RESEARCH ARTICLE

Dark and Trigonometric Soliton Solutions in Asymmetrical Nizhnik-Novikov-Veselov Equation with (2+1)-dimensional

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ARTICLE INFO

Article History:
Received 13 February 2019
Accepted 24 July 2019
Available 03 January 2021

Keywords:
(2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation
Modified exponential function method
Dark and trigonometric solutions
Contour surfaces

AMS Classification 2010:
35Axx; 35Cxx; 34Mxx

1. Introduction

Special functions such as hyperbolic and trigonometric play an important role in nonlinear science arising in physics, applied science, mathematical physics and so on. In this sense, the hyperbolic sine arises in the gravitational potential of a cylinder while the hyperbolic tangent arises in the calculation and rapidity of special relativity [1]. In recent years, many real world problems can be symbolized with the help of special functions. Therefore, scientists investigating properties of special functions need to modify or revise the classical methods [2-45] which are not giving any solutions such problems for explaining more physical meaning of problems. Authors of [54] developed a novel numerical method that possesses the capability of a multi-scale solution of the engineering problems. They showed that their method can solve the non-linear coupled differential equations with high accuracy and precision. A novel multi-resolution method proposed by Seyedi in [55] for solving partial differential equations. He tested this method for the solution of well-known viscous Burger’s equation and the obtained results showed superior accuracy in comparison to the finite difference and boundary element methods. Some important models have been investigated by experts in [56-70]. Boiti et al. [46] has introduced a model which is an important applications in incompressible fluids defined as

\[ u_t + u_{xxx} + 3(u \int (u_x)dy)_x = 0, \tag{1} \]

after named the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation (ANNVE). Jian-Guo Liu [48] has derived new Lump-type solutions by using Hirota’s bilinear form for Eq.(1). Z.L. Zhao et al have introduced the mixed Lump stripe solutions to the Eq.(1) [49]. M.S. Osman have applied the generalized unified method for finding multi-wave solutions of Eq.(1) with fractional order [50]. This manuscript is organized

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as follows. In section 2, we present in a detailed manner the modified exponential function method (MEFM). We apply MEFM to the ANVNE to find new dark and trigonometric solutions in section 3. In the last section of paper, we present a comprehensive conclusion.

2. General facts of the MEFM

MEFM is summarized as follows [51-53];

\[ P(u, u_x, u_y, u_t, u^2, \ldots) = 0, \quad (2) \]

where \( u = u(x, y, t) \), is an unknown function, \( P \) is a polynomial in \( u(x, y, t) \).

**Step 1:** Combining the independent variables \( x, y \) and \( t \) by a dependent variable \( \xi \)

\[
\begin{align*}
  u(x, y, t) &= U(\xi), \xi = kx + wy - ct, \\
  \frac{\partial u}{\partial x} &= ku'(\xi), \quad \frac{\partial u}{\partial t} = -cu'(\xi), \\
  \vdots
\end{align*}
\]

where \( k, w, c \) are real constants and non-zero. Putting Eq.(3) into Eq.(2) produces the nonlinear ordinary differential equation (NODE) as following,

\[
N(U, U', U'', U^2, \ldots) = 0, \quad (5)
\]

where \( N \) is a polynomial of \( U = U(\xi) \).

**Step 2:** We suppose the solution form of Eq.(5) in the following form;

\[
U(\xi) = \sum_{i=0}^{N} A_i (e^{-\Omega(\xi)})^i \\
\]

\[
\sum_{j=0}^{M} B_j (e^{-\Omega(\xi)})^j,
\]

in which \( A_i \) \( (0 \leq i \leq N) \) and \( B_j \) \( (0 \leq j \leq M) \)

are real-constants with \( A \neq 0, B \neq 0 \). Here, \( \Omega = \Omega(\xi) \) satisfies the following differential;

\[
\Omega' = \mu e^\Omega + e^{-\Omega} + \lambda,
\]

where \( \mu, \lambda \) are real constants.

Eq.(7) is of the following results under the several conditionals defined as;

**Family 1** If \( \mu \neq 0, \lambda^2 - 4\mu > 0 \),

\[
\Omega(\xi) = \ln(-\frac{\sqrt{\lambda^2 - 4\mu} \tanh(\frac{\lambda(\xi + c_1)}{2\mu}))}{\lambda - 2\mu}).
\]

**Family 2** If \( \mu \neq 0, \lambda^2 - 4\mu < 0 \),

\[
\Omega(\xi) = \ln(-\frac{\sqrt{-\lambda^2 + 4\mu} \tanh(\frac{\lambda(\xi + c_1)}{2\mu}))}{\lambda + 2\mu}).
\]

**Family 3** When \( \mu = 0, \lambda \neq 0, \lambda^2 - 4\mu > 0 \),

\[
\Omega(\xi) = -\ln(-\frac{\lambda}{\exp(\lambda(\xi + c_1)) - 1}).
\]

**Family 4** Once \( \mu \neq 0, \lambda \neq 0, \lambda^2 - 4\mu = 0 \),

\[
\Omega(\xi) = \ln(-\frac{2\lambda(\xi + c_1) + 4}{\lambda^2(\xi + c_1)}).
\]

**Family 5** If \( \mu = 0, \lambda = 0, \lambda^2 - 4\mu = 0 \),

\[
\Omega(\xi) = \ln(\xi + c_1).
\]

**Step 3:** Setting Eqs. (3,4) into the Eq.(2), afterward, we can find the polynomial of \( e^\Omega \). Considering all the coefficients of the same power of \( e^{-\Omega} \) to zero gives a system. By solving this system via various computational programs, we can obtain the values of parameters. This process gives many solutions to the model considered.

3. Implementation of MEFM

In this section, MEFM has been successfully considered to the ANNVE to find more and novel dark and trigonometric function traveling wave solutions. Our aim is to obtain a new hyperbolic function traveling wave solution by using an expansion method of the Eq.(1). We take the travelling wave transformation as following

\[ u = u(x, y, t) = U(\xi), \quad \xi = kx + wy - ct, \quad (11) \]

where \( k, w, c \) are real constants and non-zero. Substituting Eq.(11) into Eq.(1) along with easily calculations, we find an equation between and as

\[ N = M + 2, \quad (12) \]

**Case 1:** If we choose \( M = 1 \) and \( N = 3 \), we can write follows;

\[
U = \frac{A_0 e^{-\Omega(\xi)} + A_1 e^{-2\Omega(\xi)} + A_2 e^{-3\Omega(\xi)}}{B_0 + B_1 e^{-\Omega(\xi)}} = \frac{\gamma}{\Psi'},
\]

and

\[
U' = \frac{\gamma'\Psi - \gamma\Psi'}{\Psi'^2},
\]

\[
U'' = \cdots,
\]

where \( A_3 \neq 0, B_1 \neq 0 \). After simple calculation, we can use the following coefficients for new dark and trigonometric function traveling wave soliton solutions as

**Case-1.1** If we select following coefficients,

\[
A_0 = \frac{-2k\omega}{\lambda A_2 + 2k\omega B_0}, \quad A_1 = \frac{2k\omega B_0}{\lambda A_2 + 2k\omega B_0}, \quad A_2 = \frac{-A_2 - 2k\omega B_0}{2k\omega \lambda},
\]

\[ A_3 = \frac{-A_2 - 2k\omega B_0}{2k\omega \lambda}, \quad B_1 = \frac{-A_2 - 2k\omega B_0}{2k\omega \lambda}, \quad (16) \]
\[
c = \frac{\lambda k^3(-4A_1 + \lambda(A_2 - 6kwB_0))}{A_2 + 2kwB_0},
\]
\[
\mu = \frac{\lambda}{A_2 + 2kwB_0}(A_1 + 2kw\lambda B_0),
\]
we have the new dark solution as following under the Family-1 condition,
\[
u_1(x, y, t) = \varpi - \frac{8\lambda^2 k w (A_1 + 2kw\lambda B_0)^2}{(A_2 + 2kwB_0)^2 f^2(x, y, t)} \times (1 + \frac{A_2 + 2kwB_0}{2A_1 + 4kwB_0} f(x, y, t)),
\]
(17)

where \(\varpi = \frac{-2\lambda kw(A_1 + 2kw\lambda B_0)}{A_2 + 2kwB_0}, f(x, y, t) = -\lambda - \tau \tanh(\frac{1}{2}\tau(kx + wy - ct)), \tau = \sqrt{\lambda^2 - \frac{4\lambda(A_1 + 2kw\lambda B_0)}{A_2 + 2kwB_0}}.\)

For a better understanding of the physical meaning of Eq.(17), 3D and 2D figures along with contour graphs may be seen in Figures (1), (2) and (3) for suitable values of parameters as follows;

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Case-1.2a When
\[
A_0 = \frac{B_0}{B_1^2}(-\lambda A_3 B_0 + A_1 B_1), A_2 = A_3(\lambda + \frac{B_0}{B_1}),
\]
\[
\mu = \frac{-\lambda^2}{2} + 3\frac{A_1}{A_3} - 3\lambda \frac{B_0}{B_1},
\]
\[
c = 3k^3(\frac{4A_1}{A_3} + \lambda(-\lambda - \frac{4B_0}{B_1})), w = -\frac{A_3}{2kB_1},
\]
(18)

we have the new dark solution as following under the Family-1 condition,
\[
u_2 = \frac{\kappa \omega + \omega \lambda \tau A_3 \sqrt{3} B_1 \tanh(\frac{\sqrt{3}}{2} \tau f(x, y, t))}{A_3 B_1^2(\lambda + \tau \sqrt{3} \tanh(\frac{\sqrt{3}}{2} \tau f(x, y, t)))^2} \times \frac{3\omega(-\lambda A_3 B_0 + A_1 B_1)\tanh^2(\frac{\sqrt{3}}{2} \tau f(x, y, t))}{A_3 B_1^2(\lambda + \tau \sqrt{3} \tanh(\frac{\sqrt{3}}{2} \tau f(x, y, t)))^2},
\]
(19)

where \(\omega = -4A_1 B_1 + \lambda A_3(4B_0 + \lambda B_1), \kappa = \frac{-9A_1 B_1 + \lambda A_3(9B_0 + 2\lambda B_1)}{A_3 B_1^2}, \tau = \sqrt{\frac{-4A_1 A_3}{A_3^2} + \lambda(\lambda + \frac{4B_0}{B_1})}, f(x, y, t) = c_1 + kx + 3k^3 \tau^2 t - \frac{A_3}{2kB_1} y.\)

3D and 2D figures along with contour graphs may be seen in Figures (4),(5) and (6).
in which $\omega = -4A_1B_1 + \lambda A_3(4B_0 + \lambda B_1)$, 
$\kappa = -9A_1B_1 + \lambda A_3(9B_0 + 2\lambda B_1)$, 
$\tau = \sqrt{\frac{12A_1}{A_3} + \lambda(-\lambda - \frac{4B_0}{B_1})}$, 
$\varpi = -\frac{4A_1}{A_3} + \lambda(\lambda + \frac{4B_0}{B_1})$,
$f(x, y, t) = c_1 + kx + 3k^3\varpi t - \frac{A_3}{2kB_1}y$.

3D and 2D figures along with contour graphs may be seen in Figures (7), (8) and (9).

Figure 4. The 3D surface of Eq.(19).

Figure 5. The contour surface of Eq.(19).

Figure 6. The 2D surface of Eq.(19).

Case-1.2b When we reconsider Eq.(18) under the terms of Family-2 condition, we can find new trigonometric function traveling wave soliton solution

$$u_3 = \frac{\kappa\omega - \omega\lambda\tau\sqrt{3}A_3B_1\tan(\frac{1}{2}\tau f(x, y, t))}{A_3B_4^2(\lambda + \tau\sqrt{3}\varpi\tan^2(\frac{1}{2}\tau f(x, y, t)))^2}$$

$$+ \frac{3\omega(\lambda A_3B_0 - A_1B_1)\tan^2(\frac{1}{2}\tau f(x, y, t))}{A_3B_4^2(\lambda + \tau\sqrt{3}\varpi\tan^2(\frac{1}{2}\tau f(x, y, t)))^2}$$

(20)
4. Conclusions

With the help of MEFM, we have successfully obtained new dark and trigonometric function travelling soliton solutions. For deeper investigating of physical meanings of solutions found in this paper, 2D and 3D graphs along with contour simulations have been plotted. The alternative perspective view of the solutions Eqs. (17, 19, 20) can be viewed from the 3D, 2D graphs along with contour simulations can be also viewed from the Figs. (1,2,3,4,5,6).

Comparing the results produced in this paper with the existing paper in literature, it can be viewed that the results found in this paper are entirely new dark and trigonometric function travelling soliton solutions to the Eq.(1). To the best of our knowledge, the application of MEFM to the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation has been not submitted previously. With the help of MEFM, we have successfully obtained new dark and trigonometric function travelling soliton solutions. For deeper investigating of physical meanings of solutions found in this paper, 2D and 3D graphs along with contour simulations can be also viewed from the Figs. (1,2,3,4,5,6).

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Acknowledgments

This study was supported by Scientific Research Projects Unit of Harran University with project number K19015.

References


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