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RESEARCH ARTICLE

Obtaining triplet from quaternions

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ABSTRACT

In this study, we obtain triplets from quaternions. First, we obtain triplets from real quaternions. Then, as an application of this, we obtain dual triplets from the dual quaternions. Quaternions, in many areas, it allows ease in calculations and geometric representation. Quaternions are four dimensions. The triplets are in three dimensions. When we express quaternions with triplets, our study is conducted even easier. Quaternions are very important in the display of rotational movements. Dual quaternions are important in the expression of screw movements. Reducing movements from four dimensions to three dimensions makes our study easier. This simplicity is achieved by obtaining triplets from quaternions.

1. Introduction

A real quaternion is defined that

$$Q = w + xi + yj + zk$$

where w, x, y, z are real numbers and

$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = k, jk = i, ki = j$$
$$ji = -k, kj = -i, ik = -j.$$

The norm of a real quaternion Q is

$$|Q|^2 = Q\overline{Q} = w^2 + x^2 + y^2 + z^2.$$

The set of quaternions is indicated by H.

A dual quaternion $\hat{Q}$ is defined by

$$\hat{Q} = \hat{w} + \hat{x}i + \hat{y}j + \hat{z}k$$

where $\hat{w}, \hat{x}, \hat{y}, \hat{z} \in \widehat{D}$ ($\widehat{D}$ is dual number set). If $\hat{x} \in \widehat{D}$ then

$$\hat{x} = x + \varepsilon x^*$$

where $x, x^* \in R$ (R is real number set) and $\varepsilon^2 = 0$. Thus, we can write

$$\hat{Q} = Q + \varepsilon Q^*,$$

where Q is real quaternion component and Q^* is pure dual quaternion component. The norm of a dual quaternion $\hat{Q}$ is

$$|\hat{Q}|^2 = \hat{Q}\,\overline{\hat{Q}}$$
$$= (Q + \varepsilon Q^*)(\bar{Q} + \varepsilon \bar{Q}^*)$$
$$= Q\bar{Q} + \varepsilon (Q\bar{Q}^* + Q^*\bar{Q})$$
$$= \hat{w}^2 + \hat{x}^2 + \hat{y}^2 + \hat{z}^2.$$

The dual quaternion set is indicated by $\widehat{H}$ [3].

The triplets is in a three-dimensional space. They can be obtained from arbitrary quaternions in four-dimensional space. So, we can make our study easier.

2. Triplets and real quaternions

The polar form of a quaternion containing complex module and complex argument was expressed by Sangwine and Bihan (see [1] for detailed information) Let's take

$$Q = w + xi + yj + zk$$

real quaternion. Let

$$A = |Q|\frac{\zeta}{|\zeta|}$$

where $\zeta = w + xi$. Q can be expressed as

$$Q = Ae^{Bj}$$

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where $B = c + di$ is a complex number. Simply,

$$e^{Bj} = \alpha + \beta j + \gamma k$$

triplet is obtain by multiplying on the left by $\hat{A}^{-1}$ of $\hat{Q}$.

It is known that

$$Bj = (c + di)j = cj + dk$$

and

$$|Bj| = |B|.$$

Then it can be written that

$$Bj = |\, Bj \,| \frac{Bj}{|\, Bj \,|} = |\, B \,| \frac{Bj}{|\, B \,|}.$$

Accordingly,

$$e^{Bj} = e^{|\, B \,| \frac{Bj}{|\, B \,|}}$$

$$= \cos|\, B \,| + \frac{Bj}{|\, B \,|} \sin|\, B \,|$$

$$= \cos|\, B \,| + \frac{c}{|\, B \,|} \sin|\, B \,| j + \frac{d}{|\, B \,|} \sin|\, B \,| k$$

$$= \alpha + \beta j + \gamma k$$

where $\alpha = \cos|\, B \,|$, $\beta = \frac{c}{|\, B \,|}\sin|\, B \,|$, $\gamma = \frac{d}{|\, B \,|}\sin|\, B \,|$ (for detailed information see [1, 3, 4]). We can write

$$q = e^{Bj} = e^{|\, B \,| \frac{Bj}{|\, B \,|}} = e^{\xi j_{\varphi}}$$

where $|\, B \,| = \xi, \frac{Bj}{|\, B \,|} = j_{\varphi}$. Accordingly, we can express Q quaternion with $Q = Aq$ where A is a complex number and q is a triplet. Reverse the process, let's take $\alpha + \beta j + \gamma k$ unit quaternion. Since $\cos|B| = \alpha$ equality, $|B| = \arccos\alpha$ and $\sin|B| = \sqrt{1 - \alpha^2}$. Then $B = c + di$ can be writen where $c = \frac{|B|\beta}{\sin|B|}$, $d = \frac{|B|\gamma}{\sin|B|}$.

A leaf is determined with a unit vector j_{φ}. This vector makes a angle φ with the positive j direction in the (j, k)-plane. This unit vector can be expressed as

$$j_{\varphi} = (cos\varphi)\, j + (sin\varphi)\, k$$

and $j_{\varphi}^2 = -1$. This is a pure unit quaternion. In this leaf, let triplet q be a non-zero. This leaf is spanned by 1 and j_{φ}. We can write q in terms of its components along these vectors. Namely, we can write that

$$e^{Bj} = q = (\cos\xi)1 + (\sin\xi)j_{\varphi} = e^{\xi j_{\varphi}}$$

where ξ be the angle that q makes with the positive vector 1 direction in the leaf.

Thus, with the help of equation $Bj = \xi j_{\varphi}$, B can be calculated [2, 5]. According to Figure 1, $q \epsilon Sp\{1, j_{\varphi}\}$.

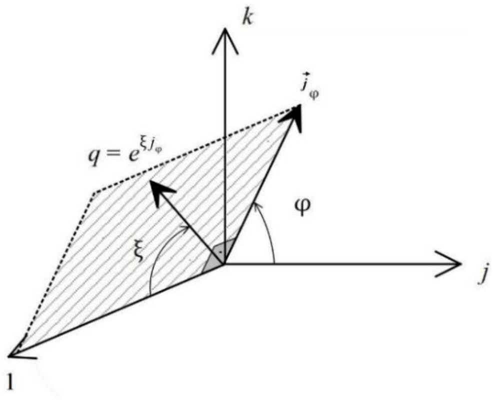

Figure 1. Triplet q.

Example 1. Consider $X = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$. Then

$$X = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k\right)$$

$$= \cos\xi + (\sin\xi)j_{\varphi}$$

$$= e^{\xi j_{\varphi}}$$

where $j_{\varphi} = \frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k$, $\cos\xi = \frac{1}{\sqrt{3}}$ and $\sin\xi = \frac{\sqrt{2}}{\sqrt{3}}$. Then $|B| = \xi = \arccos(\frac{1}{\sqrt{3}})$. So,

$$\xi j_{\varphi} = Bj$$

$$= \arccos(\frac{1}{\sqrt{3}})(\frac{1}{\sqrt{2}}\, j + \frac{1}{\sqrt{2}}\, k)$$

$$= \left(\frac{1}{\sqrt{2}}\arccos\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{2}}\arccos\left(\frac{1}{\sqrt{3}}\right)i\right)j$$

and

$$B = \frac{1}{\sqrt{2}}\arccos(\frac{1}{\sqrt{3}}) + \frac{1}{\sqrt{2}}\arccos(\frac{1}{\sqrt{3}})i.$$

3. Dual triplets and dual quaternions

Similar to the polar form of real quaternions, we can also express the polar form of dual quaternions.

Theorem 1. Let's take

$$\hat{Q} = \hat{w} + \hat{x}i + \hat{y}j + \hat{z}k$$

dual quaternion. Let

$$\hat{A} = |\, \hat{Q} \,| \frac{\hat{\zeta}}{|\, \hat{\zeta} \,|}$$

where $\hat{\zeta} = \hat{w} + \hat{x}i$. $\hat{Q}$ can be written in the form

$$\hat{Q} = \hat{A}e^{\hat{B}j}$$

where $\hat{B} = \hat{c} + \hat{d}i$ is a dual complex number.

Proof. Simply,

$$e^{\hat{B}j} = \hat{\alpha} + \hat{\beta}j + \hat{\gamma}k$$

triplet is obtain by multiplying on the left by $\hat{A}^{-1}$ of $\hat{Q}$.

It is known that

$$\hat{B}j = (\hat{c} + \hat{d}i)j = \hat{c}j + \hat{d}k$$

and $|\hat{B}j| = |\hat{B}|$. Then it can be written that

$$\hat{B}j = |\hat{B}j|\frac{\hat{B}j}{|\hat{B}j|} = |\hat{B}|\frac{\hat{B}j}{|\hat{B}|}.$$

Accordingly,

$$e^{\hat{B}j} = e^{|\hat{B}|\frac{\hat{B}j}{|\hat{B}|}}$$

$$= \cos|\hat{B}| + \frac{\hat{B}j}{|\hat{B}|}\sin|\hat{B}|$$

$$= \cos|\hat{B}| + \frac{\hat{c}}{|\hat{B}|}\sin|\hat{B}|j + \frac{\hat{d}}{|\hat{B}|}\sin|\hat{B}|k$$

$$= \hat{\alpha} + \hat{\beta}j + \hat{\gamma}k$$

where $\hat{\alpha} = \cos|\hat{B}|$, $\hat{\beta} = \frac{\hat{c}}{|\hat{B}|}\sin|\hat{B}|$, $\gamma = \frac{\hat{d}}{|\hat{B}|}\sin|\hat{B}|$ (see [1], [3] and [4]). We can write

$$\hat{q} = e^{\hat{B}j} = e^{|\hat{B}|\frac{\hat{B}j}{|\hat{B}|}} = e^{\hat{\xi}\, j_{\hat{\varphi}}}$$

where $|\hat{B}| = \hat{\xi}$, $\frac{\hat{B}j}{|\hat{B}|} = j_{\hat{\varphi}}$.

Accordingly, we can express $\hat{Q}$ dual quaternion with

$$\hat{Q} = \hat{A}\hat{q}$$

where $\hat{A}$ and $\hat{q}$ are dual complex number and dual triplet respectively.

Reverse the process, let's take $\hat{\alpha} + \hat{\beta}j + \hat{\gamma}k$ unit quaternion. Since $\cos|\hat{B}| = \hat{\alpha}$ equality, $|\hat{B}| = \arccos\hat{\alpha}$ and $\sin|\hat{B}| = \sqrt{1-\hat{\alpha}^2}$. Then $\hat{B} = \hat{c} + \hat{d}i$ can be writen where

$$\hat{c} = \frac{|\hat{B}|\hat{\beta}}{\sin|\hat{B}|} = \frac{(\arccos\hat{\alpha})\hat{\beta}}{\sin(\arccos\hat{\alpha})}$$

and

$$\hat{d} = \frac{|\hat{B}|\hat{\gamma}}{\sin|\hat{B}|} = \frac{(\arccos\hat{\alpha})\hat{\gamma}}{\sin(\arccos\hat{\alpha})}.$$

A leaf is determined by a unit dual vector $j_{\hat{\varphi}}$. This dual vector make a dual angle $\hat{\varphi}$ with the positive j direction in the (j,k)-plane making. This unit vector can be expressed as

$$j_{\hat{\varphi}} = (\cos\hat{\varphi})j + (\sin\hat{\varphi})k$$

and $j_{\hat{\varphi}}^2 = -1$. This is a pure unit dual quaternion. In this leaf, let triplet $\hat{q}$ be a non-zero. This leaf is spanned by 1 and $j_{\hat{\varphi}}$. We can write $\hat{q}$ in terms of its components along these vectors. Namely, we can write that

$$e^{\hat{B}j} = \hat{q} = (\cos\hat{\xi})\,1 + (\sin\hat{\xi})\,j_{\hat{\varphi}} = e^{\hat{\xi} j_{\hat{\varphi}}}.$$

where $\hat{\xi}$ be the dual angle that $\hat{q}$ makes with the positive vector 1 direction in the leaf.

Thus, with the help of equation $\hat{B}\,j = \hat{\xi}\,j_{\hat{\varphi}}$, $\hat{B}$ can be calculated [2, 5]. According to the Figure 2, $\hat{q} \in Sp\{1, j_{\hat{\varphi}}\}$ and $\hat{A} \in Sp\{1, i\}$.

For every unit dual quaternion, we can write that

$$\hat{Q} = \cos\hat{\theta} + \hat{\mu}\sin\hat{\theta}$$

$$= \cos(\theta + \varepsilon\,\theta^*) + \hat{\mu}\sin(\theta + \varepsilon\,\theta^*)$$

where θ is rotation angle and θ^* is translation component about the dual axis $\hat{\mu}$. Accordingly, $\hat{Q}$ is a screw operator. Because $\hat{Q}$ makes dual angular displacement about dual vector axis [3].

From [6], the dual angle $\hat{\theta} = \theta + \varepsilon\,\theta^*$ makes rotation as θ and translation as θ^* about the dual axis $\hat{\mu}$ where

$$\cos\hat{\theta} = \cos\theta - \varepsilon\,\theta^*\sin\theta$$

and

$$\sin\hat{\theta} = \sin\theta + \varepsilon\,\theta^*\cos\theta.$$

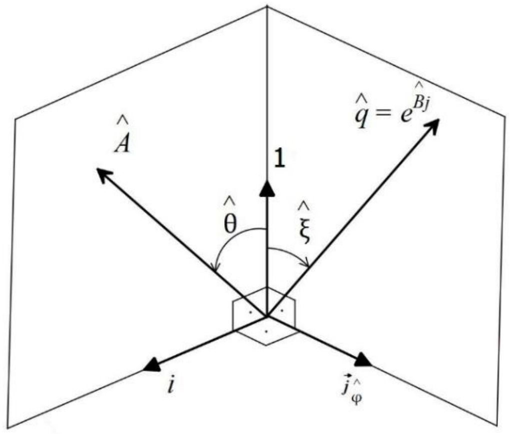

Figure 2. Dual triplet $\hat{q}$ and dual complex number $\hat{A}$.

Example 2. Let $\hat{Q} = (1+\varepsilon)i + (1-\varepsilon)j + k$ then $|\hat{Q}| = \sqrt{3} = |\hat{A}|$ where

$$\hat{A} = \sqrt{3}\frac{(1+\varepsilon)i}{1+\varepsilon} = \sqrt{3}i$$

and

$$\hat{A}^{-1} = \frac{-\sqrt{3}i}{3} = \frac{-i}{\sqrt{3}}.$$

So,

$$\hat{Q} = \hat{A}\,e^{\hat{\xi} j_{\hat{\varphi}}}$$

$$(1+\varepsilon)\,i + (1-\varepsilon)\,j + k = \sqrt{3}\,i\,e^{\hat{\xi} j_{\hat{\varphi}}}.$$

Let multiply $\hat{Q}$ by the inverse of $\hat{A}$ on the left. Then

$$\hat{q} = e^{\hat{\xi} j_{\hat{\varphi}}}$$
$$= \frac{-i}{\sqrt{3}} [(1+\varepsilon)\, i + (1-\varepsilon)\, j + k]$$
$$= \frac{1+\varepsilon}{\sqrt{3}} + j_{\hat{\varphi}} \frac{\sqrt{2-2\varepsilon}}{\sqrt{3}}$$

and

$$\cos\hat{\xi} = \frac{1+\varepsilon}{\sqrt{3}}$$
$$\sin\hat{\xi} = \frac{\sqrt{2-2\,\varepsilon}}{\sqrt{3}}$$

where $\hat{\xi}$ is dual angle. So,

$$\cos\hat{\xi} = \cos(\xi + \varepsilon\, \xi^*) = \cos\xi - \varepsilon\, \xi^* \sin\xi = \frac{1}{\sqrt{3}} + \varepsilon \frac{1}{\sqrt{3}}$$

where $\cos\xi = \frac{1}{\sqrt{3}}$, $\sin\xi = \frac{-\sqrt{2}}{\sqrt{3}}$, $\xi^* = \frac{1}{\sqrt{2}}$ and

$$\hat{\xi} = \arccos(\frac{1}{\sqrt{3}} + \varepsilon \frac{1}{\sqrt{3}}).$$

Furthermore, we can write that

$$j_{\hat{\varphi}} = \frac{\frac{1}{\sqrt{3}} j - \frac{1-\varepsilon}{\sqrt{3}} k}{\frac{\sqrt{2-2\varepsilon}}{\sqrt{3}}}$$
$$= \frac{j - (1-\varepsilon)k}{\sqrt{2-2\varepsilon}}$$

and

$$\hat{B} j = \hat{\xi} j_{\hat{\varphi}}$$
$$= \arccos(\frac{1}{\sqrt{3}} + \varepsilon \frac{1}{\sqrt{3}})(\frac{j - (1-\varepsilon)k}{\sqrt{2-2\varepsilon}})$$
$$= [\arccos(\frac{1}{\sqrt{3}} + \varepsilon \frac{1}{\sqrt{3}})(\frac{1 - (1-\varepsilon)i}{\sqrt{2-2\varepsilon}})] j.$$

Thus, $\hat{B} = \arccos(\frac{1}{\sqrt{3}} + \varepsilon \frac{1}{\sqrt{3}})(\frac{1-(1-\varepsilon)i}{\sqrt{2-2\varepsilon}})$. Checking the result:

$$\hat{A}\hat{q} = \hat{A} e^{\hat{\xi} j_{\hat{\varphi}}}$$
$$= \sqrt{3} i \exp[\arccos(\frac{1}{\sqrt{3}} + \varepsilon \frac{1}{\sqrt{3}})(\frac{j - (1-\varepsilon)k}{\sqrt{2-2\varepsilon}})]$$
$$= \sqrt{3} i (\frac{1+\varepsilon}{\sqrt{3}} + \frac{1}{\sqrt{3}} j + \frac{\varepsilon - 1}{\sqrt{3}} k)$$
$$= (1+\varepsilon)i + (1-\varepsilon)j + k$$
$$= \hat{Q}\ .$$

4. Conclusion

Here, triplets are obtained from the real quaternions. Then, dual triplets are obtained from the dual quaternions. Thus, it is believed that the quaternions which have an important place in motion geometry in general are made more useful. This will contribute to the understanding of some concepts such as rotation, translation, displacement and screw movement.

We know that a Q unit real quaternion is rotation operator. Thus, a unit quaternion can be expressed by two rotation operators.

The displacement of a rigid body is screw displacement. This displacement can be made with screw operator. Every $\hat{Q}$ unit dual quaternion is screw operator. A unit dual quaternion $\hat{Q}$ can be expressed as $\hat{Q} = \hat{A}\hat{q}$, where $\hat{A}$ is a unit dual complex number and $\hat{q}$ is a unit dual triplet. Thus, two screw operators can be expressed with a unit dual quaternion.

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