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RESEARCH ARTICLE

Behaviour of the first-order q-difference equation

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ABSTRACT

Since the need to investigate many aspects of q-difference equations cannot be ruled out, this article aims to explore response of the mechanism modelled by linear and nonlinear q-difference equations. Therefore, analysis of an important bundle of nonlinear q-difference equations, in particular the q-Bernoulli difference equation, has been developed. In this context, capturing the behaviour of the q-Bernoulli difference equation as well as linear q-difference equations are considered to be a significant contribution here. Illustrative examples related to the difference equations are also presented.

1. Introduction

Discretization of differential equations is an essential and necessary step in capturing the discrete behavior of the processes governed by the corresponding equations. One can consider an equally effective q-discretization based on geometric progression rather than conventional discretization regarding arithmetic progression. This approach gives rise to q-difference equations in which differential equations are encountered as $q \to 1$. q-Difference equations are important models for representing a large number of physical events encountered in various fields of science [1–8]. The q-difference equations were considered at the beginning of the nineteenth century [9–14]. Interested readers can check references [15–19] for historical development of the subject. Although there are important studies in the literature [20–26] on linear and nonlinear q-difference equations, the need to investigate many aspects of these topics cannot be ignored. Note that, especially, analysis of the nonlinear q-difference equations is still in the initial stages and many aspects of this analysis need to be discovered. Therefore, albeit small, an important bundle of nonlinear q-difference equations, in particular q-Bernoulli difference equation, will form the main backbone of this study. To the best knowledge of the authors, the analysis of most nonlinear q-difference equations is yet to be developed. Therefore, the q-Bernoulli difference equation, as well as linear q-difference equations, has been considered in order to make a significant contribution here.

2. Preliminaries

Let us recall some basic concepts of q-calculus [17–19, 27].

For $0 < q < 1$, we define the q-derivative of a real valued function $f(x)$ as

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad D_q f(0) = \lim_{x \to 0} D_q f(x).$$

The higher order q-derivatives are given by

$$D_q^0 f(x) = f(x), \quad D_q^n f(x) = D_q D_q^{n-1} f(x),$$

where $n \in \mathbb{N}$.

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The formulas for the q-derivative of a sum, a product and a quotient of functions are, respectively;

$$D_q\Big(f(x)+g(x)\Big) = D_q(f(x)) + D_q(g(x))$$

$$D_q\Big(f(x)g(x)\Big) = f(qx)D_q(g(x)) + g(x)D_q(f(x))$$

$$D_q\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x)D_q(f(x)) - f(x)D_q(g(x))}{g(x)g(qx)}$$

The q-analogue of any real number t is defined as $[t]_q = \frac{1-q^t}{1-q}$ and in the case, if t is a positive integer we have $[t]_q = \frac{1-q^t}{1-q} = 1 + q + \cdots + q^{t-1}$. Furthermore, the q-analogue of factorial, denoted by $[n]_q!$, is defined as

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q[n-1]_q \dots [1]_q & \text{if } n = 1, 2, \dots. \end{cases}$$

The q-analogue of $(a+x)^n$, denoted by $(a+x)_q^n$, is defined as

$$(a+x)_q^n = \begin{cases} 1 & n = 0, \\ \prod_{m=0}^{n-1}(a + xq^m) & n = 1, 2, \dots, \end{cases} \quad (1)$$

and it is also defined for any complex number α as

$$(a+x)_q^\alpha = \frac{(a+x)_q^\infty}{(a+q^\alpha x)_q^\infty},$$

where $(a+x)_q^\infty := \lim_{n\to\infty} \prod_{m=0}^{n}(a + xq^m)$ and the principal value of q^α is taken and it is assumed that $0 < q < 1$. The q-Taylor series expansion of (1) about $x = 0$ is

$$(a+x)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k}x^k q^{\binom{k}{2}} \quad (2)$$

where

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

are called q-binomial coefficients. Formula (2) is called Gauss's q-binomial formula. The interested reader can find some important identities involving q-binomial coefficients in [28].

Two q-exponential functions are defined as

$$e_q^x = \frac{1}{(1-(1-q)x)_q^\infty} = \sum_{n=0}^{\infty} \frac{1}{[n]_q!}x^n, \quad |x| < 1. \quad (3)$$

$$E_q^x = (1+(1-q)x)_q^\infty = \sum_{n=0}^{\infty} \frac{1}{[n]_q!}x^n q^{\binom{n}{2}}, \quad x \in \mathbb{C}. \quad (4)$$

One can see that $e_q^x E_q^{-x} = 1$ and $e_{q^{-1}}^x = E_q^x$.

The q-derivative of these two q-exponential functions can be found as follows:

$$D_q e_q^{ax} = a\, e_q^{ax}.$$

$$D_q E_q^{ax} = a\, E_q^{aqx}.$$

Let for some $0 \le \alpha < 1$, the function $|f(x)x^\alpha|$ is bounded on the interval $(0, A]$, then Jakson integral defines as

$$\int f(x)d_qx = (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x),$$

converges to a function $F(x)$ on $(0, A]$, which is a q-antiderivative of $f(x)$. For entire functions f(x) one can easily see that this q-integral approaches the Riemann integral as $q \to 1$, and also that the q-differentiation and q-integration are inverse to each other

$$D_q \int f(x)d_qx = f(x),$$

$$\int D_q f(x)d_qx = f(x) - f(0).$$

Suppose $0 < a < b$, the definite q-integral is defined as

$$\int_0^b f(x)d_qx = (1-q)b\sum_{j=0}^{\infty} q^j f(q^j b),$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

3. q-Difference equations

Since some natural processes are represented by linear or mostly nonlinear q-difference equations, a broad range of attention has been paid on the discovery of the behavior of the corresponding processes. In this context, we will first examine linear models and then more realistic nonlinear ones as follows.

3.1. First-order linear q-difference equations

A first order q-difference equation is linear if it has the form

$$D_q y(x) + p(x)y(x) = g(x) \quad (5)$$

or

$$D_q y(x) + p(x)y(qx) = g(x) \quad (6)$$

for some functions $p(x)$ and $g(x)$. Equation (5) can be obtain from (6) by replacing x by $q^{-1}x$ and then q by q^{-1} and vice versa. Therefore we consider to solve a first order q-difference equation which is in the form of (5) and also $p(x)$ be polynomial function.

To easily solve the first order q-difference equations, we need to give the following definitions.

Definition 1. *For any arbitrary function f(x),*

$$A_q(f(x)) = \prod_{j=0}^{\infty}(1+(1-q)f(q^j x)),$$

$$B_q(f(x)) = \prod_{j=0}^{\infty}\frac{1}{(1-(1-q)f(q^j x))}.$$

Remark 1. *In particular, $A_q(0) = B_q(0) = 1$ and for arbitrary c, $A_q(cx) = E_q^{cx}$ and $B_q(cx) = e_q^{cx}$. In case of $f(x) = x^\alpha$, one can see that $A_q(x^\alpha) = E_{q^\alpha}^{x^\alpha/[\alpha]_q}$ and $B_q(x^\alpha) = e_{q^\alpha}^{x^\alpha/[\alpha]_q}$ where $\alpha \neq 0$. Note that $A_q(-f(x)) = 1/B_q(f(x))$.*

With the selections of $f(x) = 1$ and $f(x) = -1$, one can see that $A_q(1) = \prod_{j=0}^{\infty}(2-q)$ and $A_q(-1) = \prod_{j=0}^{\infty} q$, respectively. Here

$$A_q(1) = \begin{cases} 0 & \text{if } q \in (1,3), \\ 1 & \text{if } q = 1, \\ \text{diverges} & \text{otherwise,} \end{cases}$$

and

$$A_q(-1) = \begin{cases} 0 & \text{if } q \in (-1,1), \\ 1 & \text{if } q = 1, \\ \text{diverges} & \text{otherwise.} \end{cases}$$

Proposition 1. *For any non-constant function f(x),*

$$D_q\Big(A_q(f(x))\Big) = \frac{f(x)}{x}A_q(f(qx)),$$

$$D_q\Big(B_q(f(x))\Big) = \frac{f(x)}{x}B_q(f(x)).$$

Proof. Let us use the definition to find the q-derivative of $A_q(f(x))$. Then

$$\begin{aligned} D_q\Big(A_q(f(x))\Big) &= \frac{A_q(f(qx)) - A_q(f(x))}{(q-1)x} \\ &= \frac{\prod_{j=1}^{\infty}(1+(1-q)f(q^j x))}{(q-1)x} \\ &\quad - \frac{\prod_{j=0}^{\infty}(1+(1-q)f(q^j x))}{(q-1)x} \\ &= \frac{1-(1+(1-q)f(x))}{(q-1)x} \\ &\quad \times \prod_{j=1}^{\infty}(1+(1-q)f(q^j x)) \\ &= \frac{f(x)}{x}A_q(f(qx)). \end{aligned}$$

In finding the q-derivative of $B_q(f(x))$, consideration of the property $A_q(-f(x)) = 1/B_q(f(x))$ leads to

$$\begin{aligned} D_q\Big(B_q(f(x))\Big) &= D_q\left[\frac{1}{A_q(-f(x))}\right] \\ &= \frac{-D_q A_q(-f(x))}{A_q(-f(x))A_q(-f(qx))}. \end{aligned}$$

Substitution of the q-derivative of $A_q(f(x))$ in the last result yields

$$\frac{\frac{f(x)}{x}A_q(-f(qx))}{A_q(-f(x))A_q(-f(qx))} = \frac{f(x)}{x}B_q(f(x)).$$

□

A generalized approach is presented here for solving the first order q-difference equations in (5). Let us then multiply $D_q y(x) + p(x)y(x) = g(x)$ by $A_q(qxp(qx))$ to reach

$$\begin{aligned} g(x)A_q(qxp(qx)) = &A_q(qxp(qx))D_q y(x) \\ &+ p(x)A_q(qxp(qx))y(x). \end{aligned}$$

Notice that $D_q\big(A_q(xp(x))\big) = p(x)A_q(qxp(qx))$, and thus

$$\begin{aligned} g(x)A_q(qxp(qx)) = &A_q(qxp(qx))D_q y(x) \\ &+ y(x)D_q A_q(xp(x)). \end{aligned}$$

The left hand side of the last equation is the q-derivative of $A_q(xp(x))y(x)$. The corresponding q-difference equation has become

$$D_q\Big(A_q(xp(x))y(x)\Big) = g(x)A_q(qxp(qx))$$

which can be q-integrated to obtain

$$A_q(xp(x))y(x) = \int g(x)A_q(qxp(qx))d_q x + c.$$

Therefore this equation can be solved for y:

$$\begin{aligned} y(x) = &B_q(-xp(x))\int g(x)A_q(qxp(qx))d_q x \\ &+ c\, B_q(-xp(x)). \end{aligned} \tag{7}$$

In similar manner, equation (6) can also be solved. Then let us multiply $D_q y(x) + p(x)y(qx) = g(x)$ by $B_q(xp(x))$. At the end of this process, the solution for y is found to be:

$$\begin{aligned} y(x) = &A_q(-xp(x))\int g(x)B_q(xp(x))d_q x \\ &+ cA_q(-xp(x)). \end{aligned} \tag{8}$$

In order to illustrate the previous discussion, attention can be paid on the following q-difference equations:

Example 1. *Let us consider the equation*

$$D_q y(x) - y(x) = x^m$$

where $m \geq 0$. Use of (7) for this equation gives rise to the solution

$$\begin{aligned} y(x) &= B_q(x)\int g(x)A_q(-qx)d_qx + c\,B_q(x) \\ &= e_q^x \int x^m E_q^{-qx} d_qx + c\,e_q^x \\ &= e_q^x\,\Gamma_q(m+1) + c\,e_q^x \end{aligned}$$

where $\Gamma_q(m+1) = \int x^m E_q^{-qx} d_qx$.

Example 2. *One can now take*

$$D_q y(x) + y(x) = \alpha e_q^{\beta x}$$

where α, β are real numbers and $\beta \neq -1$. Utility of (7) for the current equation leads to

$$\begin{aligned} y(x) &= B_q(-x)\int g(x)A_q(qx)d_qx + c\,B_q(-x) \\ &= \alpha\,e_q^{-x}\int e_q^{\beta x}E_q^{qx}d_qx + c\,e_q^{-x} \\ &= \tfrac{\alpha}{\beta+1}e_q^{\beta x} + c\,e_q^{-x}. \end{aligned}$$

Example 3. *Let us now take into account*

$$x^{1-n}D_q y(x) + [n]_q\,y(x) = 1$$

where $n \neq 0$. Consideration of (7) gives

$$\begin{aligned} y(x) &= B_q(-[n]_q\,x^n)\int x^{n-1}A_q([n]_q\,q^n x^n)d_qx \\ &\quad + c\,B_q(-[n]_q\,x^n) \\ &= e_{q^n}^{-x^n}\int x^{n-1}E_{q^n}^{(qx)^n}d_qx + c\,e_{q^n}^{-x^n} \\ &= \tfrac{1}{[n]_q} + c\,e_{q^n}^{-x^n}. \end{aligned}$$

Example 4. *Consider the following q-difference equation*

$$x^{1-n}D_q y(x) + y(x) = ax^{1-n}$$

where n is a positive real number. Then again

$$\begin{aligned} y(x) &= B_q(-x^n)\int aA_q(q^n x^n)d_qx + cB_q(-x^n) \\ &= a\,e_{q^n}^{-\frac{x^n}{[n]_q}}\int E_{q^n}^{\frac{q^n x^n}{[n]_q}}d_qx + c\,e_{q^n}^{-\frac{x^n}{[n]_q}} \\ &= a\,e_{q^n}^{-\frac{x^n}{[n]_q}}\,x\sum_{k=0}^{\infty}\frac{(1-q)_{q^n}^k}{(1-q^n)_{q^n}^k(1-q^{n+1})_{q^n}^k} \\ &\quad \times((1-q)(qx)^n)^k q^{n\binom{k}{2}} + c\,e_{q^n}^{-\frac{x^n}{[n]_q}} \\ &= a\,e_{q^n}^{-\frac{x^n}{[n]_q}}\,x\;{}_1\phi_1(q;q^{n+1};q^n,-(1-q)(qx)^n) \\ &\quad + c\,e_{q^n}^{-\frac{x^n}{[n]_q}} \end{aligned}$$

where ${}_r\phi_s$ is a q-hypergeometric series (see [18], page 4).

Example 5. *Consider the following q-difference equation*

$$D_q y(x) + x^{n-1}y(x) = x^{n-1}\prod_{i=1}^{n-1}E_{q^n}^{\frac{q^i x^n}{[n]_q}}$$

where n is a positive integer greater than 1. Thus

$$\begin{aligned} y(x) &= B_q(-x^n)\int x^{n-1}\prod_{i=1}^{n-1}E_{q^n}^{\frac{q^i x^n}{[n]_q}}A_q(q^n x^n)d_qx \\ &\quad + cB_q(-x^n) \\ &= e_{q^n}^{-\frac{x^n}{[n]_q}}\int x^{n-1}\prod_{i=1}^{n}E_{q^n}^{\frac{q^i x^n}{[n]_q}}d_qx + c\,e_{q^n}^{-\frac{x^n}{[n]_q}} \\ &= e_{q^n}^{-\frac{x^n}{[n]_q}}\int x^{n-1}E_q^{qx^n}d_qx + c\,e_{q^n}^{-\frac{x^n}{[n]_q}} \\ &= e_{q^n}^{-\frac{x^n}{[n]_q}}\,x^n\sum_{k=0}^{\infty}\frac{(qx^n)^k}{[k]_q![n(k+1)]_q}q^{\binom{k}{2}} + c\,e_{q^n}^{-\frac{x^n}{[n]_q}} \\ &= e_{q^n}^{-\frac{x^n}{[n]_q}}\,x^n E_q^{\frac{(1-q^n)_{q^n}}{(1-q^{2n})_{q^n}}qx^n} + c\,e_{q^n}^{-\frac{x^n}{[n]_q}}. \end{aligned}$$

where $E_q^{[n]_q x} = \prod_{i=0}^{n-1}E_{q^n}^{q^i x}$ and $E_q^{(a+b)_q x} = \sum_{k=0}^{\infty}\frac{(a+b)_q^k\,x^k}{[k]_q!}q^{\binom{k}{2}}$.

3.2. q-Bernoulli difference equations

This section is to focus on more natural processes described by the q-difference equations in the form,

$$D_{q^{n-1}}y(x) + p(x)y(x) = g(x)\prod_{j=0}^{\infty}\frac{y(q^j x)}{y(q^{n+j}x)} \quad (9)$$

and

$$D_{q^{n-1}}y(x) + p(x)y(q^{n-1}x) = g(x)\prod_{j=0}^{\infty}\frac{y(q^j x)}{y(q^{n+j}x)} \quad (10)$$

where $p(x)$ is a polynomial, $g(x)$ is a continuous function and n is any real number. Note that $n \neq 0$ and $n \neq 1$. Notice also that, as $q \to 1$, those q-difference equations (9) and (10) lead to the usual Bernoulli equations.

To solve q-difference equation (9), it is first divided by $\prod_{j=0}^{\infty}\frac{y(q^j x)}{y(q^{n+j}x)}$,

$$D_{q^{n-1}}y(x)\prod_{j=0}^{\infty}\frac{y(q^{n+j}x)}{y(q^j x)} + p(x)\prod_{j=0}^{\infty}\frac{y(q^{n+j}x)}{y(q^{j+1}x)} = g(x). \quad (11)$$

Substitution of $v(x) = \prod_{j=0}^{\infty}\frac{y(q^{n-1+j}x)}{y(q^j x)}$ is used to convert the above equation into a q-difference equation in terms of $v(x)$. Yet, one can find the q-derivative of $v(x)$ as follows

$$\begin{aligned} D_q v(x) &= D_q\Big[\prod_{j=0}^{\infty}\frac{y(q^{n-1+j}x)}{y(q^j x)}\Big] \\ &= \frac{\prod_{j=0}^{\infty}\frac{y(q^{n+j}x)}{y(q^{j+1}x)} - \prod_{j=0}^{\infty}\frac{y(q^{n-1+j}x)}{y(q^j x)}}{(q-1)x} \end{aligned}$$

$$= \frac{q^{n-1}-1}{q-1} \frac{y(x)-y(q^{n-1}x)}{(q^{n-1}-1)x} \prod_{j=0}^{\infty} \frac{y(q^{n+j}x)}{y(q^j x)}$$
$$= -[n-1]_q D_{q^{n-1}} y(x) \prod_{j=0}^{\infty} \frac{y(q^{n+j}x)}{y(q^j x)}.$$

Now, plugging of this result and $v(x)$ value into equation (11) gives the following solvable linear form,

$$\frac{-1}{[n-1]_q} D_q v(x) + p(x)v(qx) = g(x). \qquad (12)$$

This linear q-difference equation is solved for $v(x)$ as previously carried out. Then the solution $v(x)$ is used to find the required solution $y(x)$ to the original q-difference equation by plugging $v(x)$ back into

$$v(x) = \prod_{j=0}^{\infty} \frac{y(q^{n-1+j}x)}{y(q^j x)}.$$

Let us rewrite it as

$$y(x) = \begin{cases} y(q^{n-1}x)\frac{v(qx)}{v(x)} & n \neq 2, \\ \frac{1}{v(x)} & n = 2. \end{cases}$$

Then the previous expression can be rewritten for $y(x)$ as follows

- For $0 < q < 1,\ n > 1$ and $n \neq 2$;
$$y(x) = y(0) \prod_{j=0}^{\infty} \frac{v(q^{j(n-1)+1}x)}{v(q^{j(n-1)}x)}$$
- For $q > 1$ and $n < 1$;
$$y(x) = y(0) \prod_{j=0}^{\infty} \frac{v(q^{j(n-1)+1}x)}{v(q^{j(n-1)}x)}$$
- For $n = 2$;
$$y(x) = \frac{1}{v(x)}$$
- Otherwise
$$y(x) = y(\infty) \prod_{j=0}^{\infty} \frac{v(q^{j(n-1)+1}x)}{v(q^{j(n-1)}x)}$$

In a similar way, the q-difference equation in (10) is converted to the following q-difference equation in terms of $v(x)$

$$\frac{-1}{[n-1]_q} D_q v(x) + p(x)v(x) = g(x) \qquad (13)$$

and then the required solution is found to be

- For $0 < q < 1,\ n > 1$ and $n \neq 2$;
$$y(x) = y(0) \prod_{j=0}^{\infty} \frac{v(q^{j(n-1)+1}x)}{v(q^{j(n-1)}x)}$$
- For $q > 1$ and $n < 1$;
$$y(x) = y(0) \prod_{j=0}^{\infty} \frac{v(q^{j(n-1)+1}x)}{v(q^{j(n-1)}x)}$$
- For $n = 2$;
$$y(x) = \frac{1}{v(x)}$$
- Otherwise
$$y(x) = y(\infty) \prod_{j=0}^{\infty} \frac{v(q^{j(n-1)+1}x)}{v(q^{j(n-1)}x)}$$

Example 6. *Let us take the following q-difference equation with the initial value*

$$D_q y(x) + y(qx) = x^2 y(x)y(qx), \quad y(0) = q^{-1}.$$

This is a particular case of the q-Bernoulli difference equation for $n = 2$. Let $v(x) = \frac{1}{y(x)}$. Now the q-difference equation is transformed, depending on $v(x)$, to

$$D_q v(x) - v(x) = -x^2.$$

Reconsideration of Example 1 gives the solution $v(x)$ as

$$v(x) = e_q^x\ (c - [2]_q).$$

Now plugging the obtained result into $v(x) = \frac{1}{y(x)}$ with the initial value condition leads us to find the solution as $y(x) = q^{-1}E_q^{-x}$. Illustrative behaviour of the required solution $y(x) = q^{-1}E_q^{-x}$ is presented as seen in Figure 1.

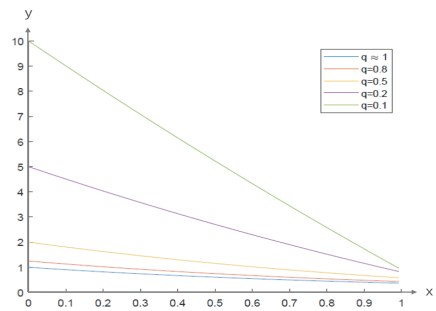

Figure 1. Behaviour of the response for various q values in Example 6.

Example 7. *Now let us consider the following special case of the q-Bernoulli difference equation with the initial value*

$$D_{q^{\frac{1}{2}}} y(x) + y(q^{\frac{1}{2}}x) = x^2 \prod_{j=0}^{\infty} \frac{y(q^j x)}{y(q^{\frac{3}{2}+j}x)}, \quad y(0) = q.$$

Here, $n = \frac{3}{2}$. Let $v(x) = \prod_{j=0}^{\infty} \frac{y(q^{\frac{1}{2}+j}x)}{y(q^j x)}$. Then the Bernoulli equation in term of $v(x)$ is transformed to

$$D_q v(x) - v(x) = -x^2$$

and the solution $v(x)$ is found to be

$$v(x) = e_q^x \, (c - [2]_q).$$

Following the previous procedure results in the solution $y(x)$ as

$$y(x) = y(0) \prod_{j=0}^{\infty} \frac{v(q^{j(n-1)+1}x)}{v(q^{j(n-1)}x)}$$

$$= q \prod_{j=0}^{\infty} \frac{e_q^{xq^{\frac{1}{2}j+1}}}{e_q^{xq^{\frac{1}{2}j}}}$$

$$= qE_q^{-x} E_q^{-xq^{\frac{1}{2}}}.$$

Physical behaviour represented by the required solution $y(x) = qE_q^{-x} E_q^{-xq^{\frac{1}{2}}}$ is depicted illustratively in Figure 2.

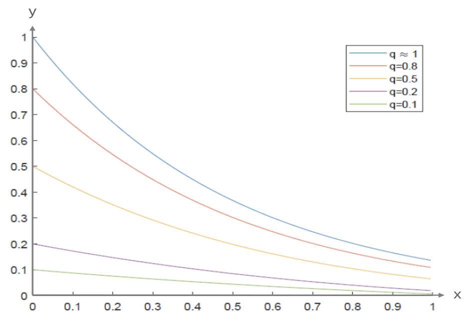

Figure 2. Behaviour of the response for various q values in Example 7.

4. Conclusions and Recommendation

The curiosity to explore many natural processes led us to discover the behaviour represented by linear and nonlinear q-difference equations. In particular, some part of the natural behaviour defined by the corresponding equations has been observed. Note that, the explanatory behaviour of the q-Bernoulli difference equation as well as linear q-difference equations has been seen to be significantly captured. For the sake of making them understandable, some illustrative examples have also been presented. In an upcoming study, with the similar approach, more realistic events taking place in more complex environments can be analyzed.

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