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RESEARCH ARTICLE

Some integral inequalities for multiplicatively geometrically P-functions

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ABSTRACT

In this manuscript, by using a general identity for differentiable functions we can obtain new estimates on a generalization of Hadamard, Ostrowski and Simpson type inequalities for functions whose derivatives in absolute value at certain power are multiplicatively geometrically P-functions. Some applications to special means of real numbers are also given.

1. Preliminaries

Let function $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex defined on an interval I of real numbers and $\zeta, \eta \in I$ with $\zeta < \eta$. The following

$$\psi\left(\frac{\zeta+\eta}{2}\right) \leq \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)du \leq \frac{\psi(\zeta)+\psi(\eta)}{2}. \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions [1]. Both inequalities hold in the reversed direction if the function ψ is concave.

Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $\zeta, \eta \in I^{\circ}$ with $\zeta < \eta$. If $|\psi'(x)| \leq M$ for all $x \in [\zeta, \eta]$, then we hold the following inequality

$$\left|\psi(x) - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(t)dt\right| \leq \frac{M}{\eta-\zeta}\left[\frac{(x-\zeta)^2+(\eta-x)^2}{2}\right]$$

for all $x \in [\zeta, \eta]$. This inequality is known as the Ostrowski inequality [2].

The following inequality is well known as Simpson's inequality .

Let $\psi : [\zeta, \eta] \to \mathbb{R}$ be a four-times continuously differentiable mapping on (ζ, η) and $\left\|\psi^{(4)}\right\|_{\infty} = \sup_{x\in(\zeta,\eta)}\left|\psi^{(4)}(x)\right| < \infty$. Then the following inequality

$$\left|\frac{1}{3}\left[\frac{\psi(\zeta)+\psi(\eta)}{2} + 2\psi\left(\frac{\zeta+\eta}{2}\right)\right] - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)du\right| \leq \frac{1}{2880}\left\|\psi^{(4)}\right\|_{\infty}(\eta-\zeta)^4.$$

holds.

Definition 1. *A nonnegative function $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is called P-function if*

$$\psi\left(t\zeta + (1-t)\eta\right) \leq \psi(\zeta) + \psi(\eta)$$

holds for all $\zeta, \eta \in I$ and $t \in (0,1)$.

We will denote by $P(I)$ the set of P-function on the interval I. Note that $P(I)$ contains all non-negative convex and quasi-convex functions.

In [3], Dragomir et al. proved the following inequality of Hadamard type for class of P-functions.

Theorem 1. *Let $\psi \in P(I)$, $\zeta, \eta \in I$ with $\zeta < \eta$ and $\psi \in L[\zeta, \eta]$. Then*

$$\psi\left(\frac{\zeta+\eta}{2}\right) \le \frac{2}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(u)du \le 2\left[\psi(\zeta) + \psi(\eta)\right].$$

Definition 2 ([4]). *Let $I \neq \emptyset$. The function $\psi : I \to [0, \infty)$ is called multiplicatively P-function (or* log-*P-function), if the inequality*

$$\psi\left(t\zeta + (1-t)\eta\right) \le \psi(\zeta)\psi(\eta)$$

holds for all $\zeta, \eta \in I$ and $t \in [0,1]$.

We will denote by $MP(I)$ the class of all multiplicatively P-convex functions on interval I. Clearly, $\psi : I \to [0, \infty)$ is multiplicatively P-function if and only if $\log \psi$ is P-function. We state that the range of the multiplicatively P-functions is greater than or equal to 1. In recent years many authors have studied P-functions and multiplicatively P-function, see [3, 5–8] and therein.

In [4], Kadakal proved the following inequalities of Hermite-Hadamard type integral inequalities for class of multiplicatively P-functions.

Theorem 2. *Let the function $\psi : I \to [1, \infty)$ be a multiplicatively P-function. If $\psi \in L[\zeta, \eta]$, then the following inequalities hold:*

$$\begin{aligned} &i) \quad \psi\left(\frac{\zeta+\eta}{2}\right) \\ \le\ & \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)\psi(\zeta+\eta-u)\,du \le \left[\psi(\zeta)\psi(\eta)\right]^2 \\ &ii) \quad \psi\left(\frac{\zeta+\eta}{2}\right) \\ \le\ & \psi(\zeta)\psi(\eta)\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)du \le \left[\psi(\zeta)\psi(\eta)\right]^2 \end{aligned}$$

In [9], Kadakal et al. gave the following definition in the literature.

Definition 3. *Let $I \neq \emptyset$ be an interval in $(0, \infty) \subseteq \mathbb{R}$. The function $\psi : I \subseteq (0, \infty) \to [0, \infty)$ is said to be multiplicatively geometrically P-function, if the following inequality*

$$\psi\left(\zeta^t \eta^{1-t}\right) \le \psi(\zeta)\psi(\eta)$$

holds for all $\zeta, \eta \in I$ and $t \in [0,1]$.

We will denote by $MGP(I)$ the class of all multiplicatively geometrically P-convex functions on interval I. Clearly, $\psi : I \subseteq (0, \infty) \to [0, \infty)$ is multiplicatively geometrically P-function if and only if $log\psi$ is P-GA-function. The range of the multiplicatively geometrically P-functions is greater than or equal to 1.

Lemma 1 ([10]). *Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $\zeta, \eta \in I$ with $\zeta < \eta$. If $\psi' \in L[\zeta, \eta]$, then*

$$\begin{aligned} &\psi\left(\sqrt{\zeta\eta}\right) - \frac{1}{\ln\eta - \ln\zeta}\int_{\zeta}^{\eta}\frac{\psi(u)}{u}du \\ =\ & \frac{\ln\eta-\ln\zeta}{4}\left[\zeta\int_0^1 t\left(\frac{\eta}{\zeta}\right)^{\frac{t}{2}} f'\left(\zeta^{1-t}(\zeta\eta)^{\frac{t}{2}}\right)dt \right. \\ &\left. -\eta\int_0^1 t\left(\frac{\zeta}{\eta}\right)^{\frac{t}{2}} f'\left(\eta^{1-t}(\eta)^{\frac{t}{2}}\right)dt\right] \end{aligned}$$

and

$$\begin{aligned} &\frac{\psi(\zeta)+\psi(\eta)}{2} - \frac{1}{\ln\eta-\ln\zeta}\int_{\zeta}^{\eta}\frac{\psi(u)}{u}du \\ =\ & \frac{\ln\eta-\ln\zeta}{2}\left[\zeta\int_0^1 t\left(\frac{\eta}{\zeta}\right)^{t} f'\left(\zeta^{1-t}\eta^{t}\right)dt \right. \\ &\left. -\eta\int_0^1 t\left(\frac{\zeta}{\eta}\right)^{t} f'\left(\eta^{1-t}\zeta^{t}\right)dt\right] \\ =\ & \zeta\frac{\ln\eta-\ln\zeta}{2}\int_0^1 (2t-1)\left(\frac{\eta}{\zeta}\right)^{t} f'\left(\zeta^{1-t}\eta^{t}\right)dt. \end{aligned}$$

The aim of this paper is to obtain the general integral inequalities giving the Hermite-Hadamard, Ostrowsky and Simpson type inequalities for the multiplicatively geometrically P-function in the special case using the above lemma.

2. Main results for the Lemma

Theorem 3. *Let the function $\psi : I \subseteq [1, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in L[\zeta, \eta]$, where $\zeta, \eta \in I^\circ$ with $\zeta < \eta$ and $\theta, \lambda \in [0,1]$. If $|\psi'|^q$ is multiplicatively P-function on $[\zeta, \eta]$, $q \ge 1$, then following holds:*

$$\begin{aligned} & \Bigg|(1-\theta)\left(\lambda\psi(\zeta)+(1-\lambda)\,\psi(\eta)\right) \\ & +\theta\psi((1-\lambda)\,\zeta+\lambda\eta)-\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)du\Bigg| \\ \leq\ & (\eta-\zeta)A_1(\theta)\left|\psi'\left(A_\lambda\right)\right| \\ & \times\left(\lambda^2\left|\psi'\left(\zeta\right)\right|+(1-\lambda)^2\left|\psi'\left(\eta\right)\right|\right) \end{aligned} \tag{2}$$

where

$$A_1(\theta)=\theta^2-\theta+\frac{1}{2}$$

and $A_\lambda=(1-\lambda)\,\zeta+\lambda\eta$.

Proof. Let $q\geq 1$ and $A_\lambda=(1-\lambda)\,\zeta+\lambda\eta$. Using the Lemma 1 and power-mean integral inequality,

$$\begin{aligned} & \Bigg|(1-\theta)\left(\lambda\psi(\zeta)+(1-\lambda)\,\psi(\eta)\right)+\theta\psi(A_\lambda) \\ & -\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)du\Bigg| \\ \leq\ & (\eta-\zeta)\Bigg|\lambda^2\int_0^1|t-\theta|\left|\psi'\left(t\zeta+(1-t)\,A_\lambda\right)\right|dt \\ & +(1-\lambda)^2\int_0^1|t-\theta|\left|\psi'\left(t\eta+(1-t)\,A_\lambda\right)\right|dt\Bigg| \\ \leq\ & (\eta-\zeta)\Bigg\{\lambda^2\left(\int_0^1|t-\theta|\,dt\right)^{1-\frac{1}{q}} \\ & \left(\int_0^1|t-\theta|\left|\psi'\left(\zeta+(1-t)\,A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}} \\ & +(1-\lambda)^2\left(\int_0^1|t-\theta|\,dt\right)^{1-\frac{1}{q}} \\ & \left(\int_0^1|t-\theta|\left|\psi'\left(t\eta+(1-t)\,A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\} \end{aligned} \tag{3}$$

is obtain. Since $|\psi'|^q$ is multiplicatively P-function on $[\zeta,\eta]$, we know that for $t\in[0,1]$

$$\left|\psi'\left(t\zeta+A_\lambda\left(1-t\right)\right)\right|^q\leq\left|\psi'\left(\zeta\right)\right|^q\left|\psi'\left(A_\lambda\right)\right|^q \tag{4}$$

and

$$\left|\psi'\left(t\eta+A_\lambda\left(1-t\right)\right)\right|^q\leq\left|\psi'\left(\eta\right)\right|^q\left|\psi'\left(A_\lambda\right)\right|^q. \tag{5}$$

By simple computation

$$\begin{aligned} & \int_0^1|t-\theta|\left|\psi'\left(t\zeta+(1-t)\,A_\lambda\right)\right|^q dt \\ \leq\ & \int_0^1|t-\theta|\left|\psi'\left(\zeta\right)\right|^q\left|f'\left(A_\lambda\right)\right|^q dt \\ =\ & \left|\psi'\left(\zeta\right)\right|^q\left|\psi'\left(A_\lambda\right)\right|^q\int_0^1|t-\theta|\,dt \\ =\ & \left|\psi'\left(\zeta\right)\right|^q\left|\psi'\left(A_\lambda\right)\right|^q\int_0^1|t-\theta|\,dt \end{aligned} \tag{6}$$

and similarly

$$\begin{aligned} & \int_0^1|t-\theta|\left|\psi'\left(t\eta+(1-t)\,A_\lambda\right)\right|^q dt \qquad (7) \\ \leq\ & \left|\psi'\left(\eta\right)\right|^q\left|\psi'\left(A_\lambda\right)\right|^q\left[\theta^2-\theta+\frac{1}{2}\right] \end{aligned}$$

and

$$\int_0^1|t-\theta|\,dt=\theta^2-\theta+\frac{1}{2}. \tag{8}$$

Thus, using (6-8) in (3), we get the inequality (2). □

Corollary 1. *Using the conditions of Theorem 3 for $\theta=1$, then the following generalized midpoint type inequality is obtained:*

$$\begin{aligned} & \left|\psi((1-\lambda)\,\zeta+\lambda\eta)-\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)du\right| \\ \leq\ & \frac{\eta-\zeta}{2}\left|\psi'\left(A_\lambda\right)\right|\left(\lambda^2\left|\psi'\left(\zeta\right)\right|+(1-\lambda)^2\left|\psi'\left(\eta\right)\right|\right). \end{aligned}$$

Corollary 2. *Using the conditions of Theorem 3 for $\theta=1$, if $|\psi'(x)|\leq M$, $x\in[\zeta,\eta]$, then the following Ostrowski type inequality is obtained*

$$\begin{aligned} & \left|\psi(x)-\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\psi(u)du\right| \\ \leq\ & M^2\left[\frac{(x-\zeta)^2+(\eta-x)^2}{2\left(\eta-\zeta\right)}\right] \end{aligned} \tag{9}$$

for each $x\in[\zeta,\eta]$.

Proof. For each $x \in [\zeta, \eta]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)\zeta + \lambda_x \eta$. Hence, we have $\lambda_x = \frac{x-\zeta}{\eta-\zeta}$ and $1-\lambda_x = \frac{\eta-x}{\eta-\zeta}$. Therefore for each $x \in [\zeta, \eta]$, from the inequality (2), (9) is obtained. □

Corollary 3. *Using the conditions of Theorem 3 for $\theta = 0$, then the following generalized trapezoid type inequality is obtained:*

$$\begin{aligned} &\left| \lambda\psi(\zeta) + (1-\lambda)\psi(\eta) - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(u)du \right| \\ \leq\ & \frac{\eta-\zeta}{2}\left|\psi'(A_\lambda)\right| \left(\lambda^2\left|\psi'(\zeta)\right| + (1-\lambda)^2\left|\psi'(\eta)\right|\right). \end{aligned}$$

Corollary 4. *Using the conditions of Theorem 3 for $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then the following Simpson type inequality is obtained*

$$\begin{aligned} &\left| \frac{1}{6}\left[\psi(\zeta) + 4\psi\left(\frac{\zeta+\eta}{2}\right) + \psi(\eta)\right] \right. \\ &\left. - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(u)du \right| \\ \leq\ & \frac{5}{36}(\eta-\zeta)\left|\psi'\left(\frac{\zeta+\eta}{2}\right)\right| A\left(\left|\psi'(\zeta)\right|, \left|\psi'(\eta)\right|\right), \end{aligned}$$

where A is arithmetic mean.

Corollary 5. *Using the conditions of Theorem 3 for $\lambda = \frac{1}{2}$ and $\theta = 1$, then the following midpoint type inequality is obtained*

$$\begin{aligned} &\left| \psi\left(\frac{\zeta+\eta}{2}\right) - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(u)du \right| \\ \leq\ & \frac{\eta-\zeta}{4}\left|\psi'\left(\frac{\zeta+\eta}{2}\right)\right| A\left(\left|\psi'(\zeta)\right|, \left|\psi'(\eta)\right|\right), \end{aligned}$$

where A is arithmetic mean.

Corollary 6. *Using the conditions of Theorem 3 for $\lambda = \frac{1}{2}$, and $\theta = 0$, then the following trapezoid type inequality is obtained*

$$\begin{aligned} &\left| \frac{\psi(\zeta) + \psi(\eta)}{2} - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(u)du \right| \\ \leq\ & \frac{\eta-\zeta}{4}\left|\psi'\left(\frac{\zeta+\eta}{2}\right)\right| A\left(\left|\psi'(\zeta)\right|, \left|\psi'(\eta)\right|\right), \end{aligned}$$

where A is arithmetic mean.

We will give another result for the considered multiplicatively P-functions as follows using Lemma 1

Theorem 4. *Let $\psi : I \subseteq [1,\infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in L[\zeta, \eta]$, where $\zeta, \eta \in I^\circ$ with $\zeta < \eta$ and $\theta, \lambda \in [0,1]$. If $|\psi'|^q$ is multiplicatively P-function on $[\zeta, \eta]$, $q > 1$, then*

$$\begin{aligned} &\left| (1-\theta)\left(\lambda\psi(\zeta) + (1-\lambda)\psi(\eta)\right) \right. \\ &\left. + \theta\psi((1-\lambda)\zeta + \lambda\eta) - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(u)du \right| \\ \leq\ & (\eta-\zeta)\left(\frac{\theta^{p+1} + (1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left|\psi'(A_\lambda)\right| \\ & \left[\lambda^2\left|\psi'(\zeta)\right| + (1-\lambda)^2\left|\psi'(\eta)\right|\right]. \end{aligned} \tag{10}$$

holds, where $A_\lambda = (1-\lambda)\zeta + \lambda\eta$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $A_\lambda = (1-\lambda)\zeta + \lambda\eta$. From Lemma 1 and by Hölder's inequality, we obtain

$$\begin{aligned} &\left[(1-\theta)\left(\lambda\psi(\zeta) + (1-\lambda)\psi(\eta)\right) \right. \\ &\left. + \theta\psi(A_\lambda) - \frac{1}{\eta-\zeta}\int_{\zeta}^{\eta} \psi(u)du \right| \\ &\leq (\eta-\zeta)\left[\lambda^2\int_0^1 |t-\theta|\left|\psi'(t\zeta + (1-t)A_\lambda)\right|dt \right. \\ &\left. + (1-\lambda)^2\int_0^1 |t-\theta|\left|\psi'(t\eta + (1-t)A_\lambda)\right|dt\right] \\ \leq\ & (b-a)\left\{\lambda^2\left(\int_0^1 |t-\theta|^p dt\right)^{\frac{1}{p}} \right. \\ & \left(\int_0^1 \left|\psi'(t\zeta + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}} \\ & + (1-\lambda)^2\left(\int_0^1 |t-\theta|^p dt\right)^{\frac{1}{p}} \\ & \left.\left(\int_0^1 \left|\psi'(t\eta + (1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{11}$$

Because $|\psi'|^q$ is multiplicatively P-function on $[\zeta, \eta]$, the inequalities (4) and (5) holds. Hence, by simple computation

$$\int_0^1 \left|\psi'\left(t\zeta+(1-t)A_\lambda\right)\right|^q dt \le \left|\psi'(\zeta)\right|^q \left|\psi'(A_\lambda)\right|^q \tag{12}$$

$$\int_0^1 \left|\psi'\left(t\eta+(1-t)A_\lambda\right)\right|^q dt \le \left|\psi'(\eta)\right|^q \left|\psi'(A_\lambda)\right|^q \tag{13}$$

and

$$\int_0^1 |t-\theta|^p dt = \frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1} \tag{14}$$

thus, using (12)-(14) in (11), (10) is obtained. □

Corollary 7. *Using the conditions of Theorem 4 with $\theta = 1$, then the following generalized midpoint type inequality is obtained*

$$\begin{aligned}&\left|\psi((1-\lambda)\zeta+\lambda\eta)-\frac{1}{\eta-\zeta}\int_\zeta^\eta \psi(u)du\right| \\ \le\ & (\eta-\zeta)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left|\psi'(A_\lambda)\right| \\ &\times\left[\lambda^2\left|\psi'(\zeta)\right|+(1-\lambda)^2\left|\psi'(\eta)\right|\right].\end{aligned}$$

where $A_\lambda = (1-\lambda)\zeta+\lambda\eta$ and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 8. *Using the conditions of Theorem 4 for $\theta = 0$, then the following generalized trapezoid type inequality is obtained*

$$\begin{aligned}&\left|\lambda\psi(\zeta)+(1-\lambda)\psi(\eta)-\frac{1}{\eta-\zeta}\int_\zeta^\eta \psi(u)du\right| \\ \le\ & \frac{(\eta-\zeta)}{(p+1)^{\frac{1}{p}}}\left|\psi'(A_\lambda)\right| \\ &\times\left[\lambda^2\left|\psi'(\zeta)\right|+(1-\lambda)^2\left|\psi'(\eta)\right|\right],\end{aligned}$$

where $A_\lambda = (1-\lambda)\zeta+\lambda\eta$ and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 9. *Using the conditions of Theorem 4 for $\theta = 1$, if $|\psi'(x)| \le M$, $x \in [\zeta,\eta]$, then the following Ostrowski type inequality is obtained*

$$\begin{aligned}&\left|\psi(x)-\frac{1}{\eta-\zeta}\int_\zeta^\eta \psi(u)du\right| \\ \le\ & \frac{M^2}{(p+1)^{\frac{1}{p}}}\left[\frac{(x-\zeta)^2+(\eta-x)^2}{\eta-\zeta}\right]\end{aligned} \tag{15}$$

for each $x \in [\zeta,\eta]$.

Proof. For each $x \in [\zeta,\eta]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)\zeta+\lambda_x\eta$. Hence we have $\lambda_x = \frac{x-\zeta}{\eta-\zeta}$ and $1-\lambda_x = \frac{\eta-x}{\eta-\zeta}$. Therefore, for each $x \in [\zeta,\eta]$, from the inequality (10), the inequality (15) is obtained. □

Corollary 10. *Using the conditions of Theorem 4 for $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then the following Simpson type inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(\zeta)+4\psi\left(\frac{\zeta+\eta}{2}\right)+\psi(b)\right]\right. \\ &\left.-\frac{1}{\eta-\zeta}\int_\zeta^\eta \psi(u)du\right| \le \frac{\eta-\zeta}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \\ &\times\left|\psi'\left(\frac{\zeta+\eta}{2}\right)\right| A\left(\left|\psi'(\zeta)\right|,\left|\psi'(\eta)\right|\right),\end{aligned}$$

is obtained, where A is the arithmetic mean.

Corollary 11. *Using the conditions of Theorem 4 for $\lambda = \frac{1}{2}$ and $\theta = 1$, then the following midpoint type inequality*

$$\begin{aligned}&\left|\psi\left(\frac{\zeta+\eta}{2}\right)-\frac{1}{\eta-\zeta}\int_\zeta^\eta \psi(u)du\right| \\ \le\ & \frac{\eta-\zeta}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left|\psi'\left(\frac{\zeta+\eta}{2}\right)\right| \\ &\times A\left(\left|\psi'(\zeta)\right|,\left|\psi'(\eta)\right|\right),\end{aligned}$$

is obtained, where A is the arithmetic mean.

Corollary 12. *Using the conditions of Theorem 4 for $\lambda = \frac{1}{2}$ and $\theta = 0$, then the following trapezoid type inequality*

$$\begin{aligned}&\left|\frac{\psi(\zeta)+\psi(\eta)}{2}-\frac{1}{\eta-\zeta}\int_\zeta^\eta \psi(u)du\right| \\ \le\ & \frac{\eta-\zeta}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left|\psi'\left(\frac{\zeta+\eta}{2}\right)\right| \\ &\times A\left(\left|\psi'(\zeta)\right|,\left|\psi'(\eta)\right|\right),\end{aligned}$$

is obtained, where A is the arithmetic mean.

3. Some applications for special means

(1) The weighted arithmetic mean

$$A_\alpha(\zeta,\eta) \quad : \quad = \alpha\zeta + (1-\alpha)\eta, \\ \alpha \in [0,1], \quad \zeta,\eta \in \mathbb{R}.$$

(2) The weighted geometric mean

$$G_\alpha(\zeta,\eta) := \zeta^\alpha \eta^{1-\alpha}, \quad \zeta,\eta > 0.$$

(3) The Logarithmic mean

$$L(\zeta,\eta) := \frac{\eta-\zeta}{\ln\eta - \ln\zeta}, \quad \zeta \neq \eta, \quad \zeta,\eta > 0.$$

Considering the results in Section 2, some inequalities can be obtained for the means given above.

Proposition 1. *Let $\zeta,\eta \in \mathbb{R}$ with $0<\zeta<\eta$ and $\lambda,\theta \in [0,1]$ we have the following inequality:*

$$\left|(1-\theta)A_\lambda\left(e^\zeta,e^\eta\right) + \theta G_\lambda\left(e^\zeta,e^\eta\right) - L\left(e^\zeta,e^\eta\right)\right|$$

$$\leq (\eta-\zeta)A_1(\theta)e^{A_\lambda(\zeta,\eta)}\left(\lambda^2 e^\zeta + (1-\lambda)^2 e^\eta\right)$$

where $A_1(\theta)$ is defined as in Theorem 3.

Proof. Using the Theorem 3 for the function $\psi(t) = e^t$, $t \in [0,\infty)$, the assertion is easily seen. □

Proposition 2. *Let $\zeta,\eta \in \mathbb{R}$ with $0<\zeta<\eta$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\lambda,\theta \in [0,1]$, following inequality*

$$\left|(1-\theta)A_\lambda\left(e^\zeta,e^\eta\right) + \theta G_\lambda\left(e^\zeta,e^\eta\right) - L\left(e^\zeta,e^\eta\right)\right|$$

$$\leq (b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \times e^{A_\lambda(\zeta,\eta)}\left(\lambda^2 e^\zeta + (1-\lambda)^2 e^\eta\right)$$

is obtained, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the Theorem 4 for the function $\psi(t) = e^t$, $t \in [0,\infty)$, the assertion is easily seen. □

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