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RESEARCH ARTICLE

A new auxiliary function approach for inequality constrained global optimization problems

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ABSTRACT

In this study, we deal with the nonlinear constrained global optimization problems. First, we introduce a new smooth exact penalty function for solving constrained optimization problems. We combine the exact penalty function with the auxiliary function in regard to constrained global optimization. We present a new auxiliary function approach and the adapted algorithm in order to solve non-linear inequality constrained global optimization problems. Finally, we illustrate the efficiency of the algorithm on some numerical examples.

1. Introduction

We consider the following continuous constrained optimization problem

$$(P) \qquad \min_{x\in\mathbb{R}^n} f(x) \quad s.t. \quad g_j(x) \leq 0, \quad j = 1, 2, ..., m,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j(x) : \mathbb{R}^n \to \mathbb{R}$, $j \in J = \{1, 2, ..., m\}$ are continuously differentiable functions. The problem (P) is considered in many problems of engineering and natural sciences [1–4] and it is studied in many papers [6,7].

There exists a very rich theory for the solution of the problem (P) [5]. One of the traditional but effective method to solve the problem (P) is the penalty function method [8]. The penalty function method has been proposed in order to transform a constrained optimization problem to an unconstrained optimization problem. The method offers constructing a barrier on the boundary of the set of feasible solutions which is defined as $D_0 := \{x \in \mathbb{R}^n : g_j(x) \leq 0,\ j = 1, 2, \ldots, m\}$ and it is assumed that D_0 is not empty. In order to construct a barrier the "$b(t) = -\log(-t)$", "$b(t) = \max(t, 0)$" functions are used. The penalized objective function is defined as

$$F(x, \rho) = f(x) + \rho \sum_{j=1}^{m} b(g_j(x)), \qquad (1)$$

and problem (P) re-stated as

$$(P_\rho) \qquad \min_{x\in\mathbb{R}^n} F(x, \rho),$$

where $\rho > 0$ is a penalty parameter. If $b(t) = \max(t, 0)$ is in the formula (1), the penalty function is called as exact penalty function according to Zangwill [9]. It can be observed that the exact penalty function may be non-smooth. When the penalty function approach is non-smooth, one of the conventional approaches is constructing a smoothing approach. The smoothing approach is based on modifying the objective function or approximating the objective function by smooth functions [10]. In order to improve the smoothing approaches, different types of valuable techniques and algorithms are developed [11–14]. In recent years, the smoothing approaches have been used for many non-smooth problems such as min-max [15,16], exact penalty [17–20] and etc. [21].

If the problem (P) or (P_ρ) has just one minimizer, then many local optimization methods can be used to solve with penalty method, but if it has multiple local minimizers, most of the well-known methods are not available to solve [22]. The studies on global optimization have become extensively increase among the other research areas of optimization [23, 24]. There are many valuable studies on global optimization depending on deterministic, stochastic and heuristic approaches [25, 26]. Most of the global optimization techniques are proposed to solve unconstrained problems, but by combining the penalty function method with a global optimization algorithm the global solution of the problem (P) can be obtained. One of the important global optimization approaches is the auxiliary function approach which includes the Tunneling Method (Algorithm) [27], Filled Function Method [28,29], Global Descent Method [30] and Cut-Peak Function Method [31]. These methods are established on finding the lower minimizer than the current one by making a suitable modification on the objective function. The modified function is generally called as auxiliary function (Filled Function, Tunneling Function and etc.) [33].

In the next section, we give some preliminary definitions. In section 3, we introduce a new penalty function in order to transform the problem (P) into an unconstrained problem. In Section 4, we present a minimization algorithm and convergence results. In Section 5, we apply the algorithms on the important test problems. In the last section, we give some concluding remarks.

2. Preliminaries

We assume that the set D_0 is closed and bounded and the function f has a finite number of local minimizers in D_0. Throughout the paper, we use x_k^* to denote the $k-$th local minimizer of f whereas by x^* we mean the global minimizer. $\|x\| = \sqrt{\sum_{k=1}^n x_k^2}$ denotes the Euclidean norm in $\mathbb{R}^n$.

Definition 1. *[13] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. The function $\tilde{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $f(x)$, if $\tilde{f}(\cdot, \beta)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed β, and for any $x \in \mathbb{R}^n$,*

$$\lim_{z\to x,\beta\to 0} \tilde{f}(z, \beta) = f(x).$$

Definition 2. *[19] Let $\varepsilon > 0$, a point x_ε is called $\varepsilon-$feasible solution for the problem (P), if*

$$g_j(x) \le \varepsilon, \quad j = 1, 2, \dots, m.$$

3. A New Penalty Function

In this section, we present a new penalty approach for the problem (P). Let us define the sets $D_j = \{x \in \mathbb{R}^n : g_j(x) \le 0\}$ for $j = 1, 2, \dots, m$. It can be observed that $\cap_{j=1}^m D_j = D_0$. The main idea in exact penalty function approach is to construct a barrier at the boundary of D_0 such that any local (global) solver can not find a point outside the set D_0. Based on this idea, we define a new penalty function as

$$F(x, \rho) = f(x) + \rho \left(\sum_{j=1}^m \chi_{D_j^c}(x) \right) \|x - x_0\|^2,$$

where $\rho > 0$, $x_0 \in D_0$ and

$$\chi_{D_j^c}(x) = \begin{cases} 0, & x \in D_j, \\ 1, & x \notin D_j, \end{cases}$$

for $j = 1, 2, \dots, m$. Since the function $\chi_{D_j^c}(x)$ is non-smooth, we apply the smoothing approach to this function in order to make it smooth. We design the following function

$$\tilde{\chi}_{D_j^c}(x, \varepsilon) = \begin{cases} 0, & t \le 0, \\ R_1(t), & 0 \le t \le \varepsilon, \\ 1, & t \ge \varepsilon, \end{cases} \tag{2}$$

where $\varepsilon > 0$ and

$$R_1(t) = \frac{-2}{\varepsilon^3} t^3 + \frac{3}{\varepsilon^2} t^2,$$

for $t = g_j(x)$, $j = 1, 2, \dots, m$. By using R_1 in formula (2), the obtained smoothing function is continuously differentiable. If the following function

$$R_2(t) = \frac{6}{\varepsilon^5} t^5 - \frac{15}{\varepsilon^4} t^4 + \frac{10}{\varepsilon^3} t^3,$$

is used in formula (2) instead of R_1, the obtained smoothing function is second order continuously differentiable. The function R_i, $(i = 1, 2, \dots, k)$ is called the smooth transition function. Now, we obtain surrogate problem $(\tilde{P}_\rho)$ as follows:

$$(\tilde{P}_\rho) \qquad \min_{x\in\mathbb{R}^n} F(x, \rho, \varepsilon), \tag{3}$$

where

$$F(x, \rho, \varepsilon) = f(x) + \rho \left(\sum_{j=1}^m \tilde{\chi}_{D_j^c}(x, \varepsilon) \right) \|x - x_0\|^2.$$

Theorem 1. *Let x^* be a solution for $(\tilde{P}_\rho)$ for sufficiently large $\rho > 0$ then $x^* \in D_0$.*

Proof. Suppose that $x^* \notin D_0$. Then, there exists j such that $t = g_j(x^*) > 0$. We have two cases:
Case 1. Let $t \ge \varepsilon$ then, we have

$$F(x^*, \rho, \varepsilon) = f(x^*) + \rho\|x^* - x_0\|^2,$$

and

$$\nabla F(x^*, \rho, \varepsilon) = \nabla f(x^*) + 2\rho(x^* - x_0) = 0.$$

Therefore, we obtain

$$\rho = -\nabla f(x^*)(2(x^* - x_0))^{-1}.$$

Since $f(x)$ continuous differentiable $\|f(x)\| < \infty$ and $x^* \neq x_0$, it can be concluded that ρ is finite. If anyone chooses $\rho_1 > \rho$, the $\nabla F(x^*, \rho_1, \varepsilon) \neq 0$.

Case 2. Let $0 < t \leq \varepsilon$ then, we have

$$F(x^*, \rho, \varepsilon) = f(x^*) + \rho \left(\frac{-2}{\varepsilon^3} t^3 + \frac{3}{\varepsilon^2} t^2 \right) \times \|x^* - x_0\|^2,$$

and

$$\nabla F(x^*, \rho, \varepsilon) = \nabla f(x^*) + \rho A(x^*, \varepsilon),$$

where

$$A(x^*, \varepsilon) = \left(\left(\frac{-2}{\varepsilon^3} t^3 + \frac{3}{\varepsilon^2} t^2 \right) \|x^* - x_0\|^2 + \left(\frac{-2}{\varepsilon^3} t^3 + \frac{3}{\varepsilon^2} t^2 \right) (2(x^* - x_0)) \right).$$

Thus, we obtain

$$\rho = -\nabla f(x^*) A(x^*, \varepsilon)^{-1}.$$

It can be seen that ρ is finite. If anyone chooses $\rho_2 > \rho$, the $\nabla F(x^*, \rho_2, \varepsilon) \neq 0$.

As a consequence, if anyone chose the parameter ρ in (1) as $\rho > \max\{\rho_1, \rho_2\}$, the point x^* cannot be outside of D_0. □

Corollary 1. *Let x^* be a solution for $(\tilde{P}_\rho)$ for sufficiently large ρ then x^* is a solution for (P).*

Proof. From Theorem 1, we have $x^* \in D_0$. Then, we obtain

$$\begin{aligned} f(x^*) &= F(x^*, \rho, \varepsilon) \\ &= F(x^*, \rho, 0) \\ &\leq F(x, \rho, 0) \\ &= f(x). \end{aligned}$$

This completes the proof. □

4. Algorithms for Minimization Procedure

In this section, we propose our new algorithm to find the global optimal point by considering the problem $(\tilde{P}_\rho)$.

Algorithm

Step 1 Determine x^0, $\rho_0 = 10$, $\varepsilon_0 > 0$, $N > 1$, $0 < \eta < 1$ and let $j = 1$ and go to Step 2.

Step 2 Use x^{j-1} as an initial point and apply one of the global optimization algorithms to solve the problem $(\tilde{P}_\rho)$. Let x_j is the solution.

Step 3 If $x_j \in \text{int}D_0$ then stop the algorithm and x^j is the optimal solution else go to Step 4.

Step 4 If x^j is $\varepsilon-$feasible for (P), then stop and x^j is the optimal solution. Otherwise, take $\rho_j = N\rho_{j-1}$, $\varepsilon_j = \eta\varepsilon_{j-1}$ and $j = j + 1$, then go to Step 2.

In Step 2 of algorithm x_j is the global optimal solution of the problem $(\tilde{P}_\rho)$ depending on the parameter ε. In order to obtain the global solution, any of the global optimization methods can be used. We use the auxiliary function based global optimization method studied in [21,33]. The Auxiliary Function Method (AFM) is very effective in terms of numerical results which is illustrated in [21]. Our auxiliary function is defined as follows:

$$\tilde{\phi}(x, x_k^*, \beta, \alpha) = f_k^* + (f(x) - f_k^*)\, \tilde{\chi}_{A_k}(t, \beta) + \alpha H(\|x - x_k^*\|^2),$$

where α and β are real parameters. The function $\tilde{\chi}_{A_k}(t, \beta)$ is defined by

$$\tilde{\chi}_{A_k}(t, \beta) = \begin{cases} 0, & t > \beta, \\ q(t, \beta), & -\beta \leq t \leq \beta, \\ 1, & t < -\beta, \end{cases}$$

where

$$q(t, \beta) = \frac{1}{4\beta^3} t^3 - \frac{3}{4\beta} t + \frac{1}{2},$$

and the function H is defined on $\mathbb{R}_+$ and it satisfies the following properties:

i. $H(u) > 0$,
ii. $H'(u) < 0$,
iii. $\lim_{u \to \infty} H(u) = 0$.

At Step 3 and 4, the feasibility of the solution is checked and the stopping conditions are declared. In order to guarantee that the algorithm is worked straightly, we prove the following theorems.

Theorem 2. *Assume that the sequence $\{x^j\}$ is produced by the Algorithm has a limit point x^*, then $x^* \in D_0$.*

Proof. Assume x^* is a limit point of $\{x^j\}$. Then there exists set $J \subset \mathbb{N}$, such that $x^j \to \overline{x}$ for $j \in J$. Let us consider the contrary that $x^* \notin D_0$, i.e. for sufficiently large $j \in J$, there exist $\delta_0 > 0$ and $i_0 \in \{1, 2, \ldots, m\}$ such that:

Case 1. $g_{i_0}(x^j) \geq \delta_0 \geq \varepsilon > 0$. Since x^j is the global minimum according $j-$th values of the parameters ρ_j, ε_j, for any $x \in D_0$ we have

$$F(x^j, \rho_j, \varepsilon_j) = f(x^j) + \rho_j \|x^j - x_0\|.$$

If $j \to \infty$ then, $\rho_j \to \infty$ and $\rho_j \|x^j - x_0\| \to \infty$ (since $x^j \notin D_0$ and $\|x^j - x_0\| > 0$). Thus, $f(x)$ takes infinite values on D_0 and it contradicts with the boundedness of f on D_0.

Case 2. $t = g_{i_0}(x^j) \geq \varepsilon \geq \delta_0 > 0$. Since x^j is the global minimum according to $j-$th values of the parameters ρ_j, ε_j, for any $x \in D_0$ we have

$$\begin{aligned} F(x^j, \rho_j, \varepsilon_j) =& f(x^j) + \rho_j \left(\frac{-2}{\varepsilon^3} t^3 + \frac{3}{\varepsilon^2} t^2 \right) \|x^j - x_0\| \\ \geq & f(x^j) + \rho_j \|x^j - x_0\|. \end{aligned}$$

If $j \to \infty$ then, $\rho_j \to \infty$ and $\rho_j \|x^j - x_0\| \to \infty$ (since $x^j \notin D_0$ and $\|x^j - x_0\| > 0$). Thus, $f(x)$ takes infinite values on D_0 and it contradicts with the boundedness of f on D_0. From the Cases 1 and 2, we obtain the result. □

Theorem 3. *Assume that for $\varepsilon \in (0, \varepsilon_0]$ the set*

$$\underset{x \in \mathbb{R}^n}{argmin}\ F(x, \rho, \varepsilon) \neq \emptyset.$$

Let x^j is generated by Algorithm when $\eta N < 1$. If $\{x^j\}$ has a limit point, then the limit point of x^j is the solution for (P).

Proof. Let x^* be a limit point of $\{x^j\}$. From Theorem 2, we have $x^* \in D_0$. Then, we obtain

$$\begin{aligned} f(x^*) &= F(x^*, \rho, \varepsilon) \\ &= F(x^*, \rho, 0) \\ &\leq F(x, \rho, 0) \\ &= f(x). \end{aligned}$$

This completes the proof. □

5. Numerical Examples

In this section, we apply our algorithm to test problems. The proposed algorithm is programmed in Matlab. Numerical results show the efficiency of this method. The detailed results are presented in the tables for all problems. For these tables, we use some symbols in order to abbreviate the expressions. The meanings of these symbols are as follows:

j :The number of iterations,

x^j :the local minimum point of the jth iteration,

ε_j :smoothing parameter of the jth iteration,

$g(x^j)$:the value of the point x^j under the constraint functions,

$F(x^j, \rho_j, \varepsilon_j)$:the value of the point x^j under F,

$f(x^j)$:the value of the point x^j under f.

Problem 1. *Let us consider the Example in [34]*

$$\begin{aligned} \min f(x) \quad &= \quad x_1^2 + x_2^2 - \cos(17x_1) - \cos(17x_2) + 3, \\ s.t. \quad & \quad g_1(x) = (x_1 - 2)^2 + x_2^2 \leq 1.6^2, \\ & \quad g_2(x) = x_1^2 + (x_2 - 3)^2 \leq 2.7^2, \\ & \quad 0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 2. \end{aligned}$$

We choose $x^0 = (1, 1)$ as a starting point $\rho_0 = 10$, $\varepsilon_0 = 0.01$, $\eta_0 = 0.1$ and $N = 3$. The results are shown in the Table 1. Considering $(\tilde{P}_\rho)$ the global minimum is obtained at a point $x^* = (0.7254, 0.3993)$ with the corresponding value 1.8376. In the paper [34], the obtained global minimum point is $x^* = (0.72540669, 0.3992805)$ with the corresponding value 1.837623. Our algorithm finds the correct point as in [34].

Problem 2. *Let us consider the Example in [35]*

$$\begin{aligned} \min f(x) &= -x_1 - x_2, \\ s.t.\ & x_2 - 2x_1^4 + 8x_1^3 - 6x_1^2 \leq 2, \\ & x_2 - 4x_1^4 + 32x_1^3 - 88x_1^2 + 96x_1 \leq 36, \\ & 0 \leq x_1 \leq 3, \quad 0 \leq x_2 \leq 4. \end{aligned}$$

We choose $x^0 = (0, 0)$ as a starting point $\rho_0 = 10$, $\varepsilon_0 = 0.01$, $\eta_0 = 0.1$ and $N = 3$. The results are shown in the Table 2. The global minimum is obtained at a point $x^* = (2.3295, 3.1783)$ with the corresponding value -5.5079. In the papers [35, 36], the obtained global minimum point is $x^* = (2.3289, 3.1883)$ with the corresponding value -5.5091. Our algorithm find the correct point as in [35, 36].

Problem 3. *Let us consider the example in [34],*

$$\begin{aligned} \min f(x) =& 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3, \\ s.t.\ & g_1(x) = \sum_{i=1}^{3} x_i^2 = 25, \\ & g_2(x) = (x_1 - 5)^2 + \sum_{i=2}^{3} x_i^2 = 25, \\ & g_3(x) = \sum_{i=1}^{3} (x_i - 5)^2 \leq 25. \end{aligned}$$

We choose $x^0 = (2, 2, 2)$ as a starting point $\rho_0 = 10$, $\varepsilon_0 = 0.01$, $\eta_0 = 0.1$ and $N = 3$. The results are shown in the Table 3. The global minimum is obtained at a points $x^* = (2.5000, 4.2196, 0.9721)$ with the corresponding value 944.2157. In the papers [34], the obtained global minimum point is $x^* = (2.500000, 4.221305, 0.964666)$ with the corresponding value 944.2157. Our algorithm finds the correct solution with the lower iteration numbers in comparison with the algorithm in [34].

Problem 4. *Consider the example in [36],*

$$\begin{aligned} \min f(x) \quad &= \quad -x_1^2 + x_2^2 + x_3^2 - x_1, \\ s.t. \quad & \quad g_1(x) = x_1^2 + x_2^2 + x_3^2 \leq 4, \\ & \quad g_2(x) = \min\{x_2 - x_3, x_3\} \leq 0. \end{aligned}$$

We choose $x^0 = (-1.6, -1, 0.2)$ as a starting point $\rho_0 = 10$, $\varepsilon_0 = 0.01$, $\eta_0 = 0.1$ and

Table 1. Table of minimization process of the Problem 1.

j	x^j	ρ_j	ε_j	$g_1(x^j)$	$g_2(x^j)$	$F(x^j, \rho_j, \varepsilon_j)$	$f(x^j)$
1	$(0.7249, 0.4007)$	10	0.01	-0.7737	-0.0083	1.8774	1.8522
2	$(0.7252, 0.3996)$	30	0.001	-0.7753	-0.0018	1.8446	1.8408
3	$(0.7253, 0.3993)$	90	0.0001	-0.7758	-0.0003	1.8388	1.8382
4	$(0.7253, 0.3992)$	270	$1e-05$	-0.7758	$-6.2645e-05$	1.8378	1.8377
5	$(0.7253, 0.3992)$	810	$1e-06$	-0.7758	$-1.0667e-05$	1.8376	1.8376

Table 2. Table of minimization process of the Problem 2.

j	x^j	ρ_j	ε_j	$g_1(x^j)$	$g_2(x^j)$	$F(x^j, \rho_j, \varepsilon_j)$	$f(x^j)$
1	$(2.3307, 3.1477)$	10	0.01	3.9592	508.57	-5.4454	-5.4784
2	$(2.3297, 3.173)$	30	0.001	3.9928	507.95	-5.4973	-5.5027
3	$(2.3296, 3.1775)$	90	0.0001	3.9988	507.85	-5.5062	-5.5071
4	$(2.3295, 3.1783)$	270	$1e-05$	3.9998	507.83	-5.5079	-5.5079

Table 3. Table of minimization process of the Problem 3.

j	x^j	ρ_j	ε_j	$g_1(x^j)$	$g_2(x^j)$	$g_3(x^j)$	$F_p(x^j, \rho_j, \varepsilon_j)$	$f(x^j)$
1	$(2.5002, 4.2214, 0.9650)$	10	0.01	0.0022	0.0001	-1.864	944.4134	944.2108
2	$(2.5000, 4.2212, 0.9650)$	30	0.001	$7.17e-05$	$3.26e-06$	-1.8625	944.2756	944.2155
3	$(2.5000, 4.2212, 0.9650)$	90	0.0001	$1.73e-05$	$-2.62e-05$	-1.8625	944.2341	944.2156
4	$(2.5000, 4.2212, 0.9650)$	270	$1e-05$	$3.92e-06$	$-4.27e-06$	-1.8625	944.2157	944.2157

Table 4. Table of minimization process of the Problem 4.

j	x^j	ρ_j	ε_j	$g_1(x^j)$	$g_2(x^j)$	$F(x^j, \rho_j, \varepsilon_j)$	$f(x^j)$
1	$(1.995, -0.0300, 0.0300)$	10	0.01	-0.0180	-0.0601	-5.9393	-5.9733
2	$(1.9991, -0.0094, 0.0094)$	30	0.001	-0.0033	-0.0188	-5.9902	-5.9954
3	$(1.9998, -0.0029, 0.0029)$	90	0.0001	-0.0005	-0.0058	-5.9984	-5.9992
4	$(2.0000, -0.0009, 0.0009)$	270	$1e-05$	-0.0001	-0.0018	-5.9997	-5.9999
5	$(2.0000, -0.0009, 0.0009)$	810	$1e-06$	$-1.6e-05$	-0.0018	-6.0000	-6.0000

$N = 3$. The results are shown in the Table 4. The global minimum is obtained at a point $x^* = (2, -0.0009, 0.0009)$ with the corresponding value -6.0000. In the papers [35, 36], the obtained global minimum point is $x^* = (1.9889, -0.0001, -0.0111)$ with the corresponding value -5.9446. Our algorithm finds the correct point as in [35, 36].

Problem 5. *The Rosen-Suzuki problem in [34]*

$$\min f(x) = \sum_{i=1}^{4} x_i^2 - 5x_1 - 21x_3 + 7x_4,$$

$$s.t.\ g_1(x) = 2x_1^2 + \sum_{i=2}^{3} x_i^2 + 2x_1 + x_2 + x_4 \le 5,$$

$$g_2(x) = \sum_{i=1}^{4} x_i^2 + x_1 - x_2 + x_3 - x_4 \le 8,$$

$$g_3(x) = \sum_{i=1}^{2} \left(x_{2i-1}^2 + 2x_{2i}^2\right) - x_1 - x_4 \le 10.$$

First, we choose $x^0 = (0, 0, 0, 0)$, $\rho_0 = 10$, $\varepsilon_0 = 0.01$, $\eta_0 = 0.1$ and $N = 3$. The results are shown in the Tables 5. The global minimum is obtained at a point $x^* = (0.1697, 0.8358, 2.0084, -0.9651)$ with the corresponding value -44.2338. In the paper [19], the obtained global minimum point is $x^* = (0.1684621, 0.8539065, 2.000167, -0.9755604)$ with the corresponding value -44.23040. In [34], the obtained global minimum point is $x^* = (0.170189, 0.835628, 2.008242, -0.95245)$ with the corresponding value -44.2338. It can be seen that our algorithm present numerically better result than the algorithm in [34].

Table 5. Table of minimization process of the Problem 5.

j	x^j	ρ_j	ε_j	$g_1(x^j)$	$g_2(x^j)$	$g_3(x^j)$	$F(x^j,\rho_j,\varepsilon_j)$	$f(x^j)$
1	$(0.1682, 0.8338, 2.0070, -0.9661)$	10	0.01	-0.0161	-0.0075	-1.8886	-44.1675	-44.2068
2	$(0.1694, 0.8351, 2.0083, -0.9651)$	30	0.001	-0.0030	-0.0017	-1.885	-44.2217	-44.2281
3	$(0.1697, 0.8354, 2.0085, -0.9649)$	90	0.0001	-0.0005	-0.0003	-1.8839	-44.2317	-44.2328
4	$(0.1697, 0.8354, 2.0086, -0.9649)$	270	$1e-05$	$-9.36e-05$	$-5.85e-05$	-1.8837	-44.2335	-44.2337
5	$(0.1697, 0.8354, 2.0086, -0.9649)$	810	$1e-06$	$-5.60e-05$	$-1.20e-05$	-1.8837	-44.2338	-44.2338

Problem 6. *Consider the example in [35],*

$$\min f(x) = \frac{\pi}{n}\left[10\sin^2 \pi x_1 + h(x) + (x_n-1)^2\right],$$
$$s.t. -10 \le x_i \le 10 \quad i = 1,2,\ldots,n,$$

where $h(x) = \sum_{i=1}^{n-1}\left[(x_i-1)^2(1+10\sin^2 \pi x_{i+1})\right]$. *For* $n = 3,5,7$ *we choose* $x^0 = (6,6,\ldots,6)$ *as a starting point* $\rho_0 = 10$, $\varepsilon_0 = 0.01$, $\eta_0 = 0.1$ *and* $N = 3$. *The results are shown in the Table* 6. *The global minimum is obtained at a point* $x^* = (1,1,\ldots,1)$ *with the corresponding value* 0. *In the paper [35], the obtained global minimum point is* $x^* = (1,1,\ldots,1)$ *with the corresponding value* 0. *Our algorithm finds the correct point as in [35].*

Problem 7. *Let us consider the Example in [34]*

$$\min f(x) = 10x_2 + 2x_3 + x_4 + 3x_3 + 4x_6,$$
$$s.t.\ g_1(x) = x_1 + x_2 = 10,$$
$$g_2(x) = -x_1 + x_3 + x_4 + x_5 = 0,$$
$$g_3(x) = -x_2 - x_3 + x_5 + x_6 = 0,$$
$$g_4(x) = 10x_1 - 2x_3 + 3x_4 - 2x_5 \le 16,$$
$$g_5(x) = x_1 + 4x_3 + x_5 \le 10,$$
$$0 \le x_1 \le 12,\ 0 \le x_2 \le 18,$$
$$0 \le x_3 \le 5,\ 0 \le x_4 \le 12,$$
$$0 \le x_5 \le 1, 0 \le x_6 \le 16.$$

We choose $x^0 = (0,0,...,0)$ *as a starting point* $\rho_0 = 10$, $\varepsilon_0 = 0.01$, $\eta_0 = 0.1$ *and* $N = 4$ *for the Algorithm. The results are shown in the Table* 7. *In [34], in which three algorithms are offered for a new smoothing technique, approximate solution is found with* 4, 3 *and* 13 *iterations in the Algorithms I, II and III, respectively. Note that the solution is not found in Algorithm II of [34]. Whereas, an approximate solution is found with* 4 *iterations in our Algorithm.*

6. Conclusion

In this study, we propose a new exact penalty function and a new algorithm for continuous constrained optimization. By considering this new penalty function approach, we construct a new minimization algorithm. We apply the algorithm on test problems and obtain satisfactory results. We also propose a new smoothing approach for non-smooth penalty functions and it provides good approximations to the non-smooth penalty functions. Moreover, it is easy applicable and has easy formulation.

The results convince that the Algorithm can be used for large scale optimization problems. By applying the minimization algorithm, the optimum value is found rapidly and the algorithm presents high accuracy in finding the optimum point. We use the auxiliary function method in the algorithm as a global optimizaiton method but anyone can use any other algorithms such as DIRECT [38], Kriging-based techniques [39] or heuristic algorithms [40, 41].

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Table 6. Table of minimization process of the Problem 6.

n	j	x^j	ρ_j	ε_j	$F(x^j, \rho_j, \varepsilon_j)$	$f(x^j)$
3	1	$(1.0000, 1.0000, 1.0000)$	10	0.01	$5.8448e-13$	$5.8448e-13$
	2	$(1.0000, 1.0000, 1.0000)$	30	0.001	$5.7929e-13$	$5.7929e-13$
5	1	$(1.0000, 1.0000, \ldots, 1.0000)$	10	0.01	$1.6646e-12$	$1.6646e-12$
	2	$(1.0000, 1.0000, \ldots, 1.0000)$	30	0.001	$1.4643e-12$	$1.4643e-12$
7	1	$(1.0000, 1.0000, \ldots, 1.0000)$	10	0.01	$2.5965e-14$	$2.5965e-14$
	2	$(1.0000, 1.0000, \ldots, 1.0000)$	30	0.001	$1.1379e-15$	$1.1379e-15$

Table 7. Table of minimization process of the Problem 7.

j	x^j	ρ_j	ε_j	$g_1(x^j)$	$g_2(x^j)$	$g_3(x^j)$	$g_4(x^j)$	$g_5(x^j)$	$F(x^j, \rho_j, \varepsilon_j)$	$f(x^j)$
1	$(1.9216, 8.0745, 0.7949, 0.1248, 1.0020, 7.8680)$	40	0.001	-0.0039	$-5.33e-07$	0.0006	-0.0032	-3.8969	117.2736	116.9372
2	$(1.9208, 8.0792, 0.7945, 0.1265, 0.9998, 7.8739)$	160	0.0001	$-8.99e-06$	$-2.56e-06$	$-2.57e-06$	-0.0007	-3.9015	117.0525	117.0021
3	$(1.9208, 8.0792, 0.7945, 0.1265, 0.9998, 7.8739)$	640	$1e-05$	$-3.71e-07$	$-2.49e-07$	$-1.76e-07$	-0.0006	-3.9015	117.0215	117.0022
4	$(1.9208, 8.0792, 0.7945, 0.1265, 0.9998, 7.8739)$	2560	$1e-06$	$-3.05e-07$	$2.79e-08$	$5.72e-09$	-0.0006	-3.9015	117.0100	117.0022

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