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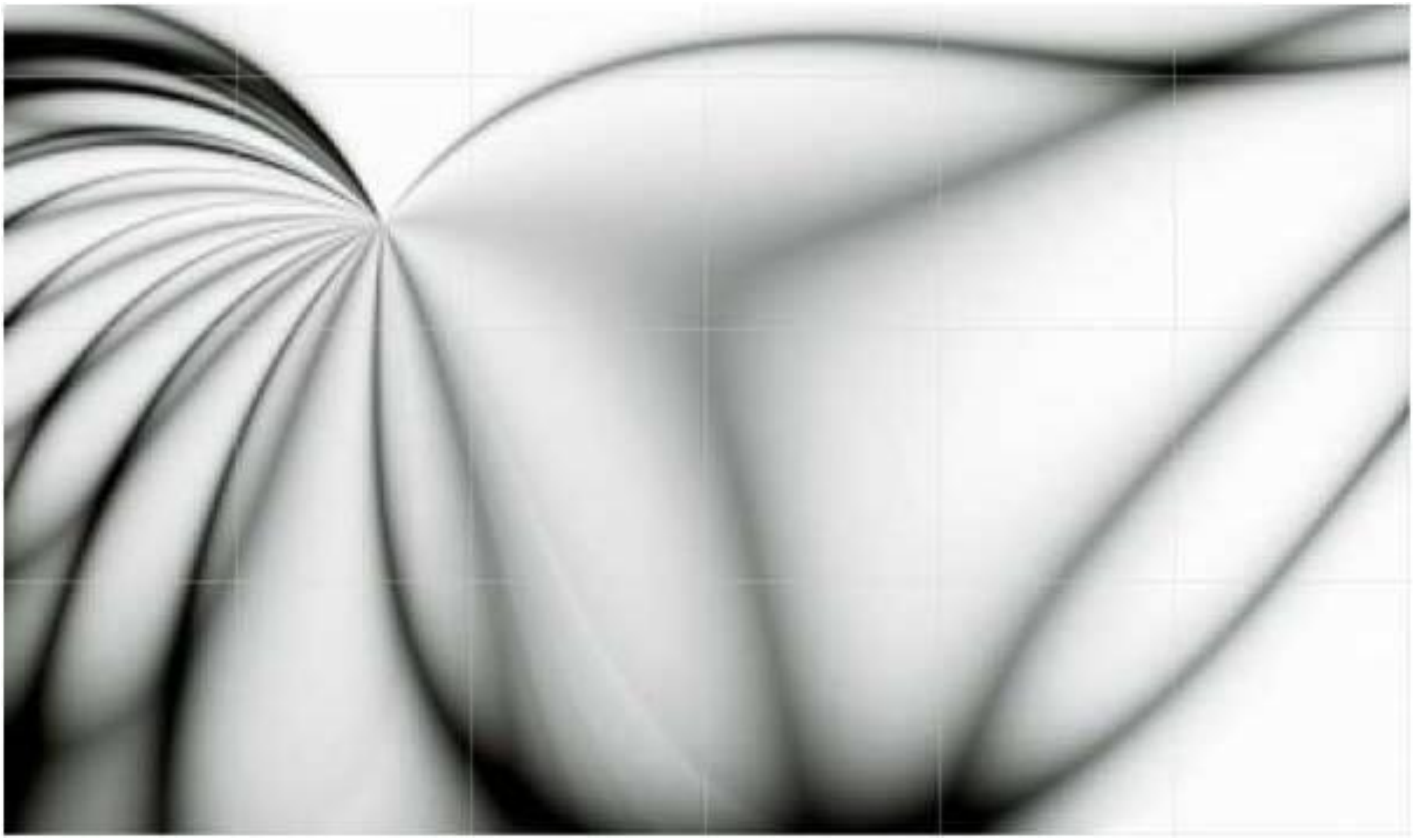

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RESEARCH ARTICLE

Magnetic field diffusion in ferromagnetic materials: fractional calculus approaches

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ABSTRACT

The paper addresses diffusion approximations of magnetic field penetration of ferromagnetic materials with emphasis on fractional calculus applications and relevant approximate solutions. Examples with applications of time-fractional semi-derivatives and singular kernel models (Caputo time fractional operator) in cases of field independent and field-dependent magnetic diffusivities have been developed: Dirichlet problems and time-dependent boundary condition (power-law ramp). Approximate solutions in all theses case have been developed by applications of the integral-balance method and assumed parabolic profile with unspecified exponents. Tow version of the integral method have been successfully implemented: SDIM (single integration applicable to time-fractional semi-derivative model) and DIM (double-integration model to fractionalized singular memory models). The fading memory approach in the sense of the causality concept and memory kernel effect on the model constructions have been discussed.

1. Introduction

There are many natural phenomena which can be modelled in diffusion approximations. Here magnetic field diffusions in solid ferromagnetics is considered with attempts to apply approximate solution based on synergism of fractional calculus and the integral-balance method in different versions. The main idea is to demonstrate the feasibility of both the fractional calculus approach and the integral solution.

In the context of the main idea of this communication magnetic diffusion of a field with parallel lines (see Figure 1) is taken as example. Two basic cases considering filed-independent and field-dependent diffusions with fixed (Dirichlet) and time dependent (power-law) boundary conditions are chosen as test examples. Moreover, the problem of magnetic field diffusion with memory is discussed with either the common time fractional operator of Caputo with singular kernel or from the more fundamental fading memory principle allowing different memory functions to be used.

1.1. Magnetic field transport in conducting media

The field transport in magnetizable and conducting media can be presented as superposition of the fundamental processes of advection and diffusion as key parts of describing behaviour of magnetic field in materials. In homogeneous (and ideal) materials, the magnetic field $\mathbf{B}$, the electric field $\mathbf{E}$ and the material velocity (mainly in the case of plasma) $\mathbf{v}$ are related by the following constitutive relationship [1]

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \qquad (1)$$

It is worthy to mention, that if the material is not ideal,that is when the material resistance is finite then the right-hand side of (1) we have [1]

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$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = D_\mu (\nabla \times \mathbf{B}) \tag{2}$$

where $D_\mu = \sigma/\mu$ (σ is the material resistivity, μ is magnetic permeability) is the magnetic diffusivity. In such a case the magnetic field induction equation is [1]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times (D_\mu \nabla \times \mathbf{B}) \tag{3}$$

If a pure resistive magnetic diffusion is considered then Eq. (3) reduces to [1]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (D_\mu \nabla \times \mathbf{B}) \tag{4}$$

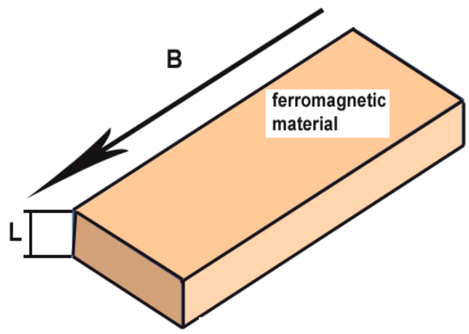


Figure 1. Schematic presentation of magnetic field with straight lines applied to a ferromagnetic material

The physics behind these relationships means that the changes in the magnetic field lines in time can be due to two principle causes: magnetic field advection (if the material is flowing as plasma or highly conductive fluid) and its diffusion through the material. Hence, as in the classical transport theory we assume a superposition of two transport mechanism: advection and diffusion. If the magnetic diffusivity D_μ is uniform (spatially independent), then it is possible to express (3), as [1]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + D_\mu \nabla^2 \mathbf{B} \tag{5}$$

That is, the magnetic field flux velocity $\mathbf{w}$ is related to the temporal change of the induction $\mathbf{B}$ by the relation [1]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}) \tag{6}$$

and $\mathbf{w}$ is termed *flux transporting velocity* [1] In a particular case considered in this article if the pure resistive material is at issue, then $\mathbf{E} = \eta \nabla \times \mathbf{B}$ and the ideal Ohm law holds (see (1)) the magnetic flux velocity is practically equal to the velocity of the medium (flowing conductive medium) $\mathbf{v}$ and (1) becomes

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \nabla F \tag{7}$$

and F is an arbitrary function of integration [1]

1.2. Magnetic field diffusion with straight field lines

1.2.1. Medium with field independent permeability

If one-dimensional case is considered then equation (4) reduces to the following diffusion equation [1–5] with constant magnetic diffusivity.

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial x}\left(D_\mu \frac{\partial B}{\partial x}\right) \tag{8}$$

With uniform magnetic diffusivity $D_\mu = D_{\mu 0} = \sigma/\mu = const.$ (σ is the resistivity of the material, $\mu = dB/dH = f(B)$ is the field dependent permeability of the material) and a sharp unit step at the boundary $x = 0$ (Dirichlet problem), that is (i.e. for the case $\mu = dB/dH = f(B) = k_B = const.$) we get

$$B(x,0) = \begin{cases} +B_0, x > 0 \\ -B_0, x < 0 \end{cases} \tag{9}$$

The case is relevant to an infinitesimally thin current sheet [1]. If the field is maintained fixed at two boundary points of a finite domain ($\pm L$) obeying the conditions $B(L,t) = -B(-L,t) = B_0$, the solution of (8) with $D_\mu = D_{\mu 0}$ is [1]

$$B(x,t) = \underbrace{B_0 \frac{x}{L}}_{stationary\ profile} + \underbrace{+ 2\frac{B_0}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left[-k^2\pi^2\left(\frac{D_{\mu 0}}{L^2}t\right)\right]\sin\left(k\pi\frac{x}{L}\right)}_{transient\quad term} \tag{10}$$

The solution means very rapidly establishment of the magnetic field stationary profile $B_0(x/L)$. Moreover, taking into account the finite Ohmic heating $\left(j^2/\sigma\right) = (D_{\mu 0}/\mu)(B_0/L)^2$ per unit length of the medium due the continuous supply of magnetic energy through the boundaries with a rate $\left(D_{\mu 0} B_0^2/\mu L\right)$ [1]

1.2.2. Medium with field dependent permeability

Commonly the power-law approximation [2] describes the magnetic field induction dependent on the field intensity, namely

$$\bar{B} = \frac{B(H)}{B_s} = \left(\frac{H}{H_s}\right)^{\gamma}, \quad 0 < \gamma < 1 \tag{11}$$

where B_s and H_S are corresponding to the point of magnetic saturation (specific characteristics for every magnetic material that can be used as characteristic scales). Actually, this permits the diffusion equation to be presented in a dimensionless form as [2, 6]

$$\frac{\partial \bar{B}}{\partial t} = D_{\mu}^{\bar{B}} \frac{\partial}{\partial x}\left(\bar{B}^{\beta} \frac{\partial \bar{B}}{\partial x}\right) \tag{12}$$

where $\beta = \frac{1-\gamma}{\gamma}$, $D_{\mu}^{B} = \frac{\sigma}{\mu_s B_s^{\beta}}$, $\mu_s = \frac{B_s}{H_s}$.

Equation (12) is a degenerate parabolic equation because of the power-law diffusivity $D_{\mu} = D_{\mu}^{B} B^{\beta}$; in such a case the solution has a finite speed in contrast to model (8) where the solution speed is infinite. Hereafter, for the sake of simplicity of the expressions we will omit the symbol $\bar{B}$ and will use only B.

1.3. Aim and motivation notes

The following part of this article demonstrates how fractional calculus can be applied to solve magnetic diffusion models with either field-independent or field-dependent diffusivity. The assumption behind these models and the approximate solutions developed is there is no changes in the material resistivity (that is, no Joules effects as result of the magnetic field changes exist). The only magnetic field effect on the material property considered is the power-law dependence of the magnetic diffusivity as implicit performance of the field dependent magnetic permeability. The fractional calculus approach envisages two directions: 1) Semi-derivative approach to the parabolic model (8), and 2) Fractionalization of the magnetic diffusion equation through a constitutive flux-gradient relationship with singular memory. In addition, the general problem of the causality principle in modelling of non-local diffusion and the fading memory approach are discussed. In general, the models and the solutions developed consider the magnetic material as a semi-infinite with a boundary condition at $x = 0$ since we are interested in the laws behind the magnetic field front propagation; before reaching the physical limit L of the medium as in solution (10). This approach allows straightforwardly seeing what would be the transient solution of the magnetic diffusion problem if memory formalism would be implemented in the diffusion model.

1.4. Paper organization

In what follows fractional semi-derivative diffusion model is developed by splitting the model (8) in section 2 and demonstrating two solutions with fixed (Dirichlet) (section 2.1.1) and time-ramp boundary condition (section 2.1.2). Further, time-fractional models of magnetic diffusions are developed (section 3) through a constitutive equation with singular memory (section 3.1) with two problem solved (section 4): Dirichlet problem (section 4.2.1) and ramp (power-law) time-dependent boundary condition (section 4.2.2) solved by application of the Double-integration Method (DIM) (section 4.1) in the general case of of field-independent magnetic diffusivity. The model counterparts with field-dependent magnetic diffusivity are solved in sections (4.3) by preliminary transform of the diffusion term in two cases: Dirichlet problem (section 4.3.1) and ramp boundary condition (section 4.3.2). The fading memory principle and the causality concept are discussed in sections 5) and 5.1.1, respectively, thus allowing to construct a more general model of magnetic diffusion (section 5.1.2) and a qualitative analysis of the different kernel functions on it (section 5.1.3).

2. Fractional calculus to magnetic diffusion problem

Here we address three principle problems :

- Fractional calculus solution by semi-derivatives of the problem with constant magnetic diffusivity with fixed and time-dependent boundary conditions
- Fractional models of magnetic diffusion with singular memories as counterparts of the integer-order models (8) and (12).
- Fractional models based on the causality principle and fading memory concept

2.1. Fractional calculus solution by semi-derivatives: general approach

Consider the model (8) which can be presented as product of two operators

$$\left(\frac{\partial^{1/2} B}{\partial t^{1/2}} - \sqrt{D_\mu}\frac{\partial B}{\partial x}\right)\times \times\left(\frac{\partial^{1/2} B}{\partial t^{1/2}} + \sqrt{D_\mu}\frac{\partial B}{\partial x}\right) = 0 \tag{13}$$

where

$$\frac{\partial^{1/2} B}{\partial t^{1/2}} = \frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_0^u \frac{B(x,t)}{\sqrt{t-u}}du - \frac{B(x,0)}{\sqrt{\pi t}} \tag{14}$$

is a Riemann-Liouville derivative of order 1/2 . In (13) only the second term has a physical meaning [7,8]. Hence, the time-fractional equivalent of (8) is [9]

$$\frac{\partial^{1/2} B}{\partial t^{1/2}} = -\sqrt{D_\mu}\frac{\partial \theta}{\partial x} \Rightarrow \frac{\partial^{1/2} B(0,t)}{\partial t^{1/2}} = -\sqrt{D_\mu}\frac{\partial B(0,t)}{\partial x} \tag{15}$$

Applying the operator $D_t^{-1/2}$ to both sides of (15) we get (16)

$$B(0,t) = -\sqrt{D_\mu}\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left[\frac{\partial B(0,t)}{\partial x}\right] \tag{16}$$

With initial condition $B(x,0) = 0$, applying a single integration with respect to the spatial coordinate x and using the Leibniz rule for differentiation under the integral sign we get

$$\frac{d}{dt}\int_0^\delta B(x,t)dx = \sqrt{D_\mu}\frac{\partial^{1/2} B(0,t)}{\partial t^{1/2}} \tag{17}$$

The upper terminal of the integral in (17) defines a sharp front of magnetic field penetration into the medium with conditions (Goodman's boundary condition [10, 11])

$$B(\delta) = 0, \quad \frac{\partial B}{\partial x}(\delta) = 0 \tag{18}$$

Equation (17) is the principle equation of **Semi-Derivative Integral Method-***single integration*(SDIM-1) [12]. The exact solution of this problem (8) is well -known [13], namely

$$B_{exact} = 1 - erf(\eta/2) \tag{19}$$

where $\eta = x/\sqrt{D_\mu t}$ is the Boltzmann similarity variable. The approximate solution developed in this work applies an assumed general parabolic profile with unspecified exponent

$$B_a = B_s(1 - x/\delta)^n \tag{20}$$

This assumed profile satisfies all boundary conditions (18) for any values of the exponent n [11].

2.1.1. SDIM solution: Dirichlet problem

With the assumed parabolic profile (20) and applying the Goodman boundary conditions we get $B_a(0,t) = B_s = 1$. Now, replacing $B(x,t)$ in the integral relation (17) by the approximate profile (20) the result is

$$\frac{d}{dt}\int_0^\delta \left(1 - \frac{x}{\delta}\right)^n dx = \sqrt{D_\mu}\frac{D^{1/2}}{\partial t^{1/2}}C \tag{21}$$

The integration of (21) with the initial condition $\delta(t=0) = 0$ yields

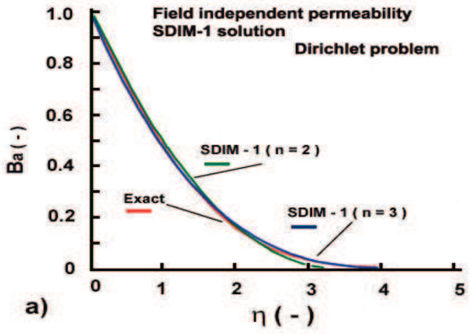
$$\frac{1}{n+1}\frac{d\delta}{dt} = \sqrt{D_\mu}\frac{1}{\sqrt{\pi t}} \Rightarrow \delta_B = \sqrt{D_\mu t}\frac{2(n+1)}{\sqrt{\pi}} \tag{22}$$

Hence the approximate distribution $B_a(x,t)$ of the magnetic field in the material is

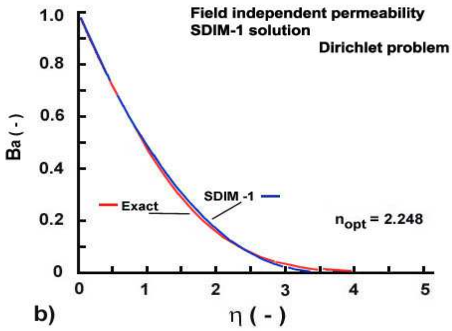
$$B_a(x,t) = B_s\left(1 - \frac{x}{\sqrt{D_\mu t}}\frac{\sqrt{\pi}}{2(n+1)}\right)^n \tag{23}$$

Hence, in terms if Boltzmann similarity variable $\eta = x/\sqrt{D_\mu t}$ the front is defined by the equality $\eta = 2(n+1)/\sqrt{\pi}$ (i.e. when $x = \delta_B$) since at this point $B_a = 0$. The optimal solution of this problem, i.e. solution with minimal mean squared error of approximation over the entire magnetic field penetration layer is $n_{opt} = 2.248$ (similar problem was resolved in [12]). Comparative numerical simulations are presented in Figure 2.

That is, the dimensionless penetration depth corresponding to the optimal solution is $\delta_B/\sqrt{D_\mu t} = \frac{2(n+1)}{\sqrt{\pi}} \approx 3.665$. Here $\sqrt{D_\mu t}$ plays a role of a length scale. Taking into account that $D_\mu = (\sigma/\mu)$ any Joule heating can change the magnetic diffusivity, as well changes in μ due to temperature effects on the material resistivity and magnetic permeability, correspondingly.



(a)



(b)

Figure 2. Approximate profiles developed by SDIM-1 approach and Dirichlet problem: for stipulated exponent $n = 2$ (a) and optimal $n_{opt} = 2.248$ (b), compared to exact solutions

2.1.2. SDIM solution: Time-dependent boundary condition

Let us consider a generalized ramp time-dependent boundary condition $b_0 t^{m/2}$ with $m \geq 0$ at $x = 0$. This problem has an exact solution [13] (Chaptert 2) expressed through the error function (in terms of the process parameters discussed here), namely

$$B_e = b_0 \Gamma\left(\frac{m}{2}+1\right)(4t)^{m/2}\, i^m \Phi\left(\frac{x}{2\sqrt{D_\mu t}}\right) \tag{24}$$

which can be applied only by either numerical solution or tabulated data.

With the generalized parabolic profile (20) and the Goodman's boundary conditions we get

$$B_a(0,t) = B_s = b_0 t^{m/2}\,, \quad B_a(\delta) = B_\infty = 0, \quad \frac{\partial B_a}{\partial x}(x=\delta) = 0 \tag{25}$$

That is, the assumed profile is

$$B_a = b_0 t^{m/2}\left(1-\frac{x}{\delta}\right)^n \tag{26}$$

Now, applying the relation (17) the result is

$$\frac{d}{dt}\int_0^\delta b_0 t^{m/2}\left(1-\frac{x}{\delta}\right)^n dx = \sqrt{D_\mu}\frac{D^{1/2}}{\partial t^{1/2}}\left(b_0 t^{m/2}\right) \tag{27}$$

The integration of eq. (27) yields

$$\frac{d}{dt}\left(b_0 t^{m/2}\frac{\delta}{n+1}\right) = = \sqrt{D_\mu}\left[b_0\frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)}t^{m/2-1/2}\right] \tag{28}$$

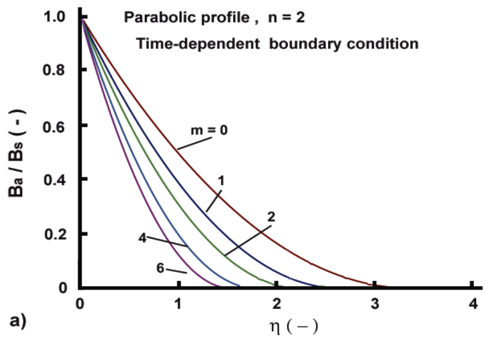
The integration of (28) with the initial condition $\delta(t=0) = 0$ yields

$$\delta_B = \sqrt{D_\mu t}\frac{2(n+1)}{(m+1)}G_m \quad \Rightarrow \frac{\delta_B}{\sqrt{D_\mu t}} = C^n_{m(B)} = \frac{2(n+1)}{(m+1)}G_m \tag{29}$$

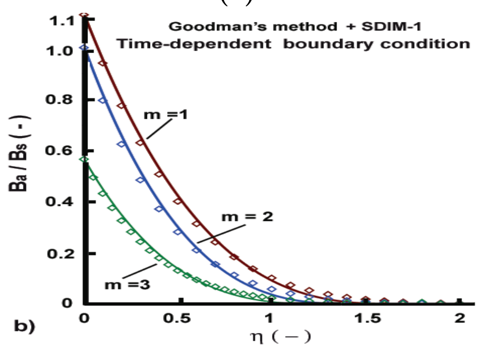
where $G_m = \frac{\Gamma(m/2+1)}{\Gamma(m/2+1/2)}$ is a constant. Then, the approximate filed induction profile is

$$B_a(x,t) = b_0 t^{m/2}\left(1-\frac{x}{\sqrt{D_\mu t}\frac{2(n+1)}{(m+1)}G_m}\right)^n = = b_0 t^{m/2}\left(1-\frac{\eta}{\frac{2(n+1)}{(m+1)}G_m}\right)^n \tag{30}$$

Hence, the front is defined by the condition $\eta = \frac{2(n+1)}{(m+1)}G_m$. The optimal values of the exponent n depend on the rate of the surface magnetization, i,e. on the values of m. The minimization of the squared mean error of approximation for different values of m yields optimal exponents summarized in Table 1. Plots of the approximate solutions are shown in Figure 3.



(a)



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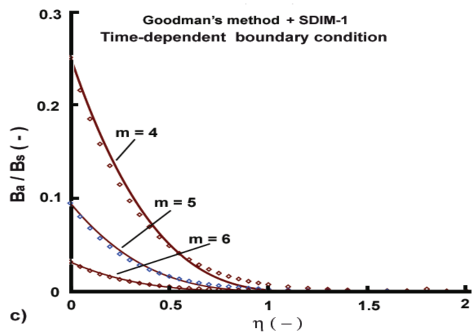


(c)

Figure 3. Normalized magnetic field profiles developed by SDIM-1 for different values of the parameter m of the surface ramping magnetization: a) SDIM-1 solution with stipulated parabolic profile exponent $n = 2$; b, c) SDIM-1 solutions with optimal exponents: Comparison with exact solutions (tabulated) from [13]

Table 1. SDIM-1:Optimal exponents for different values of m

m	1	2	3	4	5	6
n (optimal)	1.336	1.618	1.822	1.919	2.158	2.302

3. Fractional models of magnetic diffusion: simplified approach

Here we address magnetic diffusion equation with memory. Precisely, the memory function used to model is power-law with allows the fractional Caputo derivative to be applied.

3.1. Magnetic flux with memory: general approach

Let us consider a finite speed of the diffusion magnetic field into the material which cannot be assured by the parabolic model (8). In such a case following the causality principle that the reaction should follow the cause, a time shift between them can be presented through a convolution integral, that is

$$j_\mu (x,t) = \underbrace{-D_\mu \nabla B(x,t)}_{instantaneous\ (long\ times)} - \underbrace{-D_{\mu_1} \int_0^\infty R(t-\tau) \nabla B(x,t-\tau)\, d\tau}_{relaxation\ (memory\ effect)} \tag{31}$$

with a memory $R(t)$ controlled by a fractional parameter α , $0 < \alpha < 1$. In (31) the first term is relevant to long times where the relaxation disappears, known also as *instantaneous term*. If only this term is considered we get the parabolic model (8) with infinite speed of the solution. Now, we will omit this term in order to develop a model of magnetic diffusion of subdiffusion type. Applying the continuity equation

$$\frac{\partial B}{\partial t} = -\frac{\partial}{\partial x} j_\mu \tag{32}$$

as well as omitting the term $-D_\mu \nabla B(x,t)$ (and for the sake of simplicity getting $D_{\mu_1} = D_\mu$) we get a general relationship

$$\frac{\partial B}{\partial t} = D_\mu \int_0^\infty R(t-\tau) \frac{\partial^2 B(x,t-\tau)}{\partial x^2} d\tau \tag{33}$$

The function of $R(t)$ depends strongly on the physics of the magnetization and the material properties itself.

4. Fractional models of magnetic diffusion: Singular memory approach

As first example we will address a singular power law memory. In such a case the memory integral in (33) becomes a Riemann-Liouville integral of order α , namely

$$I_t^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{\partial^2 B(x,\tau)}{\partial x^2} d\tau = \\ = D_t^{-\alpha}\left[\frac{\partial^2 B(x,\tau)}{\partial x^2}\right], \quad 0<\alpha<1 \tag{34}$$

and the flux-gradient relationship can be expressed as

$$j_{\mu} = -D_{\mu} I_t^{\alpha}\left(\frac{\partial B}{\partial x}\right) \tag{35}$$

Then, the application of the continuity equation yields

$$\frac{\partial B}{\partial t} = D_{\mu} I_t^{\alpha}\left(\frac{\partial^2 B}{\partial x^2}\right) = D_{\mu}\left[D_t^{-\alpha}\left(\frac{\partial^2 B}{\partial x^2}\right)\right] \tag{36}$$

Applying the operator $D_t^{\alpha-1}$ to both sides of (36) and recalling the semi group properties of the fractional derivatives and integrals (here we consider Caputo time-fractional derivative) we get

$$\frac{\partial^{\alpha} B}{\partial t^{\alpha}} = D_{\mu}\frac{\partial^2 B}{\partial x^2} \tag{37}$$

which the well-known time-fractional diffusion (subdiffusion equation) with boundary and initial conditions

$$B(0,t) = B_s(t), \quad t \geq 0, \\ B(x,0) = B_{\infty} = 0, \quad x > 0 \tag{38}$$

Now, the magnetic diffusivity has a dimension $[D_{\mu}] = \left[m^2/\text{sec}^{\mu}\right]$. With D_{μ} independent of the time and space as well magnetic field independent, the linear problem is (37) with which the double integration method will be demonstrated next.

4.1. Double integration method (DIM)

The first step of DIM is the integration of (37) from 0 to x [14]

$$\int_x^{\delta} \frac{\partial^{\alpha} B}{\partial t^{\alpha}} dx = D_{\mu}\frac{\partial B(x,t)}{\partial x} - D_{\mu}\frac{\partial u(0,t)}{\partial x} \tag{39}$$

Taking into account that the single integration from 0 to δ can be presented as a sum $\int_0^{\delta} f(x)\,dx = \int_0^{x} f(x)\,dx + \int_x^{\delta} f(x)\,dx = -D_{\mu}\frac{\partial}{\partial x} f(x=0)$ we can obtain

$$\int_x^{\delta} \frac{\partial^{\alpha} B}{\partial t^{\alpha}} dx = -D_{\mu}\frac{\partial B(x,t)}{\partial x} \tag{40}$$

The second step of DIM is the integration of (40) from 0 to δ

$$\int_0^{\delta}\left(\int_x^{\delta} \frac{\partial^{\alpha} B}{\partial t^{\alpha}} dx\right) dx = D_{\mu} B(0,t) \tag{41}$$

Equation (41) is the principle relationship of the double integration method when the differential equation is of a fractional order [14]

4.2. Field independent magnetic permeability

Now, we will apply the integral-balance solutions to the time-fractional magnetic diffusion equation in two cases : fixed boundary condition (Dirichlet problem and time-ramping boundary condition (power-law).

4.2.1. Dirichlet problem

Now, we will apply DIM to (37) with assumed generalized parabolic profile. In this case we have

$$\frac{\partial B_a(x,t)}{\partial t} = \frac{x}{\delta^2} n\left(1-\frac{x}{\delta}\right)^{n-1}\frac{d\delta}{dt} \tag{42}$$

and incorporating this approximation in to the Caputo derivative one obtain

$$\int_0^{\delta}\left[\int_x^{\delta} {}_C D^{\alpha} B_a(x,t)\,dx\right] dx = \\ = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha}} \frac{1}{(n+1)(n+2)} \frac{d\delta^2}{dt} d\tau \tag{43}$$

$$\int_0^{\delta}\left[\int_x^{\delta} {}_C D_t^{\alpha} B_a(x,t)\,dx\right] dx = \frac{D_t^{\alpha}(\delta^2)}{N_C}, \\ N_C = (n+1)(n+2) \tag{44}$$

$${}_C D_t^{\alpha}\delta^2 = D_{\mu}\left[(n+1)(n+2)\right] \tag{45}$$

Therefore, the fractional integrations (with the physical condition $\delta(t=0)=0$) results in

$$ {}_C D_t^{\alpha} \delta(t) = \sqrt{D_\mu t^\alpha} \sqrt{\frac{(n+1)(n+2)}{\Gamma(1+\alpha)}} \tag{46} $$

Therefore the approximate solution is

$$ B_a = \left(1 - \frac{x}{\sqrt{D_\mu t^\alpha} F_n j_\alpha}\right)^n, $$

$$ F_n = \sqrt{(n+1)(n+2)}, \quad j_\alpha = 1/\sqrt{\Gamma(1+\alpha)} \tag{47} $$

The solution defines a non-Boltzmann similarity $\eta_\mu = x/\sqrt{D_\mu t^\alpha}$. Numerical simulations are presented in Figure 4 . For more details related to the optimization of the solution and the technology of DIM to fractional subdiffusion models see the extended analysis in [14].

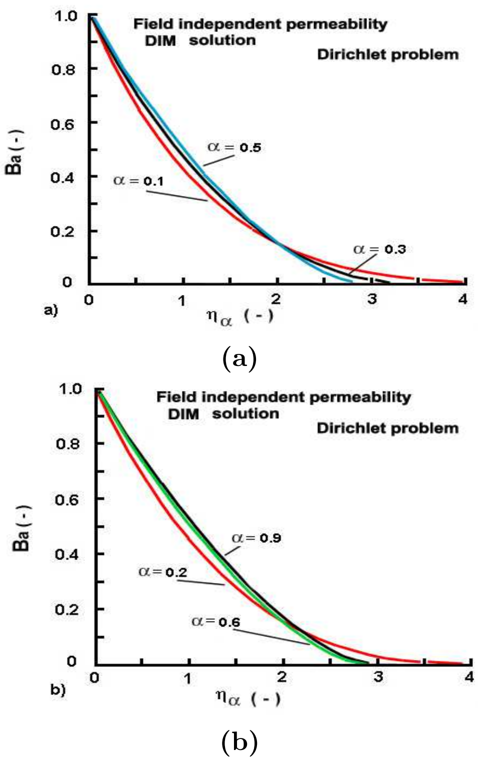


Figure 4. DIM solutions to the magnetization of field-independent material with Dirichlet Boundary condition, with : stipulated exponent of the parabolic profile $n = 2$ (a) and optimal exponent $n_{opt} = 2.248$ (b), compared to exact solutions

4.2.2. Time-dependent boundary condition

With time-dependent (power-law) boundary condition $B_a(0,t) = B_s = b_0 t^{m/2}$ the generalized parabolic profile (20) with the Goodman's boundary conditions (25) we get the assumed profile (26). Then, with the integral relation (41) we get

$$ \int_0^\delta \left(\int_x^\delta \frac{\partial^\mu B_a}{\partial t^\mu} dx \right) dx = D_\mu B(0,t) = D_\mu b_0 t^{m/2} \tag{48} $$

With (42) incorporated in (48) we have

$$ \int_0^\delta \left(\int_x^\delta \frac{\partial^\mu}{\partial t^\mu} \left[\left(b_0 t^{m/2} \right) \times \frac{\partial B_a}{\partial t} \right] dx \right) dx = \tag{49} $$
$$ = D_\mu b_0 t^{m/2} $$

The integration in left-hand side of (49) yields

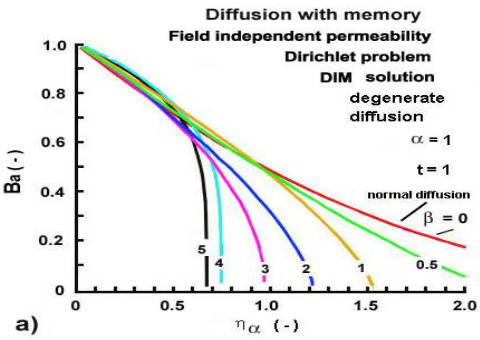
$$ {}_C D_t^{\mu} \left(\delta^2 b_0 t^{m/2} \right) = D_\mu \left(b_0 t^{m/2} \right) N_C, \quad N_C = (n+1)(n+2) \tag{50} $$

$$ \delta^2 b_0 t^{m/2} = D_\mu N_C G_m^\alpha b_0 t^{m/2+\alpha}, $$
$$ G_m^\alpha = \left(\frac{\Gamma(m/2+1)}{\Gamma(\alpha + m/2 + 1)} \right) $$
$$ \delta^2 = D_\mu t^\alpha N_C G_m^\alpha \Rightarrow \delta_m^\alpha = \sqrt{D_\mu t^\alpha} \sqrt{N_C G_m^\alpha} \tag{51} $$

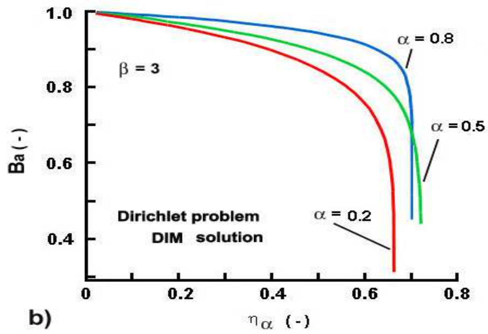
Hence , the approximate solution is

$$ B_{a,m}^\alpha = b_0 t^{m/2} \left(1 - \frac{x}{\sqrt{D_\mu t^\alpha} \sqrt{N_C} \sqrt{G_m^\alpha}} \right)^n \tag{52} $$

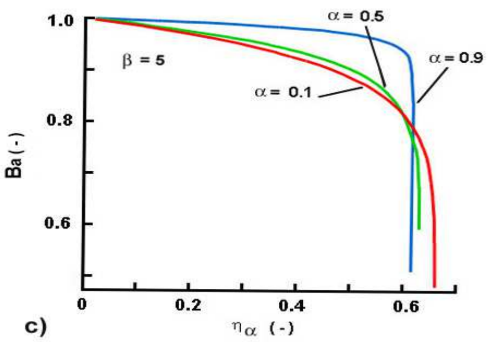
This solution defines a non-Boltzmann similarity variable $\eta_\alpha = x/\sqrt{D_\mu t^\alpha}$. Numerical simulations with various values of the fractional order α and the non-linear parameter β are shown in Figure 5.



(a)



(b)



(c)

Figure 5. Approximate profiles for stipulated exponent $n = 2$ (a) and optimal $n_{opt} = 2.248$ (b) compared to exact solutions

4.3. Field dependent magnetic permeability

The application of the integral method needs a preliminary treatment of the of the diffusion term in the right-hand side of (12), namely [15,16]

$$D_\mu^B B^m \frac{\partial u}{\partial x} = \frac{D_\mu^B}{\beta+1}\frac{\partial B^{\beta+1}}{\partial x} \tag{53}$$

Therefore, the result (53) can be considered as a non-linear counterpart of the constitutive equation (32), namely

$$\frac{\partial B}{\partial t} = \frac{D_\mu^B}{\beta+1}\int_0^\infty R(t-\tau)\frac{\partial^2 B^{\beta+1}(x,t-\tau)}{\partial x^2}d\tau \tag{54}$$

With a singular (power-law) memory function, similarly to transformations done in 3.1 we get a fractional analogue of (37) with non-linear diffusion term (similar problem was solved in [16]).

$$\frac{\partial^\alpha B}{\partial t^\alpha} = D_\mu^B \frac{1}{\beta+1}\frac{\partial^2 B^{\beta+1}}{\partial x^2} \tag{55}$$

Then applying DIM we have

$$\int_0^\delta \int_x^\delta \frac{\partial^\alpha B(x,t)}{\partial t^\alpha} dx dx = \frac{D_\mu^B}{\beta+1} B^{\beta+1}(0,t) \tag{56}$$

This is the principle DIM integral relationship when the diffusion term has a power-law non-linearity.

4.3.1. Dirichlet problem: Approximate solution

With Caputo time-fractional derivative and the assumes parabolic profile (20) as well as by help of (41) the integration in LHS of Eq.(56) yields

$${}_C D_t^\alpha \delta^2 = D_\mu^B \frac{(n+1)(n+2)}{\beta+1} \tag{57}$$

That is

$$\delta^2 = D_\mu^B \frac{N_C}{(\beta+1)}\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{58}$$

Hence, the penetration depth is

$$\delta_B^\alpha = \sqrt{D_\mu^B t^\alpha}\sqrt{\frac{N_C}{(\beta+1)}\frac{1}{\Gamma(\alpha+1)}} \tag{59}$$

and the approximate solution of (55) can be expressed as

$$B_a(x,t) = \left(1 - \frac{x}{\sqrt{D_\mu^B t^\alpha}\sqrt{\frac{(n+1)(n+2)}{(\beta+1)\Gamma(1+\alpha)}}}\right)^n \tag{60}$$

The solution defines a new similarity variable $\eta_\alpha = x/\sqrt{D_\mu^B t^\alpha}$. For $\alpha = 1$ it reduces to the classical Boltzmann similarity variable $\eta_{\alpha=1} = x/\sqrt{D_\mu^B t}$.

4.3.2. Time-dependent boundary condition: Approximate solution

From (56) it follows that the right-hand side is $\frac{D_\mu^B}{\beta+1}B^{\beta+1}(0,t)$. Then, if the boundary condition is of power law type $B_s = B(0,t) = b_0 t^{m/2}$ we have

$$\frac{D_\mu^B}{\beta+1}B^{\beta+1}(0,t) = \frac{D_\mu^B}{\beta+1}\left(b_0 t^{m/2}\right)^{\beta+1} \tag{61}$$

Therefore the DIM integral solutions is

$$\int_0^\delta \int_x^\delta \frac{\partial^\alpha B(x,t)}{\partial t^\alpha} dx dx = \frac{D_\mu^B}{\beta+1}\left(b_0 t^{m/2}\right)^{\beta+1} \tag{62}$$

Now, repeating the integration in the left-hand side of (62), as in (48) and (49) we get

$$_C D_t^\mu \left(\delta^2 b_0 t^{m/2}\right) = \frac{D_\mu^B}{\beta+1} b_0^{\beta+1} t^{m(\beta+1)/2} N_C \tag{63}$$

The fractional integration in (63) yields

$$\delta^2 b_0 t^{m/2} = \frac{D_\mu^B}{\beta+1} G_{\alpha,\beta}^m N_C b_0^{\beta+1} t^{m(\beta+1)/2+\alpha} \tag{64}$$

where

$$G_{\alpha,\beta}^m = \frac{\Gamma\left(\frac{m(\beta+1)}{2}+1\right)}{\Gamma\left(\alpha+\frac{m(\beta+1)}{2}+1\right)} \tag{65}$$

The re-arrangement in (65) results in

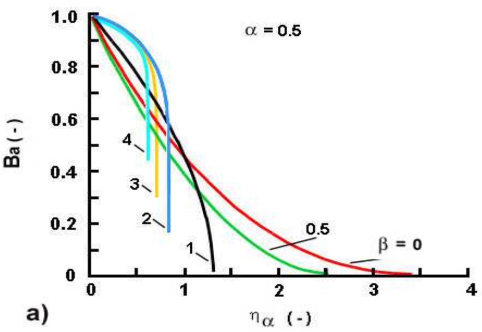
$$\delta^2 = \frac{D_\mu^B}{\beta+1} G_{\alpha,\beta}^m N_C b_0^\beta t^{\left[\frac{m}{4}(\beta-1)+\alpha\right]} \tag{66}$$

In a more useful form we have

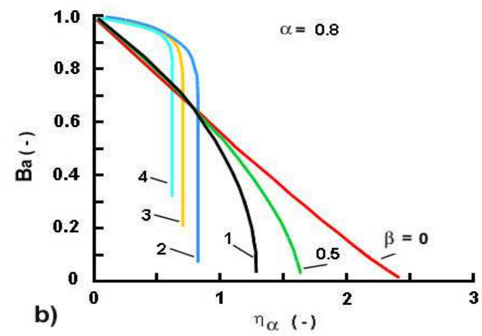
$$\delta = \sqrt{D_\mu^B t^{\frac{m(\beta-1)+4\alpha}{4}}} \sqrt{\frac{G_{\alpha,\beta}^m N_C b_0^\beta}{\beta+1}} \tag{67}$$

The exponent $\frac{m(\beta-1)+4\alpha}{4}$ in (67) should be positive since we have to assure a positive growth of the front δ. Therefore, the condition that should be obeyed is $\beta > 1 - 4\alpha/m$. Taking into account that $0 < \alpha < 1$ and $m = 1, 2, 3...$, then the condition imposed on β is satisfied.

To clarify this point, since $\beta = (1-\gamma)/\gamma$ where $0 < \gamma < 1$ ($\gamma = 0.22$ for steel [2] for example) we have always $\beta > 1$. In the particular case with steel magnetization ($\gamma = 0.22$) we get $\beta = 3.545$. In such a case the diffusion model (12) is a degenerate diffusion equation with convex solutions moving as almost sharp waves [15, 16]. In such a case the exponent of the parabolic profile (20) is $n = 1/\beta < 1$ [15]. It is noteworthy to mention that that the parabolic profile (20) with $n < 1$ generate convex profiles, while for $n > 1$ the profiles are concave. Profile of approximate solutions showing competitive actions of the sub-diffusion behaviour (trough the fractional parameter α) and the diffusion non-linearity (through the exponent β) are shown in Figure 6



(a)



(b)

Figure 6. DIM solutions to the magnetization of field-dependent material with time-dependent boundary condition and optimal exponents [16] showing how the value of the exponent β deforms the solution profile towards a rectangular wave with sharp front: a) Case with $\alpha = 0.5$, b), case with $\alpha = 0.8$

5. Fractional models by fading memory approach

5.1. Fading memory principle

For simple materials [17–21], the fading memory concept relating the flux to its gradient of a certain transported quantity A , is modelled by the following integro-differential equation

$$j_A(x,t) = -D_{A0}\frac{\partial A}{\partial x}(x,t) - -D_{A1}\int_{-\infty}^{t} R(t-\tau)\frac{\partial A}{\partial x}(x,\tau)\,d\tau \tag{68}$$

This is the Boltzmann linear superposition functional [20] with a memory kernel $R(t,z)$. In (68) D_{A0} and D_{A1} are transport (diffusion) coefficients (diffusivities). In fact, we assume a linear superposition of two fluxes

$$j_A(x,t) = \underbrace{j_{A0}}_{\text{instantaneous flux}} + \underbrace{j_{A1}}_{\substack{transient\\ flux\ with\ finite\ speed}} \tag{69}$$

Actually, the convolution integral in (68) is Stieltjes integral but because there is a restriction imposed on $R(t,z)$ to be casual function, i.e. $R(t<0,z)=0$ we may set the lower terminal to $t=0$. Thus, the gradient of the flux j_A can be presented in a general form as

$$\frac{\partial}{\partial x}j(x,t) = -D_{A0}\frac{\partial^2}{\partial x^2}A(x,t) - -D_{A1}\int_{0}^{t} R(t-\tau)\frac{\partial^2}{\partial x^2}A(x,\tau)\,d\tau \tag{70}$$

In (69) the first term is the long time, or instantaneous diffusion term, while the second is relevant to the finite sped of the diffusion wave of $A(x,t)$. This is a general linear expression of the fading memory principle since the transport coefficients are constants.

If now the quantity A is replaced by the magnetic field induction $B(x,t)$ we get the formulation (31). Moreover, if there is no flux relaxation and the speed is infinite, then the second term in (68) (as well as in (69) and (70)) is zero and the result is the classical $j_A(x,t) = -D_{A0}\frac{\partial A}{\partial x}(x,t)$ which gets different names as Fick's (diffusion) , Fourier (heat conduction) or Newton law (diffusion of momentum) laws.

The main idea behind the fading memory principle is to assure the causality of the models of dynamic systems (changing in time) as it is explained next

5.1.1. Causality principle

In all applied cases the chronological condition allows the causal relation to be satisfied (i.e. the time-shift between cause and effect) [22],i.e. *always the cause precedes the effect.* The principle conditions of the causality principle are [22]:

- ***Primitive causality***: *The effect cannot precede the cause.*
- ***Relativistic causality***: *No signal can propagate with velocity greater than the speed of the light in the vacuum.* It could be considered as a macroscopic causality condition.

Further, the causality concept means that the functions describing transients should be: *vanishing over a range of values of its arguments* (as the memory functions in the convolution integrals).

If we consider the physical system of the magnetic field diffusion with a time-dependent cause) $B_s(t)$ and the corresponding effect $B(x,t)$ the following conditions are obeyed [22].

C1: Linearity. This corresponds to the superposition principle in its simple version implying that the output is a linear functional of the input

$$B(t) = \int_{-\infty}^{\infty} R(t,\tau)B_s(\tau)\,d\tau \tag{71}$$

C2: Time-translation invariance. In this case the linear functional can be expressed as

$$B(x,t) = \int_{-\infty}^{\infty} R(t-\tau)B_s(\tau)\,d\tau = R(t) * B_s(t) \tag{72}$$

C3: Primitive causality condition. *The input cannot precede the output.* As consequence, $R(\tau)$ should be a causal function. Moreover, this is equivalent to setting the lower terminal in the (71) and (72) equal to zero, as mentioned in preceding point related to the fading memory concept.

Now, we can turn on magnetic field diffusion models with memories.

5.1.2. Fading memory in magnetic field diffusion

The fading memory concept was touched earlier with equation (31). Actually, we immediately jumped to the model where instantaneous term (long time term) does not exist thus entering into the area supported by the concept of the Continuous Time Random Walk (CTRW) where long time term does not exist. This was done especially in order to demonstrate how time-fractional Caputo derivative can be implemented in a diffusion model with respect to the non-locality, i.e. the causality principle. Moreover, the models with the Caputo derivative are more familiar and the solutions developed can be easily understood. This models, could be applied (not in the scope of this work) to composite magnetic media where small magnetic particles (of nano or macro sizes) are dispersed (almost homogeneously) is non-magnetic matrix; the gaps between the magnetic kernels are zones with high resistances with respect to the magnetic field lines, such as gaps and obstacles in porous media where fractional modelling is widely applied.

However, let us consider the case where all terms of (31) take place. In the context of the magnetic field diffusion, this precisely means that after the initial relaxation and disappearance of the send term, there is continuous magnetic energy supply through the boundary $x = 0$; the simple example is the Dirichlet problem. In such a case the complete model is

$$\begin{aligned} j_B(x,t) = -D_{B0}\frac{\partial B}{\partial x}(x,t) - \\ -D_{B1}\int_{-\infty}^{t} R(t-\tau)\frac{\partial B}{\partial x}(x,\tau)\,d\tau \end{aligned} \tag{73}$$

If the memory function is chosen to be singular power-law then the second term becomes the Riemann-Liouville fractional integral (34) and the flux-gradient relationship has be presented by an extended version of (35), namely

$$j_B = -D_{B0}\frac{\partial B}{\partial x} - D_{B1} I_t^{\alpha}\left(\frac{\partial B}{\partial x}\right) \tag{74}$$

After application of the continuity equation (32) we get

$$\frac{\partial B}{\partial t} = D_{B0}\frac{\partial^2 B}{\partial x^2} + D_{B1} I_t^{\alpha}\left(\frac{\partial^2 B}{\partial x^2}\right) \tag{75}$$

Here the non-locality is presented by the last term. This construction shows the main idea how non-locality has to be implemented at the level of constitutive equation. We will discuss a magnetic diffusion equation with exponential kernel next.

5.1.3. Memory kernel effect on the fractional model

Now, let us follows the main line drawn in the preceding point of this section and consider that flux gradient relationship contains all elements of the fading memory functional but now the convolution integral has exponential memory kernel, namely

$$\begin{aligned} j_B = -D_{B0}\int_0^t \delta_D(z)\frac{\partial B(x,z)}{\partial x}dz \\ -D_{B1}\frac{1}{\tau}\int_0^t e^{-\frac{(t-z)}{\tau}}\frac{\partial B(x,z)}{\partial x}dz \end{aligned} \tag{76}$$

where the first term is the instantaneous one since the memory kernel is the Dirac delta δ_D, while the second term has exponential memory as in the classical Cattaneo concept. This flux-gradient construction was investigated in [23] and resulted in a diffusion equation with a non-local damping term expressed through the Caputo-Fabrizio time-fractional derivative (78)

$$\frac{\partial B}{\partial t} = D_{B0}\frac{\partial^2 B}{\partial x^2} + D_{B1}(1-\alpha)\,{}^{CF}D_t^{\alpha}\left[\frac{\partial^2 B}{\partial x^2}\right] \tag{77}$$

solved in semi-infinite [25, 26] and finite domains [27].

In (77), the operator ${}^{CF}D_t^{\alpha}$ is the Caputo-Fabrizio time fractional derivative of order α [24]

$$\begin{aligned} {}^{CF}D_t^{\alpha}B(x,t) = \\ \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{dB(x,s)}{dt}ds \end{aligned} \tag{78}$$

and the relaxation time τ in (76) is related to the fractional order α as $\tau(0,\infty) = (1-\alpha)/\alpha$, $0 < \alpha < 1$ (see extended analysis in [28]) . Moreover, the concept expressed by (76) is valid even in the case when the material exhibits spatial memory and leads to a spatial Caputo-Fabrizio derivative with exponential kernel [28,29]. It is obvious, that the non-locality is not lost despite the use of exponential kernel since the last term in (77) is responsible for this.

Similarly, any other relaxation functions invoked by the type of the relaxations in the observed physical problems, may form kernels of non-local terms, but this problem is more general and beyond the scope of this work (some examples are available in [30]).

6. Conclusion

This study addressed the magnetic field diffusion model solved in various situations by tools of fractional derivatives. The main results can be outlined as:

- The semi-derivative approach to the parabolic model (8) with Dirichlet boundary condition, and especially with time ramping (power-law) boundary condition allows a direct relation between the function and the gradient, and easy integration of the boundary condition. Moreover, the approximate integral-balance solution needs only a single integration step.
- The integral-balance method by the technology of double integration (DIM) allows straightforward approximate solutions of magnetic diffusion with field-dependent diffusivity (with negligible Joules effects, i.e unchanged material resistivity). The solutions, sharp and almost rectangular waves, are moving with finite speeds (due the degenerate nature of the model).
- The magnetic field diffusion with memory was demonstrated on the basis of a singular memory kernel (power-law allowing the Caputo time-fractional derivative to be applied. This fractionalization behaviour is analogue of the CTRW concept and allows easy the approximate integral-balance solutions to be applied.
- The fading memory approach and the causality principle were used to formulate a general approach to implement nonlocality in constitutive equation, and consequently to conservation laws; in the present case to the magnetic diffusion model.
- The problems and solutions presented demonstrate a variety of approaches where the fractional calculus can be applied efficiently for solving diffusion models, and particularly to the problems related to the magnetic field (straight lines) diffusion in ferromagnetic materials. This is only a step towards solutions of more complex problems and we see the use of the fractional calculus is promising.

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RESEARCH ARTICLE

On self-similar solutions of time and space fractional sub-diffusion equations

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ABSTRACT

In this paper, we have considered two different sub-diffusion equations involving Hilfer, hyper-Bessel and Erdélyi-Kober fractional derivatives. Using a special transformation, we equivalently reduce the considered boundary value problems for fractional partial differential equation to the corresponding problems for ordinary differential equation. An essential role is played by certain properties of Erdélyi-Kober integral and differential operators. We have applied also successive iteration method to obtain self-similar solutions in an explicit form. The obtained self-similar solutions are represented by generalized Wright type function. We have to note that the usage of imposed conditions is important to present self-similar solutions via given data.

1. Introduction

Fractional calculus became one of the intensively developing theories in modern mathematics due to its wide range of applications in real life processes and also its generalized nature [1]. In particular, fractional derivative operators allow the description of memory and hereditary properties and are useful for modeling dynamic. Recently, several fractional operators have been developed to analyze the systems and models such as Caputo-Fabrizio, Hilfer, hyper-Bessel, Erdélyi-Kober fractional derivatives and many others. For instant, in recent papers [2,3], fractional differential equations are used for modeling applications in blood alcohol and fish farm models and in [4] fractional partial differential equation is used for Frankl-Type Problem.

Fractional order partial differential equations (FPDE) is one of the key objects in mathematical modeling of many diffusion-wave processes [5]. Different kind of direct and inverse problems for such equations were studied using different approaches, such as, integral transformations (Laplace, Fourier, Mellin), Green function method, method of separation of variables and etc. For PDEs, in general, one can determine special type of solutions, which are invariant under some subgroup of the full symmetry group of system. These "group-invariant" solutions are found by solving a reduced system of equations having fewer independent variables than the original system [6]. Such solutions named as self-similar solutions which play an important role in understanding of fundamental processes in mathematics and mechanics, we refer readers to [7] for application in problems of imploding shock waves and to [8] for filtration-slow groundwater motion in porous media.

The self-similarity of the solutions of partial differential equations has allowed their reduction to

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ordinary differential equations, which often simplifies the investigation. They have also served as standards in evaluating approximate methods for solving more complicated problems [8]. Moreover, they often describe the intermediate asymptotics behavior of solutions of wider classes of problems, for more details see [8].

The idea of self-similarity of solutions and Lie group analysis have been extended to fractional differential equations. For instant, in [6] and [9], the Lie group analysis of the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=d\frac{\partial^\beta u}{\partial x^\beta},\ x>0,\, t>0,\, d>0,\ \alpha,\, \beta\geq 0$$

has been discussed by Buckwar, Luchko and Gorenflo. Namely, the scale-invariant solutions were found by solving an ordinary differential equation of fractional order with a new independent variable $\eta=xt^{-\frac{\alpha}{\beta}}$. The general solution for this equation is obtained in terms of the generalized Wright function.

Furthermore, the existence and uniqueness of the space-fractional PDE with Caputo fractional derivative

$$\frac{\partial u}{\partial t}=\frac{\partial^\alpha u}{\partial x^\alpha},\ 1<\alpha\leq 2$$

was discussed, under the self-similar form

$$u(x,t)=t^\beta f\left(\frac{x}{t^{1/\alpha}}\right),\ (x,t)\in[0,X]\times[t_0,\infty),$$

where X, $t_0>0$, $\beta\in\mathbb{R}$ [10].

In [6], an admitted group dilations is found for the linear wave-diffusion equation of fractional order and these transformations are used for the construction of self-similar solutions. In [11], the methods of Lie continuous groups for symmetry analysis of FDEs were adapted and prolongation formula for fractional derivatives was proposed. Then, in [12], this formula is used for finding the exact solutions for nonlinear sub-diffusion equations with the Riemann-Liouville and Caputo fractional derivatives.

In [13], the similarity solution of the fractional diffusion equation

$$\frac{\partial^\gamma p(r,t)}{\partial t^\gamma}=\frac{1}{r^{d_s-1}}\frac{\partial}{\partial r}\left(r^{d_s-1}\frac{\partial p}{\partial 2r}\right),\ \ r>,\, t>0,$$

($\gamma=\frac{2}{d_w}$, $d_s=\frac{2d_f}{d_w}$ is the spectral dimension of the fractal) was considered and through the invariants of the group of scaling transformations, authors derived the integro-ordinary differential equation for the similarity variable.

In [14], fractional nonlinear space-time wave-diffusion equation was considered and solved by the similarity method using fractional derivatives in the Caputo, Riesz-Feller, and Riesz senses. Some particular cases are presented and the corresponding solutions are shown by means of 2-D and 3-D plots.

The following time-fractional cylindrical KdV equation with Riemann-Liouville fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{u}{2t^\alpha}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0,\ \alpha\in(0,1)$$

was reduced to the nonlinear fractional ordinary differential equation with Erdélyi-Kober fractional differential operator, using similarity transformation $u(x,t)=t^{-\frac{2\alpha}{3}}f(z)$ along with the similarity variable $z=xt^{-\frac{\alpha}{3}}$ [15].

There are other approaches, were authors have found self-similar solution by reducing considered PDEs to the hypergeometric equations. For example, Hasanov and Ruzhansky have found self-similar solutions for degenerate PDEs of the second, third and fourth orders using special method (see for details [16]). Precisely, they considered the following fourth order degenerate PDE:

$$x^n u_t - t^k u_{xxxx}=0,\qquad n,k=const>0.$$

They are looking for a solution of this equation as

$$u(x,t)=P(t)\omega(\sigma),$$

where

$$P=\left(\frac{1}{k+1}t^{k+1}\right)^{-1},\ \ \sigma=-\frac{k+1}{(n+4)^4t^{k+1}}x^{n+4}.$$

Then they have got the equation with respect to ω:

$$\begin{aligned}&x^3\omega_{xxxx}+(3+c_1+c_2+c_3)x^2\omega_{xxx}\\&+(1+c_1+c_2+c_3+c_1c_2+c_1c_3+c_2c_3)x\omega_{xx}+\\&(c_1c_2c_3-x)\omega_x-a\omega=0,\end{aligned}$$

which has special solutions represented with hypergeometric functions ${}_pF_q$.

The main motivation of the present research is the consideration of combinations of special fractional derivatives such as hyper-Bessel, Erdélyi-Kober (due to singularity) and Hilfer (due to generalized character). The obtained self-similar solutions will allow specialists in applied mathematics, who may deal with such FPDEs to study in details, since an explicit form of solutions are available. Moreover,the offered approach can be developed to conduct further investigations for more general FPDE with aforementioned fractional derivatives and also will contribute in studying the symmetry group analysis of FPDEs with these derivatives. In the present paper, we consider two problems, namely, fractional differential equation involving time and space Hilfer derivatives

$$D_{0t}^{\alpha,\delta}u(t,x) = D_{0x}^{\beta,\delta}u(t,x), \quad 0 < \alpha \leq 1,\ 1 < \beta \leq 2,$$

and fractional differential equation involving hyper-Bessel operator in time and Erdélyi-Kober fractional derivative in space variable

$$\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha} u(t,x) = x^{-\beta\rho}\frac{\partial^{\beta}}{\partial x^{\beta}}u(t,x),$$

where $1 < \beta \leq 2$, $0 < \alpha \leq 1$.

The key result is the finding of self-similar solutions of the above given equations with the specific conditions. The main tool is the reduction of considered FPDEs to the integral equations using specific transformation.

In literature, we refer some works devoted to the considered fractional derivatives, for example, hyper-Bessel operator was used to generalize the standard process of relaxation [17] and to model fractional diffusion equations governing the law of the fractional Brownian motion [18]. Also, FPDEs with hyper-Bessel operator were considered in [19] for studying direct and inverse source problems and in [20] for non-local problem of mixed type equation. Furthermore, there are different works related to applications of Erdélyi-Kober and Hilfer fractional derivatives such as fractional diffusion with Erdélyi-Kober derivative [21] and higher order partial differential equations with Hilfer fractional derivatives [22], for more details see the reference therein.

The rest of the paper is organized as follows. In the next section, we recall preliminaries related to some fractional derivatives. The main results are given in Section 3. The conclusion of the work is given in the last section.

2. Preliminaries

In this section, we present some basic definitions on fractional operators and their properties that are used further in this article.

Definition 1 ([1]). *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$I_{at}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \qquad \alpha > 0.$$

Definition 2 ([23]). *The right-sided Hilfer fractional derivative of order α and type δ is defined as*

$$D_{0t}^{\alpha,\delta}f(t) = I_{0t}^{\delta(n-\alpha)}\frac{d^n}{dt^n}I_{0t}^{(1-\delta)(n-\alpha)}f(t), \qquad (1)$$

where $n-1 < \alpha \leq n$, $0 \leq \delta \leq 1$.

For $\delta = 0$, Hilfer fractional derivative is reduced to the Riemann-Liouville fractional derivative, i.e; $D_{0t}^{\alpha,\delta}f(t) = D_{0t}^{\alpha}f(t)$.
Now, we recall the following property [24]

$$\begin{aligned} &I_{a+}^{\sigma}D_{a+}^{\sigma}f(t) = I_{a+}^{\alpha}D_{a+}^{\alpha,\delta}f(t) \\ &= f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\sigma-k-1}}{\Gamma(\sigma-k)}D_{a^+}^{n-k-1}I_{a^+}^{n-\sigma}f(a), \end{aligned} \qquad (2)$$

where $\sigma = \alpha + \delta - \alpha\delta$.

Definition 3. *([25]) The left and right-sided Erdélyi-Kober fractional integrals of order α, respectively, are defined as follows:*

$$\begin{aligned} &I_{\beta}^{\gamma,\alpha}f(t) = \\ &\frac{\beta}{\Gamma(\alpha)}t^{-\beta(\gamma+\alpha)}\int_0^t (t^{\beta}-s^{\beta})^{\alpha-1}s^{\beta(\gamma+1)-1}f(s)\,ds, \end{aligned} \qquad (3)$$

$$\begin{aligned} &J_{\beta}^{\gamma,\alpha}f(t) = \\ &\frac{\beta}{\Gamma(\alpha)}t^{\beta\gamma}\int_t^{\infty}(s^{\beta}-t^{\beta})^{\alpha-1}s^{-\beta(\gamma+\alpha-1)-1}f(s)ds, \end{aligned} \qquad (4)$$

where $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$.

Definition 4. *([25]) The left and right-sided Erdélyi-Kober fractional derivatives of order α, respectively, are given by ($n-1 < \alpha < n$, $n \in \mathbb{N}$)*

$$D_{\beta}^{\gamma,\alpha}f(t) = \prod_{j=1}^{n}\left(\gamma + j + \frac{1}{\beta}t\frac{d}{dt}\right)I_{\beta}^{\gamma+\alpha,n-\alpha}f(t), \qquad (5)$$

and

$$P_{\beta}^{\gamma,\alpha}f(t) = \prod_{j=0}^{n-1}(\gamma + j - \frac{1}{\beta}t\frac{d}{dt})\,J_{\beta}^{\gamma+\alpha,n-\alpha}f(t), \qquad (6)$$

where $\gamma \in \mathbb{R}$, $\beta > 0$.

The following property of Erdélyi-Kober fractional operators [25]

$$I_\beta^{\gamma,\alpha} x^{\lambda\beta} f(x) = x^{\lambda\beta} I_\beta^{\gamma+\lambda,\alpha} f(x), \tag{7}$$

$$I_\beta^{\gamma,\alpha} D_\beta^{\gamma,\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} c_k x^{-\beta(1+\gamma+k)}, \tag{8}$$

are true, where

$$\begin{aligned} c_k &= \frac{\Gamma(n-k)}{\Gamma(\alpha-k)} \lim_{x\to 0} x^{\beta(1+\gamma+k)} \times \\ &\prod_{i=k+1}^{n-1} (1+\gamma+i+\frac{1}{\beta} x \frac{d}{dx}) I_\beta^{\gamma+\alpha, n-\alpha} f(x). \end{aligned}$$

Furthermore, the Erdélyi-Kober fractional operators of power function are needed in the computations [26]:

$$P_\beta^{\tau,\alpha} t^p = \frac{\Gamma(\alpha+\tau-p/\beta)}{\Gamma(\tau-p/\beta)} t^p,\ \tau - p/\beta \tag{9}$$

$$J_\beta^{\tau,\alpha} t^p = \frac{\Gamma(\tau-p/\beta)}{\Gamma(\alpha+\tau-p/\beta)} t^p,\ \tau - p/\beta > 0 \tag{10}$$

$$I_\beta^{\gamma,\alpha} t^p = \frac{\Gamma(\gamma+1+p/\beta)}{\Gamma(\alpha+\gamma+1+p/\beta)} t^p,\ \gamma+1+p/\beta > 0. \tag{11}$$

Definition 5. *([27]) The hyper-Bessel operator of order order $0 < \alpha < 1$, is defined as*

$$\left(t^\theta \frac{d}{dt}\right)^\alpha f(t) = \begin{cases} (1-\theta)^\alpha t^{-(1-\theta)\alpha} I_{1-\theta}^{0,-\alpha} f(t), \text{ if } \theta < 1, \\ (\theta-1)^\alpha I_{1-\theta}^{-1,-\alpha} t^{(1-\theta)\alpha} f(t), \text{ if } \theta > 1. \end{cases} \tag{12}$$

Note that $I_\beta^{0,-\alpha} := D_\beta^{-\alpha,\alpha}$ and when $\theta = 0$, this operator coincides with the Riemann-Liouville fractional derivative.
Also, we need to recall the generalized Wright function:

Definition 6. *([9, 28]) The generalized Wright function is defined by the series expansion:*

$$W_{(\mu,a),(\nu,b)} := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)},$$

where
$\nu, \mu \in \mathbb{R}$, $a, b \in \mathbb{C}$.

3. Main Result

3.1. Fractional differential equation involving Hilfer derivative

Consider a time and space-fractional PDE

$$D_{0t}^{\alpha,\delta} u(t,x) = D_{0x}^{\beta,\delta} u(t,x),\ 0 < \alpha \le 1,\ 1 < \beta \le 2, \tag{13}$$

with the following conditions:

$$\begin{aligned} &\frac{\partial}{\partial x} I_{0x}^{2-m} u(t,0^+) = a\, t^{\gamma+\alpha(1-m)/\beta}, \\ &I_{0x}^{2-m} u(t,0^+) = b\, t^{\gamma+\alpha(2-m)/\beta}, \end{aligned} \tag{14}$$

where a, b are constants and $m = \beta + \delta - \beta\delta$.
We start by using similarity method to FPDE (13) to determine a symmetry group of scaling transformations. We introduce new independent and dependent variables

$$\bar{t} = \lambda^b t, \quad \overline{x} = \lambda x, \quad \overline{u} = \lambda^c u.$$

The time fractional derivative becomes ($\sigma_1 = \alpha + \delta - \alpha\delta$, $\delta_1 = \delta(1-\alpha)$)

$$\begin{aligned} &D_t^{\alpha,\delta} \overline{u}(\bar{t},\overline{x}) = I_t^{\delta_1} D_t^{\sigma_1} \overline{u}(\bar{t},\overline{x}) \\ &= I_t^{\delta_1} \left(\frac{1}{\Gamma(1-\sigma_1)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\sigma_1} \right) \overline{u}(\lambda^b s, \overline{x}) ds \\ &= I_t^{\delta_1} \left(\frac{\lambda^{c+b}}{\Gamma(1-\sigma_1)} \frac{\partial}{\partial \bar{t}} \int_0^{\bar{t}/\lambda^b} (\lambda^{-b}\bar{t} - s)^{-\sigma_1} \right. \\ &\qquad \left. \times \overline{u}(\lambda^b s, \overline{x})\, ds \right) \\ &= I_t^{\delta_1} \left(\frac{\lambda^{c+b\sigma_1}}{\Gamma(1-\sigma_1)} \frac{\partial}{\partial \bar{t}} \int_0^{\bar{t}} (\bar{t}-\tau)^{-\sigma_1} \right) \overline{u}(\tau,\overline{x})\, d\tau \\ &= \frac{\lambda^{c+b\sigma_1}}{\Gamma(\delta_1)} \int_0^t (t-s)^{\delta_1 - 1} D_{\bar{t}}^{\sigma_1} \overline{u}(\lambda^b s, \overline{x}) ds \\ &= \frac{\lambda^{c+b\sigma_1}}{\Gamma(\delta_1)} \int_0^{\bar{t}/\lambda^b} (\bar{t}\lambda^{-b} - s)^{\delta_1 - 1} D_{\bar{t}}^{\sigma_1} \overline{u}(\lambda^b s, \overline{x}) ds \\ &= \frac{\lambda^{c+b\sigma_1 - b - b(\delta_1 - 1)}}{\Gamma(\delta_1)} \int_0^{\bar{t}} (\bar{t}-\tau)^{\delta_1 - 1} D_{\bar{t}}^{\sigma_1} \overline{u}(\tau,\overline{x}) d\tau \\ &= \lambda^{c+b\alpha} I_{\bar{t}}^{\delta_1} D_{\bar{t}}^{\sigma_1} \overline{u}(\bar{t},\overline{x}) \\ &= \lambda^{c+b\alpha} D_{\bar{t}}^{\alpha,\delta} \overline{u}(\bar{t},\overline{x}). \end{aligned}$$

One can do the same for the space-fractional derivative, we have

$$D_x^{\beta,\delta} \overline{u}(\bar{t},\overline{x}) = \lambda^{c+\beta} D_{\overline{x}}^{\beta,\delta} \overline{u}(\bar{t},\overline{x}).$$

From the above we get

$$\begin{aligned} &D_t^{\alpha,\delta} \overline{u}(\bar{t},\overline{x}) - D_x^{\beta,\delta} \overline{u}(\bar{t},\overline{x}) \\ &= \lambda^{c+b\alpha} D_{\bar{t}}^{\alpha,\delta} \overline{u}(\bar{t},\overline{x}) - \lambda^{c+\beta} D_{\overline{x}}^{\beta,\delta} \overline{u}(\bar{t},\overline{x}) = 0, \end{aligned}$$

if $b = \dfrac{\beta}{\alpha}$. Thus, we choose the following invariant of scaling transformation

$$u(t,x) = t^{\gamma} U(\eta), \quad \eta = xt^{-\alpha/\beta}, \ \gamma > 0.$$

Now, using the above transformation, we have the following result:

Theorem 1. *The transformation*

$$u(t,x) = t^{\gamma} U(\eta), \ \eta = xt^{-\alpha/\beta} \tag{15}$$

reduces FPDE (13) to the following ODE

$$J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha} P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha} U(\eta) = D^{\beta,\delta}_{0\eta} U(\eta), \tag{16}$$

with

$$D I^{2-m}_{0\eta} U(0^+) = a \text{ and } I^{2-m}_{0\eta} U(0^+) = b, \tag{17}$$

where $\sigma_1 = \alpha + \delta - \alpha\delta$, $\delta_2 = 1 - \sigma_1$ *and* $\delta_1 = \delta(1-\alpha)$.

Proof. We begin by calculating the time-fractional derivative in terms of $U(\eta)$ using transformation (15). Using the definition of Hilfer fractional derivative (1) for $n = 1$, we have

$$D^{\alpha,\delta}_{0t} u(t,x) = I^{\delta_1} \frac{\partial}{\partial t} I^{\delta_2} t^{\gamma} U(xt^{-\alpha/\beta}). \tag{18}$$

Now, using the substitution $\tau = t\left(\frac{\eta}{s}\right)^{\beta/\alpha}$, the second integral of (18) can be reduced as follows:

$$\begin{aligned}
& I^{\delta_2} t^{\gamma} U(xt^{-\alpha/\beta}) \\
&= \frac{1}{\Gamma(\delta_2)} \int_0^t (t-\tau)^{\delta_2 - 1} \tau^{\gamma} U(x\tau^{-\alpha/\beta}) d\tau \\
&= \frac{\beta\, t^{\delta_2+\gamma} \eta^{\beta/\alpha(\gamma+1)}}{\alpha\Gamma(\delta_2)} \int_{\eta}^{\infty} (s^{\beta/\alpha} - \eta^{\beta/\alpha})^{\delta_2 - 1} \times \\
&\quad s^{-\beta/\alpha(\gamma+\delta_2)-1} U(s) ds \\
&= t^{\delta_2+\gamma} J^{\gamma+1,\delta_2}_{\beta/\alpha} U(\eta).
\end{aligned}$$

Then, taking the derivative of the above integral, we arrive to the following

$$\begin{aligned}
& \frac{d}{dt} I^{\delta_2} t^{\gamma} U(xt^{-\alpha/\beta}) \\
&= t^{\delta_2+\gamma-1} \left(\gamma + \delta_2 - \frac{\alpha}{\beta}\eta \frac{d}{d\eta}\right) J^{\gamma+1,\delta_2}_{\beta/\alpha} U(\eta) \\
&= t^{\delta_2+\gamma-1} P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha} U(\eta).
\end{aligned}$$

Using the above result and proceeding the same as above using substitution $\tau = t\left(\frac{\eta}{s}\right)^{\beta/\alpha}$ and relation $z = x\tau^{-\alpha/\beta}$, the expression in (18) becomes

$$\begin{aligned}
& I^{\delta_1} \frac{d}{dt} I^{\delta_2} t^{\gamma} U(xt^{-\alpha/\beta}) \\
&= \frac{1}{\Gamma(\delta_1)} \int_0^t (t-\tau)^{\delta_1 - 1} \tau^{\delta_2+\gamma-1} P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha} U(z) d\tau \\
&= \frac{\beta\, t^{\gamma+\delta_2+\delta_1-1}\, \eta^{\beta/\alpha(\gamma+\delta_2)}}{\alpha\Gamma(\delta_1)} \int_{\eta}^{\infty} (s^{\beta/\alpha} - \eta^{\beta/\alpha})^{\delta_1 - 1} \\
&\times s^{-\beta/\alpha(\gamma+\delta_2+\delta_1-1)-1} P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha} U(s) ds.
\end{aligned}$$

The power $\gamma + \delta_2 + \delta_1 - 1 = \gamma - \alpha$ and hence the time fractional derivative can be written as

$$D^{\alpha,\delta}_{0t} u(t,x) = t^{\gamma-\alpha} J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha} P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha} U(\eta).$$

Next, we compute the space-fractional derivative in terms of $U(\eta)$

$$D^{\beta,\delta}_{0x} u(t,x) = t^{\gamma} I^{\delta_4} \frac{\partial^2}{\partial x^2} I^{\delta_3} U(xt^{-\alpha/\beta}), \tag{19}$$

where $\delta_3 = (2-\beta)(1-\delta)$ and $\delta_4 = \delta(2-\beta)$. We use the substitution $\xi = st^{-\alpha/\beta}$, then the inner integral of (19) can be written as

$$\begin{aligned}
& I^{\delta_3} U(xt^{-\alpha/\beta}) = \frac{1}{\Gamma(\delta_3)} \int_0^x (x-s)^{\delta_3 - 1} U(st^{-\alpha/\beta}) ds \\
&= \frac{t^{\alpha\delta_3/\beta}}{\Gamma(\delta_3)} \int_0^{\eta} (\eta - \xi)^{\delta_3 - 1} U(\xi) d\xi = t^{\alpha\delta_3/\beta} I^{\delta_3} U(\eta).
\end{aligned}$$

Computing second derivative of the above gives

$$\frac{\partial^2}{\partial x^2} I^{(2-\beta)(1-\delta)} U(xt^{-\alpha/\beta}) = t^{\alpha(\delta_3-2)/\beta} \frac{d^2}{d\eta^2} I^{\delta_3} U(\eta),$$

since
$\frac{dU}{dx} = \frac{dU}{d\eta}\frac{d\eta}{dx}$.
Now, we do the same for the first integral of (19) with $\xi = st^{-\alpha/\beta}$ and $z = st^{-\alpha/\beta}$, we obtain

$$\begin{aligned}
& I^{\delta_4} \frac{\partial^2}{\partial x^2} I^{\delta_3} U(xt^{-\alpha/\beta}) \\
&= \frac{t^{\alpha(\delta_3-2)/\beta}}{\Gamma(\delta_4)} \int_0^x (x-s)^{\delta_4 - 1} \frac{d^2}{dz^2} I^{\delta_3} U(z) ds \\
&= \frac{t^{\alpha(\delta_3+\delta_4-2)/\beta}}{\Gamma(\delta_4)} \int_0^{\eta} (\eta - \xi)^{\delta_4 - 1} \frac{d^2}{d\xi^2} I^{\delta_3} U(\xi) d\xi \\
&= t^{-\alpha} I^{\delta_4} \frac{d^2}{d\eta^2} I^{\delta_3} U(\eta).
\end{aligned}$$

Thus, space-fractional derivative can be written as

$$D^{\beta,\delta}_{0x} u(t,x) = t^{\gamma-\alpha} D^{\beta,\delta}_{0\eta} U(\eta).$$

Substituting the time and space-fractional derivatives after transformation, we get the desired ordinary differential equation (16).
The solution of the fractional ordinary differential equation (16) is given in the next theorem.

Theorem 2. *The solution of FPDE (13) using transformation (15) with conditions (17) has the following form*

$$u(t,x) = t^{\gamma}\left[a\eta^{m-1}\Gamma(\gamma+1-\alpha(m-1)/\beta)\times W_{(\beta,m),(-\alpha,\gamma+1-\alpha(m-1)/\beta)}(\eta^{\beta}) + b\eta^{m-2}\Gamma(\gamma+1-\alpha(m-2)/\beta))\times W_{(\beta,m-1),(-\alpha,\gamma+1-\alpha(m-2)/\beta)}(\eta^{\beta})\right], \tag{20}$$

where $m=\beta+\delta-\delta\beta$, $\eta = xt^{-\alpha/\beta}$,

$$W_{(\beta,m),(-\alpha,\gamma+1-\alpha(m-2)/\beta)}(\eta^{\beta}) = \sum_{k=0}^{\infty}\frac{\eta^{\beta k}}{\Gamma(m+k\beta)\Gamma(\gamma+1-k\alpha-(m-1)\alpha/\beta)}$$

and

$$W_{(\beta,m-1),(-\alpha,\gamma+1-\alpha(m-2)/\beta)}(\eta^{\beta}) = \sum_{k=0}^{\infty}\frac{\eta^{\beta k}}{\Gamma(m-1+k\beta)\Gamma(\gamma+1-k\alpha-(m-2)\alpha/\beta)}.$$

Proof. Applying Riemann-Liouville fractional integral I^{β} to both sides of differential equation (16) and using property (2), we have

$$U(\eta) = \frac{a\eta^{m-1}}{\Gamma(m)} + \frac{b\eta^{m-2}}{\Gamma(m-1)} + I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}U(\eta).$$

Then, the solution can obtained using successive iterations method. We set

$$U^0(\eta) = \frac{a\eta^{m-1}}{\Gamma(m)} + \frac{b\eta^{m-2}}{\Gamma(m-1)},$$

so the nth term U^n can be written as

$$U^n(\eta) = U^0(\eta) + I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}U^{n-1}(\eta).$$

Now, we compute U^1 as follows:

$$U^1(\eta) = U^0(\eta) + I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}U^0(\eta).$$

Using properties (9)and (10), we calculate the following

$$\begin{aligned}
&I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}U^0(\eta) = \\
&I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}\left[\frac{a\eta^{m-1}}{\Gamma(m)}\frac{\Gamma(\gamma+\delta_2+\sigma_1-(m-1)\alpha/\beta)}{\Gamma(\gamma+\delta_2-(m-1)\alpha/\beta)}\right.\\
&\left.+\frac{b\eta^{m-2}}{\Gamma(m-1)}\frac{\Gamma(\gamma+\delta_2+\sigma_1-(m-2)\alpha/\beta)}{\Gamma(\gamma+\delta_2-(m-2)\alpha/\beta)}\right]\\
&= I^{\beta}\left[\frac{a\eta^{m-1}}{\Gamma(m)}\frac{\Gamma(\gamma+\delta_2+\sigma_1-(m-1)\alpha/\beta)}{\Gamma(\gamma+\delta_2+\delta_1-(m-1)\alpha/\beta)}\right.\\
&\left.+\frac{b\eta^{m-2}}{\Gamma(m-1)}\frac{\Gamma(\gamma+\delta_2+\sigma_1-(m-2)\alpha/\beta)}{\Gamma(\gamma+\delta_2+\delta_1-(m-2)\alpha/\beta)}\right]\\
&= \left[\frac{a\eta^{m+\beta-1}}{\Gamma(m+\beta)}\frac{\Gamma(\gamma+1-(m-1)\alpha/\beta)}{\Gamma(\gamma+1-\alpha-(m-1)\alpha/\beta)}+\right.\\
&\left.\frac{b\eta^{m+\beta-2}}{\Gamma(m+\beta-1)}\frac{\Gamma(\gamma+1-(m-2)\alpha/\beta)}{\Gamma(\gamma+1-\alpha-(m-2)\alpha/\beta)}\right].
\end{aligned}$$

Hence, $U^1(\eta)$ is given by

$$\begin{aligned}
U^1(\eta) &= a\left(\frac{1}{\Gamma(m)}\eta^{m-1} + \eta^{m+\beta-1}\frac{\Gamma(\gamma+1-(m-1)\alpha/\beta)}{\Gamma(m+\beta)\Gamma(\gamma+1-\alpha-(m-1)\alpha/\beta)}\right)\\
&+b\left(\frac{1}{\Gamma(m-1)}\eta^{m-2} + \eta^{m+\beta-2}\frac{\Gamma(\gamma+1-(m-2)\alpha/\beta)}{\Gamma(m+\beta-1)\Gamma(\gamma+1-\alpha-(m-2)\alpha/\beta)}\right).
\end{aligned}$$

Similarly, we compute $U^2(\eta)$

$$\begin{aligned}
U^2(\eta) &= U^0(\eta) + I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}U^1(\eta)\\
&= U^0(\eta) + I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}\times\\
&\left[U^0(\eta) + \frac{a\,\Gamma(\gamma+1-(m-1)\alpha/\beta)\eta^{m+\beta-1}}{\Gamma(m+\beta)\Gamma(\gamma+1-\alpha-(m-1)\alpha/\beta)}\right.\\
&\left.+\frac{b\,\Gamma(\gamma+1-(m-2)\alpha/\beta)\eta^{m+\beta-2}}{\Gamma(m+\beta-1)\Gamma(\gamma+1-\alpha-(m-2)\alpha/\beta)}\right].
\end{aligned}$$

One can check that

$$I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}\eta^{m+\beta-1} = \eta^{m+2\beta-1}\times\frac{\Gamma(m+\beta)\Gamma(\gamma+1-\alpha-(m-1)\alpha/\beta)}{\Gamma(m+2\beta)\Gamma((\gamma+1-2\alpha-(m-1)\alpha/\beta)}$$

and

$$I^{\beta}J^{\gamma+\delta_2,\delta_1}_{\beta/\alpha}P^{\gamma+\delta_2,\sigma_1}_{\beta/\alpha}\eta^{m+\beta-2} = \eta^{m+2\beta-2}\times\frac{\Gamma(m+\beta-1)\Gamma(\gamma+1-\alpha-(m-2)\alpha/\beta)}{\Gamma(m+2\beta-1)\Gamma(\gamma+1-2\alpha-(m-2)\alpha/\beta)}.$$

Thus,

$$U^2(\eta) = a\left(\frac{1}{\Gamma(m)}\eta^{m-1} + \eta^{m+\beta-1}\times \frac{\Gamma(\gamma+1-(m-1)\alpha/\beta)}{\Gamma(m+\beta)\Gamma(\gamma+1-\alpha-(m-1)\alpha/\beta)} + \frac{\Gamma(\gamma+1-(m-1)\alpha/\beta)\eta^{m+2\beta-1}}{\Gamma(m+2\beta)\Gamma(\gamma+1-2\alpha-(m-1)\alpha/\beta)}\right) + b\left(\frac{1}{\Gamma(m-1)}\eta^{m-2} + \eta^{m+\beta-2}\times \frac{\Gamma(\gamma+1-(m-2)\alpha/\beta)\eta^{m+\beta-2}}{\Gamma(m+\beta-1)\Gamma(\gamma+1-\alpha-(m-2)\alpha/\beta)} + \frac{\Gamma(\gamma+1-(m-2)\alpha/\beta)\eta^{m+2\beta-2}}{\Gamma(m+2\beta-1)\Gamma(\gamma+1-2\alpha-(m-2)\alpha/\beta)}\right).$$

We similarly compute $U^3(\eta)$ and get

$$U^3(\eta) = a\left(\frac{1}{\Gamma(m)}\eta^{m-1} + \frac{\Gamma(\gamma+1-(m-1)\alpha/\beta)\,\eta^{m+\beta-1}}{\Gamma(m+\beta)\Gamma((\gamma+1-\alpha-(m-1)\alpha/\beta)} + \frac{\Gamma(\gamma+1-(m-1)\alpha/\beta)\,\eta^{m+2\beta-1}}{\Gamma(m+2\beta)\Gamma(\gamma+1-2\alpha-(m-1)\alpha/\beta)} + \frac{\Gamma(\gamma+1-(m-1)\alpha/\beta)\eta^{m+3\beta-1}}{\Gamma(m+3\beta)\Gamma(\gamma+1-3\alpha-(m-1)\alpha/\beta)}\right) + b\left(\frac{1}{\Gamma(m-1)}\eta^{m-2} + \frac{\Gamma(\gamma+1-(m-2)\alpha/\beta)\,\eta^{m+\beta-2}}{\Gamma(m+\beta-1)\Gamma(\gamma+1-\alpha-(m-2)\alpha/\beta)} + \frac{\Gamma(\gamma+1-(m-2)\alpha/\beta)\,\eta^{m+2\beta-2}}{\Gamma(m+2\beta-1)\Gamma(\gamma+1-2\alpha-(m-2)\alpha/\beta)} + \frac{\Gamma(\gamma+1-(m-2)\alpha/\beta)\,\eta^{m+3\beta-2}}{\Gamma(m+3\beta-1)\Gamma(\gamma+1-3\alpha-(m-2)\alpha/\beta)}\right).$$

Now, we can write the nth term as follows:

$$U^n(\eta) = a\eta^{m-1}\Gamma(\gamma+1-(m-1)\alpha/\beta)\times \sum_{k=0}^{n}\frac{\eta^{\beta k}}{\Gamma(m+k\beta)\Gamma(\gamma+1-k\alpha-(m-1)\alpha/\beta))} + b\eta^{m-2}\Gamma(\gamma+1-(m-2)\alpha/\beta)\times \sum_{k=0}^{n}\frac{\eta^{\beta k}}{\Gamma(m+k\beta-1)\Gamma(\gamma+1-k\alpha-(m-2)\alpha/\beta))}.$$

As n goes to infinity, then

$$\begin{aligned} U(\eta) &= a\,\eta^{m-1}\Gamma(\gamma+1-(m-1)\alpha/\beta) \\ &\times W_{(\beta,m),(-\alpha,\gamma+1-(m-1)\alpha/\beta)}(\eta^{\beta}) \\ &+ b\,\eta^{m-2}\Gamma(\gamma+1-(m-2)\alpha/\beta) \\ &\times W_{(\beta,m-1),(-\alpha,\gamma+1-(m-2)\alpha/\beta)}(\eta^{\beta}). \end{aligned}$$

Substituting $U(\eta)$ in the transformation (15), we get the desired solution (20).

Remark 1. *For $\delta = 0$, Hilfer fractional derivative is reduced to Riemann-Liouville fractional derivative and this case was considered by Luchko and Gorenflo in [9]. The ordinary differential equation becomes*

$$J^{\gamma,1-\alpha}_{\beta/\alpha} P^{\gamma,\alpha}_{\beta/\alpha} U(\eta) = D^{\beta}_{x} U(\eta)$$

and one can check that the solution has the following form

$$\begin{aligned} U(\eta) &= a\,\eta^{\beta-1}\Gamma(\gamma+1-\alpha+\alpha/\beta) \\ &\times W_{(\beta,\beta),(-\alpha,\gamma+1-\alpha+\alpha/\beta)}(\eta^{\beta}) \\ &+ b\eta^{\beta-2}\Gamma(\gamma+1-\alpha+2\alpha/\beta) \\ &\times W_{(\beta,\beta-1),(-\alpha,\gamma+1-\alpha+2\alpha/\beta)}(\eta^{\beta}), \end{aligned}$$

which coincides with their result.

For $\beta = 2$ and $0 < \alpha \le 2$, this case was studied by Buckwar and Luchko, for more details see [6, 28].

Remark 2. *One may consider the same problem with $n-1 < \alpha \le n$ and $n-1 < \beta \le n$:*

$$D^{\alpha,\delta}_{0t} u(t,x) = D^{\beta,\delta}_{0x} u(t,x)$$

with conditions

$$D^{n-k-1} I^{n-m}_{0x} u(t,0^+) = c_n, \qquad m = \beta + \delta - \beta\delta,$$

and then use the same transformation in (15) to find the exact solution. Proceeding the same, the solution has the following form:

$$\begin{aligned} u(t,x) = t^{\gamma}\sum_{i=0}^{n} c_i\,\eta^{m-i-1}\times \\ \Gamma(\gamma+1-(m-i-1)\alpha/\beta) \\ \times W_{(\beta,m-i),(-\alpha,\gamma+1-(m-i-1)\alpha/\beta)}(\eta^{\beta}). \end{aligned}$$

3.2. Fractional differential equation involving hyper-Bessel operator

Consider the problem

$$\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha} u(t,x) = x^{-\beta\rho}\frac{\partial^{\beta}}{\partial x^{\beta}} u(t,x), \tag{21}$$

with the boundary conditions:

$$\lim_{x\to 0} x^{\rho(\beta-1)}(1-\beta+\frac{1}{\rho}x\frac{d}{dx}) I^{0,2-\beta}_{\rho} u(t,x) = U_0 t^{\gamma-\alpha\rho(\beta-1)/\beta} \tag{22}$$

$$\lim_{x\to 0} x^{\rho(\beta-2)}(2-\beta+\frac{1}{\rho}x\frac{d}{dx}) I^{0,2-\beta}_{\rho} u(t,x) = U_1 t^{\gamma-\alpha\rho(\beta-2)/\beta} \tag{23}$$

where $1 < \beta \leq 2$, $0 < \alpha \leq 1$, $\rho = 1-\theta$, $\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}$ stands for hyper-Bessel operator defined by (12) and $\frac{\partial^{\beta}}{\partial x^{\beta}} = D_{\rho}^{-\beta,\beta}$ represents the left-sided Erdélyi-Kober fractional derivative.

First, we use similarity method for FPDE 21 to determine a symmetry group of scaling transformations. We introduce new independent and dependent variables as before

$$\overline{t} = \lambda^{b} t, \quad \overline{x} = \lambda x, \quad \overline{u} = \lambda^{c} u.$$

The time fractional derivative becomes

$$\begin{aligned}
&\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}\overline{u}(\overline{t},\overline{x}) \\
&= \rho^{\alpha} t^{-\rho\alpha} I^{0,-\alpha}\rho\overline{u}(\overline{t},\overline{x}) \\
&= \rho^{\alpha} t^{-\rho\alpha} D_{\rho,t}^{-\alpha,\alpha}\overline{u}(\overline{t},\overline{x}) \\
&= \rho^{\alpha} t^{-\rho\alpha}\left(1-\alpha+\frac{1}{\rho}t\frac{d}{dt}\right) I_{\rho,t}^{0,1-\alpha}\overline{u}(\overline{t},\overline{x}) \\
&= \rho^{\alpha} t^{-\rho\alpha}\left(1-\alpha+\frac{1}{\rho}t\frac{d}{dt}\right)\frac{\rho\, t^{-\rho(1-\alpha)}}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} \\
&\times s^{\rho-1}\overline{u}(s\lambda^{b},\overline{x})ds \\
&= \rho^{\alpha}\lambda^{b\rho\alpha}\overline{t}^{-\rho\alpha}\left(1-\alpha+\frac{1}{\rho}\overline{t}\frac{d}{d\overline{t}}\right)\frac{\rho\,\lambda^{b\rho(1-\alpha)}\,\overline{t}^{-\rho(1-\alpha)}}{\Gamma(1-\alpha)} \\
&\times \int_0^{\overline{t}/\lambda^{b}} (\overline{t}\lambda^{-b}-s)^{-\alpha} s^{\rho-1}\overline{u}(s\lambda^{b},\overline{x})ds \\
&= \rho^{\alpha}\lambda^{b\rho\alpha}\overline{t}^{-\rho\alpha}\left(1-\alpha+\frac{1}{\rho}\overline{t}\frac{d}{d\overline{t}}\right)\frac{\rho\,\lambda^{b\rho\alpha}\, t^{-\rho(1-\alpha)}}{\Gamma(1-\alpha)} \\
&\times \int_0^{\overline{t}} (\overline{t}-\tau)^{-\alpha}\tau^{\rho-1}\overline{u}(\tau,\overline{x})d\tau \\
&= \rho^{\alpha}\lambda^{b\rho\alpha}\overline{t}^{-\rho\alpha}\left(1-\alpha+\frac{1}{\rho}\overline{t}\frac{d}{d\overline{t}}\right) I_{\rho,\overline{t}}^{0,1-\alpha}\overline{u}(\overline{t},\overline{x}) \\
&= \lambda^{b\rho\alpha}\left(\overline{t}^{\theta}\frac{\partial}{\partial\overline{t}}\right)^{\alpha}\overline{u}(\overline{t},\overline{x}).
\end{aligned}$$

Similarly, we do for the space-fractional derivative and deduce

$$\begin{aligned}
&D_{\rho,x}^{-\beta,\beta}\overline{u}(\overline{t},\overline{x}) = \\
&\prod_{k=0}^{1}\left(1-\beta+k+\frac{1}{\rho}x\frac{d}{dx}\right) I_{\rho,x}^{0,2-\beta}\overline{u}(\overline{t},\overline{x}) \\
&= \prod_{k=0}^{1}\left(1-\beta+k+\frac{1}{\rho}\overline{x}\frac{d}{d\overline{x}}\right)\frac{\rho\,\lambda^{\rho(2-\beta)}\,\overline{x}^{-\rho(2-\beta)}}{\Gamma(2-\beta)} \\
&\times \int_0^{\overline{x}/\lambda} (\overline{x}/\lambda-s)^{1-\beta}s^{\rho-1}\overline{u}(\overline{t},s\lambda)ds \\
&= \prod_{k=0}^{1}\left(1-\beta+k+\frac{1}{\rho}\overline{x}\frac{d}{d\overline{x}}\right)\frac{\rho\,\overline{x}^{-\rho(2-\beta)}}{\Gamma(2-\beta)} \\
&\times \int_0^{\overline{x}} (\overline{x}-z)^{1-\beta}z^{\rho-1}\overline{u}(\overline{t},z)dz \\
&= \prod_{k=0}^{1}\left(1-\beta+k+\frac{1}{\rho}\overline{x}\frac{d}{d\overline{x}}\right) I_{\rho,\overline{x}}^{0,2-\beta}\overline{u}(\overline{t},\overline{x}) \\
&= D_{\rho,\overline{x}}^{-\beta,\beta}\overline{u}(\overline{t},\overline{x}).
\end{aligned}$$

From the above we get

$$\begin{aligned}
&\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}\overline{u}(\overline{t},\overline{x}) - x^{\rho\beta}D_{\rho,x}^{-\beta,\beta}\overline{u}(\overline{t},\overline{x}) \\
&= \lambda^{b\rho\alpha}\left(\overline{t}^{\theta}\frac{\partial}{\partial\overline{t}}\right)^{\alpha}\overline{u}(\overline{t},\overline{x}) - \lambda^{\rho\beta}\overline{x}^{\rho\beta}D_{\rho,\overline{x}}^{-\beta,\beta}\overline{u}(\overline{t},\overline{x}) \\
&= 0,
\end{aligned}$$

if $b = \frac{\beta}{\alpha}$. Thus, we choose the following invariant of scaling transformation

$$u(t,x) = t^{\gamma}U(\eta), \quad \eta = xt^{-\alpha/\beta}.$$

The result related to equation (21) is given in the following theorem;

Theorem 3. *The transformation given by (15) reduces the FPDE (21) to the following ODE*

$$\rho^{\alpha}\eta^{\beta\rho}P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha}U(\eta) = D_{\rho}^{-\beta,\beta}U(\eta), \tag{24}$$

with

$$\lim_{\eta\to 0}\eta^{\rho(\beta-1)}(1-\beta+\frac{1}{\rho}\eta\frac{d}{d\eta})I_{\rho}^{0,2-\beta}U(\eta) = U_0 \tag{25}$$

and

$$\lim_{\eta\to 0}\eta^{\rho(\beta-2)}(2-\beta+\frac{1}{\rho}\eta\frac{d}{d\eta})I_{\rho}^{0,2-\beta}U(\eta) = U_1. \tag{26}$$

Proof. We start by rewriting the time-hyper-Bessel operator using definition (12):

$$\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}u(t,x) \quad = \rho^{\alpha}t^{-\rho\alpha}I_{\rho}^{0,-\alpha}\,t^{\gamma}U(\eta).$$

Then, make change of variable $\tau = t\left(\frac{\eta}{s}\right)^{\beta/\alpha}$ and simplify as follows

$$
\begin{aligned}
&\rho^{\alpha} t^{-\rho\alpha} D_{\rho}^{-\alpha,\alpha}\, t^{\gamma} U(\eta) \\
&= \rho^{\alpha} t^{-\rho\alpha} \left(1-\alpha+\frac{1}{\rho} t \frac{d}{dt}\right) I_{\rho}^{0,1-\alpha} t^{\gamma} U(x t^{-\alpha/\beta}) \\
&= \rho^{\alpha} t^{-\rho\alpha} \left(1-\alpha+\frac{1}{\rho} t \frac{d}{dt}\right) \frac{\rho t^{-\rho(1-\alpha)}}{\Gamma(1-\alpha)} \\
&\times \int_0^t (t^{\rho}-\tau^{\rho})^{-\alpha} \tau^{\rho-1+\gamma} U(x\tau^{-\alpha/\beta}) d\tau \\
&= \rho^{\alpha} t^{-\rho\alpha} \left(1-\alpha+\frac{1}{\rho} t \frac{d}{dt}\right) \frac{\rho\beta t^{\gamma} \eta^{\rho\beta/\alpha(1+\gamma/\rho)}}{\alpha\Gamma(1-\alpha)} \\
&\times \int_{\eta}^{\infty} (s^{\rho\beta/\alpha}-\eta^{\rho\beta/\alpha})^{-\alpha} s^{-\rho\beta(1-\alpha+\gamma/\rho)-1} U(s) ds \\
&= \rho^{\alpha} t^{\gamma-\rho\alpha} \left(1-\alpha+\frac{1}{\rho} t \frac{d}{dt}\right) J_{\rho\beta/\alpha}^{1+\gamma/\rho,1-\alpha} U(\eta).
\end{aligned}
$$

Thus, we arrive

$$
\begin{aligned}
&\rho^{\alpha} t^{-\rho\alpha} D_{\rho}^{-\alpha,\alpha}\, t^{\gamma} U(\eta) \\
&= \rho^{\alpha} t^{\gamma-\rho\alpha} \left(1-\alpha+\frac{\gamma}{\rho}-\frac{\alpha}{\rho\beta}\eta\frac{d}{d\eta}\right) J_{\rho\beta/\alpha}^{1+\gamma/\rho,1-\alpha} U(\eta) \\
&= \rho^{\alpha} t^{\gamma-\rho\alpha} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\alpha,\alpha} U(\eta),
\end{aligned}
$$

where $J_{\rho\beta/\alpha}^{1+\gamma/\rho,1-\alpha}$ and $P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\alpha,\alpha}$ are right-sided Erdélyi-Kober fractional operators. Similarly, we transform the space fractional derivative

$$
\begin{aligned}
&D_{\rho}^{-\beta,\beta} u(t,x) \\
&= \prod_{k=0}^{1} \left(1-\beta+k+\frac{1}{\rho} x \frac{d}{dx}\right) I_{\rho}^{0,2-\beta} t^{\gamma} U(x t^{-\alpha/\beta}),
\end{aligned}
$$

and substitute $s = x\left(\dfrac{z}{\eta}\right)$ in the above in the integral as follows:

$$
\begin{aligned}
&I_{\rho}^{0,2-\beta} t^{\gamma} U(x t^{-\alpha/\beta}) \\
&= \frac{\rho t^{\gamma}\, x^{-\rho(2-\beta)}}{\Gamma(2-\beta)} \int_0^x (x^{\rho}-s^{\rho})^{1-\beta} s^{\rho-1} U(s t^{-\alpha/\beta}) ds \\
&= \frac{\rho t^{\gamma} \eta^{-\rho(2-\beta)}}{\Gamma(2-\beta)} \int_0^{\eta} (\eta^{\rho}-z^{\rho})^{1-\beta} z^{\rho-1} U(z) dz \\
&= t^{\gamma} I_{\rho}^{0,2-\beta} U(\eta).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&D_{\rho}^{-\beta,\beta} u(t,x) \\
&= t^{\gamma} \prod_{k=0}^{1} \left(1-\beta+k+\frac{1}{\rho}\eta\frac{d}{d\eta}\right) I_{\rho}^{0,2-\beta} U(\eta) \\
&= t^{\gamma} D_{\rho}^{-\beta,\beta} U(\eta).
\end{aligned}
$$

Finally, substituting the transformed time and space fractional derivatives in differential equation (21), we get

$$
\rho^{\alpha} t^{\gamma-\rho\alpha} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\alpha,\alpha} U(\eta) = x^{-\rho\beta} t^{\gamma} D_{\rho}^{-\beta,\beta} U(\eta)
$$

which can be written as ordinary differential equation (24).

In the next theorem, we give the self-similar solution (invariant solution) of equation (21):

Theorem 4. *The solution of FPDE (21) using transformation (15) with conditions (25)-(26) has the following form*

$$
\begin{aligned}
u(t,x) &= t^{\gamma} \Big[\eta^{\rho(\beta-1)} \Gamma(\tfrac{\gamma}{\rho}+1+\tfrac{\alpha}{\beta}-\alpha) \\
&\times W_{(\beta,\beta),(-\alpha,\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha)}(\rho^{\alpha}\eta^{\beta}) \\
&+\eta^{\rho(\beta-2)} \Gamma(\tfrac{\gamma}{\rho}+1+\tfrac{2\alpha}{\beta}-\alpha)) \\
&\times W_{(\beta-1,\beta),(-\alpha,\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha)}(\rho^{\alpha}\eta^{\beta})\Big].
\end{aligned}
\tag{27}
$$

Proof. Applying Erdélyi-Kober fractional integral to both sides of equation (24) and using the property (8), we have

$$
\begin{aligned}
U(\eta) &= c_0\, \eta^{\rho(\beta-1)} + c_1\, \eta^{\rho(\beta-2)} \\
&+\rho^{\alpha} I_{\rho}^{-\beta,\beta} \eta^{\beta\rho} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} U(\eta),
\end{aligned}
$$

where $c_0 = \dfrac{U_0}{\Gamma(\beta)}$ and $c_1 = \dfrac{U_1}{\Gamma(\beta-1)}$.

Also, using property (7), we get

$$
\begin{aligned}
U(\eta) &= c_0\, \eta^{\rho(\beta-1)} + c_1\, \eta^{\rho(\beta-2)} \\
&+\rho^{\alpha} \eta^{\beta\rho} I_{\rho}^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} U(\eta).
\end{aligned}
$$

Note that $I_{\rho}^{-\beta,\beta} \eta^{\beta\rho} = \eta^{\beta\rho} I_{\rho}^{0,\beta}$, see [25].

To find the solution of the above equation, we use successive iteration method. We start with

$$
U^0 = c_0 \eta^{\rho(\beta-1)} + c_1 \eta^{\rho(\beta-2)}
$$

and

$$
U^n(\eta) = U^0 + \rho^{\alpha} \eta^{\beta\rho} I_{\rho}^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} U^{n-1}(\eta).
$$

The first iteration is

$$
\begin{aligned}
U^1(\eta) &= U^0 + \rho^{\alpha} \eta^{\beta\rho} I_{\rho}^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} U^0(\eta) \\
&= U^0 + \rho^{\alpha} \eta^{\beta\rho} I_{\rho}^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} \\
&\quad \left(c_0 \eta^{\rho(\beta-1)} + c_1 \eta^{\rho(\beta-2)}\right).
\end{aligned}
$$

Compute the second term of $U^1(\eta)$ using property (11), we have

$$I_\rho^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} \eta^{\rho(\beta-1)} = I_\rho^{0,\beta} \eta^{\rho(\beta-1)} \frac{\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha\right)}{\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-2\alpha\right)} = \eta^{\rho(\beta-1)} \frac{\Gamma(\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha\right)}{\Gamma(2\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-2\alpha\right)},$$

and similarly we do for the last term

$$I_\rho^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} \eta^{\rho(\beta-2)} = \eta^{\rho(\beta-2)} \frac{\Gamma(\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha\right)}{\Gamma(2\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-2\alpha\right)}.$$

Substituting back the above results in $U^1(\eta)$, we get

$$\begin{aligned} U^1(\eta) &= c_0 \left[\eta^{\rho(\beta-1)} + \rho^\alpha \eta^{\rho(2\beta-1)} \times \frac{\Gamma(\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha\right)}{\Gamma(2\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-2\alpha\right)}\right] \\ &+ c_1 \left[\eta^{\rho(\beta-2)} + \rho^\alpha \eta^{\rho(2\beta-2)} \times \frac{\Gamma(\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha\right)}{\Gamma(2\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-2\alpha\right)}\right]. \end{aligned}$$

Repeating the same procedure, one can compute $U^2(\eta)$:

$$U^2(\eta) = U^0 + \rho^\alpha \eta^{\beta\rho} I_\rho^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} U^1(\eta).$$

So,

$$I_\rho^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} \eta^{\rho(2\beta-1)} = I_\rho^{0,\beta} \eta^{\rho(2\beta-1)} \frac{\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-2\alpha\right)}{\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-3\alpha\right)} = \eta^{\rho(2\beta-1)} \frac{\Gamma(2\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-2\alpha\right)}{\Gamma(3\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-3\alpha\right)}$$

and

$$I_\rho^{0,\beta} P_{\rho\beta/\alpha}^{1-\alpha+\gamma/\rho,\alpha} \eta^{\rho(2\beta-2)} = \eta^{\rho(2\beta-2)} \frac{\Gamma(2\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-2\alpha\right)}{\Gamma(3\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-3\alpha\right)}.$$

Thus, $U^2(\eta)$ can be written as follows

$$\begin{aligned} U^2(\eta) &= c_0 \Bigg[\eta^{\rho(\beta-1)} + \rho^\alpha \eta^{\rho(2\beta-1)} \frac{\Gamma(\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha\right)}{\Gamma(2\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-2\alpha\right)} \\ &+ \rho^{2\alpha} \eta^{\rho(3\beta-1)} \frac{\Gamma(\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha\right)}{\Gamma(3\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-3\alpha\right)}\Bigg] \\ &+ c_1 \Bigg[\eta^{\rho(\beta-2)} \\ &+ \rho^\alpha \eta^{\rho(2\beta-2)} \frac{\Gamma(\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha\right)}{\Gamma(2\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-2\alpha\right)} \\ &+ \rho^{2\alpha} \eta^{\rho(3\beta-2)} \frac{\Gamma(\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha\right)}{\Gamma(3\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-3\alpha\right)}\Bigg]. \end{aligned}$$

Now, the nth iteration can be written as

$$\begin{aligned} U^n(\eta) &= U_0 \Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha\right) \sum_{k=0}^{n} \frac{\rho^{k\alpha}\eta^{\rho(k\beta-1)}}{\Gamma(k+1)\beta)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-(k+1)\alpha\right)} \\ &+ U_1 \Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha\right) \sum_{k=0}^{n} \frac{\rho^{k\alpha}\eta^{\rho(k\beta-2)}}{\Gamma(k+1)\beta-1)\Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-(k+1)\alpha\right)} \end{aligned}$$

and as n approaches infinity, we have

$$\begin{aligned} U(\eta) &= U_0 \Gamma\left(\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha\right) \eta^{\rho(\beta-1)} W_{(\beta,\beta),(-\alpha,\frac{\gamma}{\rho}+1+\frac{\alpha}{\beta}-\alpha)}(\rho^\alpha \eta^\beta) \\ &+ U_1 \Gamma\left(\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha\right) \eta^{\rho(\beta-2)} W_{(\beta-1,\beta),(-\alpha,\frac{\gamma}{\rho}+1+\frac{2\alpha}{\beta}-\alpha)}(\rho^\alpha \eta^\beta). \end{aligned}$$

Substituting $U(\eta)$ in the transformation (15), we obtain the desired solution (27).

Remark 3. *Particular case when $\rho = 1$, the fractional derivatives in (21) are reduced to Riemann-Liouville fractional derivatives and the problem was considered by Luchko and Gorenflo as mentioned above, for more details see the reference therein.*

4. Conclusion

To summarize, Fractional Partial Differential Equations (FPDEs) involving hyper-Bessel, Erdélyi-Kober and Hilfer fractional derivatives were main targets in this investigation. Using special transformation (see (15)) we first reduced the considered FPDEs to the fractional ODEs (see Theorem 1 and 3) and then we solved these ODEs using successive iterative method (see Theorem 2 and 4). The obtained self-similar solutions are expressed in terms of generalized Wright type function.

Our motivation is based on possible usage of sub-diffusion equations with such special fractional operators by specialists in applied mathematics who may deal with such sub-diffusion equations. Moreover, we believe that suggested approach can be applied for investigation of more general FPDEs and also in studying the symmetry group analysis of these of derivatives.

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RESEARCH ARTICLE

A fractional model in exploring the role of fear in mass mortality of pelicans in the Salton Sea

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ABSTRACT

The fear response is an important anti-predator adaptation that can significantly reduce prey's reproduction by inducing many physiological and psychological changes in the prey. Recent studies in behavioral sciences reveal this fact. Other than terrestrial vertebrates, aquatic vertebrates also exhibit fear responses. Many mathematical studies have been done on the mass mortality of pelican birds in the Salton Sea in Southern California and New Mexico in recent years. Still, no one has investigated the scenario incorporating the fear effect. This work investigates how the mass mortality of pelican birds (predator) gets influenced by the fear response in tilapia fish (prey). For novelty, we investigate a modified fractional-order eco-epidemiological model by incorporating fear response in the prey population in the Caputo-fractional derivative sense. The fundamental mathematical requisites like existence, uniqueness, non-negativity and boundedness of the system's solutions are analyzed. Local and global asymptotic stability of the system at all the possible steady states are investigated. Routh-Hurwitz criterion is used to analyze the local stability of the endemic equilibrium. Fractional Lyapunov functions are constructed to determine the global asymptotic stability of the disease-free and endemic equilibrium. Finally, numerical simulations are conducted with the help of some biologically plausible parameter values to compare the theoretical findings. The order α of the fractional derivative is determined using Matignon's theorem, above which the system loses its stability via a Hopf bifurcation. It is observed that an increase in the fear coefficient above a threshold value destabilizes the system. The mortality rate of the infected prey population has a stabilization effect on the system dynamics that helps in the coexistence of all the populations. Moreover, it can be concluded that the fractional-order may help to control the coexistence of all the populations.

1. Introduction

The conventional notion that predator affects the prey population only through direct killing has been changed to a great extent in recent past [1]. The population dynamics of the prey is more affected by indirect interaction with a predator as compared to direct killing [2]. In addition to killing, predators often elicit a fear response in the prey population which brings about many psychological and physiological changes in the prey [3]. The primary line of anti-predator behaviour is to avoid detection. Due to predation risk, prey may compromise with the source and choice of foraging, which ultimately affect personal or community growth and thereby affecting reproduction [2, 4]. Fear may affect the physiological condition of the juvenile prey, and this could leave a negative effect on their survival as adults [4, 5]. In the experiment conducted by Zanette et al. [6], it is observed that song sparrows had a reduced

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growth rate due to the perceived predation risk even after the absence of direct killing. So, predators frequently affect the prey, indirectly, enforcing a stressful life.

Similarly, fear-induced phenomena can also be found in different fishes in marine ecology. In rainbow trout species (*Oncorhynchus mykiss*), for example, reproduction timing may vary due to stress like disturbance or handling. However, Tilapia (*Oreochromis niloticus*), when subjected to a stressed environment, show different types of the psychology of reproduction depending on its maturity [7]. It is observed that the minimum threshold of stimulus that is required to elicit a behavioral response in prey is lower for fishes (prey individuals) who are previously exposed to higher levels of predation risk. Fish actually optimizes their feeding rate under the constraints of predation severity leading to mortality [8].

Physiological changes are observed in the body of the prey as a response to the actual presence or background knowledge of predators. These physiological changes are brought about by hormonal or neuronal changes that bring about different responses in organ systems and ultimately lead to altered reproductive capacity. Showing anti-predator behaviour costs a rebalance in energy allocation and subsequently could affect the reproduction process [9].

Salton sea is a very stressed environment for fish. From 1970 onwards, the total fish biomass of Salton sea has been crashed many times due to three physiological stressors viz. extreme temperature fluctuation, increasing salinity, and high sulphide levels and anoxia associated with mixing events [10]. This stress environment affects the vital life functions of fish, mainly population growth via reproduction [9]. Although the role of stressors directly from the environment has an active role in the life cycle of fish species at Salton sea, another stress, that is, fear of predation, can not be ruled out considering recent discussions in literature [2].

Avian botulism(*Clostridium botulinum* type C) is a regular outbreak causing sizeable mortality among the piscivorous birds of the Salton sea since the twentieth century [11]. In 1996, around 9000 white Pelicans and around 1200 brown Pelicans were killed due to this dreaded disease. However, the mortality number has dropped significantly, and white pelicans were affected less in mortality than brown pelicans [12]. Type C botulinum toxin formed inside the gastrointestinal tracts of Tilapia, infected by a variety of bacteria like vibrato, is considered to be the main cause of death among pelicans.

Mathematical modeling has a very important role in studying the interaction among the predator and prey species. After the pioneering work of Kermack and Mc Kendrick on SIRS type, epidemiological modeling has been studied widely in recent years by various researchers [13–17]. Mathematical modeling of the fear effect in prey species was first proposed by Wang et al. [18] in the year 2016. Subsequently, some fear-induced mathematical studies of predator-prey interaction have been carried out [19–21]. In their study, Hossain et al. investigated the effect of fear in a three-species intraguild predation model. Their analysis revealed that fear could stabilize the chaos produced due to omnivory predators [19]. Predators that follow cooperation strategies while hunting also creates fear upon the prey. Combining hunting cooperation (by predators) and fear effect (in prey), Pal et al. investigated a Lotka–Volterra type predator-prey model. Their study shows that an increase in the hunting cooperation induced fear may destabilize the system and produce periodic solutions via a Hopf-bifurcation [20]. Panday et al. in [21] studied the impact of fear in a tri-trophic food chain model. They observe that fear can stabilize the system from chaos to stable focus through the period halving phenomenon. Till now, we have not come across any literature where the role of fear has been analyzed in the case of the marine ecosystem. This has motivated us for the present investigation.

In recent years, researchers have shown more interest in using fractional-order differential equations (FDE) in mathematical modeling rather than integro-differential equations (IDE). FDEs can be used to model universal phenomena with greater precision [23, 24]. In [23], Heymans et al., through a series of examples, have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann–Liouville fractional derivatives. Unlike IDEs, FDEs are non-local operators where the succeeding state of any function depends not only on their existing state but also on all preceding states [25,26]. In addition, classical IDEs are incapable of providing data between two distinct integer values. To overcome such restrictions, various types of fractional-order operators were adopted in the available literature [27,28]. Moustafa et al. in [29] investigated an eco-epidemiological model with disease in the prey species in terms of Caputo fractional derivative. Khan et al. in [26] investigated a fractional SIR model with a generalized incidence rate using both Caputo and the recently developed Atangana-Baleanu-Caputo (ABC) derivative [28]. The ABC derivative having non-singular and non-local kernel contains a strong memory effect of the system. Recently, Singh et al. investigated a fractional guava fruit model involving ABC derivative for investigating the interaction between guava pests and natural enemies [30]. Tuan et al. studied the existence and uniqueness of a mild solution to an initial value problem for a fractional Rayleigh–Stokes equation driven by fractional Brownian motion [31]. Fuzzy ABC fractional derivative, fuzzy ABC fractional initial value problems, and fuzzy ABC solutions are discussed and utilized for the first time in [32]. Bonyah et al. studied a fractional optimal control model of coronavirus in ABC derivative sense [33].

Mathematical research work on large-scale mortality of pelicans in the Salton Sea was first carried out by Bairagi et al. [34] in 2001. Throughout their article, the authors presumed that pelicans only come into contact with infected tilapia. Using the same perspective, Greenhalgh et al. [36] in 2007 proposed a ratio-dependent predator-prey interaction model ignoring the predation of susceptible prey. In their research work, they adopted a purely logistic growth function of the susceptible prey. The predators do not have any alternative food resources, and they prey only on infected prey. So, their carrying capacity depends only on infected prey. In their research, Chattopadhyay et al. [35] modified the previous study by introducing an interaction between pelicans and susceptible fish. The authors presumed that the death rate of the pelicans is increased due to feeding on infected fish. Later in 2017, Greenhalgh et al. [37] modified their earlier studies in [36] by taking into account that predators feed on both the susceptible and infected preys(tilapia). Furthermore, they presumed that the diseased prey significantly influences the growth rate of susceptible prey and the carrying capacity of the predator is dependent on the total number of prey (tilapia). To make the discussions more realistic and novel, we have considered the fear effect in the prey (tilapia) due to predation (by pelicans), since Tilapia (*Oreochromis niloticus*) under stressful circumstances (like predation risk) react by boosting or completely hindering reproduction [7].

In this paper, we extend the mathematical model proposed by Greenhalgh in [37] by incorporating the fear effect in the susceptible prey (tilapia) in terms of Caputo fractional derivative. The integer-order derivative does not contain the complete memory, and it does not describe the physical behavior of the model. The memory effect in FDEs that provides data between two distinct integer values motivates us to study the model using a fractional derivative. Besides terrestrial ecosystems, the fear effect also influences marine ecosystems. Therefore we are interested in exploring the complex dynamics of the critical ecosystem of the Salton Sea (which became a dangerous habitat for birds during the 1990s) with fear effect. Additionally, the fear effect induced in the prey population due to predation risk makes the scenario a novel one and biologically more realistic and meaningful. The paper is organized as follows:

In section 2, we describe a modified predator-prey interaction model with fear effect involving Caputo fraction derivatives. In section 3, we provide some mathematical preliminaries used for analytical discussions of our model. Fundamental mathematical results like the existence, uniqueness, non-negativity and boundedness solutions of the modified model are carried out throughout section 4. In sections 5 and 6, the modified model's equilibrium points and their local stability are analyzed. In sections 7 and 8, the global stability of the disease-free equilibrium, positive equilibrium, and condition for Hopf bifurcation at the disease-free equilibrium is discussed. In section 9, numerical simulations are carried out using biologically feasible parameters. Finally, in section 10, a summary of the outcomes obtained from the current study is provided. The conclusions derived are purely on the basis of theoretical results. Experimental verification will suggest modification required in fundamental assumptions.

2. Model formulation

In this section, we discuss a modified form of a predator-prey interaction model initially forwarded by Greenhalgh et al. [37]. The model in [37] is based on the critical ecosystem of the Salton Sea located in Southern California, New Mexico, where pelicans and tilapia are the predator and prey, respectively. The disease is assumed to spread among the prey through close contact. The vibrio-infected prey is classified into susceptible and infected prey and are represented by $S(T)$ and $I(T)$ respectively. So, at any instant T, the total number of prey (tilapia) population is $N(T) = S(T) + I(T)$. Only susceptible prey (tilapia) take part in reproducing offsprings, and population growth is in logistic fashion with a carrying capacity of $k > 0$. The predators (pelicans) feed on both susceptible and infected prey, preferably the infected ones, since they are easily catchable. They assumed that the predators feed on the prey with a ratio-dependent functional response. With these basic assumptions, the model is,

$$\begin{aligned}\frac{dS}{dT} &= rS\left(1-\frac{S+I}{k}\right)-\lambda SI-\frac{pYS}{mY+S}\\ \frac{dI}{dT} &= \lambda SI-\frac{cYI}{mY+I}-\gamma I\\ \frac{dY}{dT} &= \delta Y\left(1-\frac{hY}{S+I}\right)\end{aligned} \tag{1}$$

with the initial condition $S(0) \geq 0, I(0) \geq 0, Y(0) \geq 0$ where,

r: rate of growth of the prey species in the reproducing population group,
k: total capacity of the system,
λ: the disease transmission coefficient,
p, c: catching rate of predators (pelicans) towards susceptible and infected prey (tilapia), respectively,
m: a strictly positive constant,
γ: the mortality rate of infected prey per capita,
δ: rate of growth of the predator (pelican) species per capita,
h: a constant which is related to the density-dependent death rate of the predator(pelican) population.

The modified fractional-order model of the system (1) is presented as follows

$$\begin{aligned}{}^{c}D_t^{\alpha}S(T) &= rS\left(1-\frac{S+I}{k}\right)-\lambda SI-\frac{pYS}{mY+S}\\ {}^{c}D_t^{\alpha}I(T) &= \lambda SI-\frac{cYI}{mY+I}-\gamma I\\ {}^{c}D_t^{\alpha}Y(T) &= \delta Y\left(1-\frac{hY}{S+I}\right)\end{aligned} \tag{2}$$

Since the induced fear in the prey (tilapia) due to predation risk reduces their reproduction rate, therefore we modify the first equation of system (2) by multiplying the breeding rate r with a factor $g(f, Y)$ as below,

$$\begin{aligned}{}^{c}D_t^{\alpha}S(T) &= rS\left(1-\frac{S+I}{k}\right)g(f,Y)-\lambda SI\\ &\quad -\frac{pYS}{mY+S}\end{aligned} \tag{3}$$

where Y describes the biomass of the predator (pelican) and f describes the strength of fear due to predation risk in the prey (tilapia). To make f, Y and $g(f, Y)$ biologically feasible it is appropriate to assume that [18]

$$\begin{aligned}&g(0,Y)=1, g(f,0)=1, \lim_{f\to\infty} g(f,Y)=0,\\ &\lim_{Y\to\infty} g(f,Y)=0, \frac{\partial g(f,Y)}{\partial f}<0, \frac{\partial g(f,Y)}{\partial Y}<0.\end{aligned} \tag{4}$$

Here we consider $g(f, Y) = \frac{1}{1+fY}$ which satisfies condition (4). Then the system (2) becomes:

$$\begin{aligned}{}^{c}D_t^{\alpha}S(T) &= rS\left(1-\frac{S+I}{k}\right)\left(\frac{1}{1+fY}\right)-\lambda SI\\ &\quad -\frac{pYS}{mY+S}\\ {}^{c}D_t^{\alpha}I(T) &= \lambda SI-\frac{cYI}{mY+I}-\gamma I\\ {}^{c}D_t^{\alpha}Y(T) &= \delta Y\left(1-\frac{hY}{S+I}\right)\end{aligned} \tag{5}$$

with the initial condition $S(0) \geq 0, I(0) \geq 0, Y(0) \geq 0$.
Now define $T = \lambda t, r_1 = \frac{r}{\lambda}, p_1 = \frac{p}{\lambda}, \gamma_1 = \frac{\gamma}{\lambda}, \delta_1 = \frac{\delta}{\lambda}, c_1 = \frac{c}{\lambda}$. Then the system (5) reduces to

$$\begin{aligned}{}^{c}D_t^{\alpha}S(t) &= r_1S\left(1-\frac{S+I}{k}\right)\left(\frac{1}{1+fY}\right)-SI\\ &\quad -\frac{p_1YS}{mY+S}\\ {}^{c}D_t^{\alpha}I(t) &= SI-\frac{c_1YI}{mY+I}-\gamma_1 I\\ {}^{c}D_t^{\alpha}Y(t) &= \delta_1 Y\left(1-\frac{hY}{S+I}\right)\end{aligned} \tag{6}$$

with the initial condition $S(0) \geq 0, I(0) \geq 0, Y(0) \geq 0$.

3. Mathematical preliminaries

Throughout this section, we present a few preliminary definitions as well as some important lemmas for Caputo fractional derivative [22, 24, 38, 39].

Definition 1. *[24]* Let g be any function such that $g \in \mathcal{C}^n([t_0, +\infty), \mathbb{R})$ then the Caputo fractional derivative of g having order α is defined by

$${}_{t_0}^{c}D_t^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds$$

where $\Gamma(.)$ is the Gamma function, n is a non-negative integer such that $n-1 < \alpha < n$ and $t \geq t_0$. In particular, when $0 < \alpha < 1$

$${}_{t_0}^{c}D_t^{\alpha}g(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{g'(s)}{(t-s)^{\alpha}}ds$$

Lemma 1. *[38] Let $g(t) \in \mathcal{C}[a,b]$ and ${}^{c}_{t_0}D^{\alpha}_{t}g(t) \in \mathcal{C}[a,b], 0 < \alpha \leq 1$, then*

(i) for each $t \in [a,b]$, $g(t)$ is non-decreasing provided ${}^{c}_{t_0}D^{\alpha}_{t}g(t) \geq 0$, $a < t < b$.
(ii) for each $t \in [a,b]$, $g(t)$ is non-increasing provided ${}^{c}_{t_0}D^{\alpha}_{t}g(t) \leq 0$, $a < t < b$.

Lemma 2. *[24] Consider the Cauchy problem*

$$
\begin{aligned}
{}^{c}_{a}D^{\alpha}_{t}\hat{x}(t) &= \lambda\hat{x}(t) + g(t) \\
\hat{x}(a) &= b \ (b \in \mathbb{R})
\end{aligned}
$$

with $0 < \alpha < 1$ and $\bar{\lambda} \in \mathbb{R}$. Then the solution is of the form

$$
\begin{aligned}
\hat{x}(t) &= bE_{\alpha}[\bar{\lambda}(t-a)^{\alpha}] \\
&+ \int_{a}^{t}(t-s)^{(\alpha-1)}E_{\alpha,\alpha}\left[\bar{\lambda}(t-s)^{\alpha}\right]g(s)ds
\end{aligned}
\tag{7}
$$

while the solution to the problem

$$
\begin{aligned}
{}^{c}_{a}D^{\alpha}_{t}\hat{x}(t) &= \bar{\lambda}\hat{x}(t) \\
\hat{x}(a) &= b \ (b \in \mathbb{R})
\end{aligned}
$$

is given by

$$
\hat{x}(t) = bE_{\alpha}\left[\bar{\lambda}(t-a)^{\alpha}\right]
$$

The preceding lemma is quite important to verify that the system (6) is uniformly bound which is the generalization of the Lemma 2 provided in [40].

Lemma 3. *[22] Consider a function $\bar{u}(t)$ continuous on $[t_0, +\infty)$ satisfying*

$$
\begin{aligned}
{}^{c}_{t_0}D^{\alpha}\bar{u}(t) &\leq -\bar{\lambda}\bar{u}(t) + \mu \\
\bar{u}(t_0) &= \bar{u}_{t_0}
\end{aligned}
\tag{8}
$$

where $0 < \alpha < 1, (\bar{\lambda}, \mu) \in \mathbb{R}^2$, and $\bar{\lambda} \neq 0$ and $t_0 \geq 0$ is the initial time. Then

$$
\bar{u}(t) \leq \left(\bar{u}_{t_0} - \frac{\mu}{\bar{\lambda}}\right)E_{\alpha}\left[-\bar{\lambda}(t-t_0)^{\alpha}\right] + \frac{\mu}{\bar{\lambda}} \tag{9}
$$

Lemma 4. *Consider a function $\bar{u}(t)$ continuous on $[t_0, +\infty)$ satisfying*

$$
\begin{aligned}
{}^{c}_{t_0}D^{\alpha}_{t}\bar{u}(t) &\geq \bar{\lambda}\bar{u}(t) - \mu \\
\bar{u}(t_0) &= \bar{u}_{t_0}
\end{aligned}
\tag{10}
$$

where $0 < \alpha < 1, (\bar{\lambda}, \mu) \in \mathbb{R}^2$, and $\bar{\lambda} \neq 0$ and $t_0 \geq 0$ is the initial time. Then

$$
\bar{u}(t) \geq \left(\bar{u}_{t_0} - \frac{\mu}{\bar{\lambda}}\right)E_{\alpha}\left[\bar{\lambda}(t-t_0)^{\alpha}\right] + \frac{\mu}{\bar{\lambda}} \tag{11}
$$

Proof. This lemma can be proved using the similar approach used in the proof of the lemma (3). □

Lemma 5. *[39] Consider $\hat{x}(t) \in \mathbb{R}_+$ be a continuous and derivable function. Then for any $t \geq t_0$*

$$
\begin{aligned}
&{}^{c}_{t_0}D^{\alpha}_{t}\left[\hat{x}(t) - \hat{x}^* - \hat{x}^*\ln\frac{\hat{x}(t)}{\hat{x}^*}\right] \\
&\leq \left(1 - \frac{\hat{x}^*}{\hat{x}(t)}\right){}^{c}_{t_0}D^{\alpha}_{t}\hat{x}(t), \\
&\hat{x}^* \in \mathbf{R}_+, \ \forall\, \alpha \in (0,1)
\end{aligned}
\tag{12}
$$

4. Mathematical analysis

In this section, we present the fundamental mathematical requisites like the existence, uniqueness, non-negativity, and boundedness of the solutions, as desired in any population dynamics.

4.1. Existence and Uniqueness of the system

We investigate the existence and uniqueness of the solutions of the fractional-order system (6) in the region $\mathcal{B} \times [t_0, T]$ where

$$
\begin{aligned}
\mathcal{B} = \big\{&(S,I,Y) \in \mathbb{R}^3 : \max\{|S|,|I|,|Y|\} \leq \Psi, \\
&\min\{|S|,|I|,|Y|\} \geq \Psi_0\big\} \ \text{ and } \ T < +\infty.
\end{aligned}
\tag{13}
$$

Theorem 1. For each $X_0 = (S_0, I_0, Y_0) \in \mathcal{B}$, there exists a unique solution $X(t) \in \mathcal{B}$ of the fractional-order system (6) with initial condition X_0, which is defined for all $t \geq 0$

Proof. We denote $X = (S, I, Y)$ and $\bar{X} = (\bar{S}, \bar{I}, \bar{Y})$.
Consider a mapping
$M(X) = (M_1(X), M_2(X), M_3(X))$ and

$$
\begin{aligned}
M_1(X) &= r_1 S\left(1 - \frac{S+I}{k}\right)\left(\frac{1}{1+fY}\right) - SI \\
&- \frac{p_1 Y S}{mY + S} \\
M_2(X) &= SI - \frac{c_1 Y I}{mY + I} - \gamma_1 I \\
M_3(X) &= \delta_1 Y\left(1 - \frac{hY}{S+I}\right)
\end{aligned}
\tag{14}
$$

For any $X, \bar{X} \in \mathcal{B}$ it follows from equation (14) that

$$
\begin{aligned}
&\|M(X) - M(\bar{X})\| \\
&= |M_1(X) - M_1(\bar{X})| \\
&+ |M_2(X) - M_2(\bar{X})| + |M_3(X) - M_3(\bar{X})|
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&|M_1(X) - M_1(\bar{X})| \\
&= \left| r_1 S \left(1 - \frac{S+I}{k}\right) - SI - \frac{p_1 Y S}{mY + S} - r_1 \bar{S} \left(1 - \frac{\bar{S}+\bar{I}}{k}\right) \right. \\
&\left. + \bar{S}\bar{I} + \frac{p_1 \bar{Y}\bar{S}}{m\bar{Y} + \bar{S}} \right| \\
&= \left| r_1(S - \bar{S}) - \frac{r_1}{k} S\left(S + I\right) - SI - \frac{p_1 Y S}{mY + S} - \frac{r_1}{k}\bar{S}\left(\bar{S} + \bar{I}\right) + \bar{S}\bar{I} + \frac{p_1 \bar{Y}\bar{S}}{m\bar{Y} + \bar{S}} \right| \\
&\le \left| r_1(S - \bar{S}) \right| + \frac{r_1}{k}\left| (S^2 - \bar{S}^2) \right| + \left(\frac{r_1}{k} + 1\right) \left| (SI - \bar{S}\bar{I}) \right| + \frac{p_1}{m}\left| (S - \bar{S}) \right| \\
&+ p_1 \left| (Y - \bar{Y}) \right| \\
&\le r_1|S - \bar{S}| + \frac{2r_1}{k}\Psi|S - \bar{S}| + \left(\frac{r_1}{k} + 1\right)\Psi|S - \bar{S}| + \left(\frac{r_1}{k} + 1\right)\Psi|I - \bar{I}| \\
&+ \frac{p_1}{m}|S - \bar{S}| + p_1|Y - \bar{Y}| \\
&|M_2(X) - M_2(\bar{X})| \\
&= \left| (SI - \bar{S}\bar{I}) - \gamma_1(I - \bar{I}) - \left(\frac{c_1 Y I}{mY + I}\right) + \left(\frac{c_1 \bar{Y}\bar{I}}{m\bar{Y} + \bar{I}}\right) \right| \\
&\le \Psi|S - \bar{S}| + \Psi|I - \bar{I}| + \gamma_1|I - \bar{I}| + \frac{c_1}{m}|I - \bar{I}| + c_1|Y - \bar{Y}|
\end{aligned}
$$

$$
\begin{aligned}
|M_3(X) - M_3(\bar{X})| &= \left| \delta_1 Y \left(1 - \frac{hY}{S+I}\right) - \delta_1 \bar{Y}\left(1 - \frac{h\bar{Y}}{\bar{S}+\bar{I}}\right) \right| \\
&\le \delta_1|Y - \bar{Y}| + \delta_1 h \left| \frac{Y^2}{S+I} - \frac{\bar{Y}^2}{\bar{S}+\bar{I}} \right| \\
&\le \delta_1|Y - \bar{Y}| + \delta_1 h \left| \frac{(Y^2 - \bar{Y}^2)(S+I) - Y^2(S - \bar{S}) - Y^2(I - \bar{I})}{(S+I)(\bar{S}+\bar{I})} \right| \\
&\le \delta_1|Y - \bar{Y}| + \delta_1 h \left| \frac{(Y^2 - \bar{Y}^2)}{(\bar{S}+\bar{I})} \right| + \delta_1 h \left| \frac{Y^2(S - \bar{S})}{(S+I)(\bar{S}+\bar{I})} \right| + \delta_1 h \left| \frac{Y^2(I - \bar{I})}{(S+I)(\bar{S}+\bar{I})} \right| \\
&\le \delta_1|Y - \bar{Y}| + \frac{\delta_1 h \Psi}{\Psi_0}|Y - \bar{Y}| + \frac{\delta_1 h}{4\Psi_0^2}|Y^2(S - \bar{S})| + \frac{\delta_1 h}{4\Psi_0^2}|Y^2(I - \bar{I})| \\
&\le \delta_1|Y - \bar{Y}| + \frac{\delta_1 h \Psi}{\Psi_0}|Y - \bar{Y}| + \frac{\delta_1 h \Psi^2}{4\Psi_0^2}|(S - \bar{S})| + \frac{\delta_1 h \Psi^2}{4\Psi_0^2}|Y^2(I - \bar{I})|
\end{aligned}
$$

Then equation (15) becomes,

$$
\begin{aligned}
\|M(X) - M(\bar{X})\| &\le r_1|S - \bar{S}| + \frac{2r_1}{k}\Psi|S - \bar{S}| + \left(\frac{r_1}{k} + 1\right)\Psi|S - \bar{S}| + \left(\frac{r_1}{k} + 1\right)\Psi|I - \bar{I}| \\
&+ \frac{p_1}{m}|S - \bar{S}| + p_1|Y - \bar{Y}| + \Psi|S - \bar{S}| + \Psi|I - \bar{I}| + \gamma_1|I - \bar{I}| \\
&+ \frac{c_1}{m}|I - \bar{I}| + c_1|Y - \bar{Y}| + \delta_1|Y - \bar{Y}| + 2\delta_1 h\Psi|Y - \bar{Y}| + \delta_1|Y - \bar{Y}| \\
&+ \frac{\delta_1 h \Psi}{\Psi_0}|Y - \bar{Y}| + \frac{\delta_1 h \Psi^2}{4\Psi_0^2}|(S - \bar{S})| + \frac{\delta_1 h \Psi^2}{4\Psi_0^2}|Y^2(I - \bar{I})| \\
&\le \left\{ r_1 + \left(\frac{3r_1}{k} + 2\right)\Psi + \frac{p_1}{m} + \frac{\delta_1 h \Psi^2}{4\Psi_0^2} \right\}|S - \bar{S}| + \left\{ \left(\frac{r_1}{k} + 2\right)\Psi + \gamma_1 + \frac{c_1}{m} \right. \\
&\left. + \frac{\delta_1 h \Psi^2}{\Psi_0^2} \right\}|I - \bar{I}| + \left\{ p_1 + c_1 + \delta_1 + \frac{\delta_1 h \Psi}{\Psi_0} \right\}|Y - \bar{Y}|
\end{aligned}
$$

$$
\|M(X) - M(\bar{X})\| \le L\|X - \bar{X}\|
$$

where

$$L = \max\Bigg\{ \left(r_1 + \left(\frac{3r_1}{k} + 2\right)\Psi + \frac{p_1}{m} + \frac{\delta_1 h \Psi^2}{4\Psi_0^2}\right), \left(\frac{r_1}{k} + 2\right)\Psi + \gamma_1 + \frac{c_1}{m} + \frac{\delta_1 h \Psi^2}{\Psi_0^2}, p_1 + c_1 + \delta_1 + \frac{\delta_1 h \Psi}{\Psi_0}\Bigg\}$$

Therefore $M(X)$ obeys Lipschitz condition which implies the existence and uniqueness of solution of the fractional-order system (6). □

4.2. Non-negativity and boundedness

Consider the set

$$\mathcal{B}_+ = \Big\{(S, I, Y) \in \mathcal{B} : S \in \mathbb{R}_+, I \in \mathbb{R}_+ \text{ and } Y \in \mathbb{R}_+\Big\}$$

where $\mathbb{R}_+$ is the set of all non-negative real numbers.

Theorem 2. *All the solutions of the fractional-order system* (6) *initiating in the region* $\mathcal{B}_+$ *are non-negative and bounded uniformly.*

Proof. For the proof we follow the approach used in [22].

First, we prove that the solutions $S(t)$ that initiate in $\mathcal{B}_+$ are non-negative i.e., $S(t) \geq 0$ for all $t \geq t_0$. Let us assume that is not true, then there exists $t > t_0$ such that

$$\begin{aligned} &S(t) > 0,\ t_0 \leq t < t_1 \\ &S(t_1) = 0, \\ &S(t_1^+) < 0, \quad t^+ = \{t : t \geq t_1\} \end{aligned} \tag{16}$$

Based on (16) and the first equation of system (6) we have

$$ {}_{t_0}^{c}D_{t_1}^{\alpha} S(t_1)|_{S(t_1)=0} = 0 \quad \text{where} \quad 0 < \alpha < 1 \tag{17}$$

Using *Lemma* (1), we get $S(t_1^+) = 0$, which is a contradiction as $S(t_1^+) < 0$. Hence $S(t) \geq 0$ for all $t \geq t_0$. In similar way we can get $I(t) \geq 0, Y(t) \geq 0$ for all $t \geq t_0$.

Now to prove the boundedness of all the solution of system initiated in the region $\mathcal{B}_+$, we define the function $V(t) = S(t) + I(t) + Y(t)$, then we have,

$$\begin{aligned}
&{}_{t_0}^{c}D_t^{\alpha} V(t) + \eta V(t) \\
&= r_1 S\left(1 - \frac{S+I}{k}\right)\left(\frac{1}{1+fY}\right) - SI - \frac{p_1 Y S}{mY+S} \\
&+ SI - \frac{c_1 Y I}{mY+I} - \gamma_1 I + \delta_1 Y\left(1 - \frac{hY}{S+I}\right) + \eta S \\
&+ \eta I + \eta Y \\
&\leq r_1 S - \frac{r_1}{k} S(S+I) - \frac{p_1 Y S}{mY+S} - \frac{c_1 Y I}{mY+I} - \gamma_1 I \\
&+ \delta_1 Y - \frac{\delta_1 h Y^2}{I+S} + \eta S + \eta I + \eta Y \\
&\leq (r_1+\eta)S - \frac{r_1}{k}S^2 + Y(\delta_1+\eta) - \frac{\delta_1 h}{2\Psi}Y^2 + (\eta - \gamma_1)I \\
&\leq (r_1+\eta)S - \frac{r_1}{k}S^2 + Y(\delta_1+\eta) - \frac{\delta_1 h}{2\Psi}Y^2 \\
&\leq -r_1\left(S - \frac{k(r_1+\eta)}{2r_1}\right)^2 + \frac{k(r_1+\eta)^2}{4r_1} - \frac{\delta_1 h}{2\Psi} \\
&\left(Y - \frac{\Psi(\delta_1+\eta)}{\delta_1 h}\right)^2 + \frac{\Psi(\delta_1+\eta)^2}{2\delta_1 h} \\
&\leq \frac{k(r_1+\eta)^2}{4r_1} + \frac{\Psi(\delta_1+\eta)^2}{2\delta_1 h}
\end{aligned}$$

where $\eta = \gamma_1$. By *Lemma* 3 we have,

$$\begin{aligned}
V(t) &\leq \left(V(t_0) - \frac{k(r_1+\eta)^2}{4r_1} - \frac{\Psi(\delta_1+\eta)^2}{2\delta_1 h}\right) \\
&\quad E_\alpha\Big[-\eta(t-t_0)^\alpha\Big] + \frac{k(r_1+\eta)^2}{4r_1} + \frac{\Psi(\delta_1+\eta)^2}{2\delta_1 h} \\
&\quad \to \frac{k(r_1+\eta)^2}{4r_1} + \frac{\Psi(\delta_1+\eta)^2}{2\delta_1 h}, t \to \infty
\end{aligned} \tag{18}$$

Hence, all the solution of the fractional-order system (6) which initiate in $\mathcal{B}_+$ are restricted to the region Γ, where

$$\Gamma = \left\{ \begin{array}{c} (S, I, Y) \in \bar{\Omega}_+ | S + I + Y \leq \frac{k(r_1+\eta)^2}{4r_1} \\ + \frac{\Psi(\delta_1+\eta)^2}{2\delta_1 h} + \epsilon, \epsilon > 0 \end{array} \right\} \tag{19}$$

this completes the proof of theorem. □

5. Equilibrium points

The fractional-order system (6) has the following biologically feasible equilibrium points.

(1) **The trivial or vanishing equilibrium** $E_1(0, 0, 0)$ which always exists. In an ecological sense, trivial equilibrium is important since all populations will never become extinct simultaneously.

(2) **The axial equilibrium** $E_2(k, 0, 0)$ where there is only susceptible prey, which always exists.

(3) **The disease-free equilibrium** $E_3(S_3, 0, Y_3)$ where $S_3 = \frac{hk(r_1(h+m)-p_1)}{fkp_1+hr_1(h+m)}$, $Y_3 = \frac{k(r_1(h+m)-p_1)}{fkp_1+hr_1(h+m)}$.
The disease-free equilibrium E_3 exists if and only if $r_1(h+m) > p_1$.

(4) **The predator-free equilibrium** $E_4(S_4, I_4, 0)$ where $S_4 = \gamma_1$, $I_4 = \frac{r_1(k-\gamma_1)}{k+r_1}$ which exists if and only if $\Re_0 > 1$ where $\Re_0 = \frac{k}{\gamma_1}$. $\Re_0$ is the basic reproduction number of the epidemic theory determined with help of *next generation matrix* method [41].

(5) **The positive or endemic equilibrium** $E^*(S^*, I^*, Y^*)$: From the equation of predator nullcline we obtain $Y = \frac{S+I}{h}$. Solving susceptible prey and infected prey nullcline equations and substituting Y^* gives,

$$I^* = \frac{c_1 S^* - m(S^*)^2 + \gamma_1 m S^*}{-c_1 - \gamma_1 h + hS^* - \gamma_1 m + mS^*},$$

$$Y^* = \frac{S^*(S^* - \gamma_1)}{(h+m)(S^*-\gamma_1) - c_1}.$$

Substituting I^* and Y^* in susceptible prey nullcline equation gives the following fifth degree polynomial equation,

$$\rho_0 {S^*}^5 + \rho_1 {S^*}^4 + \rho_2 {S^*}^3 + \rho_3 {S^*}^2 + \rho_4 S^* + \rho_5 = 0 \tag{20}$$

where ρ_0,ρ_1,ρ_2,ρ_3,ρ_4 and ρ_5 are given in the appendix.

I^* and Y^* are uniquely defined if S^* is a solution of the above equation (20). If $S^* \neq 0$ is a real positive solution of the polynomial equation (20), then I^* and Y^* are real and positive if,

$$\frac{c_1 + \gamma_1 h + \gamma_1 m}{h+m} < S^* < \frac{c_1 + \gamma_1 m}{m}.$$

For the parameters provided in section 9 with $f = 0.05, m = 5, h = 0.5, k = 350$ equation (20) has a unique non-zero positive real root $S^* = 41.59$ for which the corresponding $I^* = 50.57$ and $Y^* = 164.31$.

6. Local stability analysis

Throughout this section, we investigate the local stability of the equilibrium points of the fractional-order system (6). For local stability analysis of the positive equilibrium, we use the Routh-Hurwitz criterion.

6.1. Local stability of $E_1(0,0,0)$

The Jacobian matrix of the fractional-order system (6) is not well-defined at the equilibrium point $E_1(0,0,0)$. In order to show that E_1 is unstable, it is sufficient to prove that not all the trajectories initiated in the neighborhood of E_1 approach E_1. Suppose a trajectory which initiated with $Y(0) = 0$ and $S(t) > 0$, then we have $Y(t) = 0$ but $S(t) > 0 \; \forall \, t$. Hence

$$\frac{1}{S} {}^c_t D^\alpha_t S(t) = r_1\left(1 - \frac{S+I}{k}\right) - I > \frac{r_1}{2}$$

$$\implies {}^c_t D^\alpha_t S(t) > S\left(\frac{r_1}{2}\right)$$

Now, if S and I are small enough then by using Lemma 4,

$$S(t) > S_0 E_\alpha\left[\frac{r_1}{2}(t - t_0)\right]$$

for $t \geq t_0$. Therefore the trajectory cannot approaches to E_1. Hence E_1 is locally asymptotically unstable.

6.2. Local stability of $E_2(k,0,0)$

The Jacobian matrix of the fractional-order system (6) is not well defined at the axial equilibrium $E_2(k,0,0)$. To prove that E_2 is unstable we presume that E_2 is locally asymptotically stable (LAS). Now suppose a trajectory which is initiated with $Y_0 > 0$ and either $I_0 > 0$ or $S_0 > 0$. Hence either $I_t > 0$ or $S_t > 0 \; \forall \, t$. Therefore

$${}^c_t D^\alpha_t Y(t) \geq \frac{\delta_1 Y}{2}$$

Then by using Lemma 4

$$Y(t) \geq Y_0 E_\alpha\left[\frac{\delta_1}{2}(t - t_0)\right]$$

for $t \geq t_0$. Therefore the trajectory never approaches to E_2. Hence E_2 is locally asymptotically unstable.

6.3. Local stability of $E_3(S_3, 0, Y_3)$

Theorem 3. *If $\Re_0 < 1 + \frac{c_1+\chi}{\gamma_1 m}$ then the disease-free equilibrium E_3 of the fractional-order system* (6) *is locally asymptotically stable under the condition $2\delta_1 > \frac{p_1}{m}$ and $r_1 > \frac{p_1}{m}$.*

Proof. The Jacobian matrix of system (6) at E_3 is given by

$$J_{E_3} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ 0 & J_{22} & 0 \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$$

where $J_{11} = \frac{h\left((2h+fk+m)p_1 - (h+m)^2 r_1\right)}{(h+fk)(h+m)^2}$,
$J_{12} = -\frac{h((h+m)r_1 - p_1)(fkp_1 + (h+m)(k(h+fk)+hr_1))}{(h+fk)(h+m)(fkp_1+h(h+m)r_1)}$,

$J_{13} = -\frac{hp_1\left(\left(h^2+2fkh+fkm\right)r_1-fkp_1\right)}{(h+fk)(h+m)^2 r_1}$,
$J_{22} = \frac{h(h+m)r_1(m(k-\gamma_1)-c_1)-kp_1(fc_1+m(h+f\gamma_1))}{m(fkp_1+h(h+m)r_1)}$,
$J_{31} = \frac{\delta_1}{h}$, $J_{32} = \frac{\delta_1}{h}$, $J_{33} = -\delta_1$.
The eigenvalues of the Jacobian matrix of the system at the equilibrium point E_3 are the roots of the following equation

$$(J_{22} - \sigma_I)\left(\sigma^2 - A\sigma + B\right) = 0 \qquad (21)$$

where

$$A = \frac{hp_1(fk+2h+m) - (h+m)^2\left(\delta_1(fk+h)+hr_1\right)}{(h+m)^2(fk+h)},$$

$$B = \frac{\delta_1\left(r_1(h+m)-p_1\right)\left(fkp_1+hr_1(h+m)\right)}{r_1(h+m)^2(fk+h)}.$$

The characteristic equation have the following roots,

$$\sigma_I = \frac{hr_1(h+m)\left(m\left(k-\gamma_1\right)-c_1\right)-kp_1\left(c_1f+m\left(\gamma_1 f+h\right)\right)}{m\left(fkp_1+hr_1(h+m)\right)}$$

$$\sigma_2 = \frac{hp_1(fk+2h+m)-(h+m)^2\left(\delta_1(fk+h)+hr_1\right)}{2(h+m)^2(fk+h)} - \frac{\hat{\Lambda}}{2\sqrt{r_1}(h+m)^2(fk+h)}$$

$$\sigma_3 = \frac{hp_1(fk+2h+m)-(h+m)^2\left(\delta_1(fk+h)+hr_1\right)}{2(h+m)^2(fk+h)} + \frac{\hat{\Lambda}}{2\sqrt{r_1}(h+m)^2(fk+h)}$$

$$\hat{\Lambda} = \sqrt{-2p_1r_1(h+m)^2\left(\begin{array}{c} h^2r_1(fk+2h+m) \\ +\delta_1(fk+h)(fk(3h+2m)-hm)\end{array}\right) \times p_1^2\left(h^2r_1(fk+2h+m)^2+4\delta_1 fk(h+m)^2(fk+h)\right) + r_1(h+m)^4\left(hr_1-\delta_1(fk+h)\right)^2}$$

If $\Re_0 < 1 + \frac{c_1+\chi}{\gamma_1 m}$ then $|arg(\sigma_I)| = \pi > \frac{\alpha\pi}{2}$ where $\chi = \frac{kp_1(c_1f+m(\gamma_1 f+h))}{hr_1(h+m)}$.
The eigenvalues $\sigma_{2,3}$ have negative real parts if $2\delta_1 > \frac{p_1}{m}$ and $|arg(\sigma_{2,3})| = \pi > \frac{\alpha\pi}{2}$, $\forall\ 0<\alpha<1$. Therefore according to Matignon's condition [42], the disease-free equilibrium E_3 is locally asymptotically stable if $\Re_0 < 1+\frac{c_1+\chi}{\gamma_1 m}$ with the condition that $2\delta_1 > \frac{p_1}{m}$ and $r_1 > \frac{p_1}{m}$. □

6.4. Local stability of $E_4(S_4, I_4, 0)$

Theorem 4. The predator-extinction equilibrium point E_4 of the system (6) is locally asymptotically unstable if $\Re_0 > 1$.

Proof. The predator-free equilibrium E_4 exists for $\Re_0 > 1$. The Jacobian matrix of system (6) at E_4 is given by

$$J_{E_4} = \begin{pmatrix} -\frac{r_1\gamma_1}{k} & -\frac{(k+r_1)\gamma_1}{k} & \frac{fr_1\gamma_1(\gamma_1-k)-p_1(k+r_1)}{k+r_1} \\ \frac{r_1(k-\gamma_1)}{k+r_1} & 0 & -c_1 \\ 0 & 0 & \delta_1 \end{pmatrix}$$

The eigenvalues of the Jacobian matrix of the system at E_4 are the roots of the following equation

$$(\delta_1 - \Lambda_Y)\left(\Lambda^2 + \frac{r_1\gamma_1}{k} + \frac{\gamma_1 r_1\left(k-\gamma_1\right)}{k}\right) = 0$$

Clearly one eigenvalue of the characteristic polynomial is $\Lambda_Y = \delta_1$. Therefore the system (6) is locally asymptotically unstable at E_4. □

6.5. Local stability of E^*

The Jacobian matrix of system (6) at E^* is given by

$$J_{E^*} = \begin{pmatrix} V_1 & V_2 & V_3 \\ V_4 & V_5 & V_6 \\ V_7 & V_8 & V_9 \end{pmatrix}$$

where

$$V_1 = \frac{r_1\left(k-I^*-2S^*\right)}{fkY^*+k} - \frac{mp_1\left(Y^*\right)^2+I^*\left(mY^*+S^*\right)^2}{\left(mY^*+S^*\right)^2}$$

$$V_2 = -\frac{S^*\left(fkY^*+k+r_1\right)}{fkY^*+k}$$

$$V_3 = S^*\left(\frac{fr_1\left(-k+I^*+S^*\right)}{k\left(fY^*+1\right)^2} - \frac{p_1S^*}{\left(mY^*+S^*\right)^2}\right)$$

$$V_4 = I^*$$

$$V_5 = \frac{\left(S^*-\gamma_1\right)\left(mY^*+I^*\right)^2 - c_1m\left(Y^*\right)^2}{\left(mY^*+I^*\right)^2}$$

$$V_6 = -\frac{c_1\left(I^*\right)^2}{\left(mY^*+I^*\right)^2}$$

$$V_7 = \frac{\delta_1 h\left(\hat{Y}^*\right)^2}{\left(I^*+S^*\right)^2}$$

$$V_8 = \frac{\delta_1 h\left(Y^*\right)^2}{\left(I^*+S^*\right)^2}$$

$$V_9 = \frac{\delta_1\left(-2hY^*+I^*+S^*\right)}{I^*+S^*}$$

and the corresponding characteristic equation is of the form

$$\sigma^3 + \omega_1\sigma^2 + \omega_2\sigma + \omega_3 = 0 \qquad (22)$$

where ω_i, $(i = 1, 2, 3)$ are given in the Appendix. By Routh-Hurwitz stability criterion the positive equilibrium E^* will be locally asymptotically stable if $\omega_1 > 0, \omega_3 > 0$ and $\omega_1\omega_2 > \omega_3$.

7. Global stability analysis

In this section, we study the global asymptotic stability of the disease-free equilibrium point and the positive equilibrium point by constructing suitable Lyapunov functions.

Theorem 5. If $\Re_0 < k\left(S_3 - \frac{c_1}{m}\right)^{-1}$ then the disease-free equilibrium E_3 is globally asymptotically stable.

Proof. Proof of the theorem is given in the Appendix. □

Theorem 6. The positive equilibrium E^* is globally asymptotically stable with respect to solutions initiating in the interior of the region Γ if $L_2 = L_1\left(1 + \frac{r_1}{k(1+fY^*)}\right)$ *and* $\frac{r_1}{k(1+fY^*)} > \frac{p_1}{m^2 Y^*}$.

Proof. Proof of the theorem is given in the Appendix. □

8. Bifurcation analysis

Throughout this section, we analyze the possibility of occurrence of Hopf-bifurcation at the disease-free equilibrium point E_3 and positive equilibrium E^*. Oscillating behavior is one of the most frequent dynamical behavior appears in the nonlinear mathematical study of population dynamics, which lead to the Hopf-bifurcation of the system.

From the equation (21), the characteristic equation of the Jacobian matrix of the system at E_3 has a pair of purely imaginary eigenvalues for $A = 0$ and $B > 0$ which implies,

$$\begin{aligned} r_1 &> \frac{2hp_1 + mp_1}{h^2 + 2hm + m^2} \\ f &> \frac{h^2 r_1 + 2hmr_1 - 2hp_1 + m^2 r_1 - mp_1}{kp_1} \\ c_1 &> \frac{m\left(hr_1(h+m)(k-\gamma_1) - kp_1(\gamma_1 f + h)\right)}{fkp_1 + hr_1(h+m)} \\ \delta_1 &= \frac{h\left(p_1(fk+2h+m) - r_1(h+m)^2\right)}{(h+m)^2(fk+h)} \end{aligned} \tag{23}$$

Since we are discussing the effect of fear for the model, so we use rate of fear f as the bifurcation parameter. Again the characteristic equation of J_{E^*} is of the form,

$$\sigma^3 + \omega_1\sigma^2 + \omega_2\sigma + \omega_3 = 0 \tag{24}$$

The positive equilibrium E^* experiences a Hopf bifurcation for some free parameter say f at a threshold value $f = f^*$ if $\omega_1(f^*), \omega_2(f^*), \omega_3(f^*) > 0$, $\Delta = \omega_1(f^*)\omega_2(f^*) - \omega_3(f^*) = 0$ and $\frac{\partial\Delta}{\partial f}(f^*) \neq 0$. Next, we mention Matignon's criterion for the existence of a Hopf bifurcation when the order α of the fractional derivative passes through the threshold value $\alpha = \alpha^*$.

Theorem 7. *[44]* (Existence of Hopf bifurcation) When the bifurcation parameter α passes through the critical value $\alpha = \alpha^* \in (0,1)$, the fractional-order system (6) undergoes a Hopf bifurcation at any equilibrium point E if the following conditions hold

(a) the Jacobian matrix of the system at the equilibrium point E has a pair of complex conjugate eigenvalues $\hat{\lambda}_{2,3} = u \pm iv$ where $u > 0$ and one negative real root $\hat{\lambda}_1$.

(b) $\hat{m}(\alpha^*) = \alpha^*\frac{\pi}{2} - \min_{1\le i\le 3}|\arg(\hat{\lambda}_i)| = 0$.

(c) $\frac{d\hat{m}(\alpha)}{d\alpha}|_{\alpha=\alpha^*} \neq 0$ *(transversality condition)*

9. Numerical simulation

Throughout this section, we compare the analytical findings using a biologically plausible parameter set. Approximate solutions for our fractional-order system are determined using the generalized Adams–Bashforth–Moulton type predictor-corrector scheme [43]. We took the majority of our base parameter values from the eco-epidemiological study of pelicans in the Salton sea by Chattopadhyay et al. [35].

$$r = 3/day,\ c = 0.05/day,\ \gamma = 0.24/day,$$
$$\delta = 0.09/day,\ \lambda = 0.006/day,\ m = 1$$

Additionally, we take $f = 0.2$, $p = 0.03/day$, $h = 0.2$.
With these parameter values,

$$r_1 = \frac{3}{0.006},\ f = 0.5,\ p_1 = \frac{0.03}{0.006},\ m = 1,$$
$$c_1 = \frac{0.05}{0.006},\ \gamma_1 = \frac{0.24}{0.006},\ \delta_1 = \frac{0.09}{0.006},\ h = 0.2.$$

Now, we fix total capacity of the prey to be $k = 75$. For the choice of parameter values mentioned above, $S_3 = 45.76$ and $\left(\frac{c_1}{m} + \gamma_1\right) = 48.33$ satisfying $S_3 - \frac{c_1}{m} - \gamma_1 < 0$. Equivalently, $\Re_0 < k\left(S_3 - \frac{c_1}{m}\right)^{-1}$, which is the condition for global stability of E_3 obtained analytically in Theorem 5. With these set of parameter values the equilibrium points of the fractional-order system (6) are

$$E_3 = (45.7692, 0, 228.846), E_4 = (40., 30.4348, 0)$$

Under the above parameters no positive equilibrium appears. Between the two equilibria, E_4 is unstable (Theorem 4) and Theorem 5 is satisfied for global asymptotic stability of E_3. It is observed that all the trajectories of the system (6) initialed at different values approach to the disease-free equilibrium E_3, see Figure 1. Next, we consider $f = 0.05, m = 5, h = 0.5$ and $k = 350$ along with the other parameters mentioned above. For these parameters $S_3 = 328.46$, $\left(\frac{c_1}{m} + \gamma_1\right) = 41.66$ and $S_3 - \frac{c_1}{m} - \gamma_1 > 0$ which is equivalently $\Re_0 > k\left(S_3 - \frac{c_1}{m}\right)^{-1}$. The equilibrium

points of the fractional-order system (6) which exist under these parameters are
$E_2 = (350, 0, 0)$, $E_3 = (328.462, 0, 656.923)$,
$E_4 = (40, 182.353, 0)$,
$E^* = (41.5882, 40.5682, 164.313)$. Among these equilibria, E_3 is unstable since the Theorem 5 is not satisfied for the parameter set. From numerical simulations, it is found that trajectories of the system tend to the positive equilibrium E^* with the increase in time irrespective of the initial value (Figure 5). This suggests that the positive equilibrium E^* has a large domain of attraction. Again we set the parameter values $h = 0.04$, $k = 75$ together with $p_1, m, \gamma_1, \delta_1$ as mentioned in the beginning of section 9. From the first condition in equation (23),

$$r_1 > 4.9926$$

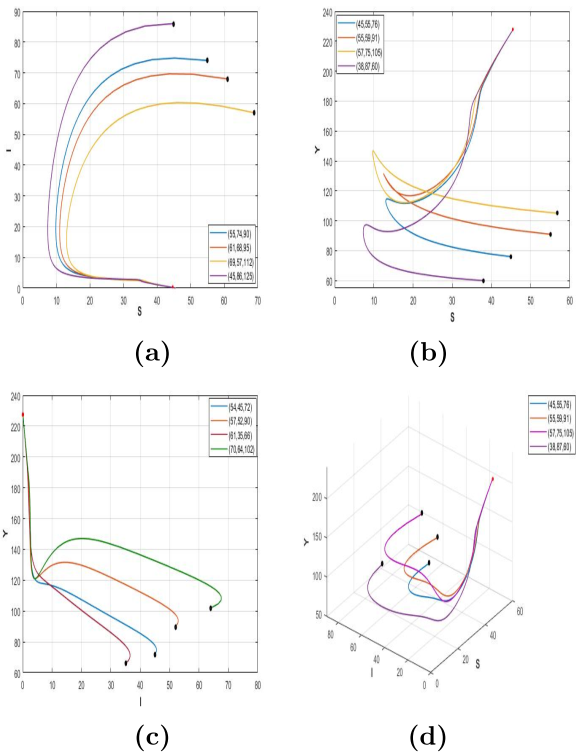
So we fix $r_1 = \frac{1}{0.006}$, then from the second condition $f > 0.466311$. Again we fix $f = 0.5$, then from third condition $c_1 > 1.59729$, so we fix $c_1 = \frac{0.05}{0.006}$. Finally for the fourth condition $\delta_1 = 0.0124456$ which implies $\delta \to 0$, i.e., the reproduction rate of predator population becomes very very small. For r_1, c_1, δ_1 as above, keeping other parameter values fixed with $f = 0.6$ and initial population $(55, 75, 190)$, the eigenvalues of the Jacobian matrix of the system at the disease-free equilibrium E_3 are $\hat{\lambda}_1 = -46.156$, $\hat{\lambda}_{2,3} = 0.0143594 \pm 0.244213i$. From the second condition of Theorem 7,

$$\alpha^* = \frac{2}{\pi}\arctan\left|\frac{0.244213}{0.0143594}\right| = 0.962611 \approx 0.962$$

and from the last condition,

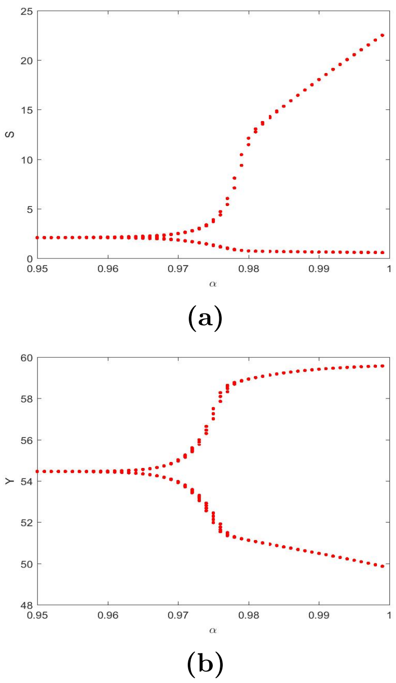
$$\left.\frac{d\hat{m}}{d\alpha}\right|_{\alpha=\alpha^*} = \frac{\pi}{2} \neq 0$$

which implies that the transversality condition holds. Hence the fractional-order system (6) at the disease-free equilibrium E_3 experiences a Hopf bifurcation when the bifurcation parameter α passes through a critical value $\alpha^* \approx 0.962$, see Figure 2. Our main interest is to discuss the fear induced in the prey as an anti-predator reaction. We fix the rate of fear f as a free parameter. The system (6) exhibits oscillatory behaviour at the disease-free equilibrium E_3 for $f = 0.7$, $r_1 = 1/0.006$, $c_1 = 0.05/0.006$, $\delta_1 = 0.0124456$ and $\alpha = 0.98$, see Figure 3. We take the initial population $(55, 75, 190)$ and $\alpha = 0.98$ with the fear coefficient f as a free parameter. It is observed that increasing the fear effect f from $f = 0.466311$ the disease-free equilibrium E_3 becomes unstable due to a Hopf bifurcation when the bifurcation parameter f passes through a critical value $f^* = 0.48$, see Figure 4. Again we set the parameter set $f = 0.05, m = 5, h = 0.5, k = 350$ together with other parameter values as mentioned in the beginning of section 9. In Figure 5, we plot the trajectories of the system (6) for $\alpha = 0.99$ with initial population $(55, 170, 450)$ which approach to $E^*(41.59, 40.57, 164.31)$. Moreover, for these parameter values $\omega_1 > 0$, $\omega_3 > 0$ and $\omega_1\omega_2 - \omega_3 > 0$, which confirms that the Routh-Hurwitz stability criterion satisfies.



(a) (b) (c) (d)

Figure 1. Phase diagram of the system (6) at the disease-free equilibrium E_3 with different initial values with $\alpha = 0.9$.



(a) (b)

Figure 2. Bifurcation diagram of the system at E_3 with respect to the bifurcation parameter α.

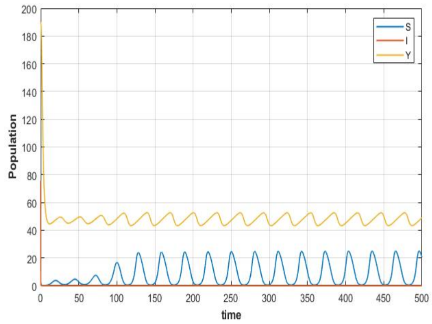
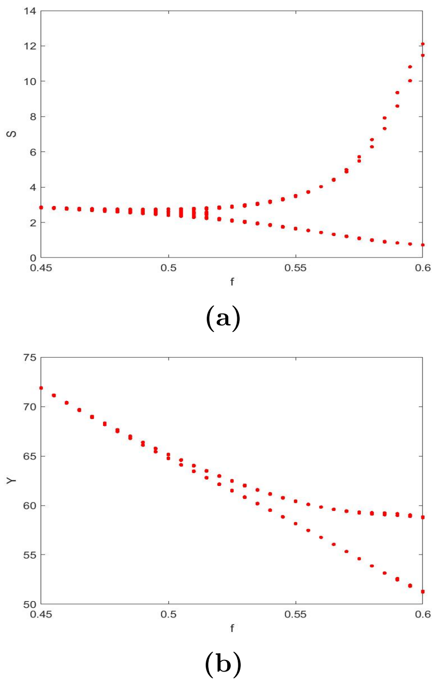


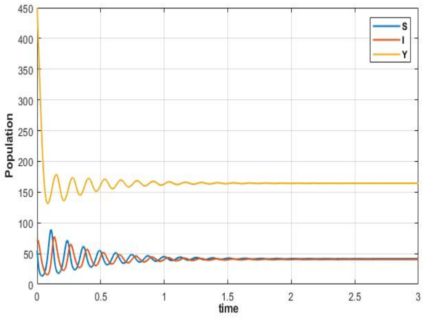
Figure 3. Time series of the fractional-order system (6) at E_3 for $f = 0.7$ and $\alpha = 0.98$



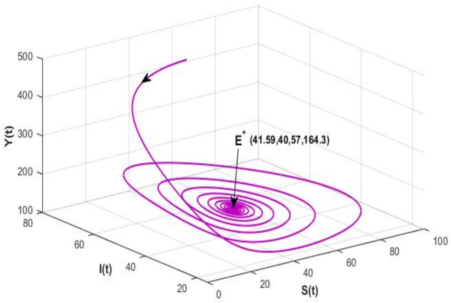
(a) (b)

Figure 4. Bifurcation diagram of the system at E_3 with respect to the fear effect f for $\alpha = 0.98$

In Figure 6, we compute orbits with above parameters and $\alpha = 0.99$ from several starting points and observe that all trajectories of the system (6) approach to the same positive equilibrium $E^*(41.59, 40.57, 164.31)$. This suggests that E^* has a large domain of attraction. We fix the parameter values $f = 0.05, m = 5, h = 0.5, k = 350$ together with the parameter values mentioned in the beginning of section 9 and keep γ_1 as free parameter. For $\gamma_1 = \frac{0.12}{0.006}$ and $\alpha = 1$ it is observed that for all the trajectories of the system (6) undergoes a Hopf bifurcation, see Figure 7 and Figure 8. Increasing γ_1 from $\gamma_1 = \frac{0.12}{0.006}$ to $\gamma_1 = \frac{0.135}{0.006}$, it is observed that the positive equilibrium E^* undergoes a backward Hopf bifurcation at $\gamma_1 = 0.1284/0.006$, see Figure 9.



(a)



(b)

Figure 5. (a) Time series of positive equilibrium point E^* for $\alpha = 0.99$, (b) Phase diagram of positive equilibrium point E^* for $\alpha = 0.99$.

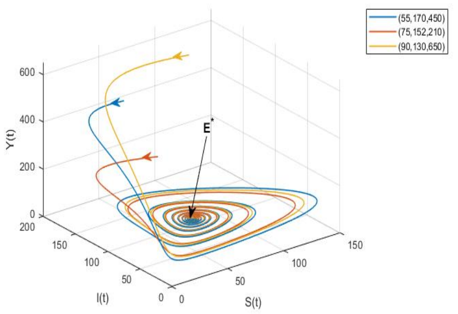


Figure 6. Phase portrait of the positive equilibrium E^* of the system (6) for $\alpha = 0.99$.

For $\gamma_1 = \frac{0.07}{0.006}$ the eigenvalues of the Jacobian matrix of the system at the positive equilibrium E^* are $\hat{\lambda}_1 = -19.0552$, $\hat{\lambda}_{2,3} = 0.849491 \pm 32.1521i$. From Theorem 7,

$$\alpha^* = \frac{2}{\pi} \arctan \left| \frac{32.1521}{0.849491} \right| = 0.983184 \approx 0.983$$

and

$$\left. \frac{d\hat{m}}{d\alpha} \right|_{\alpha=\alpha^*} = \frac{\pi}{2} \neq 0,$$

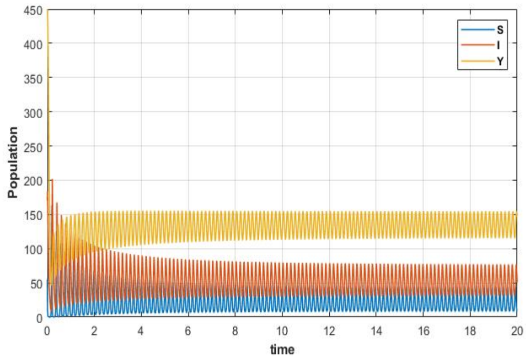


Figure 7. Time series of the fractional-order system (6) for $\gamma_1 = \frac{0.12}{0.006}$ and $\alpha = 1$

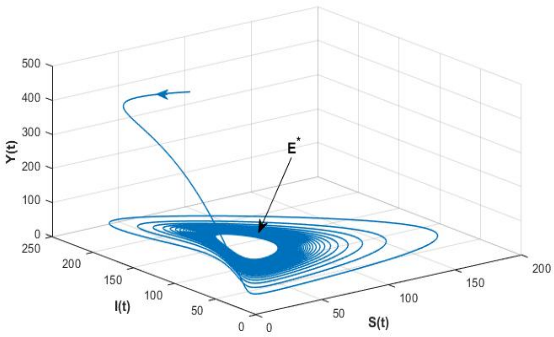
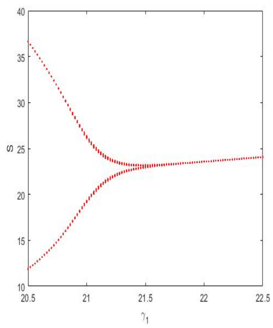
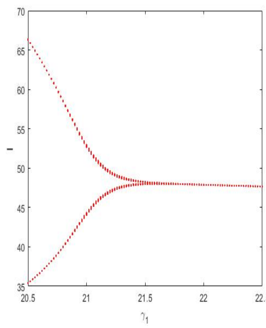


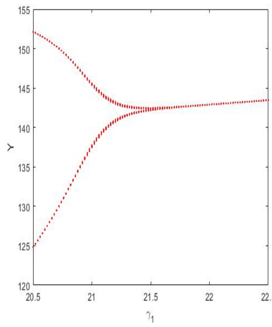
Figure 8. Phase diagram of the fractional-order system (6) for $\gamma_1 = \frac{0.12}{0.006}$ and $\alpha = 1$



(a)

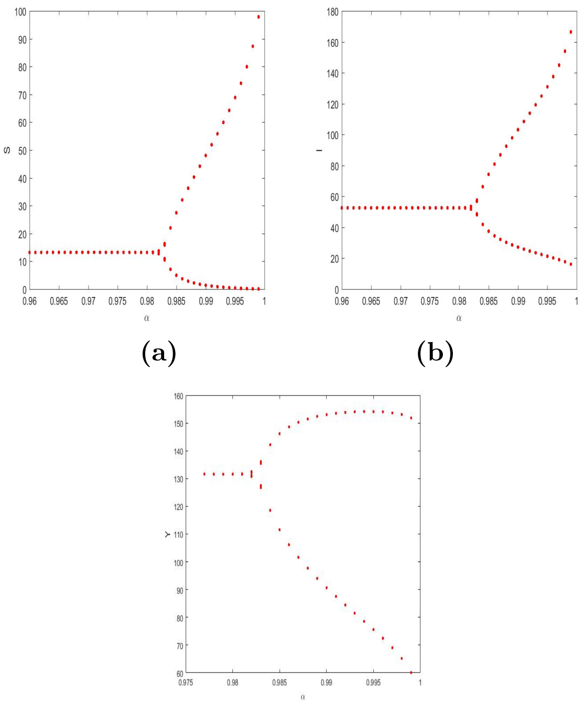


(b)



(c)

Figure 9. Bifurcation diagram of the system at E^* with respect to mortality rate of infected prey γ_1 for $\alpha = 1$.



(a) (b)

(c)

Figure 10. Bifurcation diagram of the system at E^* with respect to the bifurcation parameter α.

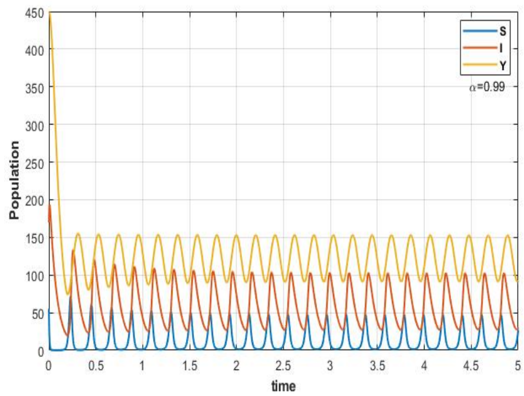


Figure 11. Time series of the system (6) for $\gamma_1 = \frac{0.07}{0.006}$ $\alpha = 0.99$.

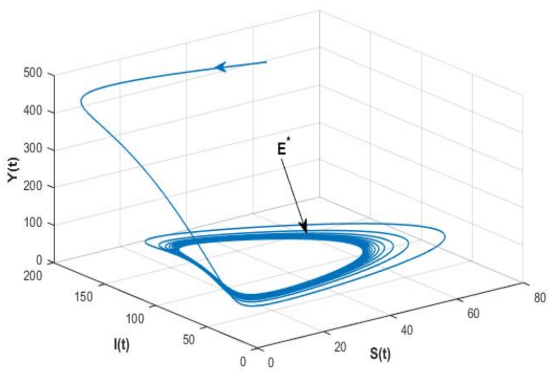
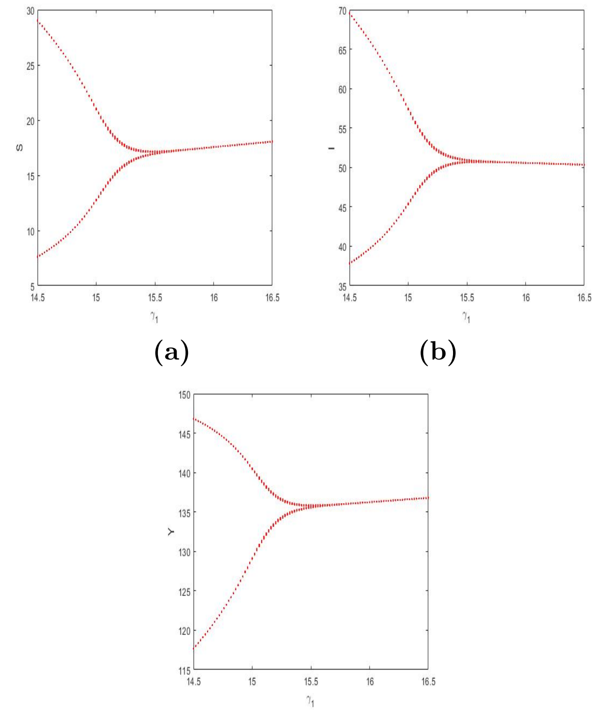


Figure 12. Phase diagram of the system for $\gamma_1 = \frac{0.07}{0.006}$ and $\alpha = 0.99$.

Hence Hopf bifurcation occurs in the system (6) at the positive equilibrium E^* when the bifurcation parameter α passes through a critical value $\alpha^* = 0.983$, see Figure 10. For $\gamma_1 = \frac{0.07}{0.006}$ and $\alpha = 0.99$ all the trajectories of the system (6) shows oscillatory behaviour via a Hopf bifurcation, see Figure 11 and Figure (12). Increasing γ_1 from $\gamma_1 = \frac{0.087}{0.006}$ to $\gamma_1 = \frac{0.099}{0.006}$, it is observed that at the positive equilibrium E^* the system (6) undergoes a backward Hopf bifurcation when the bifurcation parameter γ_1 passes through a critical value $\gamma_1^* = \frac{0.093}{0.006}$, see Figure 13.



(a) (b)

(c)

Figure 13. Bifurcation diagram of the system at E^* with respect to mortality rate of infected prey γ_1 for $\alpha = 0.99$.

For $h = 05, k = 2500, f = 0.05$ along with other parameters as mentioned in section 9 and initial population $(55, 170, 450)$ all the population coexists with population $E^*(40.93, 154.9, 39.18)$ (Figure 14). For $f = 0.12$ the eigenvalues of the Jacobian matrix of the system at the positive equilibrium E^* are $\hat{\lambda}_1 = -20.8432$, $\hat{\lambda}_{2,3} = 2.29661 \pm 70.1413i$. From Theorem 7,

$$\alpha^* = \frac{2}{\pi} \arctan \left| \frac{70.1413}{2.29661} \right| = 0.979163 \approx 0.979$$

and

$$\left. \frac{d\hat{m}}{d\alpha} \right|_{\alpha=\alpha^*} = \frac{\pi}{2} \neq 0.$$

Hence a Hopf bifurcation occurs in the fractional-order system (6) at the positive equilibrium E^* when the bifurcation parameter α passes through a critical value $\alpha^* = 0.979$, see Figure 15. With a increase in the fear coefficient all the trajectories of the system (6) at the positive equilibrium E^* undergoes a Hopf bifurcation. In Figure 16, the oscillatory behaviour of all the population is presented for $f = 0.12$ and $\alpha = 0.98$. Increasing f from $f = 0.05$ to $f = 0.18$, it is observed that the positive equilibrium undergoes a forward Hopf bifurcation when the bifurcation parameter f passes through a critical value $f^* = 0.1$, see Figure 17.

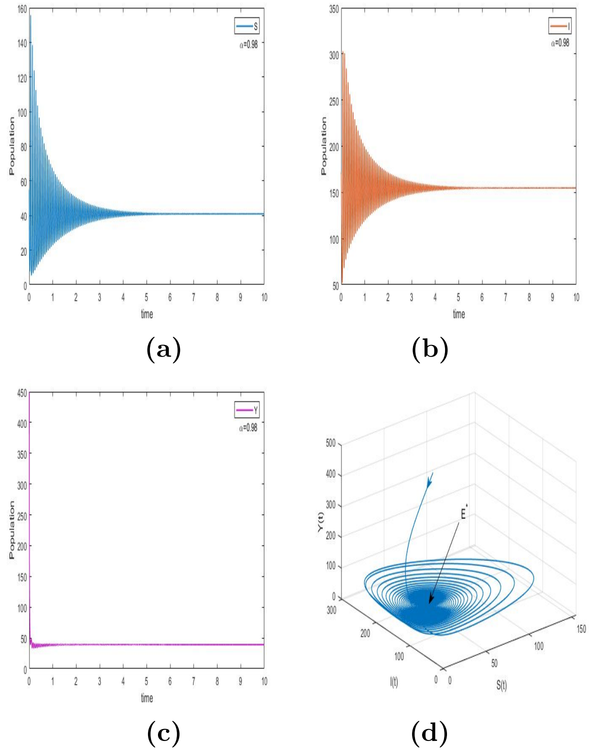


Figure 14. Time series and phase diagram of the system for $f = 0.05, \alpha = 0.98$.

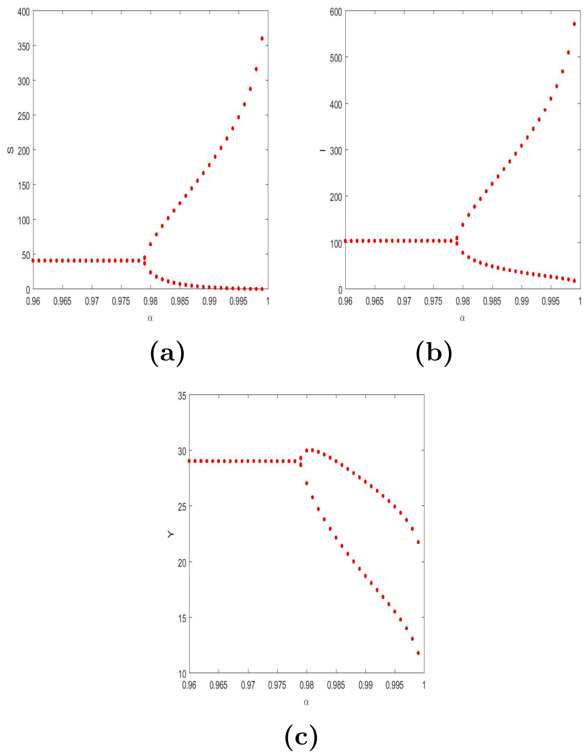


Figure 15. Bifurcation diagram of the system at E^* with respect to the bifurcation parameter α.

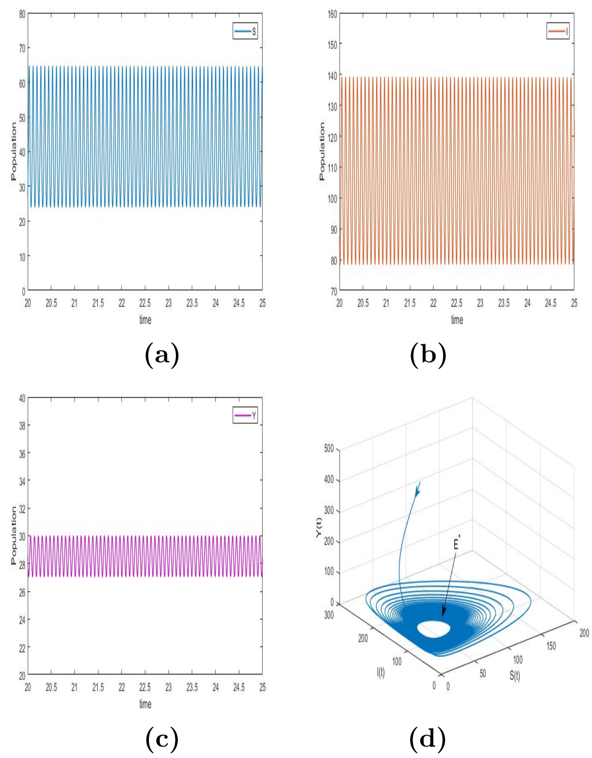
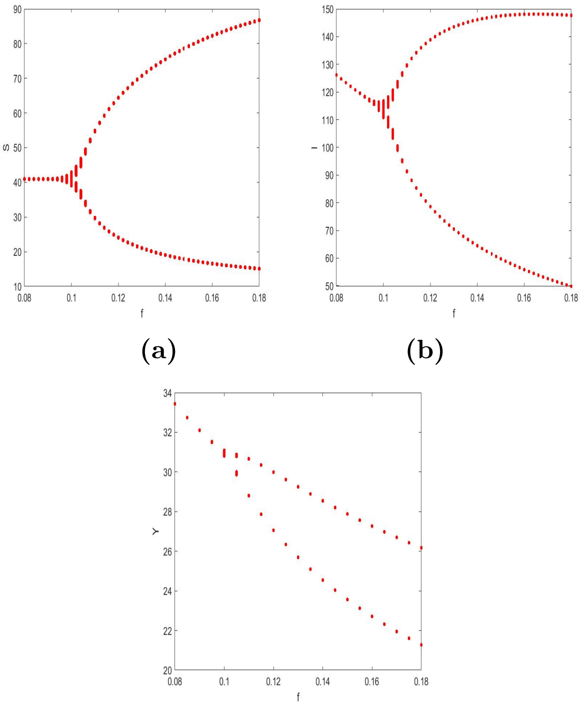


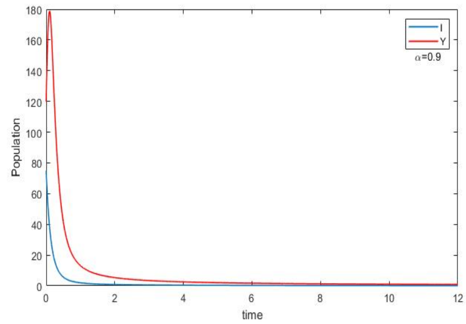
Figure 16. Time series and phase diagram of the system for $f = 0.12$ $\alpha = 0.98$.



(a) **(b)**

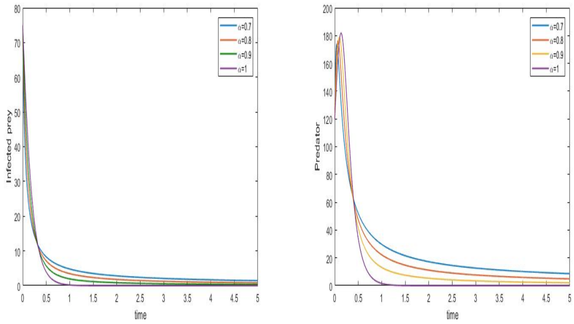
(c)

Figure 17. Bifurcation diagram of the system at E^* with respect to to fear coefficient f for $\alpha = 0.98$.



(c)

Figure 18. Time series of the system in absence of susceptible prey for $\alpha = 0.9$.



(a) **(b)**

Figure 19. (a),(b) Time series of the system when $S = 0$. Infected prey and predator approach towards extinction for different values of α.

9.1. Impact of the disease on predators in the absence of susceptible prey

In the absence of susceptible prey both the infected prey and predator populations become extinct, and the system approaches the trivial equilibrium E_1. Analytical discussion of the situation can be found in literature [37]. Here we are exploring this scenario numerically for different values α. We fix the parameter values $m = 1, c_1 = \frac{0.03}{0.006}, \gamma_1 = \frac{0.01}{0.006}, \delta_1 = \frac{0.05}{0.006}, h = 0.2$ and consider other parameters to be 0. From Figure 18, it is observed both the infected prey and predator populations approach towards extinction for the initial population $(I = 75, Y = 120)$. The behaviour of the infected prey and predator towards extinction under different values α is presented in Figure 19. Increasing the order α, the time duration of extinction for both the species become reduced.

10. Conclusion and discussion

In this paper, we investigate a modified eco-epidemiological model incorporating the fear effect. The model equations are constructed with the help of Caputo fractional-order differential equations. Fundamental requisites such as existence, uniqueness, non-negativity, and boundedness of the solutions of the system are discussed. Biologically possible equilibrium states of the model are determined. The basic reproduction number $\Re_0$ of the epidemiology theory is determined. Local and global asymptotic stabilities of the equilibrium states are presented. The disease-free equilibrium E_3 is globally asymptotically stable if $\Re_0 < k\left(S_3 - \frac{c_1}{m}\right)^{-1}$, where $\Re_0$ is the basic reproduction number of the epidemic. The global stability condition of the endemic case, i.e., the positive equilibrium, is also discussed. We determine the threshold parameter values for which the disease-free case and the endemic case become unstable. In equation (23) we present a parametric condition for which the disease-free equilibrium loses its stability due to a Hopf bifurcation. With biologically plausible parameters, we conduct numerical simulations to visualize the system's behaviour near the equilibrium points. To explore the role of the order (α) of the differential equations towards the stability of the equilibrium states, we use Matignon's theorem. Global stability of the system at the disease-free equilibrium E_3 is presented in Figure 1. The trajectories of the system at the equilibrium E_3 with α as a free parameter are presented in Figure

2. Applying Matignon's theorem, we determine the threshold value of α as $\alpha^* \approx 0.962$. When α passes through the critical value $\alpha = \alpha^*$, the endemic equilibrium E^* of the system becomes unstable via a Hopf bifurcation. An increase in the fear coefficient f through a critical value $f^* = 0.48$ (with $\alpha = 0.98$), the system shows oscillatory behaviour at the equilibrium E_3 via a Hopf bifurcation (Figure 4). From an ecological point of view, an increase in fear due to predation risk above a threshold value decreases the prey population's reproduction rate, forcing the system towards extinction. Global asymptotic stability of the positive equilibrium can be observed in Figure 6. Behaviour of the system at E^* with respect to α can be seen in Figure 10. Mass mortality of the pelicans was taking place mainly because of consuming infected tilapias in the Salton Sea. Numerically, it is observed that the parameter γ_1 could stabilize the system dynamics at the positive equilibrium when it passes through a critical value $\gamma_1 = \gamma_1^* = 0.093/0.006$ (Figure 13). For a different set of parameters, the system shows oscillatory behaviour through a Hopf bifurcation when α passes through $\alpha = \alpha^* \approx 0.979$ (Figure 15). It is also observed that the rate of fear in prey due to predation risk is responsible for the stability of the endemic equilibrium. An increase in the rate of fear due to predation risk enforces the endemic equilibrium to lose its stability via a Hopf bifurcation (Figure 17). In ecological terms, all the populations exhibit oscillatory behaviour with a certain increase in the rate of fear due to predation risk. From the numerical simulations, it is observed that below some threshold value $0 < \alpha < \alpha^*$ all the population coexists. So, it can be concluded that the system's fraction-order (α) can help to control the coexistence of all the species populations. In the absence of susceptible prey, both infected prey and predator extinct after a specific time. The effect of infected prey on the existence of the predator population is discussed numerically in subsection 9.1. It is observed that with an increase in the order α, the time of extinction for both the species get reduced (Figure 18).

We have already given brief summary of the models in [34–37] at the introduction part. As per the authors' information, fewer studies have been done in epidemic models with the fear effect. In [46], Wu et al. studied a delayed epidemic model incorporating fear effect in prey and refuge. Pal investigated a modified Lesli-Gower eco-epidemiological model with fear effect in prey [47] and observed that an increase in the fear coefficient stabilizes the system dynamics. In [49], Sha et al. investigated an eco-epidemiological model with disease in the prey. They assumed that the induced fear also reduces disease transmission along with reproduction and obtained fear-induced backward bifurcation and bi-stability. Our model differs from the model proposed by Sha et al. [49] in functional response, fear effect (no impact of fear effect in disease transmission), and type of the equations. In a fractional-order sense, Mandal et al. [48] discussed an epidemic model with the fear effect of an infectious disease. Our model is more realistic than the models studied in [35, 37] in the sense of fear induced in prey and the type of the differential equations. The reproduction rate in the prey population is affected because of the predation risk that controls the system dynamics. Again, the fractional-order of the equations may help to control the stability of the coexistence equilibrium state. Interested readers can modify this model by involving non-local, additionally, non-singular fractional derivatives such as the ABC derivative. Holling type IV functional response can be assumed as most prey shows antipredator defense mechanism. Group defense is a popular anti-predator response where the prey defends themselves by making groups; see [45]. Moreover, one can investigate the model, including the impact of fear in the disease transmission by involving fractional derivatives, see [49].

Conflict of Interests: The authors declare that there is no conflict of interest regarding the publication of this paper.

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Appendix A.

Coefficients of S^* in (20)

$$
\begin{aligned}
\rho_0 &= fkm(h+2m),\\
\rho_1 &= -c_1fk(h+3m) - fkp_1(h+m) - (h+2m)\left(hr_1(h+m) - km\left(-3\gamma_1 f + h + m\right)\right),\\
\rho_2 &= c_1\left(-k\left(-2\gamma_1 f(h+3m) + h^2 + 5hm + 5m^2\right) + fkp_1 + hr_1(2h+3m)\right) + c_1^2 fk\\
&\quad - kp_1(h+m)\left(-3\gamma_1 f + h + m\right) + (h+2m)\left(r_1(h+m)\left(3\gamma_1 h + k(h+m)\right)\right.\\
&\quad \left. - 3\gamma_1 km\left(-\gamma_1 f + h + m\right)\right),\\
\rho_3 &= c_1\left(-\left(\gamma_1 k\left(\gamma_1 f(h+3m) - 2\left(h^2 + 5hm + 5m^2\right)\right)\right) - 2kp_1\left(-\gamma_1 f + h + m\right)\right.\\
&\quad \left. + r_1\left(k\left(3h^2 + 8hm + 5m^2\right) + 2\gamma_1 h(2h+3m)\right)\right) - c_1^2\left(k\left(\gamma_1 f - 2(h+2m)\right) + hr_1\right)\\
&\quad - \gamma_1\left((h+2m)\left(\gamma_1 km\left(\gamma_1 f - 3(h+m)\right) + 3r_1(h+m)\left(\gamma_1 h + k(h+m)\right)\right)\right.\\
&\quad \left. - 3kp_1(h+m)\left(-\gamma_1 f + h + m\right)\right),\\
\rho_4 &= -\left(c_1 + \gamma_1(h+m)\right)\left(c_1\left(\gamma_1 k(h+3m) - r_1\left(\gamma_1 h + 3hk + 4km\right) + kp_1\right) + c_1^2 k\right.\\
&\quad \left. - \gamma_1\left(kp_1\left(\gamma_1 f - 3(h+m)\right) + (h+2m)\left(r_1\left(\gamma_1 h + 3k(h+m)\right) - \gamma_1 km\right)\right)\right),\\
\rho_5 &= -k\left(c_1 + \gamma_1(h+m)\right)^2\left(c_1 r_1 + \gamma_1\left(r_1(h+2m) - p_1\right)\right).
\end{aligned}
$$

Proof of Theorem 5

Proof. At the equilibrium point E_3 the system (6) reduces to,

$$
\begin{aligned}
r_1S_3\left(1-\frac{S_3+I_3}{k}\right)\left(\frac{1}{1+fY_3}\right) - S_3I_3 - \frac{p_1Y_3S_3}{mY_3+S_3} &= 0,\\
\delta_1Y_3\left(1-\frac{hY_3}{S_3+I_3}\right) &= 0.
\end{aligned}
$$

Consider the Lyapunov function,

$$
\mathcal{W}(S,I,Y) = N_1\left(S - S_3 - S_3\ln\frac{S}{S_3}\right) + N_2 I + N_3\left(Y - Y_3 - Y_3\ln\frac{Y}{Y_3}\right).
$$

We calculate the α-order derivative of $\mathcal{W}(S,I,Y)$ along the solution of the system (6) and applying Lemma 5 we get,

$$
\begin{aligned}
&{}_{t_0}^{c}D_t^{\alpha}\mathcal{W}(S,I,Y)\\
&= N_1\left(\frac{S-S_3}{S}\right){}_{t_0}^{c}D_t^{\alpha}S(t) + N_2\,{}_{t_0}^{c}D_t^{\alpha}I(t) + N_3\left(\frac{Y-Y_3}{Y}\right){}_{t_0}^{c}D_t^{\alpha}Y(t)\\
&= N_1\left(\frac{S-S_3}{S}\right)\left[r_1S\left(1-\frac{S+I}{k}\right)\left(\frac{1}{1+fY}\right) - SI - \frac{p_1YS}{mY+S}\right]\\
&+ N_2\left(SI - \frac{c_1YI}{mY+I} - \gamma_1 I\right) + N_3\left(\frac{Y-Y_3}{Y}\right)\left[\delta_1Y\left(1-\frac{hY}{S+I}\right)\right]\\
&\le N_1(S-S_3)\left[r_1\left(1-\frac{S+I}{k}\right) - I - \frac{p_1Y}{mY+S} - r_1\left(1-\frac{S_3}{k}\right)\frac{1}{1+fY_3}\right.\\
&\left. + \frac{p_1Y_3}{mY_3+S_3}\right] + N_2\left(SI - \frac{c_1YI}{mY+I} - \gamma_1 I\right) + N_3\left(Y-Y_3\right)\left[\delta_1\left(1-\frac{hY}{S+I}\right)\right.\\
&- \delta_1\left(1-\frac{hY_3}{S_3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq N_1(S-S_3)\left[\frac{r_1 f Y_3}{k(1+fY_3)}-\frac{r_1(S+I)}{k}+\frac{r_1 S_3}{k(1+fY_3)}-I-\frac{p_1 Y}{mY+S}\right.\\
&\left.+\frac{p_1 Y_3}{mY_3+S_3}\right]+N_2\left(SI-\frac{c_1 YI}{mY+I}-\gamma_1 I\right)+N_3\delta_1 h(Y-Y_3)\left[\frac{-\delta_1 hY}{S+I}+\frac{\delta_1 hY_3}{S_3+I_3}\right]\\
&\leq N_1(S-S_3)\left[\frac{r_1 f Y_3}{k(1+fY_3)}-\left(\frac{r_1}{k}\right)\frac{(S-S_3)+I+fY_3(S+I)}{(1+fY_3)}-I+\frac{p_1(S-S_3)}{m^2 Y_3}\right.\\
&\left.-\frac{p_1 S_3(S-S_3)(Y-Y_3)}{(mY+S)(mY_3+S_3)}\right]+N_2\left(SI-\frac{c_1 YI}{mY+I}-\gamma_1 I\right)+N_3\delta_1 h(Y-Y_3)\\
&\left[\frac{Y(S-S_3)+Y(I-I_3)}{(S+I)(S_3+I_3)}-\frac{(Y-Y_3)}{S_3+I_3}\right]\\
&\leq \frac{fN_1 r_1\left(S-S_3\right)Y_3}{k(fY_3+1)}-\frac{r_1}{k}(S-S_3)^2-N_1(S-S_3)\left[I+\frac{I+fY_3(S+I)}{1+fY_3}\right]\\
&+\frac{N_1 p_1}{m^2 Y_3}(S-S_3)^2-(S-S_3)(Y-Y_3)\frac{N_1 P_1 S_3}{(mY+S)(mY_3+S_3)}-\frac{N_3\delta_1 hS(Y-Y_3)^2}{(S+I)S_3}\\
&+\frac{(S-S_3)(Y-Y_3)N_3\delta_1 h}{(S+I)S_3}+\frac{N_3\delta_1 hY_3(Y-Y_3)}{S_3}+N_2\left(SI-\frac{c_1 YI}{mY+I}-\gamma_1 I\right)\\
&\leq \frac{N_1 r_1 f\left(S-S_3\right)Y_3}{k(fY_3+1)}+N_1(S-S_3)^2\left[\frac{p_1}{m^2 Y_3}-\frac{r_1}{k}\right]-\frac{N_3\delta_1 hS}{(S+I)S_3}(Y-Y_3)^2\\
&+\left[(S-S_3)(Y-Y_3)\left\{\frac{N_3\delta_1 h}{(S+I)S_3}-\frac{N_1 p_1 S}{(mY+S)(mY_3+S_3)}\right\}\right.\\
&\left.+N_1 S_3\left\{I+\frac{I+fY_3(S+I)}{1+fY_3}\right\}+\frac{N_3\delta_1 hY_3 Y}{S_3}-N_1 S\left\{I+\frac{I+fY_3(S+I)}{1+fY_3}\right\}\right]\\
&+N_2\left(SI-\frac{c_1 YI}{mY+I}-\gamma_1 I\right).
\end{aligned}
$$

Suppose $\frac{p_1}{m^2 Y_3} < \frac{r_1}{k}$ and $\theta_1 < S, I, Y < \theta_2$. We choose N_1 and N_3 such that

$$
\begin{aligned}
&\frac{N_3}{N_1} > min\left\{\frac{2\theta_2 S_3\left(\theta_2^2(m+1)\left(fY_3+2\right)\left(mY_3+S_3\right)+\theta_2 p_1 Y_3\left(fY_3+1\right)\right)}{\delta_1 h\theta_1^2(m+1)\left(fY_3+1\right)\left(mY_3+S_3\right)}, \frac{2\theta_2^3 p_1 S_3^2}{\delta_1 h\theta_1^2(m+1)Y_3}\right\},\\
&\frac{N_3}{N_1} < \frac{2\theta_2^3 S_3\left(\theta_2-fr_1 Y_3\right)}{\delta_1 h\left(fY_3+1\right)\left(Y_3\left(\theta_1^2+S_3\right)+2\theta_1^2\left(Y_3+1\right)\right)}.
\end{aligned}
$$

Then ${}^{c}_{t_0}D^{\alpha}_{t}\mathcal{W}(S,I,Y) < N_2\left(SI-\frac{c_1 YI}{mY+I}-\gamma_1 I\right)$.

Clearly, ${}^{C}_{t_0}D^{\alpha}_{t}\mathcal{W}(S,I,Y) \leq 0$ when $S_3-\frac{c_1}{m}-\gamma_1<0$ which is equivalent to $\Re_0 < k\left(S_3-\frac{c_1}{m}\right)^{-1}$

Hence the proof. □

Proof of Theorem 6:

Proof. At the equilibrium point E^* system (6) reduces to,

$$
\begin{aligned}
r_1 S^*\left(1-\frac{S^*+I^*}{k}\right)\left(\frac{1}{1+fY^*}\right)-S^* I^*-\frac{p_1 Y^* S^*}{mY^*+S^*}&=0,\\
S^* I^*-\frac{c_1 Y^* I^*}{mY^*+I^*}-\gamma_1 I^*&=0,\\
\delta_1 Y^*\left(1-\frac{hY^*}{S^*+I^*}\right)&=0.
\end{aligned}
\tag{25}
$$

To study the globally asymptotically stability of E^* the following positive definite Lyapunov function is considered:

$$
\mathcal{W}(S,I,Y)=L_1\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+L_2\left(I-I^*-I^*\ln\frac{I}{I^*}\right)+L_3\left(Y-Y^*-Y^*\ln\frac{Y}{Y^*}\right).
$$

We calculate the α-order derivative of $\mathcal{W}(S,I,Y)$ along the solution of the system (6) and applying Lemma 5 we get,

$$
\begin{aligned}
{}^{c}_{t_0}D^{\alpha}_{t}\mathcal{W}(S,I,Y) &= L_1\frac{S-S^*}{S}\,{}^{c}_{t_0}D^{\alpha}_{t}S(t) + L_2\frac{I-I^*}{I}\,{}^{c}_{t_0}D^{\alpha}_{t}I(t) + L_3\frac{Y-Y^*}{Y}\,{}^{c}_{t_0}D^{\alpha}_{t}Y(t)\\
&= L_1\frac{S-S^*}{S}\left[r_1S\left(1-\frac{S+I}{k}\right)\left(\frac{1}{1+fY}\right) - SI - \frac{p_1YS}{mY+S}\right]\\
&+ L_2\frac{I-I^*}{I}\left[SI - \frac{c_1YI}{mY+I} - \gamma_1 I\right] + L_3\frac{Y-Y^*}{Y}\left[\delta_1Y\left(1-\frac{hY}{S+I}\right)\right]\\
&\leq L_1(S-S^*)\Bigg[r_1\left(1-\frac{S+I}{k}\right) - I - \frac{p_1Y}{mY+S}\\
&- r_1\left(1-\frac{S^*+I^*}{k}\right)\left(\frac{1}{1+fY^*}\right) + I^* + \frac{p_1Y^*}{mY^*+S^*}\Bigg]\\
&+ L_2(I-I^*)\left[S - \frac{c_1Y}{mY+I} - \gamma_1 - S^* + \frac{c_1Y^*}{mY^*+I^*} + \gamma_1\right]\\
&+ L_3(Y-Y^*)\left[\delta_1\left(1-\frac{hY}{S+I}\right) - \delta_1\left(1-\frac{hY^*}{S^*+I^*}\right)\right]\\
&\leq \frac{L_1r_1fY^*}{k(1+fY^*)}(S-S^*) - \frac{L_1r_1}{k(1+fY^*)}(S-S^*)^2 - L_1\left(1+\frac{r_1}{k(1+fY^*)}\right)\\
&(S-S^*)(I-I^*) + \frac{L_1p_1}{m^2Y^*}(S-S^*)^2 - \frac{L_1p_1S}{(mY+S)(mY^*+S^*)}(Y-Y^*)(S-S^*)\\
&+ L_2(I-I^*)(S-S^*) + \frac{L_2c_1}{m^2Y^*}(I-I^*)^2 - \frac{L_2c_1I(Y-Y^*)(I-I^*)}{(mY+I)(mY^*+I^*)}\\
&+ \delta_1hL_3\left[\frac{(S-S^*)(Y-Y^*)}{(S+I)(S^*+I^*)} - \frac{(Y-Y^*)^2(S-I)}{(S+I)(S^*+I^*)} - \frac{Y(I-I^*)(Y-Y^*)}{(S+I)(S^*+I^*)}\right]\\
&\leq \frac{L_1r_1fY^*}{k(1+fY^*)}(S-S^*) - \frac{L_1r_1}{k(1+fY^*)}(S-S^*)^2 - (S-S^*)(I-I^*)\\
&\left[L_1\left(1+\frac{r_1}{k(1+fY^*)}\right) - L_2\right] + \frac{L_1p_1}{m^2Y^*}(S-S^*)^2 + \frac{L_2c_1}{m^2Y^*}(I-I^*)^2\\
&- \frac{L_1p_1S}{(mY+S)(mY^*+S^*)}(Y-Y^*)(S-S^*) - \frac{L_2c_1I(Y-Y^*)(I-I^*)}{(mY+I)(mY^*+I^*)}\\
&+ \delta_1hL_3\left[\frac{(S-S^*)(Y-Y^*)}{(S+I)(S^*+I^*)} - \frac{(Y-Y^*)^2(S-I)}{(S+I)(S^*+I^*)} - \frac{Y(I-I^*)(Y-Y^*)}{(S+I)(S^*+I^*)}\right]\\
&\leq \frac{L_1r_1fY^*}{k(1+fY^*)}(S-S^*) - \left[\frac{L_1r_1}{k(1+fY^*)} - \frac{L_1p_1}{m^2Y^*}\right](S-S^*)^2 - (S-S^*)(I-I^*)\\
&\left[L_1\left(1+\frac{r_1}{k(1+fY^*)}\right) - L_2\right] + \frac{L_2c_1}{m^2Y^*}(I-I^*)^2\\
&- \frac{L_1p_1S}{(mY+S)(mY^*+S^*)}(Y-Y^*)(S-S^*) - \frac{L_2c_1I(Y-Y^*)(I-I^*)}{(mY+I)(mY^*+I^*)}\\
&+ \delta_1hL_3\left[\frac{(S-S^*)(Y-Y^*)}{(S+I)(S^*+I^*)} - \frac{(Y-Y^*)^2(S-I)}{(S+I)(S^*+I^*)} - \frac{Y(I-I^*)(Y-Y^*)}{(S+I)(S^*+I^*)}\right]\\
&\leq -\left[\frac{L_1r_1}{k(1+fY^*)} - \frac{L_1p_1}{m^2Y^*}\right](S-S^*)^2 - (S-S^*)(I-I^*)\\
&\left(L_1\left(1+\frac{r_1}{k(1+fY^*)}\right) - L_2\right)
\end{aligned}
$$

$$+\Bigg[\frac{L_1 r_1 f Y^*}{k(1+fY^*)}(S-S^*)+\frac{L_2 c_1}{m^2 Y^*}(I-I^*)^2 - \frac{L_1 p_1 S}{(mY+S)(mY^*+S^*)}(Y-Y^*)(S-S^*)-\frac{L_2 c_1 I(Y-Y^*)(I-I^*)}{(mY+I)(mY^*+I^*)} + \delta_1 h L_3\Big\{\frac{(S-S^*)(Y-Y^*)}{(S+I)(S^*+I^*)}-\frac{(Y-Y^*)^2(S-I)}{(S+I)(S^*+I^*)}-\frac{Y(I-I^*)(Y-Y^*)}{(S+I)(S^*+I^*)}\Big\}\Bigg].$$

Suppose

$$L_2 = L_1\left(1+\frac{r_1}{k(1+fY^*)}\right)\frac{r_1}{k(1+fY^*)} > \frac{p_1}{m^2 Y^*},$$

and $\theta_1 < S, I, Y < \theta_2$.
We choose L_1 and L_3 such that

$$\frac{2\theta_2^2\theta_2 p_1 S^*\left(I^*+S^*\right)\left(S^*+Y^*\right)}{\delta_1\theta_1^4 h(m+1)\left(mY^*+S^*\right)} < \frac{L_3}{L_1} < \frac{2\theta_2 p_1 S^{*2} Y^*\left(I^*+S^*\right)}{\delta_1 h\theta_1(m+1)\left(mY^*+S^*\right)\left(Y^*\left(IY^*+S^*\right)+\theta_1^2\left(2Y^*+1\right)\right)}.$$

Then ${}_{t_0}^{c}D_t^{\alpha}\mathcal{W}(S,I,Y) < 0$.

□

Expressions of $\omega_i, (i=1,2,3)$ in equation (24),

$$\omega_1 = \frac{c_1 m z^2}{(mz+y)^2}+\gamma_1-\frac{r_1(k-2x-y)}{fkz+k}-\frac{\delta_1(-2hz+x+y)}{x+y}+\frac{m p_1 z^2}{(mz+x)^2}-x+y,$$

$$\omega_2 = -\frac{c_1 m r_1 z^2(k-2x-y)}{(fkz+k)(mz+y)^2}+\frac{c_1\delta_1 h y^2 z^2}{(x+y)^2(mz+y)^2}-\frac{c_1\delta_1 m z^2(-2hz+x+y)}{(x+y)(mz+y)^2}+\frac{c_1 m^2 p_1 z^4}{(mz+x)^2(mz+y)^2} + \frac{c_1 m y z^2}{(mz+y)^2}-\frac{\delta_1 h x z^2\left(\frac{f r_1(-k+x+y)}{k(fz+1)^2}-\frac{p_1 x}{(mz+x)^2}\right)}{(x+y)^2}-\frac{\delta_1 r_1(-k+2x+y)(-2hz+x+y)}{(x+y)(fkz+k)} - \frac{\gamma_1 r_1(k-2x-y)}{fkz+k}+\frac{xy\left(fkz+k+r_1\right)}{fkz+k}-\frac{r_1 x(-k+2x+y)}{fkz+k}-\frac{\delta_1 m p_1 z^2(-2hz+x+y)}{(x+y)(mz+x)^2} - \frac{\gamma_1\delta_1(-2hz+x+y)}{x+y}+\frac{\delta_1 x(-2hz+x+y)}{x+y}-\frac{\delta_1 y(-2hz+x+y)}{x+y}+\frac{\gamma_1 m p_1 z^2}{(mz+x)^2}-\frac{m p_1 x z^2}{(mz+x)^2} - xy+\gamma_1 y,$$

$$\omega_3 = \left(-\frac{\delta_1}{(x+y)^2}\right)\Bigg[\frac{c_1 m y z^2(x+y)(-2hz+x+y)}{(mz+y)^2}-\frac{c_1 h y^3 z^2}{(mz+y)^2}-xy(x+y)(-2hz+x+y) + \frac{c_1 m^2 p_1 z^4(x+y)(-2hz+x+y)}{(mz+x)^2(mz+y)^2}-\frac{c_1 h m p_1 y^2 z^4}{(mz+x)^2(mz+y)^2}-\frac{m p_1 x z^2(x+y)(-2hz+x+y)}{(mz+x)^2} + \frac{c_1 h r_1 y^2 z^2(k-2x-y)}{(fkz+k)(mz+y)^2}+\frac{r_1 x(x+y)(k-2x-y)(-2hz+x+y)}{fkz+k} + \frac{xy(x+y)\left(fkz+k+r_1\right)(-2hz+x+y)}{fkz+k}-\frac{c_1 m r_1 z^2(x+y)(k-2x-y)(-2hz+x+y)}{(fkz+k)(mz+y)^2} + \frac{c_1 h x y^2 z^2\left(fkz+k+r_1\right)}{(fkz+k)(mz+y)^2}-hxyz^2\left(\frac{p_1 x}{(mz+x)^2}-\frac{f r_1(-k+x+y)}{k(fz+1)^2}\right) + \frac{c_1 h m x z^4\left(\frac{f r_1(-k+x+y)}{k(fz+1)^2}-\frac{p_1 x}{(mz+x)^2}\right)}{(mz+y)^2}-hx^2z^2\left(\frac{f r_1(-k+x+y)}{k(fz+1)^2}-\frac{p_1 x}{(mz+x)^2}\right) + \frac{\gamma_1 m p_1 z^2(x+y)(-2hz+x+y)}{(mz+x)^2}+\gamma_1 y(x+y)(-2hz+x+y) + \gamma_1 h x z^2\left(\frac{f r_1(-k+x+y)}{k(fz+1)^2}-\frac{p_1 x}{(mz+x)^2}\right)-\frac{\gamma_1 r_1(x+y)(k-2x-y)(-2hz+x+y)}{fkz+k}\Bigg].$$

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RESEARCH ARTICLE

A computational approach for shallow water forced Korteweg–De Vries equation on critical flow over a hole with three fractional operators

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ABSTRACT

The Korteweg–De Vries (KdV) equation has always provided a venue to study and generalizes diverse physical phenomena. The pivotal aim of the study is to analyze the behaviors of forced KdV equation describing the free surface critical flow over a hole by finding the solution with the help of q-homotopy analysis transform technique (q-HATT). he projected method is elegant amalgamations of q-homotopy analysis scheme and Laplace transform. Three fractional operators are hired in the present study to show their essence in generalizing the models associated with power-law distribution, kernel singular, non-local and non-singular. The fixed-point theorem employed to present the existence and uniqueness for the hired arbitrary-order model and convergence for the solution is derived with Banach space. The projected scheme springs the series solution rapidly towards convergence and it can guarantee the convergence associated with the homotopy parameter. Moreover, for diverse fractional order the physical nature have been captured in plots. The achieved consequences illuminates, the hired solution procedure is reliable and highly methodical in investigating the behaviours of the nonlinear models of both integer and fractional order.

1. Introduction

Mankind is always looking for innovation, development, novelty, modernization and modification in science and technology to lead daily life in a convenient manner. In this connection, mathematics is the basic, essential and pivotal tool and which aid us to study, investigate and predict the essence of life associated with surrounding nature. Among this tool, calculus with differential and integral operators is the most efficient and favourable instrument and it has been recanalized most elegant discipline. Most of the concept in nature associated with the rate of change, variation and modification are necessitates differential calculus. Recently, many researchers came with limitations of classical concept particularly to capture power-law, non-local, non-singular, heterogeneities, exponential decay, fading memory, fatigue effect, and other functions. Later, mathematicians suggested an old tool and which is rooted soon after the classical concept, called fractional calculus (FC). Many senior pioneers prearranged the reputation of FC and proposed distinct

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notions and properties [1–15]. In a while, the fundamental theory and extensive claims of essential properties are broadly employed to model diverse physical mechanisms and everyday problems [16–22].

The essence of studying the mathematical models with differential equations of fractional and integer order is always a hot topic, and hence many researchers are magnetized towards the new approaches with numerical and analytical methods. For instance, authors in [23] find the invariant solution for Bogoyavlensky-Konopelchenko equation, the fractional-reaction diffusion trimolecular models is studied in [24], the fractional-order Gross–Pitaevskii equations are examined with the help of unified method by researchers in [25]. Similarly, authors derived interesting results for Calogero-Bogoyavlenskii-Schif [26] and coupled Korteweg–de Vries equations [27] with similar techniques. To show the essence of the Lie symmetry analysis, authors in [28] investigated about the Bratu Gelfand model, the effect of fractional derivatives are illustrated by authors to capture the stimulating results associated with fifth-order Schrodinger equation [29], COVID-19 [30], Maccari systems [31], chaotic system [32], the mathematical model of Tumour invasion and metastasis [33], and modified coupled Korteweg–de Vries system [34]. The Lump and optical solitons solutions are derived by researchers in [35] with the analytical method, and authors in [36] derived some stimulating results associated with bipartite graph and fractional operator. The projected method is hired by the scholars to investigate the system associated with Jaulent–Miodek system with energy-dependent Schrödinger potential [37], the epidemic model of childhood disease [38], liquids with gas bubbles models [39], the Zakharov–Kuznetsov equation in dusty plasma [40], and Degasperis–Procesi equations [41].

In a two-dimensional channel flow, the impact of bottom configurations on the free-surface waves is investigated with the help of the forced Korteweg-de Vries equation. The bottom topography plays a vital part in the study of shallow-water waves, and which can significantly evaluate the behaviours of wave motions [42, 43]. Shallow water or long waves are the waves in water shallower than 1/20 their actual wavelength. When the bottom configuration is more complex, the interplay between the bottom topography and solitary waves can demonstrate more stimulating dynamics of the free surface waves. When the rigid bottom of the channel has some obstacles and for an incompressible and inviscid fluid, the two-dimensional channel flow with free surface waves have been studied [44, 45]. Fluid flows over an obstacle, the forcing approximately with the KdV equation can portray the development of the free surface. The FKdV equation is very important while describing the nature sine Gordon equation as well as the nonlinear Schrödinger equation. Further, the proposed model has numerous applications in the connected branches of mathematics and physics. This equation is considered an essential tool to study the propagation of short laser pulses in optical fibres, atmosphere dynamics, geostrophic turbulence and magnetohydrodynamic waves [46, 47]. Particularly, it offers stimulating results associated to physical problems such as acoustic waves on a crystal lattice, tsunami waves over obstacles, and shallow-water waves over rocks.

Here, we consider the forced KdV equation with the free water surface elevation measured $u\left(x,t\right)$ on critical flow over a hole from undisturbed water level and which introduce and nurtured by Wu in 1987 [48], and presented as follows:

$$\frac{1}{c}\frac{\partial u}{\partial t}+\left[\left(F_r-1\right)-\frac{3}{2}\frac{u\left(x,t\right)}{h}\right]\frac{\partial u}{\partial x}-\frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}=\frac{1}{2}\frac{\partial f\left(x\right)}{\partial x},\qquad(1)$$

where F_r is Froude number and it also calls as the critical parameter, h is the sea mean water depth, $f\left(x\right)$ is the external forcing term and define as $f\left(x\right)=\frac{p_a(x)}{\rho g}+b\left(x\right)$. Here, $\frac{p_a(x)}{\rho g}$ is the surface air pressure, and $b\left(x\right)$ is rigid bottom topography and is defined by $b\left(x\right)=-0.1e^{-\frac{x^\beta}{4}}-1$. The Froude number $\left(F_r\right)$ plays an important role in Eq. (1), since its value elucidates the kind of critical flows over the localised obstacle. Specifically, for >1, $=1$ and <1 respectively represent the flow is considered as supercritical, trans-critical and subcritical. In the rigid bottom topography $b\left(x\right)$, two different kinds of hole examined, namely for $\beta=2$ and $\beta=8$. The behaviours of $b\left(x\right)$ for two distinct cases is cited in Figure 1. These cases respectively signify the hole is expected an inverse of bell-shaped and the hole is more flattened at the bottom as well as wider. Authors in [49], considered the simplified above equation by eliminating surface air pressure and presented it as follows

$$\frac{1}{c}\frac{\partial u}{\partial t}+\left[\left(F_r-1\right)-\frac{3}{2}\frac{u\left(x,t\right)}{h}\right]\frac{\partial u}{\partial x}-\frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}-\frac{1}{2}\frac{\partial b\left(x\right)}{\partial x}=0.\qquad(2)$$

In the literature, we have diverse fractional operators and the most familiarly used are including Riemann–Liouville (RL), Caputo [3], Caputo-Fabrizio (CF) [50] and Atangana–Baleanu (AB) [51] operators. However, mathematicians and scientists are always looking and searching for the tool which can help to derive and find the required consequences at a particular situation in specific context. In this regard, each earlier proposed concepts have their own confines. Including, the RL operator be unsuccessful to admit the universal truth of derivative and then M. Caputo suggested new notion which overcame this drawback. Recently, researchers cited some additional properties need to be incorporate with this operator and many new fractional operators with their own merits are suggested by mathematicians.

Recently, many researchers are hired them as generalizing tool to investigate diverse phenomena and achieved some stimulating consequences [6, 16, 43]. Particularly, these operators aid us to investigate the long-range memory, heterogeneities, exponential decay and non-local and non-singular, non-Gaussian without a steady-state and crossover behaviour. Now, we consider the fractional-order forced KdV (FF-KdV) equation by trading the time derivative with three fractional operators. Now, the FF-KdV equation is defined as follows

$$D_t^{\alpha} u(x,t) = -c\left(\left[(F_r-1) - \tfrac{3}{2}\tfrac{u(x,t)}{h}\right]\tfrac{\partial u}{\partial x} - \tfrac{h^2}{6}\tfrac{\partial^3 u}{\partial x^3} - \tfrac{1}{2}\tfrac{\partial b(x)}{\partial x}\right), \quad (3)$$

$${}_0^{CF}D_t^{\alpha} u(x,t) = -c\left(\left[(F_r-1) - \tfrac{3}{2}\tfrac{u(x,t)}{h}\right]\tfrac{\partial u}{\partial x} - \tfrac{h^2}{6}\tfrac{\partial^3 u}{\partial x^3} - \tfrac{1}{2}\tfrac{\partial b(x)}{\partial x}\right), \quad (4)$$

$${}_a^{ABC}D_t^{\alpha} u(x,t) = -c\left(\left[(F_r-1) - \tfrac{3}{2}\tfrac{u(x,t)}{h}\right]\tfrac{\partial u}{\partial x} - \tfrac{h^2}{6}\tfrac{\partial^3 u}{\partial x^3} - \tfrac{1}{2}\tfrac{\partial b(x)}{\partial x}\right), \quad (5)$$

where α ($0<\alpha\le 1$) is fractional-order. The considered model offers an interesting insight into diverse physical phenomena and hence it magnetizes researchers with different tools to present their viewpoints with corresponding derived consequences. For instance, authors in model [52] find the analytic solutions to the projected model; author in [53] presents some interesting result for the proposed model; considering the model for waves generated by topography, authors in [49,54] find the approximated analytical solution by using the HAM; authors in [55] investigated the considered problem and presented dynamics of trapped solitary waves; lines and pseudospectral solutions has been investigated by authors in [56].

The hired scheme is a blend of Laplace transform (LT) with homotopy based scheme [57, 58]. The uniqueness of q-HATT is that it does not require assumptions, perturbations, conversion of nonlinear to linear and PDE to ODE [59]. Moreover, it is the generalization of many methods (results attained by this technique is a particular case for the value of parameters associated to method). The projected algorithm has been employed due to its efficiency and accuracy to examine the extensive classes of complex as well as nonlinear models and problems and also for the system of equations [60–67]. Recently, many interesting consequences are derived by using the projected scheme while analyzing the real-world problem.

The rest of the manuscript is systematized as follows: We followed the next section by basics and essential notions of both FC and LT. In Section 3, the solution for the hired model with three fractional operators are presented and also the existence and uniqueness of solutions with two fractional operators for the model is presented using Banach space within the frame of fixed-point theory. With the aid of attained outcomes and corresponding consequences, the discussion about the results is presented in Section 4 and finally, the concluding remarks on the present study are presented in the lost segment.

2. Preliminaries

Here, we recall few basic notions of FC [3, 50, 51, 68, 69]:

Definition 1. The Caputo fractional derivative of $f \in C_{-1}^{n}$ is presented for $n \in \mathbb{N}$ as

$$D_t^{\alpha} f(t) = \frac{d^n f(t)}{dt^n}, \qquad \alpha = n \in \mathbb{N},$$
$$= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\vartheta)^{n-\alpha-1} f^{(n)}(\vartheta)\, d\vartheta,\ n-1<\alpha<n\,. \quad (6)$$

Definition 2. The fractional Caputo-Fabrizio (CF) derivative in Caputo sense for a function $f \in H^1(a,\ b)\,(b>a)$ is presented as follows [68]

$${}_0^{CFC}D_t^{\alpha}(f(t)) = \frac{\mathcal{M}[\alpha]}{1-\alpha}\int_0^t f'(\vartheta)\, exp\left[-\frac{\alpha(t-\vartheta)}{1-\alpha}\right] d\vartheta, \quad (7)$$

where $\mathcal{M}[\alpha]$ ($\mathcal{M}[0] = \mathcal{M}[1] = 1$) is a normalization function.

Definition 3. The fractional Atangana-Baleanu-Caputo derivative for $f \in H^1(a,\ b)\,(b>a)$ is

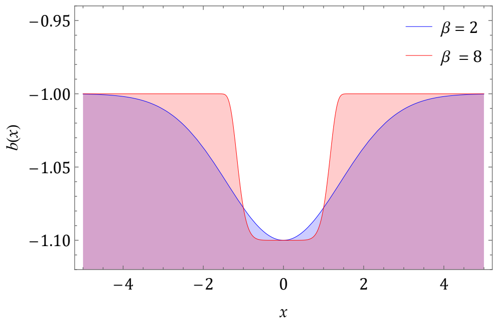


Figure 1. Nature of $b\left(x\right)$ at $\beta=2$ and 8.

$$ {}_{a}^{ABC}D_{t}^{\alpha}\left(f\left(t\right)\right) = \frac{\mathcal{M}\left[\alpha\right]}{1-\alpha}\int_{a}^{t} f^{'}\left(\vartheta\right)E_{\alpha}\left[-\frac{\alpha(t-\vartheta)^{\alpha}}{1-\alpha}\right]d\vartheta. \tag{8}$$

Definition 4. The fractional AB integral is presented as

$$ {}_{a}^{AB}I_{t}^{\alpha}\left(f\left(t\right)\right)=\frac{1-\alpha}{\mathcal{M}[\alpha]}f\left(t\right) +\frac{\alpha}{\mathcal{M}[\alpha]\Gamma(\alpha)}\int_{a}^{t}f\left(\vartheta\right)\left(t-\vartheta\right)^{\alpha-1}d\vartheta. \tag{9}$$

Definition 5. The Laplace transform (LT) for a Caputo fractional derivative $D_{t}^{\alpha}f\left(t\right)$ is defined for $\left(n-1<\alpha\leq n\right)$, as

$$\mathcal{L}\left[D_{t}^{\alpha}f\left(t\right)\right]=s^{\alpha}F\left(s\right)-\sum_{r=0}^{n-1}s^{\alpha-r-1}f^{(r)}\left(0^{+}\right), \tag{10}$$

where $F\left(s\right)$ is LT of $f(t)$.

Note: According to [68], the following must hold

$$\frac{2\left(1-\alpha\right)}{\left(2-\alpha\right)\mathcal{M}\left(\alpha\right)}+\frac{2\alpha}{\left(2-\alpha\right)\mathcal{M}\left(\alpha\right)}=1,\ \ 0<\alpha<1, \tag{11}$$

which gives $\mathcal{M}\left(\alpha\right)=\frac{2}{2-\alpha}$. By the assist of the above equation researchers in [68] proposed a novel Caputo derivative for $0<\alpha<1$ as follows

$$D_{t}^{\alpha}\left(f\left(t\right)\right)=\frac{1}{1-\alpha}\int_{0}^{t}f^{'}\left(t\right)\exp\left[\alpha\frac{t-\vartheta}{1-\alpha}\right]d\vartheta. \tag{12}$$

Definition 6. The LT for a CF derivative ${}_{0}^{CFC}\ D_{t}^{\alpha}f\left(t\right)$ is presented as below

$$\mathcal{L}\left[{}_{0}^{CFC}\ D_{t}^{(\alpha+n)}f\left(t\right)\right] = \frac{s^{n+1}\mathcal{L}[f(t)]-s^{n}f(0)-s^{n-1}f^{'}(0)-\cdots-f^{n}(0)}{s+(1-s)\alpha}. \tag{13}$$

Definition 7. The LT of AB derivative is defined by

$$\mathcal{L}\left[{}_{0}^{ABR}D_{t}^{\alpha}\left(f\left(t\right)\right)\right] = \frac{\mathcal{B}\left[\alpha\right]}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\left[f\left(t\right)\right]-s^{\alpha-1}f\left(0\right)}{s^{\alpha}+\left(\alpha/(1-\alpha)\right)}. \tag{14}$$

Theorem 1. *The Lipschitz conditions for the RL and AB derivatives are respectively held the following results [51]*

$$\left\|{}_{a}^{ABR}\ D_{t}^{\alpha}f_{1}\left(t\right)-{}_{a}^{ABR}\ D_{t}^{\alpha}f_{2}\left(t\right)\right\| < K_{1}\left\|f_{1}\left(x\right)-f_{2}\left(x\right)\right\|, \tag{15}$$

and

$$\left\|{}_{a}^{ABC}\ D_{t}^{\alpha}f_{1}\left(t\right)-{}_{a}^{ABC}\ D_{t}^{\alpha}f_{2}\left(t\right)\right\| < K_{2}\left\|f_{1}\left(x\right)-f_{2}\left(x\right)\right\|. \tag{16}$$

Theorem 2. *The fractional-order differential equation* ${}_{a}^{ABC}\ D_{t}^{\alpha}f_{1}\left(t\right)=s\left(t\right)$ *has a unique solution [51] and which is*

$$f\left(t\right)=\frac{1-\alpha}{\mathcal{B}[\alpha]}s\left(t\right) +\frac{\alpha}{\mathcal{B}[\alpha]\Gamma(\alpha)}\int_{0}^{t}s\left(\varsigma\right)\left(t-\varsigma\right)^{\alpha-1}d\varsigma. \tag{17}$$

3. Solution for FKDV equation

The considered solution procedure is presented for the FKDV equation with three fractional operators to find the series solution. Further, for both CF and AB fractional operators existence and uniqueness is derived with fixed point theory.

3.1. Caputo Sense

Consider the equation defined in Eq. (3)

$$D_{t}^{\alpha}u\left(x,t\right)+c\left(\left[\left(F_{r}-1\right)-\frac{3}{2}\frac{u}{h}\right]\frac{\partial u}{\partial x} -\frac{h^{2}}{6}\frac{\partial^{3}u}{\partial x^{3}}-\frac{1}{2}\frac{\partial b(x)}{\partial x}\right)=0, \tag{18}$$

with

$$u\left(x,0\right)=-\frac{2e^{x}}{\left(1+e^{x}\right)^{2}}. \tag{19}$$

Taking LT on Eq. (18) and using Eq. (19), we get

$$\mathcal{L}\left[u\left(x,t\right)\right]=\tfrac{1}{s}\left(-\tfrac{2e^{x}}{(1+e^{x})^{2}}\right) \tag{20}$$
$$-\tfrac{c}{s^{\alpha}}\mathcal{L}\left\{\left[(F_{r}-1)-\tfrac{3}{2}\tfrac{u}{h}\right]\tfrac{\partial u}{\partial x}-\tfrac{h^{2}}{6}\tfrac{\partial^{3}u}{\partial x^{3}}-\tfrac{1}{2}\tfrac{\partial b(x)}{\partial x}\right\}.$$

Now, $\mathcal{N}$ is presented as

$$\mathcal{N}\left[\varphi\left(x,t;q\right)\right]=\mathcal{L}\left[\varphi\left(x,t;q\right)\right]-\tfrac{1}{s}\left(-\tfrac{2e^{x}}{(1+e^{x})^{2}}\right)$$
$$+\tfrac{c}{s^{\alpha}}\mathcal{L}\{\left[(F_{r}-1)-\tfrac{3}{2}\tfrac{\varphi(x,t;q)}{h}\right]\tfrac{\partial\varphi(x,t;q)}{\partial x}$$
$$-\tfrac{h^{2}}{6}\tfrac{\partial^{3}\varphi(x,t;q)}{\partial x^{3}}-\tfrac{1}{2}\tfrac{\partial b(x)}{\partial x}\}. \tag{21}$$

At $\mathcal{H}(x,t)=1$, the deformation equation presented as

$$\mathcal{L}\left[u_{m}\left(x,t\right)-k_{m}u_{m-1}\left(x,t\right)\right]=\hbar\mathfrak{R}_{m}\left[\overrightarrow{u}_{m-1}\right], \tag{22}$$

where

$$\mathfrak{R}_{m}\left[\overrightarrow{u}_{m-1}\right]=\mathcal{L}\left[u_{m-1}\left(x,t\right)\right]$$
$$-\left(1-\frac{k_{m}}{n}\right)\left\{\frac{1}{s}\left(-\frac{2e^{x}}{(1+e^{x})^{2}}\right)\right\}$$
$$+\frac{c}{s^{\alpha}}\mathcal{L}\{(F_{r}-1)\frac{\partial u_{m-1}}{\partial x}-\frac{3}{2h}\sum_{i=0}^{m-1}u_{i}\frac{\partial u_{m-1-i}}{\partial x}$$
$$-\frac{h^{2}}{6}\frac{\partial^{3}u_{m-1}}{\partial x^{3}}-\frac{1}{2}\frac{\partial b\left(x\right)}{\partial x}\}. \tag{23}$$

On employing inverse LT on Eq. (22), one can get

$$u_{m}\left(x,t\right)=k_{m}u_{m-1}\left(x,\ t\right)+\hbar\mathcal{L}^{-1}\left\{\mathfrak{R}_{m}\left[\overrightarrow{u}_{m-1}\right]\right\}. \tag{24}$$

On simplifying the above equations by assist of $u_{0}\left(x,t\right)=-\frac{2e^{x}}{(1+e^{x})^{2}}$ we can evaluate terms of series

$$u\left(x,t\right)=u_{0}\left(x,t\right)+\sum_{m=1}^{\infty}u_{m}\left(x,t\right)\left(\frac{1}{n}\right)^{m}. \tag{25}$$

as

$$u_{1}\left(x,\ t\right)=\frac{\hbar t^{\alpha}}{\Gamma\left[\alpha+1\right]}(c(\frac{6\mathrm{e}^{2x}(-1+\mathrm{e}^{x})}{(1+\mathrm{e}^{x})^{5}h}$$
$$-\frac{\mathrm{e}^{x}(-1+11\mathrm{e}^{x}-11\mathrm{e}^{2x}+\mathrm{e}^{3x})h^{2}}{3(1+\mathrm{e}^{x})^{5}}-0.025\mathrm{e}^{-\frac{x^{2}}{4}}x$$
$$+\frac{2\mathrm{e}^{x}(-1+\mathrm{e}^{x})(-1+F_{r})}{(1+\mathrm{e}^{x})^{3}})),$$
$$\vdots$$

3.2. Caputo-Fabrizio Sense

Consider the equation defined in Eq. (4)

$${}_{0}^{CF}D_{t}^{\alpha}u\left(x,t\right)+c(\left[(F_{r}-1)-\frac{3}{2}\frac{u}{h}\right]$$
$$\frac{\partial u}{\partial x}-\frac{h^{2}}{6}\frac{\partial^{3}u}{\partial x^{3}}-\frac{1}{2}\frac{\partial b\left(x\right)}{\partial x})=0, \tag{26}$$

with initial conditions Eq. (19). Taking LT on Eq. (26) and by the assist of Eq. (19), we get

$$\mathcal{L}\left[u\left(x,t\right)\right]=\frac{1}{s}\left(-\frac{2e^{x}}{(1+e^{x})^{2}}\right)-c\frac{s+(1-s)\alpha}{s}$$
$$\mathcal{L}\left\{\left[(F_{r}-1)-\frac{3}{2}\frac{u}{h}\right]\frac{\partial u}{\partial x}-\frac{h^{2}}{6}\frac{\partial^{3}u}{\partial x^{3}}-\frac{1}{2}\frac{\partial b\left(x\right)}{\partial x}\right\}. \tag{27}$$

Now, $\mathcal{N}$ is defined as

$$\mathcal{N}\ \left[\varphi\left(x,t;q\right)\right]=\mathcal{L}\left[\varphi\left(x,t;q\right)\right]-\tfrac{1}{s}\left(-\tfrac{2e^{x}}{(1+e^{x})^{2}}\right)$$
$$+c\ \tfrac{s+(1-s)\alpha}{s}\mathcal{L}\{\left[(F_{r}-1)-\tfrac{3}{2}\tfrac{\varphi(x,t;q)}{h}\right]$$
$$\tfrac{\partial\varphi(x,t;q)}{\partial x}-\tfrac{h^{2}}{6}\tfrac{\partial^{3}\varphi(x,t;q)}{\partial x^{3}}-\tfrac{1}{2}\tfrac{\partial b(x)}{\partial x}\}. \tag{28}$$

At $\mathcal{H}(x,t)=1$, the deformation equation presented as

$$\mathcal{L}\left[u_{m}\left(x,t\right)-k_{m}u_{m-1}\left(x,t\right)\right]=\hbar\mathfrak{R}_{m}\left[\overrightarrow{u}_{m-1}\right], \tag{29}$$

where

$$\mathfrak{R}_{m}\left[\overrightarrow{u}_{m-1}\right]=\mathcal{L}\left[u_{m-1}\left(x,t\right)\right]-\left(1-\frac{k_{m}}{n}\right)$$
$$\left\{\frac{1}{s}\left(-\frac{2e^{x}}{(1+e^{x})^{2}}\right)\right\}+c\ \frac{s+(1-s)\alpha}{s} \tag{30}$$
$$\mathcal{L}\{(F_{r}-1)\frac{\partial u_{m-1}}{\partial x}-\frac{3}{2h}\sum_{i=0}^{m-1}u_{i}\frac{\partial u_{m-1-i}}{\partial x}$$
$$-\frac{h^{2}}{6}\frac{\partial^{3}u_{m-1}}{\partial x^{3}}-\frac{1}{2}\frac{\partial b\left(x\right)}{\partial x}\}.$$

Now, by the help of the initial condition, we can derive the terms of Eq. (19) as

$$u_{1}\left(x,\ t\right)=\hbar(1-\alpha+\alpha t)(c(\frac{6\mathrm{e}^{2x}\left(-1+\mathrm{e}^{x}\right)}{(1+\mathrm{e}^{x})^{5}h}$$
$$-\frac{\mathrm{e}^{x}\left(-1+11\mathrm{e}^{x}-11\mathrm{e}^{2x}+\mathrm{e}^{3x}\right)h^{2}}{3(1+\mathrm{e}^{x})^{5}}-0.025\mathrm{e}^{-\frac{x^{2}}{4}}x$$
$$+\frac{2\mathrm{e}^{x}\left(-1+\mathrm{e}^{x}\right)\left(-1+F_{r}\right)}{(1+\mathrm{e}^{x})^{3}})),$$
$$\vdots$$

Here, the existence and uniqueness are illustrated with CF operator for the considered Eq. (26) as

$${}_{0}^{CF}D_{t}^{\alpha}\left[u\left(x,\ t\right)\right]=\mathcal{Q}\left(x,t,u\right), \tag{31}$$

Now, using Eq. (31) and results derived in [46], we obtained

$$u(x,\ t)-u(x,\ 0) = {}_0^{CF}I_t^{\alpha}\{-c(\left[(F_r-1)-\frac{3}{2}\frac{u(x,t)}{h}\right]\frac{\partial u}{\partial x} -\frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x})\}, \tag{32}$$

Then we have from [41] as follows

$$u(x,\ t)-u(x,\ 0)=\frac{2(1-\alpha)}{\mathcal{M}(\alpha)}\mathcal{Q}(x,t,u) +\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t \mathcal{Q}(x,\ \zeta,u)\,d\zeta. \tag{33}$$

Theorem 3. *The kernel* $\mathcal{Q}$ *admits the Lipschitz condition and contraction if* $0 \le \left(c(F_r-1)\Lambda-\frac{3}{4h}\Lambda(a_1+a_2)-\frac{h^2}{6}\Lambda^3-\frac{1}{2}\Lambda\xi\right) < 1$ *satisfies.*

Proof. Consider the two functions A and A_1 to prove the theorem, then

$$\begin{aligned}
&\|\mathcal{Q}(x,t,u)-\mathcal{Q}(x,\ t,\ u_1)\| \\
&=\|c((F_r-1)\frac{\partial}{\partial x}[u(x,t)-u(x,t_1)] \\
&-\frac{3}{2h}\left[u(x,t)\frac{\partial u(x,t)}{\partial x}-u(x,t_1)\frac{\partial u(x,t_1)}{\partial x}\right] \\
&-\frac{h^2}{6}\frac{\partial^3}{\partial x^3}[u(x,t)-u(x,t_1)]-\frac{1}{2}\frac{\partial b(x)}{\partial x}\| \\
&=\|c((F_r-1)\frac{\partial}{\partial x}[u(x,t)-u(x,t_1)] \\
&-\frac{3}{2h}\left[\frac{1}{2}\frac{\partial}{\partial x}\left[u^2(x,t)-u^2(x,t_1)\right]\right] \\
&-\frac{h^2}{6}\frac{\partial^3}{\partial x^3}[u(x,t)-u(x,t_1)]-\frac{1}{2}\frac{\partial b(x)}{\partial x}\| \\
&\le \|c(F_r-1)\Lambda-\frac{3}{4h}\Lambda[u(x,t)-u(x,t_1)] \\
&-\frac{h^2}{6}\Lambda^3-\frac{1}{2}\frac{\partial b(x)}{\partial x}\|\,\|u(x,t)-u(x,t_1)\| \\
&\le c((F_r-1)\Lambda-\frac{3}{4h}\Lambda(a_1+a_2) \\
&-\frac{h^2}{6}\Lambda^3-\frac{1}{2}\Lambda\xi)\,\|u(x,t)-u(x,t_1)\|,
\end{aligned} \tag{34}$$

where $a_1 = \|u\|$ and $a_2 = \|u_1\|$ be the bounded function and $\|b(x)\| = \xi$ is also a bounded function. Set $\Psi = c\left((F_r-1)\Lambda-\frac{3}{4h}\Lambda(a_1+a_2)-\frac{h^2}{6}\Lambda^3-\frac{1}{2}\Lambda\xi\right)$ in Eq. (34), then

$$\|\mathcal{Q}(x,\ t,u)-\mathcal{Q}(x,\ t,u_1)\| \le \Psi\,\|u(x,\ t)-u(x,\ t_1)\|. \tag{35}$$

Eq. (35) provides the Lipschitz condition for $\mathcal{Q}$. Similarly, we can see that if $0 \le c\left((F_r-1)\Lambda-\frac{3}{4h}\Lambda(a_1+a_2)-\frac{h^2}{6}\Lambda^3-\frac{1}{2}\Lambda\xi\right) < 1$, then it implies the contraction. By the assist of the above equations, Eq. (33) simplifies to

$$u(x,\ t)=u(x,\ 0)+\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{Q}(x,\ t,u) +\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t \mathcal{Q}(x,\ \zeta,u)\,d\zeta. \tag{36}$$

Then we get the recursive form as follows

$$u_n(x,\ t)=\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\mathcal{Q}(x,\ t,u_{n-1}) +\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t \mathcal{Q}(x,\ \zeta,u_{n-1})\,d\zeta. \tag{37}$$

Now, between the terms the successive difference is defined as

$$\begin{aligned}
&\phi_n(x,\ t)=u_n(x,\ t)-u_{n-1}(x,\ t) \\
&=\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\left(\mathcal{Q}(x,t,u_{n-1})-\mathcal{Q}(x,t,u_{n-2})\right) \\
&+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t (\mathcal{Q}(x,t,u_{n-1}) \\
&-\mathcal{Q}(x,t,u_{n-2}))d\zeta.
\end{aligned} \tag{38}$$

Notice that

$$u_n(x,\ t)=\sum_{i=1}^{n}\phi_i(x,\ t). \tag{39}$$

Then we have

$$\begin{aligned}
&\|\phi_n(x,\ t)\|=\|u_n(x,\ t)-u_{n-1}(x,\ t)\| \\
&=\left\|\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}(\mathcal{Q}(x,\ t,u_{n-1}))\right. \\
&-\mathcal{Q}(x,\ t,u_{n-2})+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \\
&\left.\int_0^t (\mathcal{Q}(x,\ t,u_{n-1})-\mathcal{Q}(x,\ t,u_{n-2}))\,d\zeta\right\|.
\end{aligned} \tag{40}$$

Application of the triangular inequality, Eq. (40) reduces to

$$\begin{aligned}
&\|\phi_n(x,\ t)\| = \|u_n(x,\ t) - u_{n-1}(x,\ t)\| \\
&= \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} \\
&\|(\mathcal{Q}(x,\ t, u_{n-1}) - \mathcal{Q}(x,\ t, u_{n-2}))\| \\
&+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \\
&\left\|\int_0^t (\mathcal{Q}(x,\ t, u_{n-1}) - \mathcal{Q}(x,\ t, u_{n-2}))\, d\zeta\right\|.
\end{aligned} \tag{41}$$

The Lipschitz condition satisfied by the kernel t_1, then

$$\begin{aligned}
&\|\phi_n(x,\ t)\| = \|u_n(x,\ t) - u_{n-1}(x,\ t)\| \\
&\leq \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\Psi \left\|\phi_{(n-1)}(x,\ t)\right\| \\
&+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\Psi \int_0^t \left\|\phi_{(n-1)}(x,\ t)\right\| d\zeta.
\end{aligned} \tag{42}$$

□

By the aid of the above result, we state the following result:

Theorem 4. *If we have specific t_0, then the solution for Eq. (26) will exist and unique. Further, we have*

$$\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\Psi + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\Psi t_0 < 1.$$

Proof. Let $u(x,t)$ is the bounded functions admitting the Lipschitz condition. Then, we get by Eqs . (41) and (42)

$$\begin{aligned}
&\|\phi_i(x,\ t)\| \leq \|u_n(x,0)\| \\
&\left[\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\Psi + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\Psi\ t\right]^n.
\end{aligned} \tag{43}$$

Therefore, for the obtained solution, continuity and existence are verified. Now, to prove the Eq. (43) is a solution for Eq. (26), we consider

$$u(x,t) - u(x,0) = u_n(x,\ t) - \mathcal{K}_n(t). \tag{44}$$

Let us consider

$$\begin{aligned}
&\|\mathcal{K}_n(t)\| = \|\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}(\mathcal{Q}(x,\ t, u) - \mathcal{Q}(x,\ t,\ u_{n-1})) \\
&+\ \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t (\mathcal{Q}(x,\ \zeta, u) - \mathcal{Q}(x,\ \zeta, u_{n-1}))\, d\zeta\| \\
&\leq\ \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\|(\mathcal{Q}(x,\ t, u) - \mathcal{Q}(x,\ t,\ u_{n-1}))\| \\
&+\ \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t \|(\mathcal{Q}(x,\ \zeta, u) - \mathcal{Q}(x,\ \zeta, u_{n-1}))\|\, d\zeta \\
&\leq \tfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\Psi \|u - u_{n-1}\| \\
&+\tfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\Psi \|u - u_{n-1}\|\, t.
\end{aligned} \tag{45}$$

This process gives

$$\begin{aligned}
&\|\mathcal{K}_n(t)\| \\
&\leq \left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}t\right)^{n+1} \\
&\Psi^{n+1}M.
\end{aligned}$$

Similarly, at t_0 we can obtain

$$\begin{aligned}
&\|\mathcal{K}_n(t)\| \leq \\
&\left(\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}t_0\right)^{n+1}\Psi^{n+1}M.
\end{aligned} \tag{46}$$

As $n \to \infty$, from Eq. (46), $\|\mathcal{K}_n(t)\| \to 0$ provided $\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)} + \frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}t_0 < 1$. Next, for the solution of the projected model, we prove the uniqueness. Suppose $u^*(x,\ t)$ is another solution, then

$$\begin{aligned}
&u(x,t) - u^*(x,t) \\
&= \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}(\mathcal{Q}(x,t,u) - \mathcal{Q}(x,t,\ u^*)) \\
&+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\int_0^t (\mathcal{Q}(x,\zeta,u) - \mathcal{Q}(x,\zeta,u^*))\, d\zeta.
\end{aligned} \tag{47}$$

Now, employing the norm on the above equation we get

$$\begin{aligned}
&\|u(x,t) - u^*(x,t)\| \\
&= \left\|\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}(\mathcal{Q}(x,t,u) - \mathcal{Q}(x,\ t,\ u^*))\right. \\
&\quad+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)} \\
&\left.\int_0^t (\mathcal{Q}(x,\zeta,u) - \mathcal{Q}(x,\zeta,u^*))\, d\zeta\right\| \\
&\leq \frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\Psi \|u(x,t) - u^*(x,t)\| \\
&+\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\Psi t \|u(x,t) - u^*(x,t)\|.
\end{aligned} \tag{48}$$

On simplification

$$\begin{aligned}
&\|u(x,t) - u^*(x,t)\| \\
&\left(1 - \tfrac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\Psi - \tfrac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\Psi\ t\right) \leq 0.
\end{aligned} \tag{49}$$

From the above condition, it is clear that $u(x,t) = u^*(x,t)$, if

$$\left(1-\frac{2(1-\alpha)}{(2-\alpha)\mathcal{M}(\alpha)}\Psi-\frac{2\alpha}{(2-\alpha)\mathcal{M}(\alpha)}\Psi t\right)\geq 0. \tag{50}$$

Hence, Eq. (50) proves our required result. □

3.3. Atangana-Baleanu Sense

Consider the equation defined in Eq. (5)

$$ {}_{a}^{ABC}D_t^{\alpha}u(x,t)+c(\left[(F_r-1)-\frac{3}{2}\frac{u(x,t)}{h}\right]\frac{\partial u}{\partial x} -\frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x})=0,\ 0<\alpha\leq 1, \tag{51}$$

with initial conditions (19). Taking LT on Eq. (51) and using Eq. (19), we have

$$L[u(x,t)]=\frac{1}{s}\left(-\frac{2e^x}{(1+e^x)^2}\right)-\frac{c}{\mathcal{B}[\alpha]} \left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{\left[(F_r-1)-\frac{3}{2}\frac{u}{h}\right]\frac{\partial u}{\partial x} -\frac{h^2}{6}\frac{\partial^3 u}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x}\}. \tag{52}$$

Now, $\mathcal{N}$ is defined as

$$\begin{aligned}&\mathcal{N}[\varphi(x,t;q)]\\ &=L[\varphi(x,t;q)]+\frac{1}{s}\left(\frac{2e^x}{(1+e^x)^2}\right)\\ &+\frac{c}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)\\ &L\{\left[(F_r-1)-\frac{3}{2}\frac{\varphi(x,t;q)}{h}\right]\frac{\partial\varphi(x,t;q)}{\partial x}\\ &-\frac{h^2}{6}\frac{\partial^3\varphi(x,t;q)}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x}\}.\end{aligned} \tag{53}$$

The deformation equation at $\mathcal{H}(x,t)=1$, is given as follows

$$L[u_m(x,t)-k_m u_{m-1}(x,t)]=\hbar\mathfrak{R}_m[\overrightarrow{u}_{m-1}], \tag{54}$$

where

$$\begin{aligned}&\mathfrak{R}_m[\overrightarrow{u}_{m-1}]\\ &=L[u_{m-1}(x,t)]+\left(1-\frac{k_m}{n}\right)\left\{\frac{1}{s}\left(\frac{2e^x}{(1+e^x)^2}\right)\right\}\\ &+\frac{c}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{(F_r-1)\frac{\partial u_{m-1}}{\partial x}\\ &-\frac{3}{2h}\sum_{i=0}^{m-1}u_i\frac{\partial u_{m-1-i}}{\partial x}-\frac{h^2}{6}\frac{\partial^3 u_{m-1}}{\partial x^3}-\frac{1}{2}\frac{\partial b(x)}{\partial x}\}.\end{aligned} \tag{55}$$

Now, by the help of the initial condition, we can derive

$$u_1(x,\ t)=\hbar\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma[\alpha+1]}\right)$$

$$(c(\frac{6e^{2x}(-1+e^x)}{(1+e^x)^5 h}-\frac{e^x\left(-1+11e^x-11e^{2x}+e^{3x}\right)h^2}{3(1+e^x)^5}$$

$$-0.025e^{-\frac{x^2}{4}}x+\frac{2e^x(-1+e^x)(-1+F_r)}{(1+e^x)^3})),$$

$$\vdots$$

In the segment, the existence and uniqueness are illustrated for the considered equation associated with AB operator. We have from Eq. (51),

$$ {}_{a}^{ABC}D_t^{\alpha}u(x,t)=\mathcal{G}(x,t,u), \tag{56}$$

and the above equation is considered as

$$ {}_{0}^{ABC}D_t^{\alpha}[u(x,t)]=\mathcal{G}(x,t,u). \tag{57}$$

We have from Eq. (57) and Theorem 2

$$u(x,t)-u(x,0)=\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}(x,t,u) +\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t\mathcal{G}(x,\zeta,u)(t-\zeta)^{\alpha-1}d\zeta. \tag{58}$$

Theorem 5. *The kernel $\mathcal{G}$ admits the Lipschitz condition and contraction if* $0\leq\left(c\left((F_r-1)\delta-\frac{3}{4h}\delta(a+b)-\frac{h^2}{6}\delta^3-\frac{1}{2}\delta\xi\right)\right)<1$ *satisfies.*

Proof. To prove the theorem, let us consider the two functions u and u_1, then

$$\begin{aligned}&\|\mathcal{G}(x,t,\ u)-\mathcal{G}(x,t,\ u_1)\|\\ &=\|\ c((F_r-1)\frac{\partial}{\partial x}[u(x,t)-u(x,t_1)]\\ &-\frac{3}{2h}\left[u(x,t)\frac{\partial u(x,t)}{\partial x}-u(x,t_1)\frac{\partial u(x,t_1)}{\partial x}\right]\\ &-\frac{h^2}{6}\frac{\partial^3}{\partial x^3}[u(x,t)-u(x,t_1)]-\frac{1}{2}\frac{\partial b(x)}{\partial x}\bigg\|\\ &\leq\bigg\|c(F_r-1)\delta-\frac{3}{4h}\delta[u(x,t)-u(x,t_1)]\\ &-\frac{h^2}{6}\delta^3-\frac{1}{2}\frac{\partial b(x)}{\partial x}\bigg\|\ \|u(x,t)-u(x,t_1)\|\\ &\leq c\left((F_r-1)\delta-\frac{3}{4h}\delta(a+b)-\frac{h^2}{6}\delta^3-\frac{1}{2}\delta\xi\right)\\ &\times\|u(x,t)-u(x,t_1)\|,\end{aligned} \tag{59}$$

where $a=\|u\|$, $b=\|u_1\|$ (since u and u_1 are the bounded functions) and $\|b(x)\|=\xi$ is also a bounded function. Putting $\eta=$

$c\left(\left(F_r-1\right)\delta-\frac{3}{4h}\delta\left(a+b\right)-\frac{h^2}{6}\delta^3-\frac{1}{2}\delta\xi\right)$ in Eq. (59), then

$$\begin{aligned}&\left\|\mathcal{G}\left(x,t,u\right)-\mathcal{G}\left(x,t,u_1\right)\right\|\\&\leq\eta\left\|u\left(x,t\right)-u(x,t_1)\right\|.\end{aligned}\tag{60}$$

By the assist of forgoing relation, the Lipschitz condition is achieved for $\mathcal{G}$. Further, we can see that if $0\ \leq\left(\frac{\sigma^2}{2}\delta^2+\lambda\left(a^2+b^2+ab\right)\right)<1$, which leads to contraction. The recursive form of Eq. (60) is presented as

$$\begin{aligned}u_n\left(x,t\right)=&\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\mathcal{G}\left(x,t,u_{n-1}\right)\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\int_0^t\mathcal{G}\left(x,\zeta,u_{n-1}\right)\left(t-\zeta\right)^{\alpha-1}d\zeta,\end{aligned}\tag{61}$$

and initial condition

$$u\left(x,0\right)=u_0\left(x,t\right).\tag{62}$$

The successive difference between the terms is presented as

$$\begin{aligned}\phi_n\left(x,t\right)&=u_n\left(x,t\right)-u_{n-1}\left(x,t\right)\\&=\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\left(\mathcal{G}_1\left(x,t,u_{n-1}\right)-\mathcal{G}\left(x,t,u_{n-2}\right)\right)\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\int_0^t\mathcal{G}\left(x,\zeta,u_{n-1}\right)\left(t-\zeta\right)^{\alpha-1}d\zeta.\end{aligned}\tag{63}$$

Notice that

$$u_n\left(x,t\right)=\sum_{i=1}^{n}\phi_i\left(x,t\right).\tag{64}$$

Plugging the norm on the Eq. (63), and by the assist of Eq. (58), we get

$$\begin{aligned}\left\|\phi_n\left(x,t\right)\right\|\leq&\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\eta\left\|\phi_{(n-1)}\left(x,t\right)\right\|\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\eta\int_0^t\left\|\phi_{(n-1)}\left(x,\zeta\right)\right\|d\zeta.\end{aligned}\tag{65}$$

□

By the assist of the above result, we prove the following theorem.

Theorem 6. *The solution for Eq. (51) will exist and unique if there exist a t_0 then*

$$\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\eta+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\eta<1.$$

Proof. Let us consider the bounded function $u\left(x,t\right)$ satisfying the Lipschitz condition. Then, we get by Eq. (63) and Eq. (65), one can get

$$\begin{aligned}\left\|\phi\left(x,t\right)\right\|\ \leq\ &\left\|u_n\left(x,0\right)\right\|\\&\left[\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\eta+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\eta\right]^n.\end{aligned}\tag{66}$$

Therefore, for the obtained solutions, continuity and existence are verified. Now, to prove the Eq. (66) is a solution for Eq. (51), we consider

$$u\left(x,t\right)-u\left(x,0\right)=u_n\left(x,t\right)-\mathcal{K}_n\left(x,t\right).\tag{67}$$

Now, we consider

$$\begin{aligned}\left\|\mathcal{K}_n\left(x,t\right)\right\|=&\|\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\left(\mathcal{G}\left(x,t,u\right)-\mathcal{G}(x,t,\ u_{n-1})\right)\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\\&\int_0^t\left(t-\zeta\right)^{\mu-1}\left(\mathcal{G}\left(x,\zeta,u\right)-\mathcal{G}\left(x,\zeta,u_{n-1}\right)\right)d\zeta\|\\\leq&\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\left\|\mathcal{G}\left(x,\zeta,u\right)-\mathcal{G}\left(x,\zeta,u_{n-1}\right)\right\|\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\int_0^t\left\|\mathcal{G}\left(x,\zeta,u\right)-\mathcal{G}\left(x,\zeta,u_{n-1}\right)\right\|d\zeta\\\leq&\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\eta_1\left\|u-u_{n-1}\right\|\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\eta_1\left\|u-u_{n-1}\right\|t.\end{aligned}\tag{68}$$

Similarly, at t_0 we can obtain

$$\begin{aligned}\left\|\mathcal{K}_n\left(x,t\right)\right\|\ \leq\ &\left(\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}+\frac{\alpha\ t_0}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\right)^{n+1}\\&\eta^{n+1}M.\end{aligned}\tag{69}$$

As n tends to ∞, then $\left\|\mathcal{K}_n\left(x,\ t\right)\right\|$ approaches to 0 with respect to Eq. (69).

$$\begin{aligned}&u\left(x,t\right)-u^*\left(x,t\right)\\&=\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\left(\mathcal{G}\left(x,t,u\right)-\mathcal{G}\left(x,t,\ u^*\right)\right)\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\int_0^t\left(\mathcal{G}\left(x,\zeta,u\right)-\mathcal{G}\left(x,\zeta,u^*\right)\right)d\zeta.\end{aligned}\tag{70}$$

The Eq. (70) simplifies on applying norm,

$$\begin{aligned}&\left\|u\left(x,t\right)-u^*\left(x,t\right)\right\|\\&=\left\|\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\left(\mathcal{G}\left(x,t,u\right)-\mathcal{G}\left(x,t,\ u^*\right)\right)\right.\\&\left.+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\int_0^t\left(\mathcal{G}\left(x,\zeta,u\right)-\mathcal{G}\left(x,\zeta,u^*\right)\right)d\zeta\right\|\\&\leq\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\eta\left\|u\left(x,t\right)-u^*\left(x,t\right)\right\|\\&+\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\eta\ t\left\|u\left(x,t\right)-u^*\left(x,t\right)\right\|.\end{aligned}\tag{71}$$

On simplification

$$\begin{aligned}&\left\|u\left(x,t\right)-u^*\left(x,t\right)\right\|\\&\left(1-\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\eta t-\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\eta\right)\leq0.\end{aligned}\tag{72}$$

From the above condition, it is clear that $u\left(x,t\right)=u^*\left(x,t\right)$, if

$$\left(1-\frac{\alpha}{\mathcal{B}\left(\alpha\right)\Gamma\left(\alpha\right)}\eta t-\frac{\left(1-\alpha\right)}{\mathcal{B}\left(\alpha\right)}\eta\right)\geq0.\tag{73}$$

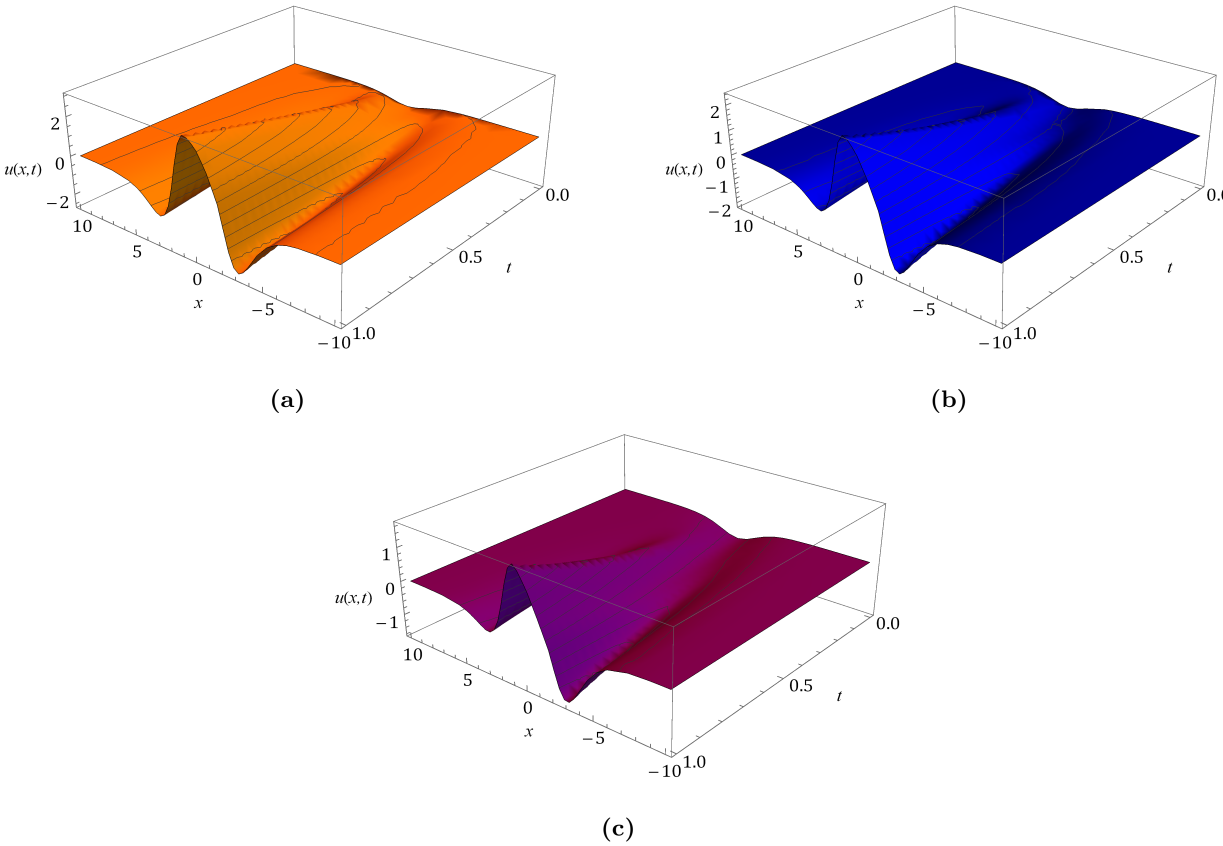


Figure 2. Surfaces of q-HATT solution for ($\boldsymbol{a}$) Caputo, ($\boldsymbol{b}$)CF and ($\boldsymbol{c}$) AB fractional operator at $n = 1,\ \hbar = -1,\ \beta = 2,\ \alpha = 1$ and $F_r = -1$.

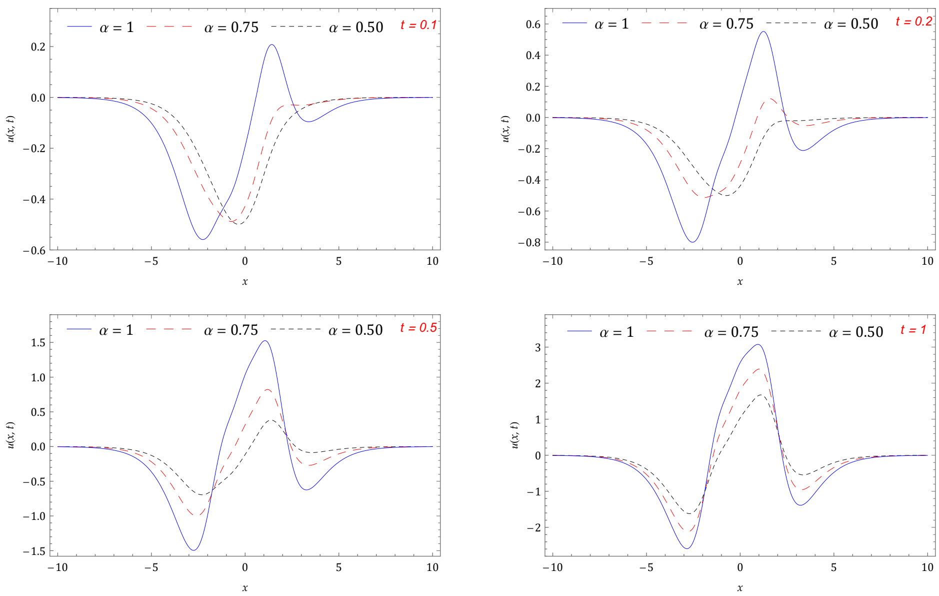


Figure 3. Response of obtained solution with distinct α and time at $n = 1,\ \hbar = -1,\ \beta = 2$ and $F_r = -1$.

Hence, Eq. (73) evidence required consequence. □

4. Results and discussion

In this section, we consider two cases as mentioned above to analyze the hired model with a hole, and presented in Figure 1. In the first case

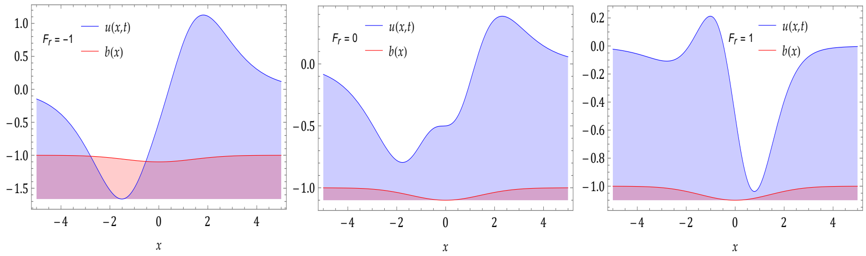
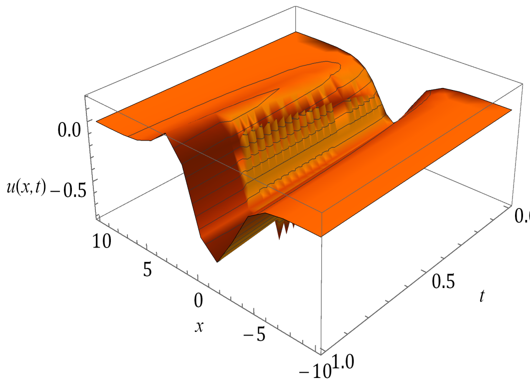
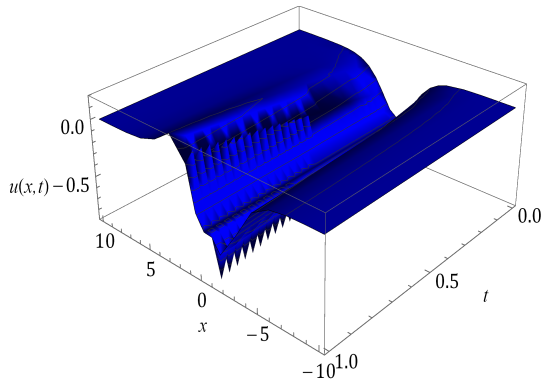


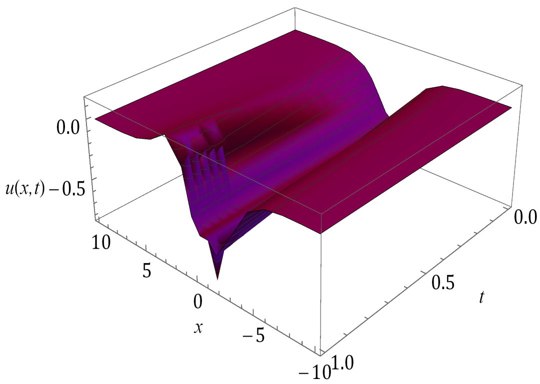
Figure 4. Nature of water elevation with $b(x)$ at $n = 1,\ \hbar = -1,\ \beta = 2,\ t = 1.5$ and $\alpha = 1$.



(a)



(b)



(c)

Figure 5. Surfaces of q-HATT solution for (***a***) Caputo, (***b***) CF and (***c***) AB fractional operator at $n = 1,\ \hbar = -1,\ \beta = 8,\ \alpha = 1$ and $F_r = -1$.

for $\beta = 2$, the behaviour of $b(x)$ is a lock-like reciprocal of bell-shape and also sharp at the bottom. For the second case (i.e., $\beta = 8$), the hole at the bottom is more flattened and wider. In the present investigation, we consider constant wave speed $c \approx \sqrt{g \times h} = \sqrt{9.8}$ with a mean water depth of the sea $h = 1$. For $\beta = 2$, the nature of archived results for the FF-KdV equation with different distinct fractional operator and fractional-order is captured respectively in Figures 2 and 3. In Figure 3 we can observe that at $x = -2$ and 2 the behaviour of water evaluation is overlapped for different value of α and moreover the change in time shows stimulating variation in the behaviours. The nature of the water elevation with sea bed topography with $\beta = 2$ and 8 are presented in Figures 4 and 7 for different Froude number in order to understand the importance of $b(x)$ and β in the obtained solution at the particular values of the time. In the same manner, for $n = 8$ surfaces for an obtained solution with a distinct fractional operator is cited in Figure 5. The response of q-HATT solution for FF-KdV equation with distinct α is dissipated in Figure 6 for $\beta = 8$. In this case, also we can notice the huge change in the behaviours with a small change in time with fractional order.

The considered method is highly noticeable for the parameters associated with the algorithm and which help to make more convergence (they are proposed based on the topological concept). To illustrate the nature of the solution obtained with homotopy parameter ($\hbar$), the $\hbar$-curves are plotted

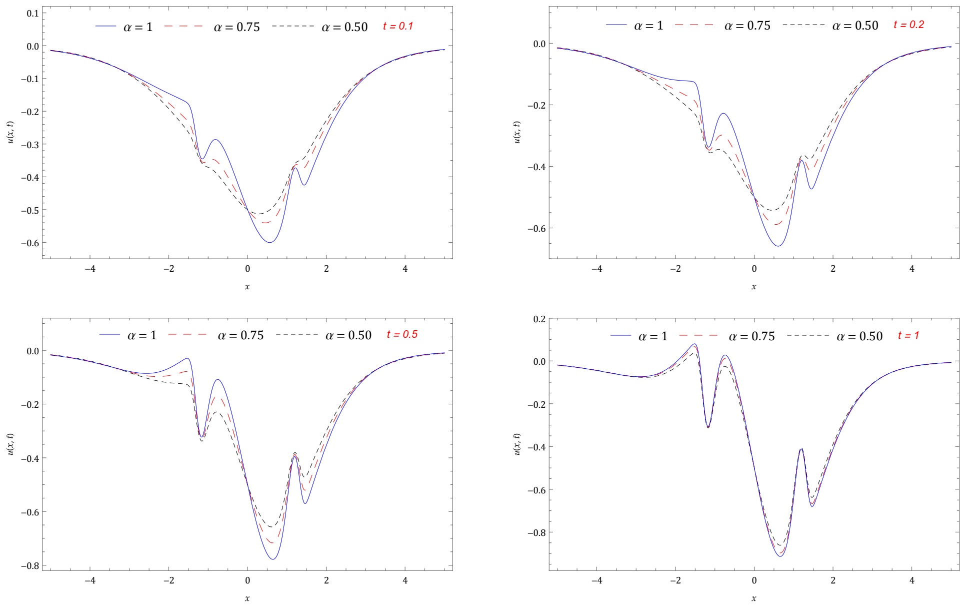


Figure 6. Response of obtained solution with distinct α and time at $n = 1,\ \hbar = -1,\ \beta = 8$ and $F_r = -1$.

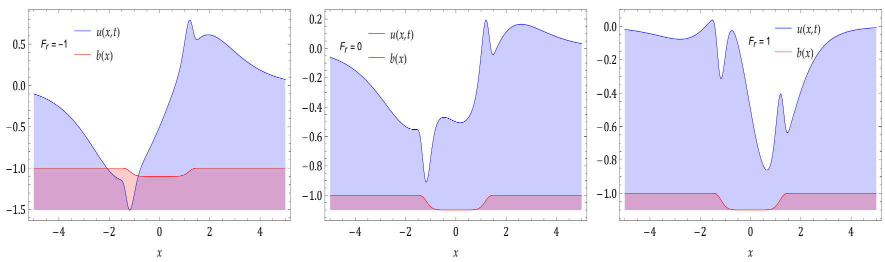
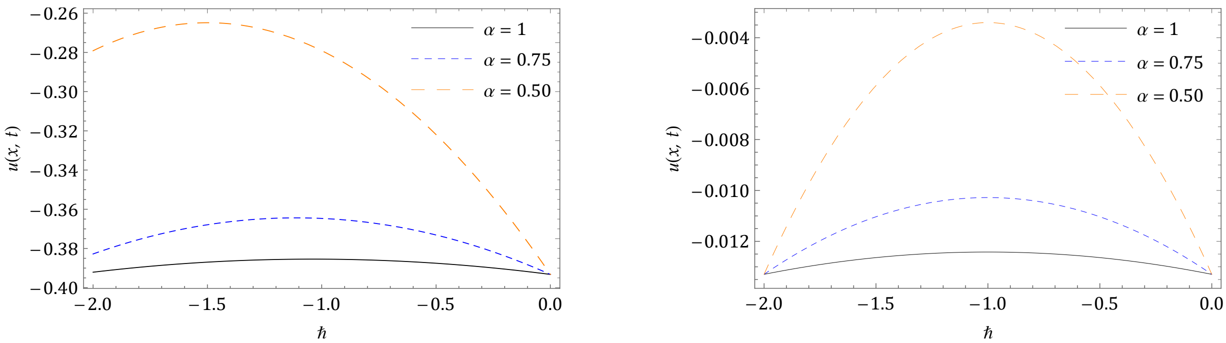


Figure 7. Nature of water elevation with $b\,(x)$ at $n = 1,\ \hbar = -1,\ \beta = 8$ and $\alpha = 1$.



(a) **(b)**

Figure 8. $\hbar$-curves for q-HATT solution with distinct α when $n = 1$ and $t = 0.01$ with $(a)\ \beta = 2$ at $x = 1$ and $(b)\,\beta = 8$ at $x = 5$.

with different α for both cases (i.e., $\beta = 2$ and 8) and are respectively captured in Figure 8. Line flat segment designates the convergence providence of the solution.

5. Conclusion

In this study, the q-HATT is applied lucratively to the analyzed effect of parameters associated with the method (rigid bottom topography and Froude number) by finding the solution for an arbitrary order shallow water forced KdV equation describing the free surface critical flow over a hole. The derived results show the effect of rigid bottom topography and Froude number with change in time and space with different fractional order. By using the considered model, two distinct kinds of hole are analyzed and which shows that for $\beta = 2$

exhibits a hole in inverse-bell shape and at $\beta = 8$ shows a hole has a sharp edge on two-sides and also it has a flattened base. The condition is derived for the considered model to illustrate the existence and uniqueness within the frame of fixed-point theory using Banach space. The effect of three fractional operators is presented and other effects are illustrated with respect to the Caputo operator. These fractional operators are playing a vital role in generalizing the models associated with power-law distribution, kernel singular, and non-local and non-singular (respectively, Caputo, CF and AB operators). Finally, the present study is to demonstrate the effect of fractional order, parameters associated with models as well as methods with their corresponding consequences.

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RESEARCH ARTICLE

Some qualitative properties of nonlinear fractional integro-differential equations of variable order

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ABSTRACT

The existence-uniqueness criteria of nonlinear fractional integro-differential equations of variable order with multiterm boundary value conditions are considered in this work. By utilizing the concepts of generalized intervals combined with the piecewise constant functions, we transform our problem into usual Caputo's fractional differential equations of constant order. We develop the necessary criteria for assuring the solution's existence and uniqueness by applying Schauder and Banach fixed point theorem. We also examine the stability of the derived solution in the Ulam-Hyers-Rassias (UHR) sense and provide an example to demonstrate the credibility of the results.

1. Introduction

The fundamental idea behind fractional calculus is simply to replace the traditional integer orders in integral and differential operators with arbitrary constant orders. Although it seems an elementary consideration, fractional order operators play an important role in describing many physical phenomena and have interesting implications. [1, 2].

The introduction of the notion of variable-order (VO) integral and differential operators together with their some main properties was firstly initialized by Samko and Ross [3] in 1993. By these operators, one can define the order of the fractional integral and derivative as a function of independent variables such as time and space variables. In view of the characterization of the non-fixed kernel, this operators allows us to designate the memory and hereditary features of natural phenomena in a better way. By virtue of its potential efficiency to model real world problems, this topic has attracted many researchers in ongoing decades. In this direction, lots of papers have been published on different branches of science and engineering such as viscoelasticity, medicine, signal processing, control systems, so on [4–7]. Since its difficulty in getting an explicit solutions for fractional differential equations of VO, many papers have been devoted to find numerical solutions for this type of problems. See [8–13] and the references cited therein. Hovewer very few paper on existence, uniqueness and stability properties of fractional variable order differential equations have been published recently [14–20].

When we conduct an overview of the literature, increasing number of authors from several areas of the scientific community have focused on investigating the existence and uniqueness of fractional

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constant order differential and integro-differential equations [21–26].

In [27], Devi et al. studied the following Caputo's fractional boundary value problem by taking into consideration the monotone iterative technique.

$$\begin{cases} {}^{C}D^{q}u = F(t,u,I^{q}(u)) + G(t,u,I^{q}(u)), \\ g\left(u\left(0\right),u\left(T\right)\right) = 0 \end{cases}$$

where $0 < q < 1$. As a consequence, it has been shown that the established monotone flows converge uniformly to the coupled extremal solutions of the considered problem.

In [28], some sort of stability results were studied for fractional integro-differential equations involving Hilfer fractional derivative ${}^{H}D_{a^+}^{\alpha,\beta;\psi}(.)$ with $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.

In particular, Bai and Kong [29] considered the existence of the solutions for the following initial value problem

$$\begin{cases} {}^{C}D_{a^+}^{\alpha}y(t) = f(t,y(t),I_{a^+}^{\alpha}y(t)), \ \ t\in[a,b], \\ y(a) = x_a, \end{cases}$$

by employing the upper and lower solution approach. The operators ${}^{C}D_{0+}^{\alpha}$ and I_{0+}^{α} stand for the Caputo-Hadamard fractional derivative and Hadamard fractional integral operators of order $\alpha \in (0,1]$, respectively.

Motivated by the preceding works, we deal with the following boundary value problem on $\mathcal{J} := [0,b]$ such that

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{u(t)}y(t) = \Phi(t,y(t),I_{0^+}^{u(t)}y(t)), \\ y(0) = 0, \ \ y(b) = 0, \end{cases} \tag{1}$$

where $1 < u(t) \leq 2$ and ${}^{C}\mathcal{D}_{0^+}^{u(t)}$, $I_{0^+}^{u(t)}$ are considered as in the sense of Caputo fractional derivative and integral of variable-order $u(t)$, respectively.

Our purpose is to investigate the existence and uniqueness of the solution of equation (1). We further show the stability of the solution in the Ulam-Hyers-Rassias (UHR) sense.

2. Mathematical Preliminaries

This part covers some fundamental concepts and lemmas that will be needed to understand the main theorems discussed in the subsequent sections.We also introduce some of the specifications for variable order operators.

Let $C(\mathcal{J},\mathbb{R})$ denote the the set of all real-valued continuous functions from $\mathcal{J}$ into $\mathbb{R}$. For an element $\chi \in C(\mathcal{J},\mathbb{R})$, define the standart norm $\|\chi\| = Sup\{|\chi(t)| : t \in \mathcal{J}\}$, and with this norm $C(\mathcal{J},\mathbb{R})$ becomes a Banach space.

For $-\infty < t_1 < t_2 < +\infty$, let the mappings to be defined $u(t) : [t_1,t_2] \to (0,+\infty)$ and $v(t) : [t_1,t_2] \to (n-1,n)$. Then, the left Riemann-Lioville(R-L) fractional integral of VO $u(t)$ ([30]) is given as

$$I_{t_1^+}^{u(t)}m(t) = \int_{t_1}^{t}\frac{(t-s)^{u(t)-1}}{\Gamma(u(t))}m(s)ds, \ \ t > t_1, \tag{2}$$

as well as the left VO Caputo derivative ([30]) is defined by

$${}^{C}\mathcal{D}_{t_1^+}^{v(t)}m(t) = \int_{t_1}^{t}\frac{(t-s)^{n-v(t)-1}}{\Gamma(n-v(t))}m^{(n)}(s)ds, \ \ t > t_1. \tag{3}$$

These definitions, as expected, correspond with the usual R-L fractional integral and Caputo fractional derivative, respectively, when $u(t)$ and $v(t)$ are constant.see e.g. [3,30,31].

Lemma 1. *([31]) Let σ_1, $\sigma_2 > 0$, $t_1 > 0$, $m \in L(t_1,t_2)$ and ${}^{C}\mathcal{D}_{t_1^+}m \in L(t_1,t_2)$. Then, differential equation*

$${}^{C}\mathcal{D}_{t_1^+}^{\sigma_1}m(t) = 0$$

has the following general solution

$$m(t) = \alpha_0+\alpha_1(t-t_1)+\alpha_2(t-t_1)^2+...+\alpha_{n-1}(t-t_1)^{n-1}$$

where $n-1 < \sigma_1 \leq n$ and α_ℓ ($\ell = 0,1,...,n-1$) are taken as arbitrary real numbers.

From that Lemma we deduce the next relation

$$I_{t_1^+}^{\sigma_1}\,{}^{C}\mathcal{D}_{t_1^+}^{\sigma_1}m(t) = m(t) + \alpha_0 + \alpha_1(t-t_1) + \alpha_2(t-t_1)^2 + ... + \alpha_{n-1}(t-t_1)^{n-1}$$

Furthermore,

$${}^{C}\mathcal{D}_{t_1^+}^{\sigma_1}I_{t_1^+}^{\sigma_1}m(t) = m(t).$$

and

$$I_{t_1^+}^{\sigma_1}I_{t_1^+}^{\sigma_2}m(t) = I_{t_1^+}^{\sigma_2}I_{t_1^+}^{\sigma_1}m(t) = I_{t_1^+}^{\sigma_1+\sigma_2}m(t).$$

Remark 1. *([32]) It's worth noting that the semigroup property isn't mostly satisfied by general functions $u(t)$, $v(t)$, i.e.,*

$$I_{t_1^+}^{u(t)}I_{t_1^+}^{v(t)}m(t) \neq I_{t_1^+}^{u(t)+v(t)}m(t).$$

Definition 1. *A function $\mu \in C(\mathcal{J},\mathbb{R})$ is said to be a C_δ class function if it belong to the set*

$$C_\delta(\mathcal{J},\mathbb{R}) = \left\{\mu \in C((0,b],\mathbb{R}) : t^\delta\mu \in C(\mathcal{J},\mathbb{R})\right\}$$

for $0 \leq \delta \leq 1$.

Lemma 2. *[13] Assume that $u : \mathcal{J} \to (1,2)$ is a continuous function and $m \in C_\delta(\mathcal{J},\mathbb{R})$. Then the fractional integral $I_{0^+}^{u(t)}m(t)$ of variable order exists for each point on $\mathcal{J}$.*

Lemma 3. *([13]) Let $u \in C(\mathcal{J},(1,2))$ and $m \in C(\mathcal{J},\mathbb{R})$ then $I_{0^+}^{u(t)}m(t) \in C(\mathcal{J},\mathbb{R})$.*

We now give the well-known Schauder fixed-point result.

Theorem 1. *([31]) Assume that E is a Banach space and Q is a nonempty convex subset of E and moreover $F : Q \longrightarrow Q$ is compact, and continuous map. Then, there exist fixed points of F in Q.*

Definition 2. *([23]) The equation (1) is called Ulam-Hyers-Rassias* **(UHR)** *stable with respect to the function $\psi \in C(\mathcal{J}, \mathbb{R}^+)$ if there exists $c_\Phi > 0$, such that for any $\epsilon > 0$ and for each solution $z \in C^1(\mathcal{J}, \mathbb{R})$ of the inequality*

$$|{}^C\mathcal{D}_{0^+}^{u(t)} z(t) - \Phi(t, z(t), I_{0^+}^{u(t)} z(t))| \leq \epsilon\psi(t),\ t \in \mathcal{J},$$

there exists a solution $y \in C(\mathcal{J}, \mathbb{R})$ of equation (1) with

$$|z(t) - y(t)| \leq c_\Phi \epsilon\psi(t),\ t \in \mathcal{J}.$$

3. Existence Results

Let us begin with introducing the following assumptions:

(H1): Let $\mathcal{P} = \{\mathcal{J}_1 := [0, b_1], \mathcal{J}_2 := (b_1, b_2], \mathcal{J}_3 := (b_2, b_3], ...\mathcal{J}_n := (b_{n-1}, b]\}$ be a partition of the interval $\mathcal{J}$, and let $u(t) : \mathcal{J} \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

$$u(t) = \sum_{\ell=1}^{n} u_\ell I_\ell(t) = \begin{cases} u_1, & if\ t \in \mathcal{J}_1, \\ u_2, & if\ t \in \mathcal{J}_2, \\ . \\ . \\ . \\ u_n, & if\ t \in \mathcal{J}_n, \end{cases}$$

where $1 < u_\ell \leq 2$ are constants, and I_ℓ is the indicator of the interval $\mathcal{J}_\ell := (b_{\ell-1}, b_\ell]$, $\ell = 1, 2, ..., n$, (with $b_0 = 0,\ b_n = b$) such that

$$I_\ell(t) = \begin{cases} 1, & for\ t \in \mathcal{J}_\ell, \\ 0, & for\ elsewhere. \end{cases}$$

(H2): Let $t^\delta \Phi : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function $(0 \leq \delta \leq 1)$, there exist constants $K,\ L > 0$, satisfying the inequality

$$t^\delta |\Phi(t, w_1, z_1) - \Phi(t, w_2, z_2)| \leq K|w_1 - w_2| + L|z_1 - z_2|,$$

For each $\ell \in \{1, 2, ..., n\}$, the set $E_\ell = C(\mathcal{J}_\ell, \mathbb{R})$, represents the Banach space of continuous functions $y : \mathcal{J}_\ell \to \mathbb{R}$ equipped with the sup norm

$$\|y\|_{E_\ell} = \sup_{t \in \mathcal{J}_\ell} |y(t)|,$$

where $\ell \in \{1, 2, ..., n\}$

We now analyze BVP defined in (1). On account of (3), the solution of (1) can be stated as

$$\int_0^t \frac{(t-s)^{1-u(t)}}{\Gamma(2-u(t))} y''(s)ds = \Phi(t, y(t), I_{0^+}^{u(t)} y(t)), \quad (4)$$

for $t \in \mathcal{J}$. If we employ $(H1)$, the foregoing equation(4) can be written as

$$\int_0^{b_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} y''(s)ds + ... + \int_{b_{\ell-1}}^t \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} y''(s)ds = \Phi(t, y(t), I_{0^+}^{u_\ell} y(t)) \quad (5)$$

for $t \in \mathcal{J}_\ell, \ell = 1, 2, ..., n$.
The solution to the BVP (1) will be introduced in the following definition.

Definition 3. *BVP (1) has a solution, if there are functions $y_\ell, \ell = 1, 2, ..., n$, such that $y_\ell \in C([0, b_\ell], \mathbb{R})$ satisfying equation (5) and boundary conditions $y_\ell(0) = 0 = y_\ell(b_\ell)$.*

Based on the preceding observation, BVP (1) can be represented as in (4) and, with considering $\mathcal{J}_\ell, \ell \in \{1, 2, ..., n\}$ as in (5).
Since we define $y(t)$ identically 0 for $t \in [0, b_{\ell-1})$, then the equation (5) is expressed as

$${}^C\mathcal{D}_{b_{\ell-1}^+}^{u_\ell} y(t) = \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)), \quad t \in \mathcal{J}_\ell.$$

We shall deal with following BVP

$$\begin{cases} {}^C\mathcal{D}_{b_{\ell-1}^+}^{u_\ell} y(t) = \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)), \\ y(b_{\ell-1}) = 0, y(b_\ell) = 0, \end{cases} \quad (6)$$

for $t \in \mathcal{J}_\ell$. On the way to achieve our purpose, the upcoming lemma will play an important role.

Lemma 4. *A function $y \in E_\ell$ establishes a solution for (6) if and only if y fulfills the integral equation*

$$y(t) = -\frac{t - b_{\ell-1}}{b_\ell - b_{\ell-1}} \left[I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \right]_{t=b_\ell} + I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)). \quad (7)$$

Proof. We first assume that $y \in E_\ell$ is solution of the problem (6). If we apply the fractional operator $I_{b_{\ell-1}^+}^{u_\ell}$ to both sides of (6) and considering Lemma 1, we obtain

$$y(t) = \alpha_1 + \alpha_2(t - b_{\ell-1}) + \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^t (t-s)^{u_\ell - 1} \times \Phi\big(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)\big) ds,$$

for $t \in \mathcal{J}_\ell$. By $y(b_{\ell-1}) = 0$, we get $\alpha_1 = 0$.
Taking into account another boundary condition

$y(b_\ell) = 0$, it follows that

$$\begin{aligned} 0 &= \alpha_2(b_\ell - b_{\ell-1}) \\ &+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\ &\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds \\ \alpha_2 &= -(b_\ell - b_{\ell-1})^{-1} I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \Big|_{t=b_\ell} \end{aligned}$$

Then, we observe that

$$\begin{aligned} y(t) &= -(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1}) \\ &\times \left[I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \right]_{t=b_\ell} \\ &+ I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)), \quad t \in \mathcal{J}_\ell. \end{aligned}$$

On the contrary, let $y \in E_\ell$ be the solution of integral equation (7). Taking into account the continuity of function $t^\delta \Phi$ and using Lemma (1), we conclude that y is the solution of the problem (6). □

We can now show our first existence result which is based on Theorem (1)

Theorem 2. *Assume that conditions (H1), (H2) hold, and if*

$$\frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \left(K + L \frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\right) < 1,$$

then, there exist at least one solution for the problem (6) on $\mathcal{J}$.

Proof. Let us set the operator $W : E_\ell \to E_\ell$ such that for $t \in \mathcal{J}_\ell$

$$\begin{aligned} Wy(t) &= -(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1}) \\ &\times \left[I_{b_{\ell-1}^+}^{u_\ell} \Phi(t, y(t), I_{b_{\ell-1}^+}^{u_\ell} y(t)) \right] \Big|_{t=b_\ell} \\ &+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \\ &\times \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) ds. \end{aligned}$$

The operator $W : E_\ell \to E_\ell$ described in 3.1 is well defined, as seen by the properties of fractional integrals and the continuity of function $t^\delta \Phi$.
Let

$$R_\ell \geq \frac{\frac{2\eta_0 (b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)}}{1 - \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \left(K + L \frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\right)},$$

with

$$\eta_0 = \sup_{t \in \mathcal{J}_\ell} |\Phi(t, 0, 0)|.$$

We generate the set

$$B_{R_\ell} = \{y \in E_\ell : \ \|y\|_{E_\ell} \leq R_\ell\}.$$

It is clear that B_{R_ℓ} is nonempty, closed, convex and bounded.

Now, we will see that W satisfies the claims of the Theorem (1). We demonstrate it by using following stages.

STEP 1: We show that $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.
For $y \in B_{R_\ell}$ and by (H2), we get

$$\begin{aligned} |Wy(t)| &\leq \frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \\ &\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\ &\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds \\ &+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \\ &\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds \\ &\leq \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\ &\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))| ds \\ &= \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\ &\times |\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s)) - \Phi(s, 0, 0)| ds \\ &+ \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} |\Phi(s, 0, 0)| ds \\ &\leq \frac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} s^{-\delta} \\ &\times (K|y(s)| + L|I_{b_{\ell-1}^+}^{u_\ell} y(s)|) ds \\ &+ \frac{2\eta_0 (b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\ &\leq \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} \\ &\times \left(K + L \frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\right) |y(s)| ds \\ &+ \frac{2\eta_0 (b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\ &\leq \frac{2(b_\ell - b_{\ell-1})^{u_\ell - 1}(b_\ell^{1-\delta} - b_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \\ &\times \left(K + L \frac{(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)}\right) R_\ell \\ &+ \frac{2\eta_0 (b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\ &\leq R_\ell, \end{aligned}$$

which yields that $W(B_{R_\ell}) \subseteq B_{R_\ell}$.

STEP 2: W is continuous.
We assume that the sequence (y_n) converges to y

in E_ℓ. Then,

$$|(Wy_n)(t)-(Wy)(t)|$$
$$\leq \frac{(b_\ell-b_{\ell-1})^{-1}(t-b_{\ell-1})}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}$$
$$\times|\Phi(s,y_n(s),I^{u_\ell}_{b^+_{\ell-1}}y_n(s))-\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t}(t-s)^{u_\ell-1}$$
$$\times|\Phi(s,y_n(s),I^{u_\ell}_{b^+_{\ell-1}}y_n(s))-\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))|ds$$
$$\leq \frac{2}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}$$
$$\times|\Phi(s,y_n(s),I^{u_\ell}_{b^+_{\ell-1}}y_n(s))-\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))|ds$$
$$\leq \frac{2}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{b_\ell}s^{-\delta}(b_\ell-s)^{u_\ell-1}$$
$$\times(K|y_n(s)-y(s)|+LI^{u_\ell}_{b^+_{\ell-1}}|y_n(s)-y(s))|)ds$$
$$\leq \frac{2K}{\Gamma(u_\ell)}\|y_n-y\|_{E_\ell}\int_{b_{\ell-1}}^{b_\ell}s^{-\delta}(b_\ell-s)^{u_\ell-1}ds$$
$$+\frac{2L}{\Gamma(u_\ell)}\|I^{u_\ell}_{b^+_{\ell-1}}(y_n-y)\|_{E_\ell}\int_{b_{\ell-1}}^{b_\ell}s^{-\delta}(b_\ell-s)^{u_\ell-1}ds$$
$$\leq \frac{2K}{\Gamma(u_\ell)}\|y_n-y\|_{E_\ell}\int_{b_{\ell-1}}^{b_\ell}s^{-\delta}(b_\ell-s)^{u_\ell-1}ds$$
$$+\frac{2L(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\|y_n-y\|_{E_\ell}\int_{b_{\ell-1}}^{b_\ell}s^{-\delta}(b_\ell-s)^{u_\ell-1}ds$$
$$\leq\left(\frac{2K}{\Gamma(u_\ell)}+\frac{2L(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\right)\|y_n-y\|_{E_\ell}$$
$$\times\int_{b_{\ell-1}}^{b_\ell}s^{-\delta}(b_\ell-s)^{u_\ell-1}ds$$
$$\leq\frac{2(b_\ell-b_{\ell-1})^{u_\ell-1}({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$$
$$\times\left(K+L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\|y_n-y\|_{E_\ell}$$ i.e., we obtain

$$\|(Wy_n)-(Wy)\|_{E_\ell}\to 0 \quad as \quad n\to\infty.$$

As a consequence, the operator W is a continuous on E_ℓ.

STEP 3: W is compact.
We will prove that $W(B_{R_\ell})$ is relatively compact, which means that W is compact. In view of step 1, $W(B_{R_\ell})$ is uniformly bounded. Namely, we have $W(B_{R_\ell})=\{W(y): y\in B_{R_\ell}\}\subset B_{R_\ell}$ thus for each $y\in B_{R_\ell}$ we get $\|W(y)\|_{E_\ell}\leq R_\ell$ showing that $W(B_{R_\ell})$is bounded. Finally, It must be demonstrated the equicontinuity of $W(B_{R_\ell})$.
For $t_1,t_2\in\mathcal{J}_\ell,\ t_1<t_2$ and $y\in B_{R_\ell}$, we write

$$|(Wy)(t_2)-(Wy)(t_1)|$$
$$=\Big|-\frac{(b_\ell-b_{\ell-1})^{-1}(t_2-b_{\ell-1})}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}$$
$$\times\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t_2}(t_2-s)^{u_\ell-1}\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))ds$$
$$+\frac{(b_\ell-b_{\ell-1})^{-1}(t_1-b_{\ell-1})}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}$$
$$\times\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))ds$$
$$-\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t_1}(t_1-s)^{u_\ell-1}\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))ds\Big|$$
$$\leq\frac{(b_\ell-b_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$$
$$\times\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)$$
$$\times|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))|ds$$
$$\leq\frac{(b_\ell-b_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$$
$$\times\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}\big|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))$$
$$-\Phi(s,0,0)\big|ds$$
$$+\frac{(b_\ell-b_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$$
$$\times\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}|\Phi(s,0,0)|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)$$
$$\times|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))-\Phi(s,0,0)|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)$$
$$\times|\Phi(s,0,0)|ds+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}$$
$$\times|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))-\Phi(s,0,0)|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}|\Phi(s,0,0)|ds$$
$$\leq\frac{(b_\ell-b_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$$
$$\times\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))$$
$$-\Phi(s,0,0)|ds$$
$$+\frac{(b_\ell-b_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$$
$$\times\int_{b_{\ell-1}}^{b_\ell}(b_\ell-s)^{u_\ell-1}|\Phi(s,0,0)|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)$$
$$\times|\Phi(s,y(s),I^{u_\ell}_{b^+_{\ell-1}}y(s))-\Phi(s,0,0)|ds$$
$$+\frac{1}{\Gamma(u_\ell)}\int_{b_{\ell-1}}^{t_1}\Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)$$
$$\times|\Phi(s,0,0)|ds+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}$$

$\times |\Phi(s, y(s), I^{u_\ell}_{b^+_{\ell-1}} y(s)) - \Phi(s,0,0)|ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2-s)^{u_\ell-1} |\Phi(s,0,0)|ds.$

$\leq \frac{(b_\ell-b_{\ell-1})^{-1}}{\Gamma(u_\ell)}((t_2-b_{\ell-1})-(t_1-b_{\ell-1}))$

$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell-s)^{u_\ell-1} s^{-\delta}(K|y(s)|+L|I^{u_\ell}_{b^+_{\ell-1}} y(s)|)ds$

$+ \frac{\eta_0(b_\ell-b_{\ell-1})^{-1}}{\Gamma(u_\ell)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$

$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell-s)^{u_\ell-1}ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} s^{-\delta}\Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)$

$\times (K|y(s)|+L|I^{u_\ell}_{b^+_{\ell-1}} y(s)|)ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t_1} \Big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\Big)ds$

$+ \frac{1}{\Gamma(u_\ell)} \int_{t_1}^{t_2} s^{-\delta}(t_2-s)^{u_\ell-1}$

$\times (K|y(s)|+L|I^{u_\ell}_{b^+_{\ell-1}} y(s)|)ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)} \int_{t_1}^{t_2} (t_2-s)^{u_\ell-1}ds$

$\leq \frac{(b_\ell-b_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$

$\times (K\|y\|_{E_\ell}+L\|I^{u_\ell}_{b^+_{\ell-1}} y\|_{E_\ell}) \int_{b_{\ell-1}}^{b_\ell} s^{-\delta}ds$

$+ \frac{\eta_0(b_\ell-b_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$

$+ \frac{1}{\Gamma(u_\ell)}(K\|y\|_{E_\ell}+L\|I^{u_\ell}_{b^+_{\ell-1}} y\|_{E_\ell})$

$\times \int_{b_{\ell-1}}^{t_1} s^{-\delta}((t_2-t_1)^{u_\ell-1})ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)}\Big(\frac{(t_2-b_{\ell-1})^{u_\ell}}{u_\ell} - \frac{(t_2-t_1)^{u_\ell}}{u_\ell} - \frac{(t_1-b_{\ell-1})^{u_\ell}}{u_\ell}\Big)$

$+ \frac{(t_2-t_1)^{u_\ell-1}}{\Gamma(u_\ell)}(K\|y\|_{E_\ell}+L\|I^{u_\ell}_{b^+_{\ell-1}} y\|_{E_\ell}) \int_{t_1}^{t_2} s^{-\delta}ds$

$+ \frac{\eta_0}{\Gamma(u_\ell)}\frac{(t_2-t_1)^{u_\ell}}{u_\ell}$

$\leq \frac{(b_\ell-b_{\ell-1})^{u_\ell-2}({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$

$\times ((t_2-b_{\ell-1})-(t_1-b_{\ell-1}))$

$\times (K\|y\|_{E_\ell}+L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|y\|_{E_\ell})$

$+ \frac{\eta_0(b_\ell-b_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$

$+ \Big(\frac{({t_1}^{1-\delta}-{b_{\ell-1}}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\Big)$

$\times (K\|y\|_{E_\ell}+L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|y\|_{E_\ell}) + \frac{\eta_0}{\Gamma(u_\ell+1)}$

$\times \Big((t_2-b_{\ell-1})^{u_\ell}-(t_2-t_1)^{u_\ell}-(t_1-b_{\ell-1})^{u_\ell}\Big)$

$+ \frac{({t_2}^{1-\delta}-{t_1}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}$

$\times (K\|y\|_{E_\ell}+L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|y\|_{E_\ell}) + \frac{\eta_0(t_2-t_1)^{u_\ell}}{\Gamma(u_\ell+1)}$

$\leq \Big(\frac{(b_\ell-b_{\ell-1})^{u_\ell-2}({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K+L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})$

$\times \|y\|_{E_\ell} + \frac{\eta_0(b_\ell-b_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\Big)$

$\times \Big((t_2-b_{\ell-1})-(t_1-b_{\ell-1})\Big)$

$+\Big(\frac{{t_2}^{1-\delta}-{b_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(K+L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|y\|_{E_\ell}\Big)$

$\times (t_2-t_1)^{u_\ell-1}$

$+ \frac{\eta_0}{\Gamma(u_\ell+1)}\Big((t_2-b_{\ell-1})^{u_\ell}-(t_1-b_{\ell-1})^{u_\ell}\Big)$

Hence $\|(Wy)(t_2)-(Wy)(t_1)\|_{E_\ell} \to 0$ as $|t_2-t_1| \to 0$. It implies that $W(B_{R_\ell})$ is equicontinuous.

As a consequence of the Theorem (1), the problem (6) has at least a solution $\widetilde{y_\ell}$ in B_{R_ℓ}.
Let

$$y_\ell = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \widetilde{y_\ell}, & t \in \mathcal{J}_\ell, \end{cases} \tag{8}$$

We know that $y_\ell \in C([0,b_\ell],X)$ defined by (8) satisfies the equation

$$\int_0^{b_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} y''_\ell(s)ds + ...$$

$$+ \int_{b_{\ell-1}}^t \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} {y_\ell}''(s)ds = \Phi(t, y_\ell(t), I^{u_\ell}_{0^+} y_\ell(t)),$$

for $t \in \mathcal{J}_\ell$, concluding that y_ℓ is a solution of (5) with $y_\ell(0)=0$, $y_\ell(b_\ell)=\widetilde{y}_\ell(b_\ell)=0$.

Then,

$$y(t) = \begin{cases} y_1(t), & t \in \mathcal{J}_1, \\ y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1, \\ \widetilde{y}_2, & t \in \mathcal{J}_2 \end{cases} \\ . \\ . \\ . \\ y_n(t) = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \widetilde{y}_\ell, & t \in \mathcal{J}_\ell \end{cases} \end{cases} .$$

constitutes a solution for BVP(1).
The principle of Banach contraction is used to arrive at the following result. □

Theorem 3. *Assume that the assumptions* $(H1)$, $(H2)$ *hold and if*

$$\frac{2({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})(b_\ell-b_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)} \times \Big(K + \frac{L(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\Big) < 1 \tag{9}$$

then the problem (6) has at most one solution in E_ℓ.

Proof. The Banach contraction concept will be used to demonstrate the unique fixed point for W specified in Theorem (3).
For $y_1(t), y_2(t) \in E_\ell$, it follows that

$$\begin{aligned}
&|(Wy_1)(t) - (Wy_2)(t)| \\
&\le \tfrac{(b_\ell - b_{\ell-1})^{-1}(t-b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\
&\times |\Phi(s, y_1(s), I^{u_\ell}_{b^+_{\ell-1}} y_1(s)) - \Phi(s, y_2(s), I^{u_\ell}_{b^+_{\ell-1}} y_2(s))| ds \\
&+ \tfrac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t-s)^{u_\ell - 1} \\
&\times |\Phi(s, y_1(s), I^{u_\ell}_{b^+_{\ell-1}} y_1(s)) - \Phi(s, y_2(s), I^{u_\ell}_{b^+_{\ell-1}} y_2(s))| ds \\
&\le \tfrac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\
&\times |\Phi(s, y_1(s), I^{u_\ell}_{b^+_{\ell-1}} y_1(s)) - \Phi(s, y_2(s), I^{u_\ell}_{b^+_{\ell-1}} y_2(s))| ds \\
&\le \tfrac{2}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} s^{-\delta} \\
&\times \left(K|y_1(s) - y_2(s)| + L I^{u_\ell}_{b^+_{\ell-1}} |y_1(s) - y_2(s))| \right) ds \\
&\le \tfrac{2K}{\Gamma(u_\ell)} \|y_1 - y_2\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds \\
&+ \tfrac{2L}{\Gamma(u_\ell)} \|I^{u_\ell}_{b^+_{\ell-1}} (y_1 - y_2)\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds \\
&\le \tfrac{2K}{\Gamma(u_\ell)} \|y_1 - y_2\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds \\
&+ \tfrac{2L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)} \|y_1 - y_2\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds \\
&\le \left(\tfrac{2K}{\Gamma(u_\ell)} + \tfrac{2L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)} \right) \|y_1 - y_2\|_{E_\ell} \\
&\times \int_{b_{\ell-1}}^{b_\ell} s^{-\delta} (b_\ell - s)^{u_\ell - 1} ds \\
&\le \tfrac{2({b_\ell}^{1-\delta} - {b_{\ell-1}}^{1-\delta})(b_\ell - b_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \\
&\times \left(K + \tfrac{L(b_\ell - b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} \right) \|y_1 - y_2\|_{E_\ell}
\end{aligned}$$

Therefore, by considering (9), the operator W is a contraction. Employing Banach contraction mapping, we result in that W has only one fixed point, say it $\widetilde{y}_\ell \in E_\ell$, which also concludes that the problem (6) has got unique solution.
We let

$$y_\ell = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \widetilde{y}_\ell, & t \in \mathcal{J}_\ell, \end{cases} \tag{10}$$

We know that $y_\ell \in C([0, b_\ell], \mathbb{R})$ defined by (10) satisfies the equation

$$\int_0^{b_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} y_\ell''(s) ds + ... + \int_{b_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} {y_\ell}''(s) ds = \Phi(t, y_\ell(t), I^{u_\ell}_{0^+} y_\ell(t)),$$

for $t \in \mathcal{J}_\ell$, which yields that y_ℓ is a unique solution of (5) with $y_\ell(0) = 0,\ y_\ell(b_\ell) = \widetilde{y}_\ell(b_\ell) = 0$.

This led us to

$$y(t) = \begin{cases} y_1(t), \quad t \in \mathcal{J}_1, \\ y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1, \\ \widetilde{y}_2, & t \in \mathcal{J}_2 \end{cases} \\ . \\ . \\ . \\ y_n(t) = \begin{cases} 0, & t \in [0, b_{\ell-1}], \\ \widetilde{x}_\ell, & t \in \mathcal{J}_\ell \end{cases} \end{cases} .$$

which is the unique solution of the boundary value problem (1). □

4. Ulam-Hyers-Rassias stability

Theorem 4. *Suppose that the conditions (H1), (H2), together with (9) hold. Assume further that*

(H3): *The function $\psi \in C(\mathcal{J}_\ell, \mathbb{R}_+)$ have increasing property and there exists $\lambda_\psi > 0$ such that*

$$I^{u_\ell}_{b_{\ell-1}^+} \psi(t) \le \lambda_\psi \psi(t)$$

then, under these assumptions, the equation (1) *has* **UHR** *stability with respect to ψ*

Proof. Suppose that $z \in C(\mathcal{J}_\ell, \mathbb{R})$ is a solution of the following inequality

$$\left| {}^C\mathcal{D}^{u_\ell}_{b_{\ell-1}^+} z(t) - \Phi(t, z(t), I^{u_\ell}_{b_{\ell-1}^+} z(t)) \right| \le \epsilon \psi(t), \tag{11}$$

for $t \in \mathcal{J}_\ell$. Let us denote $y \in C(\mathcal{J}_\ell, \mathbb{R})$ to be the unique solution of the problem

$$\begin{cases} {}^C\mathcal{D}^{u_\ell}_{b_{\ell-1}^+} y(t) = \Phi(t, y(t), I^{u_\ell}_{b_{\ell-1}^+} y(t)), \ t \in \mathcal{J}_\ell \\ y(b_{\ell-1}) = 0, \ y(b_\ell) = 0 \end{cases}$$

By using Lemma (4), we have

$$\begin{aligned} y(t) &= -\frac{(b_\ell - b_{\ell-1})^{-1}(t - b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell - 1} \\ &\times \Phi(s, y(s), I^{u_\ell}_{b^+_{\ell-1}} y(s)) ds \\ &+ \frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t-s)^{u_\ell - 1} \\ &\times \Phi(s, y(s), I^{u_\ell}_{b^+_{\ell-1}} y(s)) ds \end{aligned}$$

By integrating both sides of (11) and utilizing (H3), we find

$$\Big|z(t) + \frac{(b_\ell-b_{\ell-1})^{-1}(t-b_{\ell-1})}{\Gamma(u_\ell)}$$
$$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell-1}\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s))ds$$
$$-\frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t-s)^{u_\ell-1}\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s))ds\Big|$$
$$\le \epsilon \int_{b_{\ell-1}}^{t} \frac{(t-s)^{u_\ell-1}}{\Gamma(u_\ell)}\psi(s)ds$$
$$\le \epsilon\lambda_\psi\psi(t)$$

In addition, we get for each $t \in \mathcal{J}_\ell$

$$|z(t) - y(t)|$$
$$= \Big|z(t) + \frac{(b_\ell-b_{\ell-1})^{-1}(t-b_{\ell-1})}{\Gamma(u_\ell)}$$
$$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell-1}\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))ds$$
$$-\frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t-s)^{u_\ell-1}\Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y(s))ds\Big|$$
$$\le \Big|z(t) + \frac{(b_\ell-b_{\ell-1})^{-1}(t-b_{\ell-1})}{\Gamma(u_\ell)}$$
$$\times \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell-1}\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s))ds$$
$$-\frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t-s)^{u_\ell-1}\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z(s))ds\Big|$$
$$+\frac{(b_\ell-b_{\ell-1})^{-1}(t-b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell-1}$$
$$\times|\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z) - \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y)|ds$$
$$+\frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t-s)^{u_\ell-1}$$
$$\times|\Phi(s, z(s), I_{b_{\ell-1}^+}^{u_\ell} z) - \Phi(s, y(s), I_{b_{\ell-1}^+}^{u_\ell} y)|ds$$
$$\le \lambda_\psi\epsilon\psi(t)$$
$$+\frac{(b_\ell-b_{\ell-1})^{-1}(t-b_{\ell-1})}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{b_\ell} (b_\ell - s)^{u_\ell-1}s^{-\delta}$$
$$\times(K|z(s) - y(s)| + LI_{b_{\ell-1}^+}^{u_\ell}|z(s) - y(s)|)ds$$
$$+\frac{1}{\Gamma(u_\ell)} \int_{b_{\ell-1}}^{t} (t-s)^{u_\ell-1}s^{-\delta}$$
$$\times(K|z(s) - y(s)| + LI_{b_{\ell-1}^+}^{u_\ell}|z(s) - y(s)|)ds$$
$$\le \lambda_\psi\epsilon\psi(t) + \frac{(b_\ell-b_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)})$$
$$\times(K\|z-y\|_{E_\ell} + L\|I_{b_{\ell-1}^+}^{u_\ell}(z-y)\|_{E_\ell} \int_{b_{\ell-1}}^{b_\ell} s^{-\delta}ds$$
$$+\frac{(b_\ell-b_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}(K\|z-y\|_{E_\ell} + L\|I_{b_{\ell-1}^+}^{u_\ell}(z-y)\|_{E_\ell})$$
$$\times \int_{b_{\ell-1}}^{t} s^{-\delta}ds$$
$$\le \lambda_\psi\epsilon\psi(t) + \frac{(b_\ell-b_{\ell-1})^{u_\ell-1}({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$$
$$\times(K\|z-y\|_{E_\ell} + L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z-y\|_{E_\ell})$$
$$+\frac{(b_\ell-b_{\ell-1})^{u_\ell-1}(t^{1-\delta}-{b_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$$
$$\times(K\|z-y\|_{E_\ell} + L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z-y\|_{E_\ell})$$
$$\le \lambda_\psi\epsilon\psi(t) + \frac{2(b_\ell-b_{\ell-1})^{u_\ell-1}({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}$$
$$\times(K + L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\|z-y\|_{E_\ell}$$

which gives

$$\|z-y\|_{E_\ell}$$
$$\times\Big(1 - \frac{2({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})(b_\ell-b_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}$$
$$\times(K + L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})\Big)$$
$$\le \lambda_\psi\epsilon\psi(t)$$

For each $t \in \mathcal{J}_\ell$, we arrive at the following relation

$$\|z-y\|_{E_\ell}$$
$$\le \frac{\lambda_\psi\epsilon\psi(t)}{(1-\frac{2({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})(b_\ell-b_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}(K+L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}))}$$
$$= [1 - \frac{2({b_\ell}^{1-\delta}-{b_{\ell-1}}^{1-\delta})(b_\ell-b_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}$$
$$\times(K + L\frac{(b_\ell-b_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})]^{-1}\lambda_\psi\epsilon\psi(t)$$
$$:= c_\Phi\epsilon\psi(t)$$

which concludes that the equation (6) admits **UHR** stability with respect to ψ for each $i \in \{1, 2, ..., n\}$.

Consequently, main problem (1) has **UHR** stability with respect to ψ. □

5. Example

Consider the fractional boundary value problem that follows:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{u(t)}y(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{u(t)}y(t)|)}, \\ y(0)=0, \ \ y(2)=0. \end{cases} \tag{12}$$

for $t \in \mathcal{J} := [0,2]$,
Let

$$\Phi(t,y,z) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+y+z)},$$

$(t,y,z) \in [0,2]\times[0,+\infty)\times[0,+\infty)$ and

$$u(t) = \begin{cases} \frac{3}{2}, & t \in \mathcal{J}_1 := [0,1], \\ \frac{9}{5}, & t \in \mathcal{J}_2 :=]1,2]. \end{cases} \tag{13}$$

Then, we have

$$\begin{aligned} & t^{\frac{1}{3}}|\Phi(t,w_1,z_1)-\Phi(t,w_2,z_2)| \\ &= \left|\frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)}\left(\frac{1}{1+w_1+z_1}-\frac{1}{1+w_2+z_2}\right)\right| \\ &\le \frac{e^{-t}(|w_1-w_2|+|z_1-z_2|)}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+w_1+z_1)(1+w_2+z_2)} \\ &\le \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)}(|w_1-w_2|+|z_1-z_2|) \\ &\le \frac{1}{(e+5)}|w_1-w_2| + \frac{1}{(e+5)}|z_1-z_2|. \end{aligned}$$

As a result, with $\delta = \frac{1}{3}$ and $K = L = \frac{1}{e+5}$, the assumption (H2) is satisfied.
By (13), solution of the given problem (12) can be split into two parts as follows

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\frac{3}{2}}y(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{3}{2}}y(t)|)}, \\ t \in \mathcal{J}_1, \\ {}^{C}\mathcal{D}_{1^+}^{\frac{9}{5}}y(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{9}{5}}y(t)|)}, \\ t \in \mathcal{J}_2. \end{cases}$$

For $t \in \mathcal{J}_1$, we begin by looking at the following boundary value problem:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\frac{3}{2}}y(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{3}{2}}y(t)|)}, \\ y(0)=0, \ \ y(1)=0. \end{cases} \tag{14}$$

We are in position to check whether the condition (9) is satisfied or not

$$\frac{({b_1}^{1-\delta}-{b_0}^{1-\delta})(b_1-b_0)^{u_1-1}}{(1-\delta)\Gamma(u_1)}\left(2K + \frac{2L(b_1-b_0)^{u_1}}{\Gamma(u_1+1)}\right)$$

$$= \frac{2}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})}\left(1+\frac{1}{\Gamma(\frac{5}{2})}\right) \simeq 0.7685 < 1$$

Let $\psi(t) = t^{\frac{1}{2}}$

$$\begin{aligned} I_{0^+}^{u_1}\psi(t) &= \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}}s^{\frac{1}{2}}ds \\ &\le \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}}ds \\ &\le \frac{2}{3\Gamma(\frac{3}{2})}\psi(t) := \lambda_\psi\psi(t). \end{aligned}$$

It shows that the assumption (H3) holds with $\psi(t) = t^{\frac{1}{2}}$ and $\lambda_\psi = \frac{2}{3\Gamma(\frac{3}{2})}$.
Regarding Theorem (3), the problem (14) has a unique solution $y_1 \in E_1$, and from Theorem (4) the solution of (14) is **UHR** stable.
For $t \in \mathcal{J}_2$, the problem (12) can be written in the following way

$$\begin{cases} {}^{C}\mathcal{D}_{1^+}^{\frac{9}{5}}y(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|y(t)|+|I_0^{\frac{9}{5}}y(t)|)}, \\ y(1)=0, \ \ y(2)=0. \end{cases} \tag{15}$$

We see that

$$\frac{({b_2}^{1-\delta}-{b_1}^{1-\delta})(b_2-b_1)^{u_2-1}}{(1-\delta)\Gamma(u_2)}\left(2K + \frac{2L(b_2-b_1)^{u_2}}{\Gamma(u_2+1)}\right)$$
$$= \frac{2^{\frac{2}{3}}-1}{\frac{2}{3}\Gamma(\frac{9}{5})}\frac{2}{e+5}\left(1+\frac{1}{\Gamma(\frac{14}{5})}\right) \simeq 0.3913 < 1$$

Thus, the condition (9) is satisfied.
Also

$$\begin{aligned} I_{1^+}^{u_2}\psi(t) &= \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-s)^{\frac{4}{5}}s^{\frac{1}{2}}ds \\ &\le \frac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-s)^{\frac{4}{5}}ds \\ &\le \frac{5}{9\Gamma(\frac{9}{5})}\psi(t) \\ &:= \lambda_\psi\psi(t). \end{aligned}$$

Therefore, the condition (H3) is satisfied with $\psi(t) = t^{\frac{1}{2}}$ and $\lambda_\psi = \frac{5}{9\Gamma(\frac{9}{5})}$.
Taking into account of Theorem (3), the problem (15) has a unique solution $\widetilde{y}_2 \in E_2$, and from Theorem (4) the equation (15) has **UHR** stability.
It is known that

$$y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1 \\ \widetilde{y}_2(t), & t \in \mathcal{J}_2. \end{cases}$$

Hence, by considering definition (3), the boundary value problem (12) has got a unique solution

$$y(t) = \begin{cases} y_1(t), & t \in \mathcal{J}_1, \\ y_2(t) = \begin{cases} 0, & t \in \mathcal{J}_1, \\ \widetilde{y}_2(t), & t \in \mathcal{J}_2. \end{cases} \end{cases}$$

Eventually, according to Theorem (4), the equation (12) is **UHR** stable with respect to ψ.

6. Conclusion

We study some qualitative properties for a class of nonlinear fractional boundary value problems involving variable order operators. Since the existence and uniqueness as well as stability results to variable-order equations is rarely discussed in the literature, all of the outcomes in this paper have a great deal of potential for contributing to future researches.

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Author, A. (Year). Title of book. Publisher, Place of Publication.

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Author, A. (Year). Title of chapter. In: A. Editor and B. Editor, eds. Title of book. Publisher, Place of publication, pages.

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Internet document

Author, A. (Year). Title of document [online]. Source. Available from: URL [Accessed (date)].

Holland, M. (2004). Guide to citing Internet sources [online]. Poole, Bournemouth University. Available from: http://www.bournemouth.ac.uk/library/using/guide_to_citing_internet_sourc.html [Accessed 4 November 2004].

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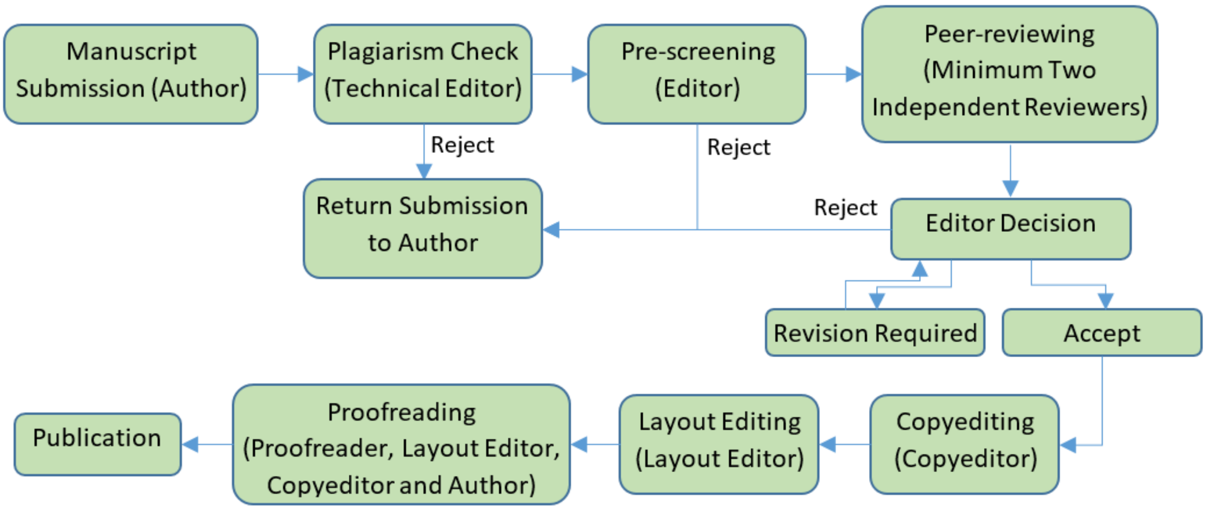

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