Special issue of the International Conference on Applied Mathematics in Engineering (ICAME'18)

An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.9, No.3, pp.1-5 (2019)
http://doi.org/10.11121/ijocta.01.2019.00678

RESEARCH ARTICLE

On the numerical solution for third order fractional partial differential equation by difference scheme method

Mahmut Modanlı

Harran University, Faculty of Art and Science, Department of Mathematics, Turkey
mmodanli@harran.edu.tr

ARTICLE INFO

Article History:
Received 15 August 2018
Accepted 10 December 2018
Available 19 March 2019

Keywords:
Third order fractional partial differential equation
Exact solution
Stability estimate
Difference scheme

AMS Classification 2010:
34A08; 34K37; 74G15

ABSTRACT

The third order fractional partial differential equations is obtained the exact solution depending on initial-boundary value problem. The exact solution and its stability estimates theorem is proved for this equation. Difference schemes are presented for the third order fractional partial differential equation. The stabilities of these difference schemes for this problem are given. The numerical solutions of the third order fractional partial differential equation defined by Caputo fractional derivative for fractional orders $\alpha = 0.1, 0.5, 0.9$ are calculated by these methods. The exact solutions are compared with the numerical results and it is shown that the given method is effective.

1. Introduction

The theory of fractional differential equations becomes one of the most interesting and attractive topics and has shown an increasing development. Differential equations involving fractional order derivatives are used to model a variety of systems has important applied sciences and engineering aspects. In applied sciences, this frame of derivatives are used to model a variety of systems, of which the important applications lie in field of viscoelasticity, electrode-electrolyte polarization, heat conduction, electromagnetic waves, diffusion equation and so on [1, 2].

Finite difference methods in particular became very popular and a large number of schemes has been published very recently. Consequently it becomes important to understand how they compare in terms of accuracy, stability and computing times. In [3–7], fractional differential transform method (FDTM) and modified fractional differential transform method (MFDTM) to solve third-order dispersive partial differential equations were studied by various authors. Third order partial differential equations were investigated in [8], [9], and [10]. In [11], the initial value problem for the third order partial differential equation with time delay with self adjoint positive operator of a Hilbert space was investigated. Finally, some paper implemented several on the numerical solutions of the fractional differential equations in recent years [12–18].

Now, we shall give the following basic definitions for this study.

Definition 1. The Caputo fractional derivative $D_t^{\alpha}u(t,x)$ of order α depended on time is defined as:

$$\frac{\partial^{\alpha}u(t,x)}{\partial t^{\alpha}} = D_t^{\alpha}u(t,x) \tag{1}$$
$$= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{1}{(t-p)^{\alpha-n+1}}\frac{\partial^{\alpha}u(p,x)}{\partial p^{\alpha}}dp,$$
$$(n-1<\alpha<n),$$

and for $\alpha = n \in N$ defined as:

$$D_t^{\alpha}u(t,x)=\frac{\partial^{\alpha}u(t,x)}{\partial t^{\alpha}}=\frac{\partial^{n}u(t,x)}{\partial t^{n}}.$$

Definition 2. First-order approximation method computing the problem (1) given by the formula:

$$D_t^{\alpha}U_n^k\cong g_{\alpha,\tau}\sum_{j=1}^{k}w_j^{(\alpha)}(U_n^{k-j+1}-U_n^{k-j}),\qquad (2)$$

where $g_{\alpha,\tau}=\frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}$ and $w_j^{(\alpha)}=j^{1-\alpha}-(j-1)^{1-\alpha}$. From the above facts, we have the following approximation [19]:

$$\frac{\partial^{\alpha}u(t_k,x_n)}{\partial t^{\alpha}} = g_{\alpha,\tau}\left[w_1U_n^k-w_kU_n^0+\sum_{j=1}^{k-1}\left(w_{k-j+1}-w_{k-j}\right)u_n^j\right].\qquad (3)$$

In this work, we consider the third order fractional partial differential equation depend on initial boundary value problem

$$\begin{cases}\frac{\partial^3u(t,x)}{\partial t^3}+\frac{\partial^{\alpha}u(t,x)}{\partial t^{\alpha}}-\frac{\partial^2u(t,x)}{\partial x^2}+u(t,x)=f(t,x),\\ 0<x<L,\ 0<t<T,\ 0<\alpha<1,\\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x),\\ u_{tt}(0,x)=\sigma(x),\ 0\le t\le T,\\ u(t,X_L)=u(t,X_R)=0,\ X_L<x<X_R.\end{cases}\qquad (4)$$

For the problem (4), basic definitions are given. The exact solution of the problem (4) and its stability inequalities are investigated. The first order of difference schemes of the problem (4) are constructed. The theorem of stability estimates for the solution of difference schemes for initial-boundary value problem for this partial differential equation are obtained. The results of numerical experiments are presented and are compared with exact solutions. These results obtained with Matlab programming showed that the method gives good results for this problem.

2. The exact solution and stability for third order fractional partial diferential eqation

Consider the equation (4) the following abstract form

$$\begin{cases}\frac{d^3u(t)}{dt^3}+Au(t)=F(t),\ (0<t<T),\\ u(0)=\varphi,\ u'(0)=\psi,\ u''(0)=\sigma,\end{cases}\qquad (5)$$

in a Hilbert space $H=L_2[0,L]$. Here $f(t)=f(t,x)$ is abstract function defined on $[0,T]$ with values in $H=L_2[0,L]$. $\varphi=\varphi(x)$ and $\psi=\psi(x)$ are the elements of $H=L_2[0,L]$. $u(t)=u(t,x)$ is unknown abstract function defined on $[0,T]$ with values in $H=L_2[0,L]$.

$A:D(A)\to H$ is the differential operator defined by formula

$$Au(x)=-u''(x)+u(x)$$

with domain

$$D(A)=\{u:u_x,\ u_{xx}\in L_2[0,L];\ u(0)=u(L)=0\}.$$

Here, $F(t)=f(t)-D_t^{\alpha}u(t)$.

Now, we shall get the formula for the solution of the problem (5). Using the method [8] , we write the problem (5) as the following first order linear differential equations:

$$\begin{cases}\frac{du(t)}{dt}-aBu(t)=w(t),\\ \frac{dw(t)}{dt}-\overline{a}Bw(t)=v(t),\\ \frac{dv(t)}{dt}+Bv(t)=F(t),\end{cases}\qquad (6)$$

where $B=A^{1/3}$. Using the initial conditions of the problem (5) and the formula (6), we get new initial conditions for the formula (5) as the following:

$$\begin{cases}w(0)=u'(0)-aBu(0)\\ v(0)=u''(0)-Bu'(0)+B^2(0).\end{cases}\qquad (7)$$

Here, $a=\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $\overline{a}=\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

Integrating the formula (6) and using the initial conditions of the formula (7), we obtain

$$u(t) = R_1u(0)+R_2u'(0)+R_3u''(0)+\int_0^t R_4f(s)ds-\int_0^t R_4D_s^{\alpha}u(s)ds.\qquad (8)$$

Here

$$
\begin{aligned}
R_1 &= \frac{\overline{a}e^{aA^{1/3}t}-ae^{\overline{a}A^{1/3}t}}{\overline{a}-a}+\frac{e^{-A^{1/3}t}-e^{aA^{1/3}t}}{(a+1)(\overline{a}+1)} \\
&+\frac{e^{\overline{a}A^{1/3}t}-e^{aA^{1/3}t}}{(\overline{a}+1)(\overline{a}-a)}, \\
R_2 &= \frac{e^{aA^{1/3}t}-e^{\overline{a}A^{1/3}t}}{(\overline{a}-a)A^{1/3}}-\frac{e^{-A^{1/3}t}-e^{aA^{1/3}t}}{(a+1)(\overline{a}+1)A^{1/3}} \\
&-\frac{e^{\overline{a}A^{1/3}t}-e^{aA^{1/3}t}}{(\overline{a}+1)(\overline{a}-a)A^{1/3}}, \\
R_3 &= \frac{e^{-A^{1/3}t}-e^{aA^{1/3}t}}{(a+1)(\overline{a}+1)A^{2/3}}+\frac{e^{\overline{a}A^{1/3}t}-e^{aA^{1/3}t}}{(\overline{a}+1)(\overline{a}-a)A^{2/3}}, \\
R_4 &= -\frac{e^{-A^{1/3}(t-s)}-e^{aA^{1/3}(t-s)}}{(a+1)(\overline{a}+1)A^{2/3}}.
\end{aligned}
$$

Lemma 1. *The following inequalities are satisfied:*

$$
\begin{cases} \|R_1\|_H \le M(\delta), & \|R_2\|_H \le M(\delta) \\ \|R_3\|_H \le M(\delta), & \|R_4\|_H \le M(\delta). \end{cases} \tag{9}
$$

Lemma 2. *For $t \ge 0$ of the following estimates hold:*

$$
\left\| e^{-A^{1/3}t} \right\| \le e^{-\delta^{1/3}t} \tag{10}
$$

The proof of this Lemma is supported the spectral representation of unit self-adjoint positive definite operator A in a Hilbert space H.

Lemma 3. *Suppose that $\varphi \in D(A)$, $\psi \in D(A^{2/3})$, $\sigma \in D(A^{1/3})$, $D_t^{\alpha}u(t)$ and $f(t)$ are continuously differentiable on $[0,T]$. Then, there are the following stability inequality for the formula (8)*

$$
\begin{aligned}
\|D_t^{\alpha}u(t)\|_H &\le M\left\{ \|\varphi\|_H + \left\|A^{-1/3}\psi\right\|_H \right. \\
&+ \left\|A^{-2/3}\sigma\right\|_H \\
&\left. + \max_{0\le t\le T}\left\|A^{-2/3}f(t)\right\|_H \right\}.
\end{aligned} \tag{11}
$$

Proof. Taking the first derivative of the problem (8) and using the following formula for fractional derivative of order $0<\alpha<1$, we find

$$
D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(p)dp}{(t-p)^{\alpha}}, \text{ where } u(0)=0, \tag{12}
$$

which implies that the proof of this lemma is completed. □

Theorem 1. *Let $\varphi \in D(A)$, $\psi \in D(A^{2/3})$, $\sigma \in D(A^{1/3})$ and $f(t)$ be continuously differentiable on $[0,T]$. Then, there is a unique solution of problem (5) and the following stability inequalities hold:*

$$
\begin{aligned}
&\max_{0\le t\le T}\|u(t)\|_H \\
&\le M\left\{ \|\varphi\|_H + \left\|A^{-1/3}\psi\right\|_H + \left\|A^{-2/3}\sigma\right\|_H \right. \\
&\left. + \max_{0\le t\le T}\left\|A^{-2/3}f(t)\right\|_H \right\},
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\max_{0\le t\le T}\left\|\frac{d^3u(t)}{dt^3}\right\|_H + \max_{0\le t\le T}\|Au(t)\|_H \\
&\le M\left\{ \|A\varphi\|_H + \left\|A^{2/3}\psi\right\|_H + \left\|A^{1/3}\sigma\right\|_H \right. \\
&\left. + \max_{0\le t\le T}\left\|f'(t)\right\|_H + \|f(0)\|_H \right\}
\end{aligned} \tag{14}
$$

are valid, where M is independent on $f(t)$, $t \in [0,T]$, φ, ψ, and σ.

Proof. From (9), (10) and (8), the proof of the formula (13) and (14) are completed. □

3. Constructed difference scheme and its stability

Let us choose $h = \frac{L}{M}$ for $x-$axis and $\tau = \frac{T}{N}$ for $t-$axis as grid mess in the difference scheme method. In this case, we have

$x_n = x_L + nh$; $n = 1,2,...M$, $t_k = k\tau$, $k = 1,2,...,N$. Applying the formula (2) for the fractional partial differential equation (4), we construct the following the first order difference schemes

$$
\begin{cases}
\frac{U_n^{k+2}-3U_n^{k+1}+3U_n^k-U_n^{k-1}}{\tau^3} \\
+g_{\alpha,\tau}\sum\limits_{j=1}^{k} w_j^{(\alpha)}(U_n^{k-j+1}-U_n^{k-j})+U_n^k \\
-\frac{1}{2h^2}[U_{n+1}^{k+1}-2U_n^{k+1}+U_{n-1}^{k+1}+U_{n+1}^k \\
-2U_n^k+U_{n-1}^k] \\
= f_n^k = f(t_k,x_n), \\
U_0 = \varphi, \\
\frac{U_1-U_0}{\tau} = \psi.
\end{cases} \tag{15}
$$

Theorem 2. *Suppose that the assumption $A \ge \delta$ holds and $\varphi \in D(A)$, $\psi \in D(A^{2/3})$ and $\sigma \in D(A^{1/3}$. Then, for the solution of difference scheme (15) the following stability estimates*

$$\max_{1\leq k\leq N}\|\frac{U_n^{k+2}-3U_n^{k+1}+3U_n^k-U_n^{k-1}}{\tau^3}\|_H$$
$$+\max_{1\leq k\leq N}\|Au_k\|_H$$
$$\leq \quad M(\delta)\left\{\|A\varphi\|_H+\left\|A^{2/3}\psi\right\|_H+\left\|A^{1/3}\sigma\right\|_H\right.$$
$$\left.+\max_{0\leq t\leq T}\left\|\frac{f_k-f_{k-1}}{\tau}\right\|_H+\|f_1\|_H\right\},$$

hold, where $M(\delta)$ is independent of choosing τ, φ, ψ, σ and f_k, $1\leq s\leq N-1$.

The proof of Theorem 2 is based on the formulas for the solution of difference schemes (15), on the estimates for the step operators and on the self-adjointness and positivity of operator A.

4. Numerical experiments

Example

Investigate the following third order fractional partial differential equation for initial boundary value problems

$$\begin{cases} \frac{\partial^3u(t,x)}{\partial t^3}+\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}-\frac{\partial^2u(t,x)}{\partial x^2}+u(t,x) \\ =\sin x(4+6\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}+2t^3), \\ 0<x<\pi,\ 0<t<1,\ 0<\alpha\leq 1, \\ u(0,x)=-\sin x, \\ u_t(0,x)=0, u_{tt}(0,x)=0,\ 0\leq t\leq 1, \\ u(t,0)=u(t,\pi)=0,\ 0\leq x\leq \pi. \end{cases} \tag{16}$$

This problem has the exact solution of as $u(t,x)=(t^3-1)\sin x$.

For the numerical solution of problem (16), we applied difference schemes method to (10). By the help of modified Gauss elimination method, we compute the maximum norm of error of the numerical solution as

$$\varepsilon=\max_{\substack{n=0,1,...,M\\ k=0,1,2...,N}}\left|u(t,x)-U(t_k,x_n)\right|,$$

where $U_n^k=U(t_k,x_n)$ is the numerical solution and $u(t,x)$ is the exact solution. The error analysis table gives our the error analysis for difference schemes method.

Table 1. Error analysis table.

$\tau=\frac{1}{N}, h=\frac{pi}{M}$		
The difference scheme (16)		
	α	
$N=M=40$	0.1	0.0722
	0.5	0.0711
	0.9	0.0692
$N=M=80$	0.1	0.0365
	0.5	0.0359
	0.9	0.0349
$N=M=160$	0.1	0.0183
	0.5	0.0180
	0.9	0.0188
$N=M=240$	0.1	0.0122
	0.5	0.0120
	0.9	0.0213
$N=625, M=25$	0.1	0.0047
	0.5	0.0112
	0.9	0.0247

5. Conclusion

The exact solution of the third order fractional partial differential equation is examined. The abstract theorem on the stability estimate for the solution of the initial boundary value problems for the third order fractional equations is established. The first order of accuracy difference schemes for the numerical solution of the initial-boundary value problems for the third order fractional equations are presented. Stability estimates for the solution of difference schemes for the initial-boundary value problems for the third order fractional equations are obtained. The Matlab implementation of the first order of accuracy difference schemes for the approximate solution of the initial boundary value problem for the third order fractional equations are presented. Taking into consideration the results of numerical examples, applications of the theorems are shown.

References

[1] Hilfer, R. (2000). *Applications of fractional calculus in physics*. Singapore: Word Scientific Company.

[2] Caputo, M. (1967). Linear models of dissipation whose Q is almost frequency independent Part II. *Geophysical Journal of the Royal Astronomical Society banner*, 13(5), 529-539.

[3] Djidjeli, K., & Twizell, EH. (1991). Global extrapolations of numerical methods for solving a third-order dispersive partial differential equations. *International Journal of Computer Mathematics*, 41(1-2), 81-89.

[4] Wazwaz A.M. (2003). An analytic study on the third-order dispersive partial differential equation. *Applied Mathematics and Computation*, 142(23), 511-520.

[5] Twizell, E.H. (1984). *Computational methods for partial differential equations*. Ellis Horwood, New York: Chichester and John Wiley and Sons.

[6] Lewson, J.D., & Morris, J.LI. (1978). The extrapolation of first-order methods for parabolic partial differential equations I. *SIAM Journal on Numerical Analysis*, 15(6), 1212-1224.

[7] Mengzhao, Q. (1983). Difference scheme for the dispersive equation. *Computing*, 31(3), 261-267.

[8] Ashyralyev, A., & Simsek, S.N. (2017). An operator method for a third order partial differential equation. *Numerical Functional Analysis and Optimization*, 38(10), 1341-1359.

[9] Apakov, Y.P. (2012). On the solution of a boundary-value problem for a third-order equation with multiple characteristics. *Ukrainian Mathematical Journal*, 64(1), 1-12.

[10] Belakroum, K., Ashyralyev, A., Guezane-Lakoud, A., & Lukashov, A. (2016). A note on the nonlocal boundary value problem for a third order partial differential equation, *AIP Conference Proceedings*, 1759:1, 020021, Online publication date: 10-Aug-2016.

[11] Ashyralyev, A., Hincal, E., & Ibrahim, S. (2018). Stability of the third order partial differential equations with time delay, *AIP Conference Proceedings*, 1997(1), 02086.

[12] Baskonus, H.M., & Bulut, H. (2015). On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth-Moulton method. *Open Mathematics*, 13(1).

[13] Baskonus, H.M., Mekkaoui, T., Hammouch, Z., & Bulut, H. (2015). Active control of a chaotic fractional order economic system. *Entropy*, 17(8), 5771-5783.

[14] Baskonus, H.M., & Bulut, H. (2016). Regarding on the prototype solutions for the nonlinear fractional-order biological population model. *AIP Conference Proceedings*, 1738(1), p.290004.

[15] Baskonus, H.M., Hammouch, Z., Mekkaoui, T., & Bulut, H. (2016). Chaos in the fractional order logistic delay system: circuit realization and synchronization. *AIP Conference Proceedings*, 1738(1), p.290005.

[16] Gencoglu, M.T., Baskonus, H.M., & Bulut, H. (2017). Numerical simulations to the nonlinear model of interpersonal Relationships with time fractional derivative. *AIP Conference Proceedings*, 1798(1), p.020103.

[17] Ozdemir, N., Avci, D., & Iskender, B.B. (2011). The numerical solutions of a two-dimensional space-time Riesz-Caputo fractional diffusion equation. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 1(1), 17-26.

[18] Evirgen, F. (2016). Analyze the optimal solutions of optimization problems by means of fractional gradient based system using VIM. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 6(2), 75-83.

[19] Karatay, I., Bayramoglu, S.R., & Sahin. A. (2013). A new difference scheme for time fractional heat equation based on the Crank-Nicholson method. *Fractional Calculus and Applied Analysis*, 16, 892-910.

***Mahmut Modanlı** is an Assistant Professor in Harran University of Department of Mathematics. He has been a faculty member since 2016. His research interests lie in the area of applied mathematics, Matlab programming, numerical anaysis, fractional partial differential equations. He has two sons and one daughter.*