzy variables) Fuzzy variables are more realistic than crisp variables because they reflect measurement uncertainty in experimental data. Data based on fuzzy variables offer us more accurate information regarding actual occurrences than data based on crisp variables, which is a fascinating contradiction. Albert Einstein well-articulated this crucial aspect in 1921 as in the following phrase: In terms of referring to reality, mathematical laws are unreliable. And, to the best of their knowledge, they do not refer to fact.*

The name fuzzy variable was chosen because it performs the same role in the current development as the random variable in probability theory. In contrast, we argue that this is a far better explanation for a fuzzy set on the real line [37], [38].

Now, on the value of a flexible rate, the possibility distribution function is produced by a fuzzy subset a of X, represented through the membership grade$\mu_a : X \to [0,\ 1]$. To put it another way, if $\Psi$is a fuzzy variable, then $\mu_a(N)$ is the highest level to which the actual value N may go.However, $\mu_a(N) = 0$ means that the parameter Ψcannot conclude the value N in the sense of possibility theory. With assignment $\Psi = N$, the amount $\mu_a(N)$ indicates the degree of possibility, where specific$N$ values are more likely than others. The nearest the value of $\mu_a(N)$ is 1, the more likely that N is the variable's actual value.

The possibility measure of M is determined by $Pos_\mu = Sup_{N\in M}\mu_\alpha(N)$, provided the subset $M \subset X Nec_\mu(M) = 1 - Pos_\mu(M^c)$ where M^c stands for$M$ in X X complement set and by pointing out that $Pos_\mu(M)$ is a test of the possibility of assuming values in M by the fuzzy variable.It is easily verifiable that $Pos_\mu(\phi) = 0$ and $Pos_\mu(X) = 1$ [22], [35], [36].

In addition, based on the possibility and necessity measures, we provide a third index called the credibility measure, which is as follows:

$$Cr_\mu(M) = \frac{1}{2}\left(Pos_\mu(M) + Nec_\mu(M)\right) \qquad (2)$$

To obtain a typical representation of a fuzzy variable Ψ on the credibility space, it is necessary to establish the definition of its probable or the expected value, which can be labeled as $E[\Psi] = \int_0^{+\infty} Cr_\mu\{\Psi \geq r\}\,dr - \int_{-\infty}^{0} Cr\{\Psi \leq r\}\,dr$, the expected value is not specified for any real number r,if the right-hand side of this equation is of nature $(\infty - \infty)$, that is at least one of the two integrals must be finite.It is noted that the concept of $Pos(M)$, $Nec(M)$ *and* $Cr(M)$of the fuzzy variable Ψ relies on the possibility distribution function μ_a.

Suppose a random variable replaces a fuzzy variable Ψ and the probability measure Pr replaces Cr, then we have

$$\int_0^{+\infty} Pr_\mu\{\Psi \geq r\}\,dr - \int_{-\infty}^{0} Pr\{\Psi \leq r\}\,dr = \int_{-\infty}^{+\infty} N\varphi(N)\,dN,$$

this is identically the anticipated random variable value, suggesting that the description of the expected return of a fuzzy vector is equivalent to the description of the expected return of a random vector.

Now, if we specify the quantities $\Psi'_\alpha = inf\{N : \mu_a(N) \geq \alpha\}$ and $\Psi''_\alpha = \{N : \mu_a(N) \geq \alpha\}\ \forall\ \alpha > 0$, for the non-negative set X of real numbers, then equation (2) becomes $E[\Psi] = \frac{1}{2}\int_0^1 \left(\Psi'_\alpha + \Psi''_\alpha\right) d\alpha$, given that Ψ'_α and Ψ''_α are finite and since Ψ is a fuzzy normalized variable,the α-negative value and the α-positive value of Ψ are respectively Ψ' and Ψ'' [35-38].

Definition 3. *(Fuzzy variable Transformation) The machine designing process contains a lot of uncertain data, and both random variables and fuzzy variables may be used to express uncertainty. The issue of uncertainty in the fuzzy variables might be handled using the idea of the cut-set of fuzzy mathematics by transforming from the fuzzy system to the general class [39].*

Now, related to specific subsets of the real numbers assume that$h$is a continuous function and if Ψ is a fuzzy variable, then $\rho = h(\Psi)$, and there is a simple method of describing the possibility distribution function $\mu_{h(a)}(N)$ to $h(\Psi)$ from the distribution $\mu_a(N)$of Ψ. For a given $M \subset X$, Ψ considers a value on $h^{-1}(M)$ if and only if $h(\Psi)$ considers values on M. So by interpretation, the ability to assign characters in $h^{-1}(M)$is similar to the ability of $h(\Psi)$to assign characters in M.Because of this,for the fuzzy variable$\rho = h(\Psi)$, it points out that $Pos_\rho(M) = \text{Sup}_{p\in M}\,\mu_{\hat{h}(a)}(p) = \text{Sup}_{N\in h^{-1}(M)}\,\mu_a(N) = Pos_\Psi\left(h^{-1}(M)\right)$ and hence we can achieve a function by considering

$$\mu_{\hat{h}(a)}(p) = \operatorname*{Sup}_{N\in h^{-1}(p)} \mu_a(N) \qquad (3)$$

It is observed that equation (3) is the depiction of the Zadeh extension of h. As a result, we represent the possibility distribution function of $h(\Psi)$ by $\mu_{\hat{\mathrm{h}}(a)}(p)$ [40].

Now, by [41], if a fuzzy variable $\rho = h(\Psi)$ and h be a monotonic (strictly increasing or decreasing) function and the integrals $\int_0^1 h\left(\Psi'_\alpha\right) d\alpha$ and $\int_0^1 h\left(\Psi''_\alpha\right) d\alpha$ are finite; then the expected value can be determined by

$$E\left[h\left(\Psi\right)\right] = \frac{1}{2}\int_0^1 \left[h\left(\Psi'_\alpha\right) + h\left(\Psi''_\alpha\right)\right] d\alpha. \quad (4)$$

Someone may now be confronted with the issue of ambiguity regarding various elements in dynamic tumor nature and specific procedures.Assume that,μ_{a_1} and μ_{a_2} be the possibility distribution functions for the fuzzy variables Ψ_1 and Ψ_2 accordingly defined on the set X. Such factors describe a fuzzy variable on $X \times X$, that is $\rho = (\ \Psi_1,\ \Psi_2)$ and

$$\mu_a(N,p) \ = \ min\left\{\mu_{a_1}(N), \mu_{a_2}(p)\right\} \quad (5)$$

provides its combined possibility distribution function $\mu_a : X \times X \to [0,\ 1]$.

3. Tumor growth in a fuzzy environment using the Gompertz model

The Gompertz model, probably second to the logistic model, is one of the most often used sigmoid models for fitting growth data and other data. With application to tumor growth, several dynamic growth rate functions have been discussed. Gompertz growth has been demonstrated to recreate cell growth that slows with population density and thus is appropriate to observe tumor growth slowdown with tumor size.The growth rate is calculated by taking the negative logarithm of the present size of the population and dividing it by the carrying capacity:

$$\begin{aligned} &\dot{N}\left(t\right) = -\gamma N\left(t\right) log\left(\frac{N\left(t\right)}{K}\right);\ t > 0, \\ &N\left(0\right) = n_0, \quad \gamma > 0 \ \ and\ K > n_0 \end{aligned} \quad (6)$$

here $N(t)$denotes the tumor cell concentration in the target organism, $\dot{N}\left(t\right)$ denotes the derivative of N concerning time $t \neq 0$,γ indicates the net rate of tumor replication, and $K > 0$ denotes the tumor carrying capacity or the volume at which it stabilizes when the resource supply remains constant. Even though such parameters are commonly regarded as trustworthy, it is critical in creating realistic and empirical models to assess the uncertainty associated with their inherent variance or complexity. The genesis of the Gompertz model has been disputed for years; numerous independent investigations have found a strong and a substantial connection between the Gompertz model parameters and in either experimental systems or human data, and some researchers hypothesized that this would indicate a consistent maximum tumor size across tumor kinds within a species.

The dynamics of $N(t)$ over time are defined by the Gompertz model. In this context, a significant query that frequently arises in research is when $N(t)$ approaches a particular interest value.The solution of equation (**??**) is given by

$$\delta_t\left(n_0,\ \gamma,\ K\right) = Ke^{-ln}\left(\frac{K}{N_0}\right)e^{-\gamma t} \quad (7)$$

It has already been established that dealing with parameter inaccuracy is not always suitable due to a lack of comprehensive knowledge or estimation failure. A basic technique of coping with Gompertz equation uncertainties (6) is utilized to obtain these parameter estimations by utilizing the equation(7) to calculate the average approximations and to assess the complexity [42, 43, 45–47].

Let us now suppose that the fuzzy marks constrain the parameters n_0, γ and K. In other words, we suppose that such parameters fulfill an assertion such as the fuzzy variable (Ψ) generally is a_0. So the membership grade of the fuzzy mark is roughly the probability distribution of the fuzzy variable (Ψ) as per Zadeh. As the term $\delta_t\left(n_0, \gamma, K\right)$ of equation (7) is a fuzzy variable for a specified time $t > 0$, so the termsn_0, γ and K in equation (6) are also fuzzy variables.With the help of Zadeh extension,the procedures mentioned in the earlier parts on the parameters n_0, γ and K of the possibility distribution function $\delta_t\left(n_0, \gamma, K\right)$, for a fixed $t > 0$ can be obtained.

To create a realistic and practical model, it is necessary to remember that the parameters of equation (6) are approximate owing to the assumed lack of information and the mistakes in the calculation technique inherent in the relevant issues of the tumor growth. Different approaches, such as the use of random variables, are considered to characterize these parameters.The authors occasionally evaluate the Gompertz equation (6) with changes in the carrying capacity (K).

Note: Before we go into the details of the proposed study, everyone has a question: why do we need the starting condition, coefficient, and fuzzy numbers in tumor development modeling? The reason behind this is that.

(1) **Fuzziness in the initial condition:**If we want to calculate the population of tumor cells after a particular duration of time. For this reason, we had to estimate the beginning number of the tumor cell population, which is difficult to enumerate in an exact amount due to the inaccuracy of the count. As a result, it's preferable to use the original history as a guiding parameter.

(2) **Fuzziness in the coefficient:**If the pace of growth rate at which the population of tumor cells grows is unknown, it is difficult to estimate an exact amount. The value should be treated as ambiguous for this purpose.

(3) **Fuzziness in both the initial condition and the coefficient:**If both cases are combined in a model, this case can be used as well.

(4) **Fuzziness in the carrying capacity:**As carrying capacity varies over time in response to slow environmental changes, such as climate change or ecological succession, it is seen as ambiguous.

The creation of the fuzzy parameter system based on the Gompertz growth equation (6) is demonstrated in this paper in Fig.2, as follows:

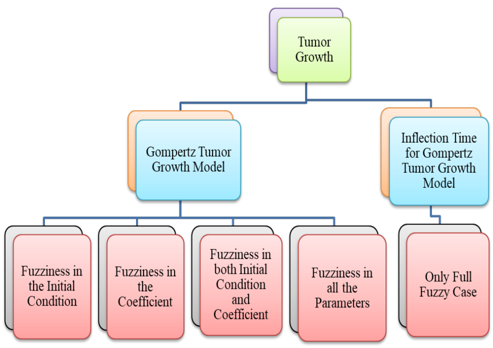

Figure 2. Fuzzy transformation mechanism of the Gompertz tumor growth model.

- *Gompertz growth equation* (6) *with fuzziness in the initial condition*n_0, *coefficient* γ *and in the initial condition* n_0 *and the coefficient* γ *together.*

The transformed model when $n_0$is taken into account as a fuzzy variable with a membership function of the possibility distribution function $\mu_{n_{01}}(N) : X \to [0,1]$, where the initial condition is limited roughly to $n_0 > 0$. However, it is suggested that the other two factors,the net growth rateγ and the carrying capacity K be specified and fixed. As a result, the Gompertz equation (6) with a fuzzy initial condition implies $\dot{N}(t)(t,\ \alpha) = -\gamma N(t)(t,\ \alpha)\ \ log\left(\frac{N(t)(t,\ \alpha)}{K}\right)$, and here $\delta_t(n_0)$ values rely on the numbers inferred by the initial condition as a fuzzy variable. Now, for a fuzzy setn_{01}, the α-*cut* of the fuzzy set $\widehat{\delta_t}(n_{01})$ having membership function $\mu_{n_{01}}(N)$ is given by the closed interval

$$\left[\widehat{\delta_t}(n_{01})\right]^{\alpha} = \left[\delta_t\left(\Psi'_{\alpha}\right),\ \ \delta_t\left(\Psi''_{\alpha}\right)\right]$$

Also, for the fuzzy variable $\delta_t(n_0)$, the expected value by using the above result can be obtained as

$$E\left[\delta_t(n_0)\right] = \frac{1}{2}\int_0^1 \left(\delta_t\left(\Psi'_{\alpha}\right) + \delta_t\left(\Psi''_{\alpha}\right)\right) d\alpha,$$

and for the membership function $\mu_{n_{01}}(N) = \chi\{[c,\ d]\}$ with the initial condition as fuzzy, the expected value is given by $E\left[\delta_t(n_0)\right] = \frac{\delta_t(c)+\delta_t(d)}{2}$. Now, the transformed model when the coefficient $\gamma$is considered a fuzzy variable and the initial condition n_0 and carrying capacity $K$is intended to be specified and fixed with a membership function of the possibility distribution function $\mu_{\gamma_1}(N) : X \to [0,1]$. Then, the Gompertz equation (6) with the fuzzy coefficient implies

$$\dot{N}(t)(t,\ \alpha) = -\gamma(\alpha)N(t)\,log\left(\frac{N(t)}{K}\right)$$

Also,the transformed model when both the initial condition n_0, coefficient γtogether are taken into account as fuzzy variables and carrying capacity$K$ is intended to be specified and fixed with a membership function of the possibility distribution function $\mu_{n_{01}}(N) : X \to [0,1]$ and $\mu_{\gamma_1}(N) : X \to [0,1]$. Then, the Gompertz equation (6) with the fuzzy initial condition$n_0$and coefficient γ implies

$$\dot{N}(t)(t,\ \alpha) = -\,\gamma(\alpha)N(t)(t,\ \alpha) \times log\left(\frac{N(t)(t,\ \alpha)}{K}\right)$$

- *Gompertz growth equation* (6) *with fuzziness in all parameters (full fuzzy case)*

Here, the transformation on all the three parameters, i.e., the initial condition n_0, the coefficient γ and the carrying capacity K are considered. Now, for the variables n_0, γ and K, let the fuzzy sets

n_{01}, γ_1 and K_1 describe the fuzzy limits on the results calculated respectively. Also, define a fuzzy array $q = n_{01},\ \gamma_1,\ K_1$; where $\mu_q(n_0,\ \gamma,\ K) = \min\{\mu_{n_{01}}(n_0),\ \mu_{\gamma_1}(\gamma),\ \mu_{K_1}(K)\}$ gives the joint possibility distribution function. Now, for the fuzzy sets $n_{01},\ \gamma_1$ *and* K_1, the *α-cuts* of the fuzzy set $\widehat{\delta}_t(n_{01},\ \gamma_1,\ K_1)$ and having membership functions $\mu_{n_{01}}(N),\ \mu_{\gamma_1}(N), and\ \mu_{K_1}(N)$ is given by the closed interval

$$\left[\widehat{\delta}_t(n_{01}, \gamma_1, K_1)\right]^{\alpha} = \left[\delta_t\left(\Psi_a^{'}, \rho_a^{'}, \nu_a^{'}\right), \delta_t\left(\Psi_a^{''}, \rho_a^{''}, \nu_a^{''}\right)\right].$$

Also, for the fuzzy variable $\delta_t(n_0,\ \gamma,\ K)$, the expected value by using the above result can be obtained as

$$E[\delta_t(n_0, \gamma, K)] = \frac{1}{2}\int_0^1 \left(\delta_t\left(\Psi_a^{'}, \rho_a^{'}, \nu_a^{'}\right), \delta_t\left(\Psi_a^{''}, \rho_a^{''}, \nu_a^{''}\right)\right) d\alpha$$

and for the membership function $\mu_{n_{01}}(N) = X_{[c,d]}(N)$, $\mu_{\gamma_1}(N) = X_{[\gamma_{01}, \gamma_{02}]}(N)$ and $\mu_{K_1}(N) = X_{[K_{01}, K_{02}]}(N)$ with an initial condition, coefficient and carrying capacity as fuzzy, the expected value is given by $E[\delta_t(n_0, \gamma, K)] = \frac{\delta_t(c, \gamma_{01}, K_{01}) + \delta_t(c, \gamma_{02}, K_{02})}{2}$.

As a result, the Gompertz equation (6) with the fuzzy parameters $n_0,\ \gamma$ *and* K implies $\dot{N}(t)(t,\ \alpha) = -\gamma(\alpha) N(t)(t,\ \alpha)\, log\left(\frac{N(t)(t,\ \alpha)}{K(\alpha)}\right)$

Note: The lower α cut and the upper α cut for the fuzzy variable with the membership function $\mu_{n_{01}}(N) = \chi[a,\ b]$ of the possibility distribution function are described as

$$\Psi_\alpha^{'} = m - (1-\alpha)\theta \text{ and } \Psi_\alpha^{''} = m + (1-\alpha)\theta \quad (8)$$

where for the fuzzy interval $\chi[a,\ b]$,m is the middle term with difference θ between the middle value and the upper value of the interval [a, b].

4. Inflection time for Gompertz tumor growth model

The tumor curve development slows (moving from concave up to downward) at an inflection point)or accelerates (moving from concave down to upward). To put it another way, the inflection point aids in emphasizing the evolution that has occurred over time. Most people know that cancerous growth has a rapid beginning proliferation, which indicates that cell alterations occur rapidly throughout time. This study looks at the tumor's development and the accomplishment of carrying capacity K over time. For example,suppose the equation describes tumor size growth over time. In that case, this timespan may indicate the saturation threshold at K, since nutrients are transferred from initial growth to cell division. When it comes to cancer cells, the decreasing return rule has a specific effect on-time behavior.

Human tumor growth was once thought to be both unpredictable and rapid. A study of the development of laboratory tumors has revealed that the majority of the time, this growth rate matches a straightforward equation, such as the Gompertz equation. As a result, scientists have devised a formula to calculate the number of tumor cells within a specific period, allowing them to determine the exact growth of each cancer.As a result of this information, oncologists can design a robust recovery strategy till the tumor has had a chance to respond to treatment. In mathematical modeling, the inflection time T_i is essential because it identifies the period (or population) at which the maximum growth rate occurs, which may be used to predict future growth rates [48–50]. To get the inflection time $T_i > 0$, using Gompertz equation (6), use the following formula

$$T_i = \frac{\ln\left(-\ln(\frac{n_0}{K})\right)}{\gamma}; \quad n_0 < K/2 \qquad (9)$$

Here, T_i is a fuzzy variable in the same way as $n_0,\ \gamma$ *and* K are fuzzy variables.

- *Fuzziness in the initial condition n_0, coefficient γ and in both the initial condition n_0 and coefficient γ together of the equation (9)*

The transformed model for the inflection time of the Gompertz equation (9) when n_0 is taking into account as fuzzy by a membership grade of the possibility distribution function $\mu_{n_{01}}(N) : X \to [0, 1]$ with the net growth rate γ and the carrying capacity K be specified implies

$$T_i = \frac{\ln\left(-\ln(\frac{n_0(\alpha)}{K})\right)}{\gamma}$$

It is also possible to examine, for the fuzzy variable $T_i(n_0)$, the expected value while evaluating the impact of fuzzy uncertainty on the maximal tumor growth period, which is given as

$$E[T_i(n_0)] = \frac{1}{2}\int_0^1 \left(T_i\left(\Psi_\alpha^{'}\right) + T_i\left(\Psi_\alpha^{''}\right)\right) d\alpha$$

and for the membership function $\mu_{n_{01}}(N) = \chi\{[c,\ d]\}$ with the initial condition as fuzzy, the expected value is given by $E\left[T_i\left(n_0\right)\right] = \frac{T_i(c)+T_i(d)}{2}$.

Now, the transformed model when the coefficient γ is taken into account as a fuzzy variable and the initial condition n_0 and carrying capacity $K$is intended to be specified and fixed with a membership function of the possibility distribution function $\mu_{\gamma_1}(N) : X \to [0,1]$. Then, the inflection time of the Gompertz equation (9) with the fuzzy coefficient implies

$$T_i = \frac{\ln\left(-\ln(\frac{n_0}{K})\right)}{\gamma(\alpha)}$$

Also, the transformed model when both the initial condition n_0, coefficient γ are taken into account as fuzzy variables and carrying capacity K is intended to be specified and fixed with a membership function of the possibility distribution function $\mu_{n_{01}}(N) : X \to [0,1]$ and $\mu_{\gamma_1}(N) : X \to [0,1]$. Then, the inflection time of the Gompertz equation (9) with the fuzzy coefficient implies

$$T_i = \frac{\ln\left(-\ln(\frac{n_0(\alpha)}{K})\right)}{\gamma(\alpha)}.$$

- *Fuzziness in all the parameters of the equation* (9) *(full fuzzy case)*

Here, the transformation on all the three parameters, i.e., the initial condition n_0, the coefficient γ and the carrying capacity K is considered by the fuzzy sets $n_{01},\ \gamma_1\ and\ K_1$ respectively. The fuzzy set $\widehat{T_i} = \widehat{T_i}\left(n_{01},\ \gamma_1,\ K_1\right)$, where

$$\mu_{\widehat{T_i}}(\tau) = sup\left\{\mu_{n_{01}}\left(n_0,\ \gamma,\ K\right) : \left(n_0,\ \gamma,\ K\right) \in {T_i}^{-1}(\tau)\right\}$$

specifies the possibility distribution function of the fuzzy variable $T_i\left(n_0,\ \gamma,\ K\right)$. Now for the fuzzy sets $n_{01},\ \gamma_1\ and\ K_1$, the α-cut set $\widehat{T_i}\left(n_{01},\ \gamma_1,\ K_1\right)$ with the membership functions $\mu_{n_{01}}(N)$, $\mu_{\gamma_1}(N)$, and $\mu_{K_1}(N)$ is given by the closed interval $\left[\widehat{T_i}\left(n_{01}, \gamma_1, K_1\right)\right]^{\alpha} = \left[T_i\left(\Psi''_a, \rho''_a, \nu''_a\right), \delta_t\left(\Psi'_a, \rho'_a, \nu'_a\right)\right]$, as the function $T_i\left(n_0,\ \gamma,\ K\right)$ is increasing in K and is decreasing in $\gamma\ and\ n_0$.

So the expected value of the fuzzy variable $T_i\left(n_0,\ \gamma,\ K\right)$ may be determined by using equation (4), and it gives $E\left[T_i\left(n_0, \gamma, K\right)\right]^{\alpha} = \frac{1}{2}\int_0^1\left(T_i\left(\Psi''_a, \rho''_a, \nu''_a\right), \delta_t\left(\Psi'_a, \rho'_a, \nu'_a\right)\right)d\alpha$ and for the membership function $\mu_{n_{01}}(N) = \chi_{[c,\ d]}(N)$, $\mu_{\gamma_1}(N) = \chi_{[\gamma_{01},\ \gamma_{02}]}(N)\ and\ \mu_{K_1}(N) = \chi_{[K_{01},\ K_{02}]}(N)$ with an initial condition, coefficient and carrying capacity as fuzzy, the expected value is given by

$$E\left[T_i\left(n_0,\ \gamma,\ K\right)\right] = \frac{T_i\left(d,\ \gamma_{02},\ K_{01}\right) + T_i\left(c,\ \gamma_{01},\ K_{02}\right)}{2}.$$

As a result, the inflection time for the Gompertz equation (6) given by equation (9) with the fuzzy parameters $n_0,\ \gamma\ and\ K$ implies

$$T_i = \frac{\ln\left(-\ln(\frac{n_0(\alpha)}{K(\alpha)})\right)}{\gamma(\alpha)}.$$

Note: The lower α-cut and the upper α-cut are given by the equation (8), i.e.

$$\Psi'_a = m - (1-\alpha)\,\theta \text{ and } \Psi''_a m + (1-\alpha)\,\theta.$$

5. Numerical simulation

Numerical simulations of the Gompertz growth model under fuzzy environment for tumor development are performed in this section using the possibility distribution function. With varying growth rates and achieving the carrying capacity at different times, the net rate of tumor cell concentration fluctuates according to the lower and upper α-cuts of the possibility distribution function for the specific parameter in the Gompertz equation. We can find out how to solve the fuzzy Gompertz equation of tumor growth using the MATLAB R2019b software.

- Gompertz equation with the fuzzy initial condition:

Take $N(t) = \chi[3,\ 11]$, $\gamma = 0.15$, $K = 100$ and $t \in [0,\ 100]$. Then, the problem becomes

$$\dot{N}(t)(t,\ \alpha) = -(0.15)\,N(t)(t,\ \alpha) \times log\left(\frac{N(t)(t,\ \alpha)}{100}\right)$$

with initial condition $N(0,\alpha) = 7 - (1-\alpha)\,4$.

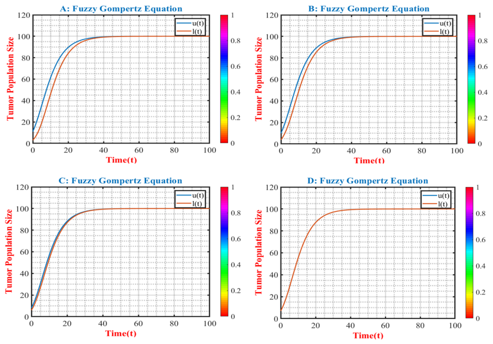

Figure 3. In this case, the system of Gompertz equation with only the initial condition as fuzzy is defined and the tumor population size as a function of time for $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.75$ and $\alpha = 1$ are illustrated.

- Gompertz equation with the fuzzy coefficient or net population rate of tumor:

Take $N(t) = 7$, $\gamma = \chi[0.1,\ 0.2]$, $K = 100$ and $t \in [0,\ 100]$.

Then, the problem becomes

$$\dot{N}(t)(t,\ \alpha) = -\left(0.15 - (1-\alpha)\, 0.05\right) N(t) \times log\left(\frac{N(t)}{100}\right)$$

with initial condition $N(0, \alpha) = 7$.

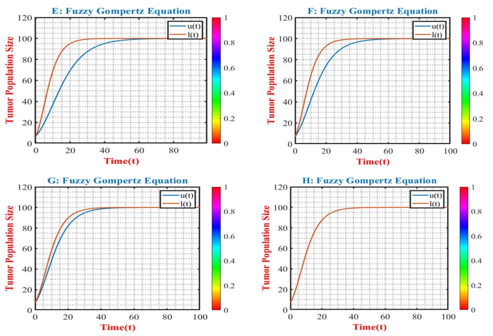

Figure 4. In this case, the system of Gompertz equation with only the coefficient as fuzzy is defined and the tumor population size as a function of time for $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.75$ and $\alpha = 1$ are illustrated.

- Gompertz equation with both the initial condition and the net population rate of the tumor as fuzzy:

Take $N(t) = \chi[3,\ 11]$, $\gamma = \chi[0.1,\ 0.2]$, $K = 100$ and $t \in [0,\ 100]$.

Then, the problem becomes

$$\dot{N}(t)(t,\ \alpha) = -\left(0.15 - (1-\alpha)\, 0.05\right) N(t)(t,\ \alpha) \times log\left(\frac{N(t)(t,\ \alpha)}{100}\right)$$

with initial condition $N(0, \alpha) = 7 + (1-\alpha)\, 4$.

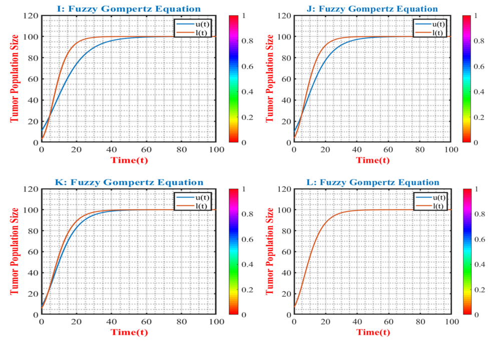

Figure 5. In this case, the system of Gompertz equation with both initial condition and the coefficient as fuzzy is defined and the tumor population size as a function of time for $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.75$ and $\alpha = 1$ are illustrated.

- Gompertz equation with all the parameters, i.e., initial condition, the net population rate of the tumor, and carrying capacity as fuzzy:

Take $N(t) = \chi[3,\ 11]$, $\gamma = \chi[0.1,\ 0.2]$, $K = \chi[90,\ 110]$ and $t \in [0,\ 100]$. Then, the problem becomes

$$\dot{N}(t)(t,\ \alpha) = -\left(0.15 - (1-\alpha)\, 0.05\right) N(t)(t,\ \alpha) \times log\left(\frac{N(t)(t,\ \alpha)}{100 - (1-\alpha)\, 10}\right)$$

with initial condition $N(0, \alpha) = 7 + (1-\alpha)\, 4$.

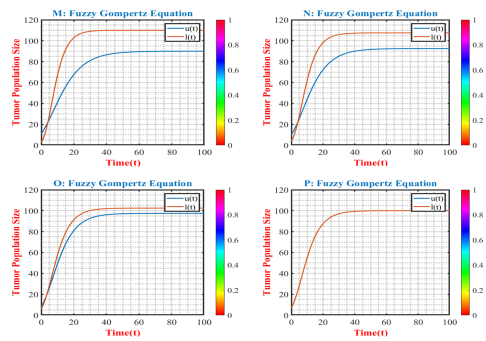

Figure 6. In this case, the system of Gompertz equation full fuzzy i.e., initial condition, coefficient and carrying capacity all are fuzzy is defined and the tumor population size as a function of time for $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.75$ and $\alpha = 1$ are illustrated.

6. Results and discussion

A fuzzy mathematical model of tumor development trajectory makes significant progress in displaying the realistic path of tumor growth with the help of a degree of accuracy.The possibility distribution function can solve the Gompertz equation for tumor development in a fuzzy environment. Upon attaining the carrying capacity at different intervals, the net tumor volume or the tumor cell concentration changes appropriately, as demonstrated by utilizing the lower and upper α-cuts for the possibility distribution function for a specific parameter of the Gompertz equation.As a result, the lower and upper α-cuts of the possibility distribution function indicate the region of ambiguity for the net tumor volume.

The absence of replication time at the beginning stages of a tumor would be attributed to a variance process or a reaction of the body to the sickness owing to some adverse effect of defensive characteristics throughout tumor cell differentiation. However, in certain situations, the defensive process would have been abducted at the start, resulting in a fast rise that is precisely proportionate to the pace of tumor development. The growth mechanism is not disrupted in the linear phase, and maximal increase happens at $K/2$, indicating that the defensive system is effectively viable. However, as tumor size increases, growth becomes more complex, and the growth rate eventually decreases, resulting in a plateau phase due to nutritional shortages, oxygen crises, and other factors.The Gompertz model extends the logistic model that includes an asymmetrical graph with an intersection point.The Gompertz model depicts the tumor's inherent phases and therefore, best deals with its development pattern.

The crisp mathematical model differs from the fuzzy mathematical model of tumor growth in the way that in the crisp model, the parameters are fixed, whereas, in the fuzzy model, the parameters are variable due to a variety of factors, including the fact that tumors are constantly evolving, resulting in changing dynamics. We can study the growth mathematically inside the binary value in a crisp mode. Still, this work shows the behavior of tumor development by modifying the initial tumor cell population, tumor net population rate, and carrying capacity of the tumor through the α-cut in a fuzzy model. This feature allows tumor load to be calculated based on the extent of accuracy, which might be extremely important for tumor staging and analysis.

Also, for the fuzzy intervals $n_0 = \chi\,[3,\ 11]\,,\ \gamma = \chi\,[0.1,\ 0.2]$ and $K = \chi\,[90,\ \ 110]$, it has been found that the inflection time $T_i(\alpha)$ for the Gompertz growth in a full fuzzy manner for $\alpha = 0$ is 12.24 and 4.17 as a lower and upper α-cut respectively, for $\alpha = 0.25$ is 8.8 and 5.44 as a lower and upper α-cut respectively, for $\alpha = 0.75$ is 6.4 and 7.2 as a lower and upper α-cut respectively. Finally, for $\alpha = 1$it is 6.52 for both lower and upper α-cut,which indicates the optimal time for tumor growth is also a fuzzy variable. Any variables of the possibility distribution function of fuzzy numbers must be kept in mind when measuring resemblance and its shape and midpoint are essential metrics. For the fuzzy intervals $n_0 = \chi\,[3,\ 11]\,,\ \gamma = \chi\,[0.1,\ 0.2]$ and $K = \chi\,[90,\ \ 110]$, the expected value of $T_i\,(n_0,\ \gamma,\ K)$ is given by $E\,[T_i\,(n_0,\ \gamma,\ K)] = -4.55$, means that there will be no tumor cell population, i.e., the margins don't contain cancerous cells.

7. Managerial insights

This study achieves an intelligent decision support system called fuzzy mathematical tumor growth modeling, which combines population models, mathematical modeling, and fuzzy logic. Possibility distribution functions can be utilized in decision-making, risk response, and optimization algorithms to solve the Gompertz equation of tumor development under fuzzy conditions. According to the fuzzy Gompertz equation, the tumor growth rate decreases linearly with volume until it approaches zero at carrying capacity. For example, it has been demonstrated that when reaching carrying capacity at different intervals of time, the net volume of tumor or tumor cell concentration fluctuates according to the lower and upper α-cuts of the possibility distribution function. Besides explaining tumor development in a fuzzy environment, the model also provides practical strategies for treating cancer. In real-life data, it may be utilized to analyze tumor distribution, cell count estimation, and tumor staging, which, in turn, leads to more precise targeting of treatment methods. As a continuation of this study, stochastic differential equations from the above-described models will be used to calculate and assess tumor development behavior in a fuzzy environment. It has become possible to understand clinical data with the use of new complicated mathematical models, as well. As a general rule, fuzzy logic reduces complexity in the tumor detection method by reducing the number of variables involved.

8. Conclusion

As a disease of extreme complexity, cancer's progression, remission, and treatment mechanisms are still unknown. Fuzzy mathematical modeling addresses its uncertainty and therefore gives a feasible way to cope with it in every phase. It is essential to replicate some real-life events using fuzzy mathematical modeling to obtain a more realistic representation of reality. Changing the model parameters in the tumor growth fuzzy model will lower the overall residuals, minimizing the uncertainty between the numerical predicting model and the actual results of medical studies. This study has interpreted the initial state, net population rate, and carrying capacity as a set of fuzzy variables whose possibility distribution function is determined by the membership grade of fuzzy sets. Because this method is based on facts, it may be utilized to determine the most effective prescription in less time by eliminating the uncertainty associated with tumor growth. Therefore, fuzzy mathematical modeling supports the resolution of ambiguities in calculation parameters, allowing for the differentiation of present and expected tumor growth modeling. It is also possible to improve tumor growth in a fuzzy environment by combining multiple derived principles with different fuzzy methods.This paper depicts the tumor development process numerically.

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