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RESEARCH ARTICLE

A new generalization of Rhoades' condition

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ABSTRACT

In this paper, our aim is to obtain a new generalization of the well-known Rhoades' contractive condition. To do this, we introduce the notion of an S-normed space. We extend the Rhoades' contractive condition to S-normed spaces and define a new type of contractive conditions. We support our theoretical results with necessary illustrative examples.

1. Introduction

Metric fixed point theory is important to find some applications in many areas such as topology, analysis, differential equations etc. So different generalizations of metric spaces were studied (see [1], [2], [3], [4], [5], [6] and [7]). For example, Mustafa and Sims introduced a new notion of "G-metric space" [6]. Mohanta proved some fixed point theorems for self-mappings satisfying some kind of contractive type conditions on complete G-metric spaces [5].

Recently Sedghi, Shobe and Aliouche have defined the concept of an S-metric space in [7] as follows:

Definition 1. *[7] Let X be a nonempty set and $\mathcal{S}: X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:*

(S1) $\mathcal{S}(x, y, z) = 0$ *if and only if* $x = y = z$,

(S2) $\mathcal{S}(x, y, z) \leq \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$.

Then $\mathcal{S}$ is called an S-metric on X and the pair $(X, \mathcal{S})$ is called an S-metric space.

Let (X, d) be a complete metric space and T be a self-mapping of X. In [8], T is called a Rhoades' mapping if the following condition is satisfied:

$$\textbf{(R25)} \quad d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for each $x, y \in X$, $x \neq y$. Any fixed point result was not given for a Rhoades' mapping in [8]. Since then, many fixed point theorems were obtained by several authors for a Rhoades' mapping (see [9], [10] and [11]). Furthermore, the Rhoades' condition was extended on S-metric spaces and new fixed point results were presented (see [12], [13] and [14]). Now we recall the Rhoades' condition on an S-metric space.

Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping of X. In [12] and [14], the present authors defined Rhoades' condition **(S25)** on $(X, \mathcal{S})$ as follows:

$$\textbf{(S25)} \quad \mathcal{S}(Tx, Tx, Ty) < max\{\mathcal{S}(x, x, y), \mathcal{S}(Tx, Tx, x), \mathcal{S}(Ty, Ty, y), \mathcal{S}(Ty, Ty, x), \mathcal{S}(Tx, Tx, y)\},$$

for each $x, y \in X$, $x \neq y$.

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In this paper, to obtain a new generalization of the Rhoades' condition, we introduce the notion of an S-normed space. We give some basic concepts and topological definitions related to an S-norm. Then, we study a new form of Rhoades' condition (**R25**) on S-normed spaces and obtain a fixed point theorem. In Section 2, we introduce the definition of an S-norm on X and investigate some basic properties which are needed in the sequel. We investigate the relationships among an S-norm and other known concepts by counter examples. In Section 3, we define Rhoades' condition (**NS25**) on an S-normed space. We study a fixed point theorem using the condition (**NS25**) and the notions of reflexive S-Banach space, S-normality, closure property and convexity. In Section 4, we investigate some comparisons on S-normed spaces such as the relationships between the conditions (**NR25**) and (**NS25**).

2. S-normed spaces

In this section, we introduce the notion of an S-normed space and investigate some basic concepts related to an S-norm. We study the relationships between an S-metric and an S-norm (resp. an S-norm and a norm).

Definition 2. *Let X be a real vector space. A real valued function $\|.,.,.\| : X \times X \times X \to \mathbb{R}$ is called an S-norm on X if the following conditions hold:*

(**NS1**) *$\|x,y,z\| \geq 0$ and $\|x,y,z\| = 0$ if and only if $x = y = z = 0$,*

(**NS2**) *$\|\lambda x, \lambda y, \lambda z\| = |\lambda| \|x,y,z\|$ for all $\lambda \in \mathbb{R}$ and $x,y,z \in X$,*

(**NS3**) *$\|x+x', y+y', z+z'\| \leq \|0,x,z'\| + \|0,y,x'\| + \|0,z,y'\|$ for all $x,y,z,x',y',z' \in X$.*

The pair $(X, \|.,.,.\|)$ is called an S-normed space.

Example 1. *Let $X = \mathbb{R}$ and $\|.,.,.\| : X \times X \times X \to \mathbb{R}$ be the function defined by*

$$\|x,y,z\| = |x| + |y| + |z|,$$

*for all $x,y,z \in X$. Then $(X, \|.,.,.\|)$ is an S-normed space. Indeed, we show that the function $\|.,.,.\|$ satisfies the conditions (**NS1**), (**NS2**) and (**NS3**).*

(**NS1**) *By the definition, clearly we have $\|x,y,z\| \geq 0$ for all $x,y,z \in X$. If $\|x,y,z\| = |x| + |y| + |z| = 0$, we obtain $x = y = z = 0$.*

(**NS2**) *Let $x,y,z \in X$ and $\lambda \in \mathbb{R}$. Then we have*

$$\begin{aligned}\|\lambda x, \lambda y, \lambda z\| &= |\lambda x| + |\lambda y| + |\lambda z| \\ &= |\lambda| |x| + |\lambda| |y| + |\lambda| |z| \\ &= |\lambda| (|x| + |y| + |z|) \\ &= |\lambda| \|x,y,z\|.\end{aligned}$$

(**NS3**) *Let $x,y,z,x',y',z' \in X$. Then we obtain*

$$\begin{aligned}\|x+x', y+y', z+z'\| &= |x+x'| + |y+y'| \\ &+ |z+z'| \\ &\leq |x| + |x'| + |y| + |y'| \\ &+ |z| + |z'| \\ &\leq |0| + |x| + |z'| \\ &+ |0| + |y| + |x'| \\ &+ |0| + |z| + |y'| \\ &= \|0,x,z'\| + \|0,y,x'\| \\ &+ \|0,z,y'\|.\end{aligned}$$

*Consequently, the function $\|.,.,.\|$ satisfies the conditions (**NS1**), (**NS2**), (**NS3**) and so $(X, \|.,.,.\|)$ is an S-normed space.*

Now, we show that every S-norm generates an S-metric.

Proposition 1. *Let $(X, \|.,.,.\|)$ be an S-normed space. Then the function $\mathcal{S} : X \times X \times X \to [0,\infty)$ defined by*

$$S(x,y,z) = \|x-y, y-z, z-x\| \quad (1)$$

is an S-metric on X.

Proof. Using the condition (**NS1**), it can be easily seen that the condition (**S1**) is satisfied. We show that the condition (**S2**) is satisfied. By (**NS3**), we have

$$\begin{aligned}S(x,y,z) &= \|x-y, y-z, z-x\| \\ &= \left\| \begin{array}{c} x-a+a-y, y-a+a-z \\ , z-a+a-x \end{array} \right\| \\ &\leq \|0, x-a, a-x\| + \|0, y-a, a-y\| \\ &+ \|0, z-a, a-z\| \\ &= S(x,x,a) + S(y,y,a) + S(z,z,a),\end{aligned}$$

for all $x,y,z,a \in X$.

Then, the function $\mathcal{S}$ is an S-metric and the pair (X,S) is an S-metric space. □

We call the S-metric defined in (1) as the S-metric generated by the S-norm $\|.,.,.\|$ and denoted by $S_{\|.\|}$.

Corollary 1. *Every S-normed space is an S-metric space.*

Example 2. *Let X be a nonempty set, (X,d) be a metric space and $\mathcal{S} : X \times X \times X \to [0,\infty)$ be the function defined by*

$$\mathcal{S}(x,y,z) = d(x,y) + d(x,z) + d(y,z),$$

for all $x,y,z \in X$. Then the function $\mathcal{S}$ is an S-metric on X [7].
Let $X = \mathbb{R}$. If we consider the usual metric d on X, we obtain the S-metric $\mathcal{S}$ defined as

$$\mathcal{S}(x,y,z) = |x-y| + |x-z| + |y-z|,$$

for all $x,y,z \in \mathbb{R}$. Using Proposition 1, we see that $\mathcal{S}$ is generated by the S-norm defined in Example 1. Indeed, we have

$$\begin{aligned} S(x,y,z) &= \|x-y, y-z, z-x\| \\ &= |x-y| + |y-z| + |z-x| \\ &= |x-y| + |x-z| + |y-z| \\ &= d(x,y) + d(x,z) + d(y,z), \end{aligned}$$

for all $x,y,z \in X$.

Lemma 1. *An S-metric $\mathcal{S}$ generated by an S-norm on an S-normed space X satisfies the following conditions*

(1) $\mathcal{S}(x+a, y+a, z+a) = \mathcal{S}(x,y,z)$,
(2) $\mathcal{S}(\lambda x, \lambda y, \lambda z) = |\lambda|\, \mathcal{S}(x,y,z)$,

for each $x,y,z,a \in X$ and every scalar λ.

Proof. The proof follows easily from the Proposition 1. □

We note that every S-metric can not be generated by an S-norm as we have seen in the following example:

Example 3. *Let X be a nonempty set and the function $S : X \times X \times X \to [0,\infty)$ be defined by*

$$S(x,y,z) = \begin{cases} 0 \;; & \text{if } x = y = z \\ 1 \;; & \text{otherwise} \end{cases},$$

for all $x,y,z \in X$. Then the function $\mathcal{S}$ is an S-metric on X. We call this S-metric is the discrete S-metric on X. The pair (X,S) is called discrete S-metric space. Now, we prove that this S-metric can not be generated by an S-norm. On the contrary, we assume that this S-metric is generated by an S-norm. Then the following equation should be satisfied :

$$S(x,y,z) = \|x-y, y-z, z-x\|,$$

for all $x,y,z \in X$.
If we consider the case $x = y \neq z$ and $|\lambda| \neq 0,1$ then we obtain

$$\begin{aligned} &S(\lambda x, \lambda y, \lambda z) \\ &= \|0, \lambda(y-z), \lambda(z-x)\| = 1 \\ &\neq |\lambda|\, S(x,y,z) \\ &= |\lambda|\, \|0, y-z, z-x\| = |\lambda|, \end{aligned}$$

which is a contradiction with (**NS2**). *Consequently, this S-metric can not be generated by an S-norm.*

We use the following result in the next section.

Lemma 2. *Let $(X, \|.,.,.\|)$ be an S-normed space. We have*

$$\|0, x-y, y-x\| = \|0, y-x, x-y\|,$$

for each $x,y \in X$.

Proof. By the condition (**NS3**), we get

$$\begin{aligned} \|0, x-y, y-x\| &\leq \|0,0,0\| + \|0,0,0\| \\ +\|0, y-x, x-y\| &= \|0, y-x, x-y\| \end{aligned} \quad (2)$$

and

$$\begin{aligned} \|0, y-x, x-y\| &\leq \|0,0,0\| + \|0,0,0\| \\ +\|0, x-y, y-x\| &= \|0, x-y, y-x\|. \end{aligned} \quad (3)$$

Using (2) and (3) we obtain $\|0, x-y, y-x\| = \|0, y-x, x-y\|$. □

We recall the definition of a norm on X as follows.
Let X be a real vector space. A real valued function $\|.\| : X \to \mathbb{R}$ is called a norm on X if the following conditions hold:

(**N1**) $\|x\| \geq 0$ for all $x \in X$.
(**N2**) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$.
(**N3**) $\|\lambda x\| = |\lambda|\, \|x\|$ for all $\lambda \in \mathbb{R}$ and $x \in X$.
(**N4**) $\|x + y\| \leq \|x\| + \|y\|$ for all $x,y \in X$.

The pair $(X, \|.\|)$ is called a normed space.
We show that every norm generates an S-norm. We give the following proposition.

Proposition 2. *Let $(X, \|.\|)$ be a normed space and the function $\|.,.,.\| : X \times X \times X \to \mathbb{R}$ be defined by*

$$\|x,y,z\| = \|x\| + \|y\| + \|z\|, \quad (4)$$

for all $x, y, z \in X$. Then $(X, \|.,.,.\|)$ is an S-normed space.

Proof. We show that the function $\|.,.,.\|$ defined in (4) satisfies the conditions (**NS1**), (**NS2**) and (**NS3**).
(**NS1**) It is clear that $\|x, y, z\| \geq 0$ and $\|x, y, z\| = 0$ if and only if $x = y = z = 0$.

(**NS2**) Let $\lambda \in \mathbb{R}$ and $x, y, z \in X$. Then we obtain

$$\begin{aligned} \|\lambda x, \lambda y, \lambda z\| &= \|\lambda x\| + \|\lambda y\| + \|\lambda z\| \\ &= |\lambda| \, \|x\| + |\lambda| \, \|y\| + |\lambda| \, \|z\| \\ &= |\lambda| \left(\|x\| + \|y\| + \|z\| \right) \\ &= |\lambda| \, \|x, y, z\|. \end{aligned}$$

(**NS3**) Let $x, y, z, x', y', z' \in X$. Then we obtain

$$\begin{aligned} &\|x + x', y + y', z + z'\| \\ &= \|x + x'\| + \|y + y'\| + \|z + z'\| \\ &\leq \|x\| + \|x'\| + \|y\| + \|y'\| + \|z\| + \|z'\| \\ &= \|0\| + \|x\| + \|z'\| + \|0\| + \|y\| \\ &+ \|x'\| + \|0\| + \|z\| + \|y'\| \\ &= \|0, x, z'\| + \|0, y, x'\| + \|0, z, y'\|. \end{aligned}$$

Consequently, the function $\|.,.,.\|$ satisfies the conditions (**NS1**), (**NS2**), (**NS3**) and so $(X, \|.,.,.\|)$ is an S-normed space. □

We have proved that every norm on X defines an S-norm on X. We call the S-norm defined in (4) as the S-norm generated by the norm $\|.\|$. For example, the S-norm defined in Example 1 is the S-norm generated by the usual norm on $\mathbb{R}$.
There exists an S-norm which is not generated by a norm as we have seen in the following example.

Example 4. *Let X be a nonempty set and the function $\|.,.,.\| : X \times X \times X \to \mathbb{R}$ be defined by*

$$\begin{aligned} \|x, y, z\| = &\, |x - 2y - 2z| + |y - 2x - 2z| \\ &+ |z - 2y - 2x| , \end{aligned}$$

for all $x, y, z \in X$. Then, the function $\|.,.,.\|$ is an S-norm on X, but it is not generated by a norm.
Now, we show that the conditions (**NS1**)*,* (**NS2**) *and* (**NS3**) *are satisfied.*

(**NS1**) *By the definition, clearly we obtain $\|x, y, z\| \geq 0$ and $\|x, y, z\| = 0$ if and only if $x = y = z = 0$ for all $x, y, z \in X$.*

(**NS2**) *We have*

$$\begin{aligned} \|\lambda x, \lambda y, \lambda z\| &= |\lambda x - 2\lambda y - 2\lambda z| \\ &+ |\lambda y - 2\lambda x - 2\lambda z| \\ &+ |\lambda z - 2\lambda y - 2\lambda x| \\ &= |\lambda| \begin{pmatrix} |x - 2y - 2z| \\ + |y - 2x - 2z| \\ + |z - 2y - 2x| \end{pmatrix} \\ &= |\lambda| \, \|x, y, z\|, \end{aligned}$$

for all $\lambda \in \mathbb{R}$ and $x, y, z \in X$.

(**NS3**) *Let $x, y, z, x', y', z' \in X$. Then we obtain*

$$\begin{aligned} &\|x + x', y + y', z + z'\| \\ &= \left| x + x' - 2y - 2y' - 2z - 2z' \right| \\ &+ \left| y + y' - 2x - 2x' - 2z - 2z' \right| \\ &+ \left| z + z' - 2y - 2y' - 2x - 2x' \right| \\ &\leq \left| 2x + 2z' \right| + \left| x - 2z' \right| \\ &+ \left| z' - 2x \right| + \left| 2y + 2x' \right| \\ &+ \left| y - 2x' \right| + \left| x' - 2y \right| \\ &+ \left| 2z + 2y' \right| + \left| z - 2y' \right| \\ &+ \left| y' - 2z \right| \\ &= \|0, x, z'\| + \|0, y, x'\| + \|0, z, y'\|. \end{aligned}$$

Consequently, the function $\|.,.,.\|$ is an S-norm on X.
On the contrary, we assume that this S-norm is generated by a norm. Then the following equation should be satisfied

$$\|x, y, z\| = \|x\| + \|y\| + \|z\|,$$

for all $x, y, z \in X$.
If we consider $\|x, 0, 0\|$ and $\|x, x, 0\|$ then we obtain

$$\begin{gathered} \|x, 0, 0\| = \|x\| = |x| + |2x| + |2x| = 5\,|x| , \\ \|x, x, 0\| = 2\,|x| = |x| + |x| + |4x| = 6\,|x| \end{gathered}$$

and so $\|x\| = 5\,|x|$ and $\|x\| = 3\,|x|$, which is a contradiction. Hence this S-norm is not generated by a norm.

Now we prove that every S-norm generate a norm.

Proposition 3. *Let X be a nonempty set, $(X, \|.,.,.\|)$ be an S-normed space and the function $\|.\| : X \to \mathbb{R}$ be defined as follows:*

$$\|x\| = \|0, x, 0\| + \|0, 0, x\|,$$

for all $x \in X$. Then the function $\|.\|$ is a norm on X and $(X, \|.\|)$ is a normed space.

Proof. Using the conditions (**NS1**) and (**NS2**), it is clear that we obtain the conditions (**N1**), (**N2**) and (**N3**) are satisfied.

Now, we show that the condition (**N4**) is satisfied.

(**N4**) Let $x, y \in X$. By the condition (**NS3**), we have

$$\begin{aligned}\|x+y\| &= \|0, x+y, 0\| + \|0, 0, x+y\| \\ &= \|0, x+y, 0\| + \|0, 0, y+x\| \\ &\leq \|0, 0, 0\| + \|0, x, 0\| + \|0, 0, y\| \\ &+ \|0, 0, x\| + \|0, 0, 0\| + \|0, y, 0\| \\ &= \|x\| + \|y\|.\end{aligned}$$

Consequently, the function $\|.\|$ is a norm on X and $(X, \|.\|)$ is a normed space. □

We call this norm as the norm generated by the S-norm $\|., ., .\|$.

Let X be a real vector space. New generalizations of normed spaces have been studied in recent years. For example, Khan defined the notion of a G-norm and studied some topological concepts in G-normed spaces [15]. Now we recall the definition of a G-norm and give the relationship between a G-norm and an S-norm.

Definition 3. *[15] Let X be a real vector space. A real valued function $\|., ., .\| : X \times X \times X \to \mathbb{R}$ is called a G-norm on X if the following conditions hold:*

(**NG1**) *$\|x, y, z\| \geq 0$ and $\|x, y, z\| = 0$ if and only if $x = y = z = 0$.*

(**NG2**) *$\|x, y, z\|$ is invariant under permutations of x, y, z.*

(**NG3**) *$\|\lambda x, \lambda y, \lambda z\| = |\lambda| \, \|x, y, z\|$ for all $\lambda \in \mathbb{R}$ and $x, y, z \in X$.*

(**NG4**) *$\|x + x', y + y', z + z'\| \leq \|x, y, z\| + \|x', y', z'\|$ for all $x, y, z, x', y', z' \in X$.*

(**NG5**) *$\|x, y, z\| \geq \|x+y, 0, z\|$ for all $x, y, z \in X$.*

The pair $(X, \|., ., .\|)$ is called a G-normed space.

Proposition 4. *Every G-normed space is an S-normed space.*

Proof. Using the conditions (**NG1**) and (**NG3**), we see that the conditions (**NS1**) and (**NS2**) are satisfied. We only show that the condition (**NS3**) is satisfied.

(**NS3**) Let $x, y, z, x', y', z' \in X$. Using the conditions (**NG2**) and (**NG4**), we obtain

$$\begin{aligned}&\|x + x', y + y', z + z'\| \\ &= \|(x + 0) + x', 0 + (y + y'), z' + z\| \\ &\leq \|x + 0, 0, 0 + z'\| + \|x', y + y', z\| \\ &= \|0, 0 + x, 0 + z'\| + \|x', y + y', z\| \\ &\leq \|0, 0, 0\| + \|0, x, z'\| + \|x', y, 0\| + \|0, y', z\| \\ &= \|0, x, z'\| + \|0, y, x'\| + \|0, z, y'\|.\end{aligned}$$

Consequently, the condition (**NS3**) is satisfied. □

The converse of Proposition 4 can not be always true as we have seen in the following example.

Example 5. *Let $X = \mathbb{R}$ and the S-norm be defined as in Example 4. If we put $x = 1$, $y = 5$ and $z = 0$, the condition (**NG5**) is not satisfied. Indeed, we have*

$$\begin{aligned}\|x, y, z\| &= |x - 2y - 2z| + |y - 2x - 2z| \\ &+ |z - 2y - 2x| \\ &= 23\end{aligned}$$

and

$$\begin{aligned}\|x + y, 0, z\| &= |x + y - 2z| + |2x + 2y + 2z| \\ &+ |z - 2y - 2x| \\ &= 30.\end{aligned}$$

Hence this S-norm is not a G-norm on $\mathbb{R}$.

Now we give the definitions of an open ball and a closed ball on an S-normed space.

Definition 4. *Let $(X, \|., ., .\|)$ be an S-normed space. For given x_0, a_1, $a_2 \in X$ and $r > 0$, the open ball $B_{a_1}^{a_2}(x_0, r)$ and the closed ball $B_{a_1}^{a_2}[x_0, r]$ are defined as follows:*

$B_{a_1}^{a_2}(x_0, r) = \{y \in X : \|y - x_0, y - a_1, y - a_2\| < r\}$

and

$B_{a_1}^{a_2}[x_0, r] = \{y \in X : \|y - x_0, y - a_1, y - a_2\| \leq r\}$.

Example 6. *Let us consider the S-normed space $(X, \|., ., .\|)$ generated by the usual norm on X, where $X = \mathbb{R}^2$ and*

$$\|x\| = \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2},$$

for all $x \in \mathbb{R}^2$. *Then the open ball* $B_{a_1}^{a_2}(x_0, r)$ *in* $\mathbb{R}^2$ *is a 3-ellipse given by*

$$B_{a_1}^{a_2}(x_0, r) = \{y \in \mathbb{R}^2 : \|y - x_0\| + \|y - a_1\| + \|y - a_2\| < r\}.$$

If we choose $y = (y_1, y_2)$, $x_0 = (1, 1)$, $a_1 = (0, 0)$, $a_2 = (-1, -1)$ *in* $\mathbb{R}^2$ *and* $r = 5$, *then we obtain*

$$B_{a_1}^{a_2}(x_0, r) = \left\{ \begin{array}{c} y \in \mathbb{R}^2 : \sqrt{(y_1 - 1)^2 + (y_2 - 1)^2} \\ +\sqrt{y_1^2 + y_2^2} \\ +\sqrt{(y_1 + 1)^2 + (y_2 + 1)^2} < 5 \end{array} \right\}, \quad (5)$$

as shown in Figure 1a.

Now we give the following example using an S-norm which is not generated by a norm.

Example 7. *Let* $X = \mathbb{R}^2$ *and the function* $\|.,.,.\| : X \times X \times X \to \mathbb{R}$ *be defined as in Example 4. Then we have*

$$\begin{aligned} &\|x, y, z\| \\ &= |x - 2y - 2z| + |y - 2x - 2z| + |z - 2y - 2x| \\ &= \sqrt{(x_1 - 2y_1 - 2z_1)^2 + (x_2 - 2y_2 - 2z_2)^2} \\ &+ \sqrt{(y_1 - 2x_1 - 2z_1)^2 + (y_2 - 2x_2 - 2z_2)^2} \\ &+ \sqrt{(z_1 - 2y_1 - 2x_1)^2 + (z_2 - 2y_2 - 2x_2)^2}, \end{aligned}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in \mathbb{R}^2$. *Then* $(\mathbb{R}^2, \|.,.,.\|)$ *is an* S*-normed space. The open ball* $B_{a_1}^{a_2}(x_0, r)$ *in* $\mathbb{R}^2$ *is*

$$B_{a_1}^{a_2}(x_0, r) = \{y \in \mathbb{R}^2 : \|y - x_0, y - a_1, y - a_2\| < r\}.$$

If we choose $y = (y_1, y_2)$, $x_0 = (1, 1)$, $a_1 = (0, 0)$, $a_2 = (-1, -1)$ *in* $\mathbb{R}^2$ *and* $r = 20$, *then we obtain*

$$B_{a_1}^{a_2}(x_0, r) = \left\{ \begin{array}{c} y \in \mathbb{R}^2 : \sqrt{(3y_1 + 3)^2 + (3y_2 + 3)^2} \\ +\sqrt{9y_1^2 + 9y_2^2} \\ +\sqrt{(3 - 3y_1)^2 + (3 - 3y_2)^2} < 20 \end{array} \right\}, \quad (6)$$

as shown in Figure 1b.

Definition 5. *Let* $(X, \|.,.,.\|)$ *be an* S*-normed space.*

(1) *A sequence* $\{x_n\}$ *in* X *converges to* x *if and only if*
$$\lim_{n \to \infty} \|0, x_n - x, x - x_n\| = 0.$$
That is, for each $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*
$$\|0, x_n - x, x - x_n\| < \varepsilon,$$
for all $n \geq n_0$.

(2) *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if*
$$\lim_{n,m,l \to \infty} \|x_n - x_m, x_m - x_l, x_l - x_n\| = 0.$$
That is, for each $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that*
$$\|x_n - x_m, x_m - x_l, x_l - x_n\| < \varepsilon,$$
for all $n, m, l \geq n_0$.

(3) *An* S*-normed space is called complete if each Cauchy sequence in* X *converges in* X.

(4) *A complete* S*-normed space is called an* S*-Banach space.*

Proposition 5. *Every convergent sequence in an* S*-normed space is a Cauchy sequence.*

Proof. Let a sequence $\{x_n\}$ in X be convergent to x. For each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|0, x_n - x, x - x_n\| < \frac{\varepsilon}{3},$$

for all $n \geq n_0$. We now show that for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|x_n - x_m, x_m - x_l, x_l - x_n\| < \varepsilon,$$

for all $n, m, l \geq n_0$. Using the condition (**NS3**), we obtain

$$\begin{aligned} &\|x_n - x_m, x_m - x_l, x_l - x_n\| \\ &= \left\| \begin{array}{c} x_n - x + x - x_m, x_m - x \\ +x - x_l, x_l - x + x - x_n \end{array} \right\| \\ &\leq \|0, x_n - x, x - x_n\| + \|0, x_m - x, x - x_m\| \\ &\quad + \|0, x_l - x, x - x_l\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Consequently, the sequence $\{x_n\}$ in X is a Cauchy sequence. □

The converse of Proposition 5 can not be always true as we have seen in the following example.

Example 8. *Let* $X = (0, 1) \subset \mathbb{R}$ *and the function* $\|.,.,.\| : X \times X \times X \to \mathbb{R}$ *be an* S*-norm generated by the usual norm on* X. *If we consider the sequence* $\{x_n\} = \left\{\frac{1}{n}\right\}$ *on* X, *then this sequence*

(a) The open ball which is corresponding to the S-norm defined in (5).

(b) The open ball which is corresponding to the S-norm defined in (6).

Figure 1. Some open balls in $(\mathbb{R}^2, \|.,.,.\|)$

is a Cauchy sequence, but it is not a convergent sequence on X.

Now we show that the sequence is a Cauchy sequence. For x_n, x_m, $x_l \in X$, we obtain

$$\begin{aligned}
&\lim_{n,m,l\to\infty} \|x_n - x_m, x_m - x_l, x_l - x_n\| \\
&= \lim_{n,m,l\to\infty} \left\| \frac{1}{n} - \frac{1}{m}, \frac{1}{m} - \frac{1}{l}, \frac{1}{l} - \frac{1}{n} \right\| \\
&= \lim_{n,m,l\to\infty} \left(\left|\frac{1}{n} - \frac{1}{m}\right| + \left|\frac{1}{m} - \frac{1}{l}\right| + \left|\frac{1}{l} - \frac{1}{n}\right| \right) = 0.
\end{aligned}$$

The sequence is convergent to 0 as follows:

$$\lim_{n\to\infty} \|0, x_n - x, x - x_n\| = \lim_{n\to\infty} \|0, \tfrac{1}{n} - 0, 0 - \tfrac{1}{n}\| = 0,$$

for all $x_n \in X$. But $0 \notin X$. Consequently, the sequence is not convergent on X.

3. A fixed point theorem on S-normed spaces

In this section, we introduce the Rhoades' condition on an S-normed space and denote it by (**NS25**). We prove a fixed point theorem using this contractive condition.

At first, we give some definitions and a proposition which are needed in the sequel.

Definition 6. *Let $(X, \|.,.,.\|)$ be an S-normed space and $E \subseteq X$. The closure of E, denoted by $\overline{E}$, is the set of all $x \in X$ such that there exists a sequence $\{x_n\}$ in E converging to x. If $E = \overline{E}$, then E is called a closed set.*

Definition 7. *Let $(X, \|.,.,.\|)$ be an S-normed space and $A \subseteq X$. The subset A is called bounded if there exists $r > 0$ such that*

$$\|0, x - y, y - x\| < r,$$

for all $x, y \in A$.

Definition 8. *Let $(X, \|.,.,.\|)$ be an S-normed space and $A \subseteq X$. The S-diameter of A is defined by*

$$\delta^s(A) = \sup\{\|0, x - y, y - x\| : x, y \in A\}.$$

If A is bounded then we will write $\delta^s(A) < \infty$.

Definition 9. *Let X be an S-Banach space, $A \subseteq X$ and $u \in X$.*

(1) *The S-radius of A relative to a given $u \in X$ is defined by*
$$r^s_u(A) = \sup\{\|0, u - x, x - u\| : x \in A\}.$$
(2) *The S-Chebyshev radius of A is defined by*
$$r^s(A) = \inf\{r^s_u(A) : u \in A\}.$$
(3) *The S-Chebyshev centre of A is defined by*
$$C^s(A) = \{u \in A : r^s_u(A) = r^s(A)\}.$$

By Definition 8 and Definition 9, it can be easily seen the following inequality:

$$r^s(A) \le r^s_u(A) \le \delta^s(A).$$

Definition 10. *A point $u \in A$ is called S-diametral if $r^s_u(A) = \delta^s(A)$. If $r^s_u(A) < \delta^s(A)$, then u is called non-S-diametral.*

Definition 11. *A convex subset of an S-Banach space X has S-normal structure if every S-bounded and convex subset of A having $\delta^s(A) > 0$ has at least one non-S-diametral point.*

Proposition 6. *If X is a reflexive S-Banach space, A is a nonempty, closed and convex subset of X, then $C^s(A)$ is nonempty, closed and convex.*

Proof. It can be easily seen by definition of $C^s(A)$. □

Now we introduce the Rhoades' condition (**NS25**) on an S-Banach space.

Definition 12. *Let $(X, \|.,.,.\|)$ be an S-Banach space and T be a self-mapping of X. We define*

$$(\mathbf{NS25}) \quad \|0, Tx - Ty, Ty - Tx\| < \max\left\{\begin{array}{c} \|0, x-y, y-x\|, \\ \|0, Tx-x, x-Tx\|, \\ \|0, Ty-y, y-Ty\|, \\ \|0, Ty-x, x-Ty\|, \\ \|0, Tx-y, y-Tx\| \end{array}\right\},$$

for each $x, y \in X$, $x \neq y$.

Lemma 3. *[16] Let X be a Banach space. Then X is reflexive if and only if for any decreasing sequence $\{K_n\}$ of nonempty, bounded, closed and convex subsets of X,*

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Lemma 4. *Let X be an S-Banach space. Then X is reflexive if and only if for any decreasing sequence $\{K_n\}$ of nonempty, bounded, closed and convex subsets of X,*

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Proof. By the definition of reflexivity the proof follows easily. □

Recall that the convex hull of a set A is denoted by $conv(A)$ and any member of this set $conv(A)$ has the form

$$\sum_{i=1}^{n} \alpha_i x_i,$$

where $x_i \in A_i, \alpha_i \geq 0$ for all $i = 1, ..., n$ and $\sum_{i=1}^{n} \alpha_i = 1$.

Now, we give the following fixed point theorem.

Theorem 1. *Let X be a reflexive S-Banach space and A be a nonempty, closed, bounded and convex subset of X, having S-normal structure. If $T : A \to A$ is a continuous self-mapping satisfying the condition (**NS25**) then T has a unique fixed point in A.*

Proof. At first, we show that the existence of the fixed point. Let $\mathcal{A}$ be the family of every nonempty, closed and convex subsets of A. Also we assume that if $F \in \mathcal{A}$ then $TF \subseteq F$. The family $\mathcal{A}$ is nonempty since $A \in \mathcal{A}$. We can partially order $\mathcal{A}$ by set inclusion, that is, if $F_1 \subseteq F_2$ then $F_1 \leq F_2$.

In $\mathcal{A}$, if we define a decreasing net of subsets

$$S = \{F_i : F_i \in \mathcal{A}, i \in I\},$$

then by reflexivity, this net S has nonempty intersection. Because it is a decreasing net of nonempty, closed, bounded and convex subsets of X. If we put $F_0 = \bigcap_{i \in I} F_i$ we have that F_0 is in $\mathcal{A}$ and is a lower bound of S.

Using Zorn's Lemma, there is a minimal element, denoted by F, in $\mathcal{A}$ as S is any arbitrary decreasing net in $\mathcal{A}$. We see that this F is a singleton.

Assume that $\delta^s(F) \neq \emptyset$. Since F is nonempty, closed and convex, $C^s(F)$ is a nonempty, closed and convex subset of F. We have that

$$r^s(F) < \delta^s(F),$$
$$\delta^s(C^s(F)) \leq r^s(F) < \delta^s(F)$$

and so $C^s(F)$ is a proper subset of F.

Let $(F_m)_{m \in \mathbb{N}}$ be an increasing sequence of subsets of F, defined by

$$F_1 = C^s(F) \text{ and } F_{m+1} = conv(F_m \cup TF_m),$$

for all $m \in \mathbb{N}$. If we denote the S-diameters of these sets F_k by $\delta_k^s = \delta^s(F_k)$, we show that

$$\delta_k^s \leq r^s(F),$$

for all $k \in \mathbb{N}$.

Using the (PMI), we obtain

(1) For $k = 1$,
$\delta_1^s = \delta^s(F_1) = \delta^s(C^s(F)) \leq r^s(F)$.

(2) If $\delta_k^s \leq r^s(F)$ for every $k = 1, ..., m$ then $\delta_{m+1}^s \leq r^s(F)$.

We note that

$$\begin{aligned} \delta_{m+1}^s = \delta^s(F_{m+1}) &= \delta^s(conv(F_m \cup TF_m)) \\ &= \delta^s(F_m \cup TF_m). \end{aligned}$$

By the definition of S-diameter, for any given $\varepsilon > 0$ there are x' and y' in $F_m \cup T(F_m)$ satisfying

$$\delta_{m+1}^s - \varepsilon < \|0, x' - y', y' - x'\| \leq \delta_{m+1}^s.$$

We obtain the following three cases for x', y':

(1) $x', y' \in F_m$ or
(2) $x' \in F_m$ and $y' \in TF_m$ or

(3) $x', y' \in TF_m$.

Redefining x' and y' as follows:

(1) $x' = x$ and $y' = y$ with $x, y \in F_m$,
(2) $x' = x$ and $y' = Ty$ with $x, y \in F_m$,
(3) $x' = Tx$ and $y' = Ty$ with $x, y \in F_m$.

We show that in any case

$$\delta^s_{m+1} - \varepsilon < r^s(F).$$

Case 1. By the definition of δ^s_m and the induction hypothesis, we obtain

$$\delta^s_{m+1} - \varepsilon < \|0, x-y, y-x\| \le \delta^s_{m+1} \le r^s(F) \quad (7)$$

and so $\delta^s_{m+1} - \varepsilon < r^s(F)$.
Case 2. We obtain

$$\delta^s_{m+1} - \varepsilon < \|0, x-Ty, Ty-x\|$$

with $x, y \in F_m$. Then by the definition of F_m, we have $x, y \in conv(F_{m-1} \cup TF_{m-1})$ and so there is a finite index set I such that $x = \sum_{i\in I} \alpha_i x_i$, with $\sum_{i\in I} \alpha_i = 1$, $\alpha_i \ge 0$ and $x_i \in F_{m-1} \cup TF_{m-1}$ for any $i \in I$. We can separate the set I in two disjoint subsets, $I = I_1 \cup I_2$, such that if $i \in I_1$ then $x_i \in F_{m-1}$ and if $i \in I_2$ then $x_i \in TF_{m-1}$.
Now redefining x_i as $x_i = Tx_i$ with $x_i \in F_{m-1}$, we obtain

$$x = \sum_{i\in I_1} \alpha_i x_i + \sum_{i\in I_2} \alpha_i Tx_i.$$

Substituting in $\|0, x-Ty, Ty-x\|$, we get

$$\|0, x-Ty, Ty-x\| \le \sum_{i\in I_1} \alpha_i \|0, x_i - Ty, Ty - x_i\| + \sum_{i\in I_2} \alpha_i \|0, Tx_i - Ty, Ty - Tx_i\|. \quad (8)$$

Applying the condition (**NS25**) to $\|0, Tx_i - Ty, Ty - Tx_i\|$, we have

$$\|0, Tx_i - Ty, Ty - Tx_i\| < \max \left\{ \begin{array}{c} \|0, x_i - y, y - x_i\|, \|0, x_i - Tx_i, Tx_i - x_i\|, \\ \|0, y - Ty, Ty - y\|, \|0, x_i - Ty, Ty - x_i\|, \\ \|0, Tx_i - y, y - Tx_i\| \end{array} \right\}. \quad (9)$$

As $x_i \in F_{m-1}$, $Tx_i, y \in F_m$, we have

$$\begin{aligned} &\|0, x_i - y, y - x_i\| \le r^s(F), \\ &\|0, x_i - Tx_i, Tx_i - x_i\| \le r^s(F), \\ &\|0, Tx_i - y, y - Tx_i\| \le r^s(F) \end{aligned}$$

and replacing in (9), we obtain

$$\|0, Tx_i - Ty, Ty - Tx_i\| < \max \left\{ \begin{array}{c} r^s(F), \\ \|0, y - Ty, Ty - y\|, \|0, x_i - Ty, Ty - x_i\| \end{array} \right\}.$$

Let us subdivide the index set I_2 in three disjoint subsets $I_2 = I_2^1 \cup I_2^2 \cup I_2^3$ such that

$$\begin{aligned} I_2^1 &= \{i \in I_2 : \|0, Tx_i - Ty, Ty - Tx_i\| < r^s(F)\}, \\ I_2^2 &= \left\{ \begin{array}{c} i \in I_2 : \|0, Tx_i - Ty, Ty - Tx_i\| \\ < \|0, x_i - Ty, Ty - x_i\| \end{array} \right\}, \\ I_2^3 &= \left\{ \begin{array}{c} i \in I_2 : \|0, Tx_i - Ty, Ty - Tx_i\| \\ < \|0, y - Ty, Ty - y\| \end{array} \right\}. \end{aligned}$$

Then using (8), we have

$$\|0, x-Ty, Ty-x\| \le \sum_{i\in I_1\cup I_2^2} \alpha_i \|0, x_i - Ty, Ty - x_i\| + \sum_{i\in I_2^1} \alpha_i r^s(F) + \sum_{i\in I_2^3} \alpha_i \|0, y - Ty, Ty - y\|. \quad (10)$$

Redefining I_1, I_2 and I_3

$$\overline{I_1} = I_1 \cup I_2^2,\ \overline{I_2} = I_2^1 \text{ and } \overline{I_3} = I_2^3. \quad (11)$$

Then we have $I = \overline{I_1} \cup \overline{I_2} \cup \overline{I_3}$, with $\overline{I_j} \cap \overline{I_k} = \emptyset$. If $j \ne k$ and $\sum_{i\in I} \alpha_i = 1$ then using (10), it becomes

$$\|0, x-Ty, Ty-x\| \le \sum_{i\in \overline{I_1}} \alpha_i \|0, x_i - Ty, Ty - x_i\| + \sum_{i\in \overline{I_2}} \alpha_i r^s(F) + \sum_{i\in \overline{I_3}} \alpha_i \|0, y - Ty, Ty - y\|. \quad (12)$$

If $A_0 = \sum_{i\in \overline{I_2}} \alpha_i$ and $B_0 = \sum_{i\in \overline{I_3}} \alpha_i$ with $\sum_{i\in \overline{I_1}} \alpha_i + A_0 + B_0 = 1$. Using (12), we have

$$\|0, x-Ty, Ty-x\| \le \sum_{i\in \overline{I_1}} \alpha_i \|0, x_i - Ty, Ty - x_i\| + A_0 r^s(F) + B_0 \|0, y - Ty, Ty - y\|. \quad (13)$$

For each $i \in \overline{I_1}$, $x_i \in F_{m-1} = conv(F_{m-2} \cup TF_{m-2})$, there is a finite set J_i, such that

$$x_i = \sum_{j\in J_i} \beta_i^j x_i^j, \quad (14)$$

with $x_i^j \in F_{m-2} \cup TF_{m-2}$, $\beta_i^j \geq 0$ and $\sum\limits_{j \in J_i} \beta_i^j = 1$ for any $j \in J_i$. Let $J_i = J_i^1 \cup J_i^2$, with $J_i^1 \cap J_i^2 = \emptyset$ such that

$$x_i = \sum_{j \in J_i^1} \beta_i^j x_i^j + \sum_{j \in J_i^2} \beta_i^j T x_i^j. \tag{15}$$

For each $i \in \overline{I_1}$ we have

$$\begin{aligned} &\|0, x_i - Ty, Ty - x_i\| \\ &\leq \sum_{j \in J_i^1} \beta_i^j \|0, x_i^j - Ty, Ty - x_i^j\| \\ &+ \sum_{j \in J_i^2} \beta_i^j \|0, Tx_i^j - Ty, Ty - Tx_i^j\|. \end{aligned} \tag{16}$$

Applying the condition (**NS25**) to $\|0, Tx_i^j - Ty, Ty - Tx_i^j\|$, we have

$$\|0, Tx_i^j - Ty, Ty - Tx_i^j\| \tag{17}$$

$$< \max \left\{ \begin{array}{c} \|0, x_i^j - y, y - x_i^j\|, \\ \|0, Tx_i^j - x_i^j, x_i^j - Tx_i^j\|, \\ \|0, y - Ty, Ty - y\|, \\ \|0, Tx_i^j - y, y - Tx_i^j\|, \\ \|0, x_i^j - Ty, Ty - x_i^j\| \end{array} \right\}.$$

Since $x_i^j \in F_{m-2}$ and $y \in F_m$, we have

$$\begin{aligned} &\|0, x_i^j - y, y - x_i^j\| \leq r^s(F), \\ &\|0, Tx_i^j - x_i^j, x_i^j - Tx_i^j\| \leq r^s(F), \\ &\|0, Tx_i^j - y, y - Tx_i^j\| \leq r^s(F). \end{aligned}$$

By (17), we obtain

$$\begin{aligned} &\|0, Tx_i^j - Ty, Ty - Tx_i^j\| \\ &< \max \left\{ \begin{array}{c} r^s(F), \|0, y - Ty, Ty - y\|, \\ \|0, x_i^j - Ty, Ty - x_i^j\| \end{array} \right\}. \end{aligned}$$

Let $J_i^2 = J_i^{2_1} \cup J_i^{2_2} \cup J_i^{2_3}$ with $J_i^{2_k} \cap J_i^{2_p} = \emptyset$ such that

$$\begin{aligned} J_i^{2_1} &= \left\{ \begin{array}{c} j \in J_i^2 : \|0, Tx_i^j - Ty, Ty - Tx_i^j\| \\ < r^s(F) \end{array} \right\}, \\ J_i^{2_2} &= \left\{ \begin{array}{c} j \in J_i^2 : \|0, Tx_i^j - Ty, Ty - Tx_i^j\| \\ < \|0, y - Ty, Ty - y\| \end{array} \right\}, \\ J_i^{2_3} &= \left\{ \begin{array}{c} j \in J_i^2 : \|0, Tx_i^j - Ty, Ty - Tx_i^j\| \\ < \|0, x_i^j - Ty, Ty - x_i^j\| \end{array} \right\}. \end{aligned}$$

Using (16), we obtain

$$\begin{aligned} &\|0, x_i - Ty, Ty - x_i\| \\ &\leq \sum_{j \in J_i^1 \cup J_i^{2_3}} \beta_i^j \|0, x_i^j - Ty, Ty - x_i^j\| \\ &+ \sum_{i \in J_i^{2_1}} \beta_i^j r^s(F) + \sum_{i \in J_i^{2_2}} \beta_i^j \|0, y - Ty, Ty - y\|. \end{aligned} \tag{18}$$

Let us denote by

$$J_1^i = J_i^1 \cup J_i^{2_3},\ J_2^i = J_i^{2_2} \text{ and } J_3^i = J_i^{2_1}.$$

Then using (18), we have

$$\begin{aligned} \|0, x_i - Ty, Ty - x_i\| &\leq \sum_{j \in J_1^i} \beta_i^j \|0, x_i^j - Ty, Ty - x_i^j\| \\ &+ \sum_{i \in J_3^i} \beta_i^j r^s(F) + \sum_{i \in J_2^i} \beta_i^j \|0, y - Ty, Ty - y\|. \end{aligned} \tag{19}$$

If $A_i = \sum\limits_{i \in J_3^i} \beta_i^j$ and $B_i = \sum\limits_{i \in J_2^i} \beta_i^j$ with $\sum\limits_{j \in J_1^i} \beta_i^j + A_i + B_i = 1$. Using (19), we obtain

$$\begin{aligned} \|0, x_i - Ty, Ty - x_i\| &\leq \sum_{j \in J_1^i} \beta_i^j \|0, x_i^j - Ty, Ty - x_i^j\| \\ &+ A_i r^s(F) + B_i \|0, y - Ty, Ty - y\|. \end{aligned} \tag{20}$$

Using (13) and $\|0, x_i - Ty, Ty - x_i\|$ by (20), we obtain

$$\begin{aligned} \|0, x - Ty, Ty - x\| &\leq \sum_{i \in \overline{I_1}} \alpha_i \sum_{j \in J_1^i} \beta_i^j \|0, x_i^j - Ty, Ty - x_i^j\| \\ &+ \left[\sum_{i \in \overline{I_1}} \alpha_i A_i + A_0 \right] r^s(F) \\ &+ \left[\sum_{i \in \overline{I_1}} \alpha_i B_i + B_0 \right] \|0, y - Ty, Ty - y\|. \end{aligned}$$

Let $A_1 = \sum\limits_{i \in \overline{I_1}} \alpha_i A_i$ and $B_1 = \sum\limits_{i \in \overline{I_1}} \alpha_i B_i$. Then we have

$$\begin{aligned} \|0, x - Ty, Ty - x\| &\leq \sum_{i \in \overline{I_1}} \alpha_i \sum_{j \in J_1^i} \beta_i^j \|0, x_i^j - Ty, Ty - x_i^j\| \\ &+ (A_1 + A_0) r^s(F) + (B_1 + B_0) \|0, y - Ty, Ty - y\|. \end{aligned} \tag{21}$$

We note that

$$\sum_{i \in \overline{I_1}} \alpha_i \sum_{j \in J_1^i} \beta_i^j + \sum_{i \in \overline{I_1}} \alpha_i A_i + A_0 + \sum_{i \in \overline{I_1}} \alpha_i B_i + B_0 = 1.$$

Let us take $K = \bigcup\limits_{i \in \overline{I_1}} \left(\bigcup\limits_{j \in J_1^i} j \right)$ and denote the scalars by ξ_k. To each k relative to the pair (i, j), x_i^j will be denoted by x_k.
Using (21), we obtain

$$\begin{aligned} \|0, x - Ty, Ty - x\| &\leq \sum_{k \in K} \xi_k \|0, x_k - Ty, Ty - x_k\| \\ &+ (A_1 + A_0) r^s(F) + (B_1 + B_0) \|0, y - Ty, Ty - y\|, \end{aligned}$$

where $\sum_{k\in K}\xi_k + A_1 + A_0 + B_1 + B_0 = 1$ and $x_k \in F_{m-2}$.

Repeating this process which is done for x_k, we get

$$\|0, x-Ty, Ty-x\| \le \sum_{p\in P}\gamma_p\|0, x_p-Ty, Ty-x_p\| + \sum_{k=0}^{m-1}A_k r^s(F) + \sum_{k=0}^{m-1}B_k\|0, y-Ty, Ty-y\|, \tag{22}$$

where $\sum_{p\in P}\gamma_p + \sum_{i=0}^{m-1}(A_i+B_i) = 1$ and $x_p \in F_1 = C^s(F)$.

Hence $\|0, x_p-Ty, Ty-x_p\| \le r^s(F)$ and using (22), we obtain

$$\|0, x-Ty, Ty-x\| \le \sum_{k=0}^{m-1}B_k\|0, y-Ty, Ty-y\| + \left(\sum_{p\in P}\gamma_p + \sum_{k=0}^{m-1}A_k\right)r^s(F). \tag{23}$$

Let us turn to $\|0, y-Ty, Ty-y\|$. Since $y \in conv(F_{m-1}\cup TF_{m-1})$, we have $y = \sum_{i\in I}\alpha_i y_i$ with $\sum_{i\in I}\alpha_i = 1$, $y_i \in F_{m-1}\cup TF_{m-1}$ and $\alpha_i \ge 0$ for all $i \in I$. Let $I = I_1\cup I_2$ such that $I_1\cap I_2 = \emptyset$. If $i \in I_1$ then $y_i \in F_{m-1}$ and if $i \in I_2$ then $y_i \in TF_{m-1}$. Let $y_i = Ty_i$. Then we can write

$$y = \sum_{i\in I_1}\alpha_i y_i + \sum_{i\in I_2}\alpha_i Ty_i,$$

with $y_i \in F_{m-1}$.

Substituting in $\|0, y-Ty, Ty-y\|$ we get

$$\|0, y-Ty, Ty-y\| \le \sum_{i\in I_1}\alpha_i\|0, y_i-Ty, Ty-y_i\| + \sum_{i\in I_2}\alpha_i\|0, Ty_i-Ty, Ty-Ty_i\|. \tag{24}$$

Using the condition (**NS25**), we obtain

$$\|0, Ty_i-Ty, Ty-Ty_i\| < \max\left\{\begin{array}{c}\|0, y_i-y, y-y_i\|,\\ \|0, y_i-Ty_i, Ty_i-y_i\|,\\ \|0, y-Ty, Ty-y\|,\\ \|0, y_i-Ty, Ty-y_i\|,\\ \|0, Ty_i-y, y-Ty_i\|\end{array}\right\}. \tag{25}$$

Since $y_i \in F_{m-1}$, $y \in F_m$, we have

$$\|0, y_i-y, y-y_i\| \le r^s(F),$$
$$\|0, y_i-Ty_i, Ty_i-y_i\| \le r^s(F),$$
$$\|0, Ty_i-y, y-Ty_i\| \le r^s(F).$$

Using (25), we obtain

$$\|0, Ty_i-Ty, Ty-Ty_i\| < \max\left\{\begin{array}{c}r^s(F), \|0, y-Ty, Ty-y\|,\\ \|0, y_i-Ty, Ty-y_i\|\end{array}\right\}.$$

Redefining the index set $I_2 = I_2^1\cup I_2^2\cup I_2^3$ with

$$I_2^1 = \left\{\begin{array}{c}i \in I_2 : \|0, Ty_i-Ty, Ty-Ty_i\|\\ < r^s(F)\end{array}\right\},$$
$$I_2^2 = \left\{\begin{array}{c}i \in I_2 : \|0, Ty_i-Ty, Ty-Ty_i\|\\ < \|0, y-Ty, Ty-y\|\end{array}\right\},$$
$$I_2^3 = \left\{\begin{array}{c}i \in I_2 : \|0, Ty_i-Ty, Ty-Ty_i\|\\ < \|0, y_i-Ty, Ty-y_i\|\end{array}\right\}.$$

Now using (24), we get

$$\|0, y-Ty, Ty-y\| \le \sum_{i\in I_1\cup I_2^3}\alpha_i\|0, y_i-Ty, Ty-y_i\| + \sum_{i\in I_2^1}\alpha_i r^s(F) + \sum_{i\in I_2^2}\alpha_i\|0, y-Ty, Ty-y\|. \tag{26}$$

We note that if $\sum_{i\in I_2^2}\alpha_i = 1$ then

$$\|0, y-Ty, Ty-y\| \le \|0, Ty_i-Ty, Ty-Ty_i\| < \|0, y-Ty, Ty-y\|,$$

which is a contradiction.

Then $\sum_{i\in I_2^2}\alpha_i < 1$ and using (26), we obtain

$$\|0, y-Ty, Ty-y\| \le \sum_{i\in I_1\cup I_2^3}\frac{\alpha_i}{1-\sum_{i\in I_2^2}\alpha_i}\|0, y_i-Ty, Ty-y_i\| + \sum_{i\in I_2^1}\frac{\alpha_i}{1-\sum_{i\in I_2^2}\alpha_i}r^s(F), \tag{27}$$

with $\sum_{i\in I_1\cup I_2^3}\frac{\alpha_i}{1-\sum_{i\in I_2^2}\alpha_i} + \sum_{i\in I_2^1}\frac{\alpha_i}{1-\sum_{i\in I_2^2}\alpha_i} = 1.$

Let $I_1 = I_1\cup I_2^3$, $I_2 = I_2^1$ and $\beta_i = \frac{\alpha_i}{1-\sum_{i\in I_2^2}\alpha_i}$.

Using (27), we obtain

$$\|0, y - Ty, Ty - y\| \leq \sum_{i\in I_1} \beta_i \|0, y_i - Ty, Ty - y_i\| + \sum_{i\in I_2} \beta_i r^s(F). \quad (28)$$

If $A_0 = \sum_{i\in I_2} \beta_i$ then using (28), we have

$$\|0, y - Ty, Ty - y\| \leq \sum_{i\in I_1} \beta_i \|0, y_i - Ty, Ty - y_i\| + A_0 r^s(F), \quad (29)$$

with $\sum_{i\in I_1} \beta_i + A_0 = 1$ and $y_i \in F_{m-1} = conv(F_{m-2} \cup TF_{m-2})$.

For each $i \in I_1$,

$$y_i = \sum_{j\in J_i^1} \gamma_i^j y_i^j + \sum_{j\in J_i^2} \gamma_i^j T y_i^j,$$

with $y_i^j \in F_{m-2}$ and $\sum_{j\in J_i^1\cup J_i^2} \gamma_i^j = 1$. So we obtain

$$\|0, y_i - Ty, Ty - y_i\| \leq \sum_{j\in J_i^1} \gamma_i^j \|0, y_i^j - Ty, Ty - y_i^j\| + \sum_{j\in J_i^2} \gamma_i^j \|0, Ty_i^j - Ty, Ty - Ty_i^j\|. \quad (30)$$

Using the condition (**NS25**), we have

$$\|0, Ty_i^j - Ty, Ty - Ty_i^j\| < \max \left\{ \begin{array}{c} \|0, y_i^j - y, y - y_i^j\|, \\ \|0, Ty_i^j - y_i^j, y_i^j - Ty_i^j\|, \\ \|0, y - Ty, Ty - y\|, \\ \|0, Ty_i^j - y, y - Ty_i^j\|, \\ \|0, y_i^j - Ty, Ty - y_i^j\| \end{array} \right\}.$$

Since $y_i^j \in F_{m-2}$, $Ty_i^j \in F_{m-1}$ and $y \in F_m$, we can write

$$\begin{array}{l} \|0, y_i^j - y, y - y_i^j\| \leq r^s(F), \\ \|0, Ty_i^j - y_i^j, y_i^j - Ty_i^j\| \leq r^s(F), \\ \|0, Ty_i^j - y, y - Ty_i^j\| \leq r^s(F) \end{array}$$

and

$$\|0, Ty_i^j - Ty, Ty - Ty_i^j\| < \max \left\{ \begin{array}{c} r^s(F), \|0, y - Ty, Ty - y\|, \\ \|0, y_i^j - Ty, Ty - y_i^j\| \end{array} \right\}.$$

Let J_i^2 be the union of the disjoint sets $J_i^2 = J_i^{2_1} \cup J_i^{2_2} \cup J_i^{2_3}$ such that

$$J_i^{2_1} = \left\{ \begin{array}{c} j \in J_i^2 : \|0, Ty_i^j - Ty, Ty - Ty_i^j\| \\ < \|0, y_i^j - Ty, Ty - y_i^j\| \end{array} \right\},$$
$$J_i^{2_2} = \left\{ \begin{array}{c} j \in J_i^2 : \|0, Ty_i^j - Ty, Ty - Ty_i^j\| \\ < r^s(F) \end{array} \right\},$$
$$J_i^{2_3} = \left\{ \begin{array}{c} j \in J_i^2 : \|0, Ty_i^j - Ty, Ty - Ty_i^j\| \\ < \|0, y - Ty, Ty - y\| \end{array} \right\}.$$

Using (30), we obtain

$$\begin{array}{l} \|0, y_i - Ty, Ty - y_i\| \\ \leq \sum_{j\in J_i^1\cup J_i^{2_1}} \gamma_i^j \|0, y_i^j - Ty, Ty - y_i^j\| \\ + \sum_{j\in J_i^{2_2}} \gamma_i^j r^s(F) + \sum_{j\in J_i^{2_3}} \gamma_i^j \|0, y - Ty, Ty - y\|. \end{array} \quad (31)$$

Now redefine the index sets $J_i^1 = J_i^1 \cup J_i^{2_1}$, $J_i^2 = J_i^{2_2}$, $J_i^3 = J_i^{2_3}$ and using (31) we can write

$$\begin{array}{l} \|0, y_i - Ty, Ty - y_i\| \\ \leq \sum_{j\in J_i^1} \gamma_i^j \|0, y_i^j - Ty, Ty - y_i^j\| \\ + \sum_{j\in J_i^2} \gamma_i^j r^s(F) + \sum_{j\in J_i^3} \gamma_i^j \|0, y - Ty, Ty - y\|, \end{array}$$

with $\sum_{j\in J_i^1} \gamma_i^j + \sum_{j\in J_i^2} \gamma_i^j + \sum_{j\in J_i^3} \gamma_i^j = 1$.

Using the (29), we obtain

$$\begin{array}{l} \|0, y - Ty, Ty - y\| \\ \leq \sum_{i\in I_1} \beta_i \sum_{j\in J_i^1} \gamma_i^j \|0, y_i^j - Ty, Ty - y_i^j\| \\ + \sum_{i\in I_1} \beta_i \sum_{j\in J_i^3} \gamma_i^j \|0, y - Ty, Ty - y\| \\ + \left[\sum_{i\in I_1} \beta_i \sum_{j\in J_i^2} \gamma_i^j + A_0 \right] r^s(F). \end{array} \quad (32)$$

If $\sum_{i\in I_1} \beta_i \sum_{j\in J_i^3} \gamma_i^j = 1$ we have

$$\|0, y - Ty, Ty - y\| < \|0, y - Ty, Ty - y\|,$$

which is a contradiction.

Hence $\sum_{i\in I_1} \beta_i \sum_{j\in J_i^3} \gamma_i^j < 1$ and using (32), we obtain

$$\|0, y - Ty, Ty - y\| \leq \frac{\sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^1} \gamma_i^j}{1 - \sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^3} \gamma_i^j} \|0, y_i^j - Ty, Ty - y_i^j\| + \frac{\sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^2} \gamma_i^j + A_0}{1 - \sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^3} \gamma_i^j} r^s(F), \tag{33}$$

with $\frac{\sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^1} \gamma_i^j}{1 - \sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^3} \gamma_i^j} + \frac{\sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^2} \gamma_i^j + A_0}{1 - \sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^3} \gamma_i^j} = 1.$

Let $A_1 = \frac{\sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^2} \gamma_i^j + A_0}{1 - \sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^3} \gamma_i^j}$ and denote the index set by $K = \bigcup\limits_{i\in I_1} \left(\bigcup\limits_{j\in J_i^1} j \right)$, write ζ_k for $k \in K$ relative to (i, j), that is

$$\zeta_k = \frac{\sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^2} \gamma_i^j}{1 - \sum\limits_{i\in I_1} \beta_i \sum\limits_{j\in J_i^3} \gamma_i^j}.$$

Also we write y_k for y_i^j. Then using (33), we obtain

$$\|0, y - Ty, Ty - y\| \leq \sum_{k\in K} \zeta_k \|0, y_k - Ty, Ty - y_k\| + (A_1 + A_0) r^s(F),$$

with $\sum\limits_{k\in K} \zeta_k + A_1 + A_0 = 1$ and $y_k \in F_{m-2}$.
Repeating this process we get

$$\|0, y - Ty, Ty - y\| \leq \sum_{p\in P} \lambda_p \|0, y_p - Ty, Ty - y_p\| + \sum_{k=0}^{m-1} A_k r^s(F),$$

where $y_p \in F_1$ and $\sum\limits_{p\in P} \lambda_p + \sum\limits_{k=0}^{m-1} A_k = 1$.
Then $\|0, y_p - Ty, Ty - y_p\| \leq r^s(F)$ and

$$\|0, y - Ty, Ty - y\| \leq \left(\sum_{p\in P} \lambda_p + \sum_{k=0}^{m-1} A_k \right) r^s(F) = r^s(F).$$

Using (23), we get

$$\|0, x - Ty, Ty - x\| \leq \left(\sum_{k=0}^{m-1} B_k + \sum_{p\in P} \gamma_p + \sum_{k=0}^{m-1} A_k \right) r^s(F),$$

with $\sum\limits_{k=0}^{m-1} B_k + \sum\limits_{p\in P} \gamma_p + \sum\limits_{k=0}^{m-1} A_k = 1$.
Consequently, we obtain $\|0, x - Ty, Ty - x\| \leq r^s(F)$ and so

$$\delta_{m+1}^s - \varepsilon < \|0, x - Ty, Ty - x\| \leq r^s(F).$$

Case 3. For $x, y \in F_m$, we have

$$\delta_{m+1}^s - \varepsilon < \|0, Tx - Ty, Ty - Tx\| < \max \left\{ \begin{array}{c} \|0, x - y, y - x\|, \\ \|0, Tx - x, x - Tx\|, \\ \|0, y - Ty, Ty - y\|, \\ \|0, x - Ty, Ty - x\|, \\ \|0, Tx - y, y - Tx\| \end{array} \right\} \tag{34}$$

and repeating what has been done in Case 2, we get

$$\delta_{m+1}^s - \varepsilon < \|0, Tx - Ty, Ty - Tx\| \leq r^s(F).$$

In all three cases we have $\delta_{m+1}^s - \varepsilon < r^s(F)$. If ε tends to 0 we get $\delta_{m+1}^s \leq r^s(F)$.
Let $F^\infty = \bigcup\limits_{n\in\mathbb{N}} F_n$. Then F^∞ is nonempty because $C^s(F) \neq \emptyset$. Since $F_k \subseteq F_{k+1}$, we obtain

$$\delta^s(F^\infty) = \lim_{k\to\infty} \delta^s(F_k) \leq r^s(F).$$

As $F_k \subset F$, $F^\infty \subseteq F$ and so $\delta^s(F^\infty) \leq r^s(F)$.
Using the S-normal structure of F we have $r^s(F) < \delta^s(F)$ and $\delta^s(F^\infty) < \delta^s(F)$. So F^∞ must be a proper subset of F. We obtain that F^∞ is convex and $TF^\infty \subseteq F^\infty$.
Let $M = \overline{convF^\infty} = \overline{F^\infty}$, its diameter is the same as F^∞. So we have

$$\delta^s(M) \leq r^s(F) < \delta^s(F)$$

and M is closed, nonempty and convex proper subset of F. Since T is continuous then M is T-invariant and

$$TM = TF^\infty \subseteq \overline{TF^\infty} \subseteq \overline{F^\infty} = M.$$

So $M \in \mathcal{A}$ and $M \subsetneqq F$ contradicting the minimality of F. Hence, it should be $\delta^s(F) = 0$. Consequently, F has a unique fixed point under T. □

4. Some comparisons on S-normed spaces

In [12], the present authors defined Rhoades' condition (**S25**) using the notion of an S-metric. Also, they investigated relationships between the conditions (**S25**) and (**R25**) in [13].

In this section, we determine the relationships between the conditions (**S25**) (resp. (**NR25**)) and (**NS25**).

At first, we recall the Rhoades' condition on normed spaces as follows [17]:

Let $(X, \|.\|)$ be a Banach space and T be a self-mapping of X.

$$(\mathbf{NR25}) \qquad \|Tx - Ty\| < \max\left\{\begin{array}{c} \|x-y\|, \|x-Tx\|, \\ \|y-Ty\|, \|x-Ty\|, \\ \|y-Tx\| \end{array}\right\},$$

for each $x, y \in X$, $x \neq y$.

Now we give the relationship between (**S25**) and (**NS25**) in the following proposition.

Proposition 7. *Let $(X, \|.,.,.\|)$ be an S-Banach space, $(X, S_{\|.\|})$ be the S-metric space obtained by the S-metric generated by $\|.,.,.\|$ and T be a self-mapping of X. If T satisfies the condition* (**NS25**) *then T satisfies the condition* (**S25**).

Proof. Assume that T satisfies the condition (**NS25**). Using the condition (**NS25**), we have

$$\begin{array}{l} S_{\|.\|}(Tx,Tx,Ty) = \|Tx-Tx, Tx-Ty, Ty-Tx\| \\ = \|0, Tx-Ty, Ty-Tx\| \\ < \max\left\{\begin{array}{c} \|0,x-y,y-x\|, \|0,Tx-x,x-Tx\|, \\ \|0,Ty-y,y-Ty\|, \|0,Ty-x,x-Ty\|, \\ \|0,Tx-y,y-Tx\| \end{array}\right\} \\ = \max\left\{\begin{array}{c} S_{\|.\|}(x,x,y), S_{\|.\|}(Tx,Tx,x), \\ S_{\|.\|}(Ty,Ty,y), S_{\|.\|}(Ty,Ty,x), \\ S_{\|.\|}(Tx,Tx,y) \end{array}\right\} \end{array}$$

and so the condition (**S25**) is satisfied by T on $(X, S_{\|.\|})$. □

Now, we give the relationship between the conditions (**NR25**) and (**NS25**) in the following proposition.

Proposition 8. *Let $(X, \|.\|)$ be a Banach space, $(X, \|.,.,.\|)$ be an S-normed space obtained by the S-norm generated by $\|.\|$ and T be a self-mapping of X. If T satisfies the condition* (**NR25**) *then T satisfies the condition* (**NS25**).

Proof. Let T satisfies the condition (**NR25**). Using the conditions (**NR25**) and (**N3**), we have

$$\begin{array}{l} \|0, Tx-Ty, Ty-Tx\| \\ = \|0\| + \|Tx-Ty\| + \|Ty-Tx\| \\ = 2\|Tx-Ty\| \\ < 2\max\left\{\begin{array}{c} \|x-y\|, \|x-Tx\|, \\ \|y-Ty\|, \|x-Ty\|, \|y-Tx\| \end{array}\right\} \\ = \max\left\{\begin{array}{c} 2\|x-y\|, 2\|x-Tx\|, \\ 2\|y-Ty\|, 2\|x-Ty\|, 2\|y-Tx\| \end{array}\right\} \\ = \max\left\{\begin{array}{c} \|x-y\| + \|y-x\|, \\ \|x-Tx\| + \|Tx-x\|, \\ \|y-Ty\| + \|Ty-y\|, \\ \|x-Ty\| + \|Ty-x\|, \\ \|y-Tx\| + \|Tx-y\| \end{array}\right\} \\ = \max\left\{\begin{array}{c} \|0,x-y,y-x\|, \\ \|0,Tx-x,x-Tx\|, \\ \|0,Ty-y,y-Ty\|, \\ \|0,Ty-x,x-Ty\|, \\ \|0,Tx-y,y-Tx\| \end{array}\right\} \end{array}$$

and so the condition (**NS25**) is satisfied. □

Finally, we give the relationship between Theorem 1 and the following theorem.

Theorem 2. *[17] Let X be a reflexive Banach space and A be a nonempty, closed, bounded and convex subset of X, having normal structure. If $T : A \to A$ is a continuous self-mapping satisfying the condition* (**NR25**) *then T has a unique fixed point in A.*

Theorem 1 and Theorem 2 coincide when X is an S-Banach space obtained by the S-norm generated by $\|.\|$. Clearly, Theorem 1 is a generalization of Theorem 2 as we have seen in Section 2 that there are S-norms which are not generated by any norm.

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