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RESEARCH ARTICLE

Approximate controllability for Riemann-Liouville fractional differential equations

Lavina Sahijwani*, Nagarajan Sukavanam

Department of Mathematics, IIT Roorkee, India
lsahijwani@ma.iitr.ac.in, n.sukavanam@ma.iitr.ac.in

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ABSTRACT

The article objectifies the approximate controllability of fractional nonlinear differential equations having Riemann-Liouville derivatives. The nonlinear term involved in the equation, also depending on the control parameter $u(\cdot)$, is considered to be locally Lipschitz. First, the existence of solutions is deduced using Lipschitz condition, semigroup theory and fixed point approach. Then sufficient condition for approximate controllability of the system is established using Cauchy convergence through iterative and approximate techniques. The theory of semigroup together with probability density function has been utilized to reach the desired conclusions. Lastly, an application is provided to support the proposed methodology.

1. Introduction

The scope of this article revolves around the underneath system:

$$D_\tau^\eta z(\tau) = Az(\tau) + Bu(\tau) + g(\tau, z(\tau), u(\tau)), \quad (1)$$
$$\tau \in (0, a], \quad 0 < \eta \leq 1$$
$$I_\tau^{1-\eta} z(\tau)|_{\tau=0} = z_0 \in Z,$$

where D_τ^η indicates the Riemann-Liouville η^{th} order derivative. $A : D(A) \subseteq Z \to Z$ generates a C_0 - semigroup $T(\tau)(\tau \geq 0)$ on Z. $z(\tau)$ and $u(\tau)$ takes value in Banach spaces Z and U respectively. The linear map B is defined from $L^q([0, a]; U)$ to $L^q([0, a]; Z)$, $q > \frac{1}{\eta}$. g is a function from $[0, a] \times Z \times U \to Z$.

The study of fractional calculus has long been admired from past three decades. The first work, exclusively committed to the study of fractional calculus, is the book by Oldham and Spanier [1], 1974. Fractional derivatives serves as an exemplary mechanism for the interpretation of hereditable properties and memory of profuse scientific, physical and engineering phenomena. On account of finer accuracy and precision over integer-order models, fractional derivatives accelerates its applications in diffusion process, biological mathematical models, aerodynamics, viscoelasticity, electrical engineering, signal and image processing, control theory, heat equation, electricity mechanics, electrodynamics of complex medium, etc. (see [2–10]).

In domain of fractional calculus, Riemann-Liouville and Caputo type derivatives have maintained to be the centre of attention for numerous analysts. Riemann-Liouville derivative shows supremacy over Caputo in the sense that it allows the function involved to bear discontinuity at origin. Also, in turn, doesn't allow the use of traditional initial conditions, the initial conditions involved in Riemann-Liouville case are integral initial conditions. Heymans and Podlubny [11] were the ones accredited for the manifestation of physical significance to the initial conditions used in regard of Riemann-Liouville fractional order viscoelastic systems.

Controllability is the qualitative property of steering any dynamical system from initial arbitrary position to any desired final position utilizing

*Corresponding Author

appropriate control functions within stipulated time. Control theory, being a multidisciplinary branch stemmed from mathematics to engineering, has wide-ranging implementation in robotics, aeronautical and automobile engineering, image processing, biomathematical modelling and appreciably more. Control theory, in spaces of infinite and finite dimensions, have thoroughly been discussed in [12] and [13] respectively. The conception of controllability was first initiated and established by Kalman [14] in 1963, and since then it is the matter of prime importance for the researchers worldwide. In due course, profuse types of controllability were examined by the researchers in the past. Approximate controllability of semilinear fractional systems involving Caputo derivative was established by Sakthivel in [9] by supposing C_0- semigroup $T(t)$ to be compact and the nonlinear function involved to be uniformly bounded and continuous, Devies in [15] established exact and null controllability for linear systems, Mahmudov in [16] designed partial approximate controllability for Caputo type frcational order systems, Klamka in [17] mannered constrained controllability. Wen & Zhou [18] discussed complete and approximate controllability of semilinear system for Caputo derivative with control in the nonlinear part. The results of existence and controllability for various differential systems of integral and fractional order involving Riemann-Liouville and Caputo derivatives have closely been demonstrated in many artefacts (refer [3,6,9,17,19–31] and references therein). The article [32] discusses about the numerical treatment of fractional heat equation. S. N. Bora [33] recently established the approximate controllability for semilinear Hilfer fractional evolution equations by relaxing the compactness of the semigroup generated. Vijayakumar, Nisar & Shukla [34–40] established important results of controllability and approximate controllability of fractional evolution systems involving other new fractional derivatives like Atangana-Baleanu derivative and Hilfer derivative.

This artefact explores the study for Riemann-Liouville differential systems involving control function in the nonlinear part and is drafted as: Section 2 gives the briefing for basic results and definitions. Results for the existence of solutions are apparent in Section 3. Section 4 accords with the sufficient assumptions and controllability conditions. Section 5 presents an application validating the proposed methodology. Section 6 concludes the article by summarizing the present findings along with discussing the futuristic scope.

2. Preliminaries

This segment revisits several fundamental concepts and definitions which are beneficial for the smooth study of the paper. The considered Banach space is

$$C_{1-\eta}([0,a];Z) = \{z : \tau^{1-\eta} z(\tau) \in C([0,a];Z)\}$$

equipped with the norm

$$\|z\|_{C_{1-\eta}} = \sup_{\tau\in[0,a]} \{\tau^{1-\eta}\|z(\tau)\|_Z\},$$

where $C([0,a];Z)$ indicates the set of all continuous functions defined from $[0,a]$ to Z. For C_0 - semigroup $T(\tau)$, let $M = \sup_{\tau\in[0,a)} \|T(\tau)\| < \infty$.

Definition 1. *[4] The Riemann-Liouville $\eta^{th}-$order fractional integral is written in terms of the following integral*

$$I^{\eta}_{\tau} z(\tau) = \frac{1}{\Gamma(\eta)} \int_0^{\tau} (\tau - r)^{\eta-1} z(r)dr, \qquad \eta > 0,$$

where Γ denotes the gamma function.

Definition 2. *[4] The fractional $\eta^{th}-$order Riemann-Liouville derivative is defined by the following expression*

$$D^{\eta}_{\tau} z(\tau) = \frac{1}{\Gamma(n-\eta)} \left(\frac{d}{d\tau}\right)^n \int_0^{\tau} (\tau - r)^{n-\eta-1} z(r)dr,$$

where $0 \le n-1 < \eta < n$.

Definition 3. *[4] A function of the complex variable w defined by*

$$E_{\eta}(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\eta i + 1)}$$

is known as the Mittag-Leffler function in one parameter.

Definition 4. *[41] A mild solution of the system (1) is a function $z \in C_{1-\eta}([0,a];Z)$ satisfying the underneath integral equation:*

$$\begin{aligned} z(\tau) &= \tau^{\eta-1} T_{\eta}(\tau) z_0 \\ &+ \int_0^{\tau} (\tau - r)^{\eta-1} T_{\eta}(\tau - r) Bu(r) dr \\ &+ \int_0^{\tau} (\tau - r)^{\eta-1} T_{\eta}(\tau - r) g(r, z(r), u(r)) dr. \end{aligned} \tag{2}$$

where

$$T_{\eta}(\tau) = \eta \int_0^{\infty} \Theta \xi_{\eta}(\Theta) T(\tau^{\eta}\Theta) d\Theta,$$

$$\xi_{\eta}(\Theta) = \frac{1}{\eta} \Theta^{-1-\dfrac{1}{\eta}} \varpi_{\eta}(\Theta^{-\frac{1}{\eta}}),$$

$$\varpi_{\eta}(\Theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \Theta^{-n\eta-1} (-1)^{n-1} \frac{\Gamma(1+n\eta)}{n!} sin(n\pi\eta)$$

with $\Theta \in (0,\infty)$ *and domain of the probability density function* $\xi_\eta(\Theta)$ *is* $(0,\infty)$, *i.e.,* $\xi_\eta(\Theta) \geq 0$ *and* $\int_0^\infty \xi_\eta(\Theta)d\Theta = 1$.

Definition 5. *Let* $z(\tau,u)$ *be a mild solution of the system(1) at time* τ *corresponding to a control* $u(.) \in U$. *The set* $K_a(g) = \{z(a,u) \in Z; u(.) \in U\}$ *is known as the reachable set for final time* a. *If* $K_a(g)$ *becomes dense in Z, the system (1) is approximately controllable on [0,a].*

Lemma 1. *[31] The operator* $T_\eta(\tau)$ *possesses the underneath properties:*

(i) *For every fixed* $\tau \geq 0$, *operator* $T_\eta(\tau)$ *is linear and bounded, which means, for any* $z \in Z$,
$$\|T_\eta(\tau)z\| \leq \frac{M}{\Gamma(\eta)}\|z\|.$$

(ii) *Operator* $T_\eta(\tau)(\tau \geq 0)$ *is strongly continuous.*

3. Existence of mild solution

This segment establishes the existence and uniqueness of mild solution for the system (1) utilizing the Banach fixed point approach along with the generalised Gronwall's inequality. The results are based on the below mentioned hypotheses:

(H1) A function $\psi(\cdot)$ exists in $L^q([0,a];\mathbb{R}^+)$, $q > \frac{1}{\eta}$, and a constant $b > 0$, such that $\|g(\tau,z,u)\| \leq \psi(\tau) + b\tau^{1-\eta}\|z\|_Z + \|u\|_U$ for a.e. $\tau \in [0,a]$ and all $z \in Z$.

(H2) A constant $\Bbbk > 0$ exists in a way satisfying $\|g(\tau,z,u) - g(\tau,y,v)\| \leq \Bbbk\big[\|z-y\|_Z + \|u-v\|_U\big]$ $\forall\ z,y \in Z$ and $\forall\ u,v \in U$.

Theorem 1. *The nonlinear system (1) admits a unique mild solution in* $C_{1-\eta}([0,a];Z)$ *for each control* $u(.) \in L^q([0,a];U)$, *provided the hypotheses H(1)-H(2) hold true.*

Proof. Consider the operator G as
$$(Gz)(\tau) = \tau^{\eta-1}T_\eta(\tau)z_0 + \int_0^\tau (\tau-r)^{\eta-1}T_\eta(t-r)\big[Bu(r) + f(r,z(r),u(r))\big]dr. \qquad (3)$$

It is unchallenging to confirm that G maps $C_{1-\eta}([0,a];Z)$ into itself under the hypotheses $H(1) - H(2)$.

It is now required to prove G^m is a contraction operator on $C_{1-\eta}([0,a];Z)$ for some $m \in \mathbb{N}$.

For any $z,y \in C_{1-\eta}([0,a];Z)$ and $\tau \in [0,a]$, it is

$$\begin{aligned}
&\tau^{1-\eta}\|(Gz)(\tau) - (Gy)(\tau)\|_{C_{1-\eta}} \\
&\leq \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)\big[g(r,z(r),u(r)) - g(r,y(r),u(r))\big]\|dr \\
&\leq \frac{\tau^{1-\eta}M}{\Gamma(\eta)}\int_0^\tau (\tau-r)^{\eta-1}\|g(r,z(r),u(r)) - g(r,y(r),u(r))\|_Z dr \\
&\leq \frac{\tau^{1-\eta}M\Bbbk}{\Gamma(\eta)}\int_0^\tau (\tau-r)^{\eta-1}r^{\eta-1}r^{1-\eta}\|z(r)-y(r)\|_Z dr \\
&\leq \frac{\tau^{1-\eta}M\Bbbk}{\Gamma(\eta)}\|z-y\|_{C_{1-\eta}}\int_0^\tau r^{\eta-1}(\tau-r)^{\eta-1}dr \\
&\leq \frac{\Gamma(\eta)M\Bbbk\tau^\eta}{\Gamma(2\eta)}\|z-y\|_{C_{1-\eta}}.
\end{aligned} \qquad (4)$$

Further, by applying induction on m and using (3), (4), it leads to

$$\tau^{1-\eta}\|(G^m z)(\tau) - (G^m y)(\tau)\| \leq \frac{\Gamma(\eta)(\Bbbk M a^\eta)^m}{\Gamma[(m+1)\eta]}\|z-y\|_{C_{1-\eta}}$$

Therefore, G^m is shown as a contraction operator on $C_{1-\eta}([0,a];Z)$ with the inequality obtained as

$$\|G^m z - G^m y\|_{C_{1-\eta}} \leq \frac{\Gamma(\eta)(\Bbbk M a^\eta)^m}{\Gamma(m+1)\eta}\|z-y\|_{C_{1-\eta}} \qquad (5)$$

where $\frac{(\Bbbk M a^\eta)^m}{\Gamma(m+1)\eta}$ becomes the m^{th} term of the two parameter Mittag-Leffler series $E_{\eta,\eta}(M\Bbbk a^\eta) = \sum_{i=0}^\infty \frac{(M\Bbbk a^\eta)^i}{\Gamma(i\eta+\eta)}$. The series converges uniformly on $[0,a]$, thus for sufficiently large m,

$$\frac{\Gamma(\eta)(\Bbbk M a^\eta)^m}{\Gamma(m+1)\eta} < 1.$$

It is evident through generalisation of Banach fixed point theorem and (5) that G possess a unique fixed point $z(\cdot)$ on $C_{1-\eta}([0,a];Z)$ which serves as the requisite solution of system (1). □

4. Controllability results

Defining the underneath operators:
The Nemytskil operator
$$\Omega_g : C_{1-\eta}([0,a];Z) \to L^q([0,a];Z)$$
is defined as

$$\Omega_g(z)(\tau) = g(\tau, z(\tau), u(\tau)),$$
$$z(\cdot) \in C_{1-\eta}([0,a];Z).$$

and the bounded linear operator

$$\mathbb{F} : L^q([0,a];Z) \to Z$$

as

$$\mathbb{F}f = \int_0^a (a-r)^{\eta-1} T_\eta(a-r) f(r)dr,$$
$$f(\cdot) \in L^q([0,a];Z).$$

The hypotheses mentioned below are made to prove the approximate controllability for the considered system (1):

(H3) A constant $\Bbbk' > 0$ exists in a way satisfying $\|g(\tau,z,u) - g(\tau,y,v)\| \le \Bbbk' \left[\tau^{1-\eta}\|z-y\|_Z + \|u-v\|_U\right] \quad \forall\ z,y \in Z,\ u,v \in U$ and $\tau \in [0,a]$.

(H4) The operator B is bounded below, i.e., a constant $\ell > 0$ exists satisfying $\|u\| \le \ell \|Bu\| \quad \forall u \in U.$

(H5) For any $\epsilon > 0$ and $\vartheta(\cdot) \in L^q([0,a],Z)$, $\exists$ a $u(\cdot) \in L^q([0,a];U)$ satisfying $\|\mathbb{F}\vartheta - \mathbb{F}Bu\|_Z < \epsilon$, $\|Bu(\cdot)\|_{L^q([0,a];Z])} < \aleph \|\vartheta(\cdot)\|_{L^q([0,a];Z)}$, where $\aleph$ is a constant independent of $\vartheta(\cdot) \in L^q([0,a];Z)$,

$$\frac{M\aleph\Bbbk'}{\Gamma(\eta)}\Big(\frac{aq-a}{q\eta-1}\Big)^{\frac{q-1}{q}} \times (1+\Bbbk'\ell)E_\eta(M\Bbbk' a) + \aleph\Bbbk'\ell < 1. \tag{6}$$

Lemma 2. *Assuming the hypotheses (H1), (H3) and (H4) hold true for the considered function g, then every mild solution of the control system (1) meets the inequalities stated below for any* $u, v \in L^q([0,a];U)$*:*

$$\|z(\cdot;0,z_0,u)\|_{C_{1-\eta}} \le kE_\eta(Mab),$$
$$\|z(\cdot) - y(\cdot)\|_{C_{1-\eta}} \le \varrho E_\eta(M\Bbbk' a)\|Bu - Bv\|_{L^q},$$

where

$$k = \frac{M}{\Gamma(\eta)}\left[\|z_0\| + \Big(\frac{q-1}{q\eta-1}\Big)^{\frac{q-1}{q}}(\|Bu\|_{L^q} + \|\psi\|_{L^q} + \|u\|_{L^q})a^{1-\frac{1}{q}}\right],$$

$$\varrho = \frac{M}{\Gamma(\eta)}\Big(\frac{q-1}{q\eta-1}\Big)^{\frac{q-1}{q}}(1+\Bbbk'\ell)a^{1-\frac{1}{q}}.$$

Proof. Let z be a mild solution of system (1) in accord with control $u(\cdot) \in L^q([0,a];U)$ on $C_{1-\eta}([0,a];Z)$, then

$$z(\tau) = \tau^{\eta-1}T_\eta(\tau)z_0 + \int_0^\tau (\tau-r)^{\eta-1}T_\eta(\tau-r)Bu(r)dr + \int_0^\tau (\tau-r)^{\eta-1}T_\eta(\tau-r)g(r,z(r),u(r))dr$$

For $\tau \in [0,a]$,

$$\begin{aligned}
\tau^{1-\eta}\|z(\tau)\| &\le \|T_\eta(\tau)z_0\| \\
&\quad + \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)Bu(r)\|dr \\
&\quad + \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)g(r,z(r),u(r))\|dr \\
&\le \frac{M}{\Gamma(\eta)}\Bigg[\|z_0\| + \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|Bu(r)\|_Z dr \\
&\quad + \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}[\psi(r) + br^{1-\eta}\|z(r)\|_Z + \|u(r)\|_U]dr\Bigg] \\
&\le \frac{M}{\Gamma(\eta)}\Bigg[\|z_0\| + \Big(\frac{aq-a}{q\eta-1}\Big)^{\frac{q-1}{q}}(\|Bu\|_{L^q} + \|\psi\|_{L^q} \\
&\quad + \|u\|_{L^q}) + ba^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}r^{1-\eta}\|z(r)\|_Z dr\Bigg].
\end{aligned} \tag{7}$$

Thus,

$$\tau^{1-\eta}\|z(\tau)\| \le k + \frac{Mba^{1-\eta}}{\Gamma(\eta)} \times \int_0^\tau (\tau-r)^{\eta-1}[r^{1-\eta}\|z(r)\|]dr.$$

Using generalised Gronwall's inequality ([42]), it concludes to

$$\tau^{1-\eta}\|z(\tau)\| \le kE_\eta(Mab).$$

Therefore,

$$\|z\|_{C_{1-\eta}} = \sup_{\tau\in[0,a]} \tau^{1-\eta}\|z(\tau)\|_Z \le kE_\eta(Mab)$$

Now,

$$
\begin{aligned}
&\tau^{1-\eta}\|z(\tau)-y(\tau)\| \\
&\leq \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)[Bu(r)-Bv(r)]\|dr \\
&\quad + \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|T_\eta(\tau-r)[g(r,z(r),u(r)) \\
&\quad - g(r,y(r),v(r))]\|dr \\
&\leq \frac{M}{\Gamma(\eta)}\Bigg[\tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\|Bu(r)-Bv(r)\|dr \\
&\quad + \tau^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}\big(\mathbb{k}'[r^{1-\eta}\|z(r)-y(r)\|_Z \\
&\quad + \|u(r)-v(r)\|_U]\big)dr\Bigg] \\
&\leq \frac{M}{\Gamma(\eta)}\Bigg[\left(\frac{q-1}{q\eta-1}\right)^{\frac{q-1}{q}}(1+\mathbb{k}'\ell)\|Bu(r)-Bv(r)\|a^{1-\frac{1}{q}} \\
&\quad + \mathbb{k}'a^{1-\eta}\int_0^\tau (\tau-r)^{\eta-1}r^{1-\eta}\|z(r)-y(r)\|dr\Bigg].
\end{aligned} \tag{8}
$$

Thus,

$$
\begin{aligned}
&t^{1-\eta}\|z(\tau)-y(\tau)\|_Z \\
&\leq \varrho\|Bu(r)-Bv(r)\|_Z \\
&\quad + \frac{M\mathbb{k}'a^{1-\eta}}{\Gamma(\eta)}\int_0^\tau (\tau-r)^{\eta-1}r^{1-\eta}\|z(r)-y(r)\|dr.
\end{aligned}
$$

Again, using generalised Gronwall's identity [42], we have

$$\tau^{1-\eta}\|z(\tau)-y(\tau)\|_Z \leq \varrho E_\eta(M\mathbb{k}'a)\|Bu-Bv\|_{L^q}$$

Hence,

$$\|z-y\|_{C_{1-\eta}} \leq \varrho E_\eta(M\mathbb{k}'a)\|Bu-Bv\|_{L^q}$$

This accomplishes the proof. □

Theorem 2. *The nonlinear control system (1) becomes approximately controllable, provided the hypotheses $(H1)$ and $(H3)-(H5)$ hold true and A generates the differentiable semigroup $T(t)$.*

Proof. It is well known that domain of A, $D(A)$ is dense in Z. Thus, to manifest approximate controllability of nonlinear control system (1), it is adequate to claim that $D(A) \subset \overline{K_a(g)}$, i.e., for any given $\epsilon > 0$ and $\lambda \in D(A)$, a control $u_\epsilon \in L^q([0,a];U)$ can be found satisfying

$$\|\lambda^* - \mathbb{F}(Bu_\epsilon) - \mathbb{F}(\Omega_g(z_\epsilon))\|_Z \leq \epsilon,$$

where $z_\epsilon(t)$ is a mild solution of system(1) in accord with the control $u_\epsilon(t)$ and

$$\lambda - a^{\eta-1}T_\eta(a)z_0 = \lambda^* \in D(A)$$

Let $\epsilon > 0$ be given and $u_1 \in L^q([0,a];U)$. Then by hypothesis (H5), there exists $u_2 \in L^q([0,a];U)$ satisfying

$$\|\lambda^* - \mathbb{F}(\Omega_g(z_1)) - \mathbb{F}(Bu_2)\|_Z \leq \frac{\epsilon}{2^2}$$

where $z_1(\tau) = z(\tau,u_1)$. Denote $z_2(\tau) = z(\tau,u_2)$, again by hypothesis (H5), $\exists\, \omega_2 \in L^q([0,a];U)$ satisfying

$$\|\mathbb{F}[\Omega_g(z_2) - \Omega_g(z_1)] - \mathbb{F}(B\omega_2)\|_Z \leq \frac{\epsilon}{2^3}$$

and

$$
\begin{aligned}
&\|B\omega_2\|_{L_p} \\
&\leq \aleph\|\Omega_g(z_2)-\Omega_g(z_1)\|_{L_p} \\
&\leq \aleph\mathbb{k}'\left[\tau^{1-\eta}\|z_2-z_1\| + \|u_2-u_1\|\right] \\
&\leq \aleph\mathbb{k}'\left[\varrho E_\eta(M\mathbb{k}'a)\|Bu_2-Bu_1\| + \ell\|Bu_2-Bu_1\|\right] \\
&\leq \Bigg[\frac{M\aleph\mathbb{k}'}{\Gamma(\eta)}\left(\frac{q-1}{q\eta-1}\right)^{1-\frac{1}{q}}(1+\mathbb{k}'\ell)a^{1-\frac{1}{q}}E_\eta(M\mathbb{k}'a) \\
&\quad + \aleph\mathbb{k}'\ell\Bigg]\|Bu_2-Bu_1\|_{L^q}.
\end{aligned}
$$

Now, define

$$u_3(\tau) = u_2(\tau) - \omega_2(\tau), \quad u_3(\tau) \in U,$$

then

$$
\begin{aligned}
&\|\lambda^* - \mathbb{F}\Omega_g(z_2) - \mathbb{F}Bu_3\|_Z \\
&\quad \leq \|\lambda^* - \mathbb{F}\Omega_g(z_1) - \mathbb{F}Bu_2\|_Z \\
&\quad\quad + \|\mathbb{F}B\omega_2 - [\mathbb{F}\Omega_g(z_2) - \mathbb{F}\Omega_g(z_1)]\|_Z \\
&\quad \leq \left(\frac{1}{2^2}+\frac{1}{2^3}\right)\epsilon
\end{aligned}
$$

By applying inductions, a sequence $\{u_n\}$ in $L^q([0,a];U)$ is obtained such that

$$
\begin{aligned}
&\|\lambda^* - \mathbb{F}\Omega_g(z_n) - \mathbb{F}Bu_{n+1}\|_Z \\
&\quad < \left(\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\ldots+\frac{1}{2^{n+1}}\right)\epsilon,
\end{aligned}
$$

where $z_n(\tau) = z(\tau, u_n(\tau))$ and

$$
\begin{aligned}
&\|Bu_{n+1} - Bu_n\|_{L^q} \\
&\quad < \Bigg[\frac{M\aleph\mathbb{k}'}{\Gamma(\eta)}\left(\frac{aq-a}{q\eta-1}\right)^{1-\frac{1}{q}}(1+\mathbb{k}'\ell)E_\eta(M\mathbb{k}'a) \\
&\quad\quad + \aleph\mathbb{k}'\ell\Bigg]\|Bu_n - Bu_{n-1}\|_{L^q}
\end{aligned}
$$

By (6), it is evident that the sequence $\{Bu_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $L^q([0,a];Z)$. Thus, for any $\epsilon > 0$, a positive integer n_0 can be found satisfying

$$\|\mathbb{F}Bu_{n_0+1} - \mathbb{F}Bu_{n_0}\|_Z < \frac{\epsilon}{2}.$$

Now,

$$\begin{aligned}
&\|\lambda^* - \mathbb{F}\Omega_g(z_{n_0}) - \mathbb{F}Bu_{n_0}\|_Z \\
&\quad \leq \|\lambda^* - \mathbb{F}\Omega_g(z_{n_0}) - \mathbb{F}Bu_{n_0+1}\|_Z \\
&\qquad + \|\mathbb{F}Bu_{n_0+1} - \mathbb{F}Bu_{n_0}\|_Z \\
&\quad \leq \left(\frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{n_0+1}}\right)\epsilon + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Hence, the approximate controllability of (1) is proved. □

5. Example

Examine the below mentioned initial value problem for $\tau \in (0,1]$ and $x \in [0,\pi]$:

$$\begin{aligned}
D_\tau^{\frac{2}{3}} z(\tau,x) &= \frac{\partial^2}{\partial x^2} z(\tau,x) + u(\tau) \\
&\quad + g(\tau, z(\tau,x), u(\tau)), \qquad (9) \\
z(\tau,0) &= z(\tau,\pi) = 0, \\
I_{0^+}^{\frac{1}{3}} z(\tau,x)|_{\tau=0} &= z_0(x),
\end{aligned}$$

Take $Z = U = L^2([0,\pi])$ and $A : D(A) \subset Z \to Z$ as

$$Az = z''$$

where

$$D(A) = \left\{ z \in Z \mid z, \frac{\partial z}{\partial x} \text{ are absolutely continuous}, \frac{\partial^2 z}{\partial x^2} \in Z \text{ and } z(0) = 0 = z(\pi) \right\}$$

Then, A can be expressed as

$$Az = \sum_{m=1}^{\infty} (-m^2)\langle z, \alpha_m\rangle \alpha_m, \quad z \in D(A)$$

where $\alpha_m(x) = \sqrt{\frac{2}{\pi}} sin\ mx$ $(m \in \mathbb{N})$ are the eigen functions corresponding to the eigen values $-m^2$ respectively and $\{\alpha_1, \alpha_2, ...\}$ is a basis of Z.
A differentiable semigroup $T(\tau)(\tau > 0)$ in Z having A as its infinitesimal genaerator is expressed as

$$T(\tau)z = \sum_{m=1}^{\infty} exp^{-m^2\tau}\langle z, \alpha_m\rangle\alpha_m, \quad z \in Z$$
$$\text{and } \|T(\tau)\| \leq e^{-1} < 1 = M.$$

Let us choose the nonlinear function g as

$$\begin{aligned}
g(\tau, z(\tau,x), u(\tau)) &= 1 + \tau^2 + \beta\tau^\gamma \\
&\quad \times [z(\tau,x) + \sin z(\tau,x) + u(\tau)],
\end{aligned}$$

where β and γ are constants with $-1 \leq \beta \leq 1$ and $\gamma \geq 1 - \eta$. Now,

$$\begin{aligned}
&\|g(\tau, z(\tau,x), u(\tau))\| \\
&\quad \leq 1 + \tau^2 + |\beta|\tau^\gamma \left[\left\|[z(\tau,x) + \sin z(\tau,x)]\right\| + \|u(\tau)\|\right] \\
&\quad \leq 1 + \tau^2 + |\beta|\tau^{\gamma+\eta-1}\tau^{1-\eta}\left[2\|z(\tau,x)\| + \|u(\tau)\|\right] \\
&\quad \leq (1+\tau^2) + 2|\beta|\tau^{1-\eta}\|z(\tau,x)\| + \|u(\tau)\|
\end{aligned}$$

and

$$\begin{aligned}
&\|g(\tau, z(\tau,x), u(\tau)) - g(\tau, y(\tau,x), v(\tau))\| \\
&\quad \leq |\beta|\tau^\gamma \left[\|z(\tau,x) - y(\tau,x) + \sin z(\tau,x) - \sin y(\tau,x)\| \right. \\
&\qquad \left. + \|u(\tau) - v(\tau)\|\right] \\
&\quad \leq |\beta|\tau^{\gamma+\eta-1}\tau^{1-\eta}\Bigg[\|z(\tau,x) - y(\tau,x)\| \\
&\qquad + \left\|2\cos\left(\frac{z(\tau,x)+y(\tau,x)}{2}\right)\sin\left(\frac{z(\tau,x)-y(\tau,x)}{2}\right)\right\| \\
&\qquad + \|u(\tau) - v(\tau)\|\Bigg] \\
&\quad \leq |\beta|\tau^{1-\eta}\left[2\|z(\tau,x) - y(\tau,x)\| + \|u(\tau) - v(\tau)\|\right] \\
&\quad \leq 2|\beta|\left[\|z(\tau,x) - y(\tau,x)\| + \|u(\tau) - v(\tau)\|\right]
\end{aligned}$$

Here, the assumptions $(H1)$ and $(H2)$ are evidently satisfied with $\psi(\tau) = 1 + \tau^2$ and $b = \Bbbk = 2|\beta|$. Moreover, assumption $(H5)$ is satisfied by choosing β sufficiently close to zero.
The abstract form of the system (1) is expressed as:

$$D_\tau^{\frac{2}{3}}\widetilde{z}(\tau) = A\widetilde{z}(\tau) + B\widetilde{u}(\tau) + g(\tau, \widetilde{z}(\tau), \widetilde{u}(\tau)), \tau \in (0,1],$$
$$I_\tau^{\frac{1}{3}}\widetilde{z}(\tau)|_{\tau=0} = \widetilde{z}_0,$$

where $\widetilde{z}(\tau) = z(\tau,\cdot)$, $\widetilde{u}(\tau) = u(\tau,\cdot)$ and $\widetilde{z}_0 = z_0(\cdot)$.
Approximate controllability of (1) accomplishes from Theorem 2 as it is seen assumptions $H(1)$-$H(5)$ are satisfied.

6. Conclusion

In this paper, thorough analysis for existence and uniqueness, and approximate controllability of the fractional nonlinear differential system has been performed in Banach spaces. The existence and uniqueness results were established using concepts of fractional calculus, definition [41], generalised Gronwall's inequality, semigroup theory and Banach's fixed point theorem. The sufficient condition for approximate controllability was derived with the aid of Lemma 2 and iterative technique. The present findings of the paper can be extended to stochastic fractional differential equations with or without delay in state or in the control term present in the nonlinear function of the system. For some idea, see [34, 38, 39].

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Lavina Sahijwani *is presently working as a Ph.D. scholar in the Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India. She has done her M.Sc. Mathematics from Maharshi Dayanand Saraswati University, Ajmer, India.*
https://orcid.org/0000-0001-7809-210X

Nagarajan Sukavanam *is a Professor & former Head of the Department at Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India. He has obtained his Ph.D. from Indian Institute of Science, Bangalore, India in the year 1985. He has supervised more than 25 scholars to obtain their doctorate degrees and has adjudicated several doctoral thesis from different Universities. He has published more than 150 research articles in International journals of high repute.*
https://orcid.org/0000-0001-9627-8653