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RESEARCH ARTICLE

Approximate controllability of nonlocal non-autonomous Sobolev type evolution equations

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ABSTRACT

The aim of this article is to investigate the existence of mild solutions as well as approximate controllability of non-autonomous Sobolev type differential equations with the nonlocal condition. To prove our results, we will take the help of Krasnoselskii fixed point technique, evolution system and controllability of the corresponding linear system.

1. Introduction

In this article, we discuss the approximate controllability of nonlocal Sobolev type non-autonomous evolution equations in a separable Hilbert space X:

$$\begin{aligned} \frac{d}{dt}[\mathbb{E}x(t)] + \mathbf{A}(t)x(t) &= \mathcal{F}(t, x(t)) + \mathbb{B}u(t), \\ & \quad t \in (0, b], \\ x(0) + \mathcal{G}(x) &= x_0, \quad x_0 \in D(\mathbb{E}), \quad (1) \end{aligned}$$

where $\mathbf{A}(t)$, $\mathbb{E}$ are X-valued linear operators with domains are subsets of X, and $\mathcal{F}$ is X-valued function defined over $J \times X$, $\mathcal{G}$ is $D(\mathbb{E})$-valued function defined over $\mathcal{C}(J, X)$, $J = [0, b]$. The control function $u \in \mathcal{L}^2(J, \mathbb{U})$, $\mathbb{U}$ is a Hilbert space and $\mathbb{B}$ is X-valued linear and bounded operator defined over $\mathbb{U}$.

The Sobolev type differential equations appears in several fields such as thermodynamics [1], fluid flow via fissured rocks [2], and mechanics of soil [3]. Brill [4] first established the existence of solution for a semilinear Sobolev differential equation in a Banach space. Lightbourne et al. [5] studied a partial differential equation of Sobolev type.

Generalization of classical initial condition which is known as nonlocal condition is more effective to obtain better results. Nonlocal Cauchy problem was first considered by Byszewski [6].

Controllability is an important issue in engineering and mathematical control theory. The problem of exact controllability is to show that there exists a control function, that steers the solution of the system from its initial state to the given final state. However in approximate controllability, it is possible to steer the solution of the system from its initial state to arbitrary small neighbourhood of the the final state. Mostly the problem of controllability for various kinds of differential, integro-differential equations and impulsive differential equations are studied for autonomous systems. For more details, we refer to [7] - [13].

The existence of mild solutions for a non-autonomous nonlocal integro-differential equation

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is investigated by Yan [14] via Banach contraction principle, Schauder's fixed point theorem and the theory of evolution families. Haloi et al. [15] generalized the above results for non-autonomous differential equations with deviated arguments by the use of theory of analytic semigroup and Banach fixed point theorem. Alka et al. [16] generalized the results of [15] for instantaneous impulsive non-autonomous differential equations with iterated deviating arguments. Hamdy [17] studied sufficient conditions for controllability of autonomous Sobolev type fractional integro-differential equations with the help of Schauder's fixed point theorem and the theory of compact semigroup. Mahmudov [18] discussed the approximate controllability of autonomous fractional Sobolev type differential system in Banach space with the help of Schauder's fixed point theorem. Recently, Haloi [19] established sufficient conditions for approximate controllability of non-autonomous nonlocal delay differential systems with deviating arguments by using theory of compact semigroup and Krasnoselskii fixed point theorem.

To the best of our knowledge, no work yet available on approximate controllability of non-autonomous Sobolev type differential systems, inspired by this, we consider the system (1) to find the sufficient conditions for the approximate controllability.

The remaining part of the article is organized as following. Section 2 is concerned with some basic notations and definitions, also we will introduce the expression for mild solutions of the system (1). In section 3, we will study our main results. In section 4, we will present an example to illustrate our results. In last section 5, we will discuss the conclusions.

2. Preliminaries

This section is concerned with some basic assumptions, definitions and theorems required to prove our objectives. For more details, we refer [7], [20] and [21]. Let us denote $\mathcal{C}(J,X)$ for the complete norm space of all continuous maps from J to X, for a finite constant $r>0$, let $\Omega_r = \{x \in \mathcal{C}(J,X) : \|x(t)\| \leqslant r, t \in J\}$. $\mathcal{L}^p(J,X)(1 \leqslant p < \infty)$ is the Banach space of all Bochner integrable functions from J to X with norm $\|x\|_{\mathcal{L}^p(J,X)} = (\int_0^b \|x(t)\|^p dt)^{\frac{1}{p}}$.

Now, we impose the following restrictions (see [4], [20], [21]).

- (A1) The operator $\mathbf{A}(t)$ is closed, domain of $\mathbf{A}(t)$ is dense in X and independent of t.
- (A2) For $Re(\vartheta) \leqslant 0$, $t \in J$, the resolvent operator of $\mathbf{A}(t)$ exists and satisfies $\|\mathcal{R}(\vartheta;t)\| \leqslant \frac{\varsigma}{|\vartheta|+1}$, for some positive constant ς.
- (A3) For each fixed $\tau_3 \in J$, there are constants $\mathcal{K} \geqslant 0, \rho \in (0,1]$ such that $\|[\mathbf{A}(\tau_1) - \mathbf{A}(\tau_2)]\mathbf{A}^{-1}(\tau_3)\| \leqslant \mathcal{K}|\tau_1 - \tau_2|^\rho$ for any $\tau_1, \tau_2 \in J$.
- (S1) $\mathbb{E}$ is closed, bijective operator, and $D(\mathbb{E}) \subset D(\mathbf{A})$.
- (S2) $\mathbb{E}^{-1} : X \to D(\mathbb{E})$ is compact.

The assumptions $(A1), (A2)$ imply that $-\mathbf{A}(t)$ generates an analytic semigroup in $B(X)$, where the symbol $B(X)$ stands for Banach space of all bounded linear operators on X. The closed graph theorem with the above assumptions imply that the linear operator $-\mathbf{A}(t)\mathbb{E}^{-1} : X \to X$ is bounded, and so for each $t \in J$, $-\mathbf{A}(t)\mathbb{E}^{-1}$ generates a semigroup of bounded linear operators and hence a unique evolution system $\{\mathcal{S}(t_1,t_2) : 0 \leqslant t_2 \leqslant t_1 \leqslant b\}$ on X, which satisfies (see [14], [20], [21]):

- (i) $\mathcal{S}(t_1,t_2) \in B(X)$ and is continuous strongly in t_1, t_2 for $0 \leqslant t_2 \leqslant t_1 \leqslant b$.
- (ii) $\mathcal{S}(t_1,t_2)x \in D(\mathbf{A})$, $x \in X$, $0 \leqslant t_2 \leqslant t_1 \leqslant b$.
- (iii) $\mathcal{S}(t_1,t_2)\mathcal{S}(t_2,t_3) = \mathcal{S}(t_1,t_3)$, $0 \leqslant t_3 \leqslant t_2 \leqslant t_1 \leqslant b$.
- (iv) $\mathcal{S}(\eta,\eta)$ is identity operator, for $\eta \in J$.
- (v) $\|\mathcal{S}(t_1,t_2)\| \leqslant \mathcal{M}$, $0 \leqslant t_2 \leqslant t_1 \leqslant b$, for some positive constant $\mathcal{M}$.
- (vi) For each fixed t_2, $\{\mathcal{S}(t_1,t_2), t_2 < t_1\}$ is uniformly continuous in uniform operator norm.
- (vii) For $0 \leqslant t_2 < t_1 \leqslant b$, the derivative $\frac{\partial \mathcal{S}(t_1,t_2)}{\partial t_1}$ exists in strong operator topology, is strongly continuous in t_1. Moreover,

$$\frac{\partial \mathcal{S}(t_1,t_2)}{\partial t_1} + \mathbf{A}(t_1)\mathcal{S}(t_1,t_2) = 0,\ 0 \leqslant t_2 < t_1 \leqslant b.$$

Theorem 1. *([4, 20]) Let $\mathcal{F}$ is a uniformly Hölder continuous function on J with exponent $\beta \in (0,1]$, and the assumptions $(A1)$-$(A3)$, $(S1)$-$(S2)$ hold, then the unique solution for the linear Cauchy problem*

$$\begin{aligned} \frac{d}{dt}[\mathbb{E}x(t)] + \mathbf{A}(t)x(t) &= \mathcal{F}(t), \quad t \in J, \\ x(0) &= x_0 \in D(\mathbb{E}), \qquad (2) \end{aligned}$$

is given by

$$x(t) = \mathbb{E}^{-1}\mathcal{S}(t,0)\mathbb{E}x_0 + \int_0^t \mathbb{E}^{-1}\mathcal{S}(t,s)\mathcal{F}(s)ds. \quad (3)$$

Definition 1. *A mild solution of (1) is a function $x \in \mathcal{C}(J,X)$ satisfying the following integral equation*

$$x(\varrho) = \mathbb{E}^{-1}\mathcal{S}(\varrho,0)\mathbb{E}(x_0 - \mathcal{G}(x)) + \int_0^\varrho \mathbb{E}^{-1}\mathcal{S}(\varrho,\eta)[\mathcal{F}(\eta,x(\eta)) + \mathbb{B}u(\eta)]d\eta, \quad \varrho \in J.$$

For the control u and initial data x_0, use $x^b(x_0,u)$ to denote the state value at time b. The set $\mathcal{R}(b,x_0) = \{x^b(x_0,u) : u \in \mathcal{L}^2(J,\mathbb{U})\}$, is called the reachable set at time b.

Definition 2. *([8]) If $\mathcal{R}(b,x_0)$ is dense in X, the system (1) is called approximately controllable on J.*

Consider the linear control system:

$$\frac{d}{dt}[\mathbb{E}x(t)] + \mathbf{A}(t)x(t) = \mathbb{B}u(t), \quad t \in J,$$
$$x(0) = x_0. \quad (4)$$

Corresponding to (4), the controllability operator is given as

$$\Gamma_0^b = \int_0^b \mathcal{V}(b,\eta)\mathbb{B}\mathbb{B}^*\mathcal{V}^*(b,\eta)d\eta, \quad (5)$$

where $\mathcal{V}(t,s) := \mathbb{E}^{-1}\mathcal{S}(t,s)$, $*$ denotes the adjoint of the operator. Notice that Γ_0^b is a bounded linear operator.

Theorem 2. *([8]) The necessary and sufficient conditions for the linear system (4) to be approximately controllable on J is that, $\delta R(\delta,\Gamma_0^b) \to 0$ as $\delta \to 0^+$ in the strong operator topology, where $R(\delta,\Gamma_0^b) := (\delta I + \Gamma_0^b)^{-1}$, $\delta > 0$.*

Now, we recall the Krasnoselskii *fixed point technique.*

Theorem 3. *([22]) Let S is a convex bounded closed subset of a Banach space X. Suppose that F_1, F_2 be be two X-valued operators defined on S such that such that $F_1x + F_2y \in S$ whenever x, $y \in S$, F_1 is continuous and compact, and F_2 is contraction map. Then F_1+F_2 has a fixed point in S.*

3. Main results

In this section, we prove the existence of mild solutions and approximate controllability of (1). For $x \in \mathcal{C}(J,X)$, consider the control function for the system (1) as following :

$$u(t) = u_\lambda(t,x) = \mathbb{B}^*\mathcal{V}^*(b,t)R(\lambda,\Gamma_0^b)p(x), \quad (6)$$

with

$$p(x) = x^b - \mathcal{V}(b,0)\mathbb{E}(x_0 - \mathcal{G}(x)) - \int_0^b \mathcal{V}(b,\eta)\mathcal{F}(\eta,x(\eta))d\eta. \quad (7)$$

For any $\lambda > 0$, define F_λ on $\mathcal{C}(J,X)$ as following:

$$(F_\lambda x)(\varrho) = (\Phi_\lambda x)(\varrho) + (\Psi_\lambda x)(\varrho), \ \varrho \in J, \quad (8)$$

where

$$(\Phi_\lambda x)(\varrho) = \mathcal{V}(\varrho,0)\mathbb{E}(x_0 - \mathcal{G}(x)) + \int_0^\varrho \mathcal{V}(\varrho,\eta)\mathcal{F}(\eta,x(\eta))d\eta,$$
$$(\Psi_\lambda x)(\varrho) = \int_0^\varrho \mathcal{V}(\varrho,\eta)\mathbb{B}u_\lambda(\eta)d\eta. \quad (9)$$

Now, we state the assumptions that are useful to prove our objective.

- (H1) *$\mathcal{S}(t,s)$, is a compact evolution system whenever $t - s > 0$ $(0 \leqslant s < t \leqslant b)$.*
- (H2) *The function $\mathcal{F}(\cdot,x)$ from J to X is Lebesgue measurable for every fixed $x \in X$, and the function $\mathcal{F}(t,\cdot)$ from X to X is continuous for every fixed $t \in J$, and for all $\varrho \in J$, $\eta_1, \eta_2, \in X$, we have*
  $$\|\mathcal{F}(\varrho,\eta_1) - \mathcal{F}(\varrho,\eta_2)\| \leqslant L_1\|\eta_1 - \eta_2\|,$$
  for some constant $L_1 > 0$.
- (H3) *The function $\mathcal{G}$ from $\mathcal{C}(J,X)$ to $D(\mathbb{E})$ is continuous and there is a constant $L_2 > 0$ such that*
  $$\|\mathbb{E}(\mathcal{G}(x_1) - \mathcal{G}(x_2))\| \leqslant L_2\|x_1 - x_2\|, \quad \forall\, x_1\,,\, x_2 \in \mathcal{C}(J,X).$$
- (H4) *(A_1)-(A_3) and $(S_1), (S_2)$ hold.*

For convenience, we use the following notations:

$$N_1 = \sup_{t\in J}\|\mathcal{F}(t,0)\|, \quad K_1 = (L_1 r + N_1)b,$$
$$\mathcal{M}_1 = \|\mathbb{B}\|, \quad \mathcal{M}_2 = \|\mathbb{E}^{-1}\|.$$

Lemma 1. *If the assumption $(H2)$ holds, then for $x \in \Omega_r$ and $\varrho \in J$ we have $\int_0^\varrho \|\mathcal{F}(\eta,x(\eta))\|d\eta \leqslant K_1$.*

Proof. By assumption $(H2)$, we get

$$\begin{aligned}\int_0^{\varrho}\|\mathcal{F}(\eta,x(\eta))\|d\eta &\leqslant \int_0^{\varrho}\Big(\|\mathcal{F}(\eta,x(\eta)) \\ &\quad -\mathcal{F}(\eta,0)\| + \|\mathcal{F}(\eta,0)\|\Big)d\eta \\ &\leqslant \int_0^{\varrho}(L_1\|x\|+N_1)d\eta \\ &\leqslant (L_1 r + N_1)b = K_1.\end{aligned}$$

□

Theorem 4. *Let the assumptions* $(H1)$*-*$(H4)$ *hold and the functions* $\mathbb{E}(\mathcal{G}(0))$ *is bounded, then a mild solution to the system* (1) *exists, provided that*

$$\Lambda := \mathcal{M}_2\mathcal{M}(L_2 + L_1 b) < 1. \tag{10}$$

Proof. The proof is divided into the following steps :

Step I: For $\lambda > 0$, we have a constant R (depends on λ), satisfying $F_\lambda(\Omega_R) \subset \Omega_R$.
For any positive constant r and $x \in \Omega_r$, if $t \in J$, then by using (6), $(H3)$ and Lemma (1), we have

$$\begin{aligned}u_\lambda(t,x) &= \mathbb{B}^*\mathcal{V}^*(b,t)R(\lambda,\Gamma_0^b)\Big[x^b \\ &\quad -\mathcal{V}(b,0)\mathbb{E}(x_0-\mathcal{G}(x)) \\ &\quad -\int_0^b \mathcal{V}(b,\eta)\mathcal{F}(\eta,x(\eta))d\eta\Big] \\ \|u_\lambda(t,x)\| &\leqslant \frac{\mathcal{M}_1\mathcal{M}_2\mathcal{M}}{\lambda}\Big[\|x^b\| + \mathcal{M}_2\mathcal{M}(\|\mathbb{E}x_0\| \\ &\quad +\|\mathbb{E}(\mathcal{G}(x)-\mathcal{G}(0))\| + \|\mathbb{E}\mathcal{G}(0)\|) \\ &\quad +\mathcal{M}_2\mathcal{M}K_1\Big] \\ &\leqslant \frac{\mathcal{M}_1\mathcal{M}_2\mathcal{M}}{\lambda}\Big[\|x^b\| + \mathcal{M}_2\mathcal{M}(\|\mathbb{E}x_0\| \\ &\quad +L_2 r + \|\mathbb{E}\mathcal{G}(0)\|) + \mathcal{M}_2\mathcal{M}K_1\Big] \\ &:= K_2, \end{aligned} \tag{11}$$

and from (8), (11), we obtain

$$\begin{aligned}(F_\lambda x)(t) &= \mathcal{V}(t,0)\mathbb{E}(x_0-\mathcal{G}(x)) \\ &\quad +\int_0^t \mathcal{V}(t,\eta)\mathcal{F}(\eta,x(\eta))d\eta \\ &\quad +\int_0^t \mathcal{V}(t,\eta)\mathbb{B}u_\lambda(\eta,x)d\eta\end{aligned}$$

$$\begin{aligned}\|(F_\lambda x)(t)\| &\leqslant \|\mathcal{V}(t,0)\|(\|\mathbb{E}(x_0)\| + \|\mathbb{E}\mathcal{G}(x)\|) \\ &\quad +\int_0^t \|\mathcal{V}(t,\eta)\|\|\mathcal{F}(\eta,x(\eta))\|d\eta \\ &\quad +\int_0^t \|\mathcal{V}(t,\eta)\|\|\mathbb{B}\|\|u_\lambda(\eta,x)\|d\eta \\ &\leqslant \mathcal{M}_2\mathcal{M}(\|\mathbb{E}x_0\| + L_2 r + \|\mathbb{E}\mathcal{G}(0)\|) \\ &\quad +\mathcal{M}_2\mathcal{M}K_1 \\ &\quad +\mathcal{M}_2\mathcal{M}\mathcal{M}_1K_2 b.\end{aligned} \tag{12}$$

This implies, for large enough $r > 0$, $F_\lambda(\Omega_r) \subset \Omega_r$ holds.

Step II: $\Phi_\lambda : \Omega_R \to \Omega_R$ is contraction.
For $x, y \in \Omega_R$ and $t \in J$, using $(H2)$ and $(H3)$ we obtain

$$\begin{aligned}\|(\Phi_\lambda x)(t) - (\Phi_\lambda y)(t)\| &\leqslant \|\mathcal{V}(t,0)\mathbb{E}(\mathcal{G}(x) \\ &\quad -\mathcal{G}(y))\| \\ &\quad +\int_0^t \|\mathcal{V}(t,s)\| \\ &\quad \|\mathcal{F}(s,x(s)) \\ &\quad -\mathcal{F}(s,y(s))\|ds \\ &\leqslant \mathcal{M}_2\mathcal{M}L_2\|x-y\| \\ &\quad +\mathcal{M}_2\mathcal{M} \\ &\quad \int_0^t L_1\|x-y\|ds \\ &\leqslant \mathcal{M}_2\mathcal{M}L_2\|x-y\| \\ &\quad +\mathcal{M}_2\mathcal{M}L_1 b\|x-y\| \\ &\leqslant \mathcal{M}_2\mathcal{M}(L_2+L_1 b) \\ &\quad \|x-y\| \\ &= \Lambda\|x-y\|.\end{aligned} \tag{13}$$

Since $\Lambda < 1$, therefore Φ_λ is contraction.

Step III: Ψ_λ is continuous in Ω_R.
Consider $\{x_n\}$ be a sequence in Ω_R with $\lim_{n\to\infty} x_n = x$ in Ω_R. From continuity of non-linear term $\mathcal{F}$ with respect to state variable, we have

$$\lim_{n\to\infty}\mathcal{F}(\eta,x_n(\eta)) = \mathcal{F}(\eta,x(\eta)), \quad \text{for each } \eta \in J.$$

So, we can conclude that

$$\sup_{\eta\in J}\|\mathcal{F}(\eta,x_n(\eta)) - \mathcal{F}(\eta,x(\eta))\| \to 0 \quad \text{as} \quad n \to \infty. \tag{14}$$

For $t \in J$, $(S1)$, $(H3)$, and (14) yield the following

$$\begin{aligned}\|p(x_n)-p(x)\| &\leqslant \mathcal{M}_2\mathcal{M}\|\mathbb{E}\mathcal{G}(x_n)-\mathbb{E}\mathcal{G}(x)\| \\ &\quad +\mathcal{M}_2\mathcal{M}\int_0^b \|\mathcal{F}(\zeta,x_n(\zeta)) \\ &\quad -\mathcal{F}(\zeta,x(\zeta))\|d\zeta \\ &\leqslant \mathcal{M}_2\mathcal{M}\|\mathbb{E}\mathcal{G}(x_n)-\mathbb{E}\mathcal{G}(x)\| \\ &\quad +\mathcal{M}_2\mathcal{M}b\sup_{\zeta\in J}\|\mathcal{F}(\zeta,x_n(\zeta)) \\ &\quad -\mathcal{F}(\zeta,x(\zeta))\| \\ &\to 0 \quad \text{as} \quad n\to\infty, \qquad (15)\end{aligned}$$

therefore (6) implies that

$$\|u_\lambda(\eta,x_n)-u_\lambda(\eta,x)\|\to 0 \quad \text{as} \quad n\to\infty, \qquad (16)$$

and so

$$\begin{aligned}\|(\Psi_\lambda x_n)(t)-(\Psi_\lambda x)(t)\| &\leqslant \mathcal{M}_2\mathcal{M}\mathcal{M}_1 b \\ &\quad \sup_{\eta\in J}\|u_\lambda(\eta,x_n) \\ &\quad -u_\lambda(\eta,x)\| \\ &\to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

which means Ψ_λ is continuous in Ω_R.

Step IV: $\Psi_\lambda:\Omega_R\to\Omega_R$ is compact. For this we need to show :

(i): The set $\{(\Psi_\lambda x)(\varrho): x\in\Omega_R\}$ is relatively compact subset of X, for each $\varrho\in J$.
For $\varrho=0$, obviously the set $\{(\Psi_\lambda x)(0): x\in\Omega_R\}=\{0\}$ is compact subset of X. For fixed $\varrho\in(0,b]$, and $\xi\in(0,\varrho)$, consider an operator Ψ_λ^ξ on Ω_R as following

$$\begin{aligned}(\Psi_\lambda^\xi x)(\varrho) &= \int_0^{\varrho-\xi}\mathcal{V}(\varrho,\eta)\mathbb{B}u_\lambda(\eta,x)d\eta \\ &= \int_0^{\varrho-\xi}\mathbb{E}^{-1}\mathcal{S}(\varrho,\varrho-\xi) \\ &\quad \mathcal{S}(\varrho-\xi,\eta)\mathbb{B}u_\lambda(\eta,x)d\eta \\ &= \mathbb{E}^{-1}\mathcal{S}(\varrho,\varrho-\xi) \\ &\quad \int_0^{\varrho-\xi}\mathcal{S}(\varrho-\xi,\eta)\mathbb{B}u_\lambda(\eta,x)d\eta \\ &= \mathbb{E}^{-1}\mathcal{S}(\varrho,\varrho-\xi)y(\varrho,\xi).\end{aligned}$$

Since $\mathbb{E}^{-1}$ and $\mathcal{S}(\varrho,\varrho-\xi)$ are compact, and $y(\varrho,\xi)$ is bounded on Ω_R, we get $\{(\Psi_\lambda^\xi x)(\varrho): x\in\Omega_R\}$ is relatively compact subset of X. Also

$$\begin{aligned}\|(\Psi_\lambda x)(\varrho)-(\Psi_\lambda^\xi x)(\varrho)\| &\leqslant \int_{\varrho-\xi}^{\varrho}\|\mathcal{V}(\varrho,\eta)\mathbb{B} \\ &\quad u_\lambda(\eta,x)\|d\eta \\ &\leqslant \mathcal{M}_2\mathcal{M}\mathcal{M}_1\xi\|u_\lambda\| \\ &\to 0 \quad as \quad \xi\to 0.\end{aligned}$$

Hence, $\{(\Psi_\lambda x)(\varrho): x\in\Omega_R\}$ is relatively compact subset of X.

(ii): Now, we show $\{\Psi_\lambda x: x\in\Omega_R\}$ is equicontinuous. For any $x\in\Omega_R$ and $0\leqslant\varrho_1<\varrho_2\leqslant b$, we have

$$\begin{aligned}\|(\Psi_\lambda x)(\varrho_2)-(\Psi_\lambda x)(\varrho_1)\| &= \Bigg\|\int_0^{\varrho_2}\mathbb{E}^{-1}\mathcal{S}(\varrho_2,\eta) \\ &\quad \mathbb{B}u_\lambda(\eta,x)d\eta \\ &\quad -\int_0^{\varrho_1}\mathbb{E}^{-1}\mathcal{S}(\varrho_1,\eta) \\ &\quad \mathbb{B}u_\lambda(\eta,x)d\eta\Bigg\| \\ &\leqslant \Bigg\|\int_0^{\varrho_1}\mathbb{E}^{-1}[\mathcal{S}(\varrho_2,\eta) \\ &\quad -\mathcal{S}(\varrho_1,\eta)] \\ &\quad \mathbb{B}u_\lambda(\eta,x)d\eta\Bigg\| \\ &\quad +\Bigg\|\int_{\varrho_1}^{\varrho_2}\mathbb{E}^{-1}\mathcal{S}(\varrho_2,\eta) \\ &\quad \mathbb{B}u_\lambda(\eta,x)d\eta\Bigg\| \\ &\leqslant J_1+J_2.\end{aligned}$$

For $\varrho_1=0$, it is easy to see that $J_1=0$. When $\varrho_1>0$, let $\varepsilon>0$ small enough, we obtain

$$\begin{aligned}J_1 &\leqslant \Bigg\|\int_0^{\varrho_1-\varepsilon}\mathbb{E}^{-1}[\mathcal{S}(\varrho_2,\eta)-\mathcal{S}(\varrho_1,\eta)] \\ &\quad \mathbb{B}u_\lambda(\eta,x)d\eta\Bigg\| \\ &\quad +\Bigg\|\int_{\varrho_1-\varepsilon}^{\varrho_1}\mathbb{E}^{-1}[\mathcal{S}(\varrho_2,\eta)-\mathcal{S}(\varrho_1,\eta)] \\ &\quad \mathbb{B}u_\lambda(\eta,x)d\eta\Bigg\| \\ &\leqslant \mathcal{M}_2\mathcal{M}_1(\varrho_1-\varepsilon)\|u_\lambda\| \\ &\quad \sup_{\eta\in[0,\varrho_1-\varepsilon]}\|\mathcal{S}(\varrho_2,\eta)-\mathcal{S}(\varrho_1,\eta)\| \\ &\quad +2\mathcal{M}_2\mathcal{M}\mathcal{M}_1\varepsilon\|u_\lambda\| \\ J_2 &\leqslant \mathcal{M}_2\mathcal{M}\mathcal{M}_1\|u_\lambda\|(\varrho_2-\varrho_1)\end{aligned}$$

Hence, $J_1, J_2 \to$ as $\varrho_2 \to \varrho_1, \varepsilon \to 0$. As a result $\|(\Psi_\lambda x)(\varrho_2) - (\Psi_\lambda x)(\varrho_1)\| \to 0$ independently of $x \in \Omega_R$ as $\varrho_2 \to \varrho_1$, which means that $\Psi_\lambda : \Omega_R \to \Omega_R$ is equicontinuous. Thus, by Arzela-Ascoli theorem, Ψ_λ is compact on Ω_R.
Therefore Krasnoselskii fixed point theorem implies that F_λ has a fixed point, which is a mild solution to the problem (1). □

Now, we are ready to discuss the approximate controllability of the system (1). In order to prove it, the following hypotheses are also required:

(H5) *$\delta R(\delta, \Gamma_0^b) \to 0$ whenever $\delta \to 0^+$ in strong operator topology.*
(H6) *There exist constants $L_3 > 0$ and $L_4 > 0$, such that*

$$\begin{aligned} \|\mathbb{E}\mathcal{G}(x)\| &\leqslant L_3, \quad \forall\, x \in \mathcal{C}(J, X), \\ \|\mathcal{F}(t, x)\| &\leqslant L_4, \quad \forall (t, x) \in J \times X. \end{aligned}$$

Theorem 5. *If the assumptions of Theorem 4 as well as hypotheses $(H5)$ and $(H6)$ are satisfied, then (1) is approximately controllable on J.*

Proof. Theorem 4 guaranteed that F_λ has a fixed point in Ω_R. Let x_λ is a mild solution of (1) under the control $u_\lambda(t, x_\lambda)$ given by (6) and satisfies

$$\begin{aligned} x_\lambda(b) &= \mathcal{V}(b,0)\mathbb{E}(x_0 - \mathcal{G}(x_\lambda)) \\ &\quad + \int_0^b \mathcal{V}(b,\eta)[\mathcal{F}(\eta, x_\lambda(\eta)) \\ &\quad + \mathbb{B}u_\lambda(\eta, x_\lambda)]d\eta \\ &= x^b - p(x_\lambda) + \int_0^b \mathcal{V}(b,\eta) \\ &\quad \mathbb{B}u_\lambda(\eta, x_\lambda)d\eta \\ &= x^b - p(x_\lambda) + \int_0^b \mathcal{V}(b,\eta) \\ &\quad \mathbb{B}\mathbb{B}^*\mathcal{V}^*(b,\eta)R(\lambda, \Gamma_0^b)p(x_\lambda)d\eta \\ &= x^b - p(x_\lambda) + \Gamma_0^b R(\lambda, \Gamma_0^b)p(x_\lambda) \\ &= x^b - [I - \Gamma_0^b(\lambda I + \Gamma_0^b)^{-1}]p(x_\lambda) \\ &= x^b - \lambda R(\lambda, \Gamma_0^b)p(x_\lambda), \end{aligned} \tag{17}$$

where

$$\begin{aligned} p(x_\lambda) &= x^b - \mathcal{V}(b,0)\mathbb{E}(x_0 - \mathcal{G}(x_\lambda)) \\ &\quad - \int_0^b \mathcal{V}(b,\eta)\mathcal{F}(\eta, x_\lambda(\eta))d\eta. \end{aligned}$$

According to the compactness of $\mathbb{E}^{-1}$, $\mathcal{S}(t,s)$, and the uniform boundedness of $\mathbb{E}\mathcal{G}$, we see that there exists a subsequence of $\{\mathcal{V}(b,0)\mathbb{E}\mathcal{G}(x_\lambda) : \lambda > 0\}$, still denoted by it, converges to some $x_g \in X$ as $\lambda \to 0$. Since $\mathcal{F}$ is uniformly bounded, we get

$$\int_0^b \|\mathcal{F}(\eta, x_\lambda(\eta))\|^2 d\eta \leqslant L_4^2 b.$$

Hence $\mathcal{F}(\cdot, x_\lambda(\cdot))$ is a bounded sequence in $L^2(J, X)$. So, $\{\mathcal{F}(\cdot, x_\lambda(\cdot)) : \lambda > 0\}$ has a subsequence, still denoted by it, converges weakly to some $\mathcal{F}(\cdot) \in L^2(J, X)$. Define

$$\varpi = x_b - \mathcal{V}(b,0)\mathbb{E}x_0 + x_g - \int_0^b \mathcal{V}(b,s)\mathcal{F}(s)ds.$$

Now, we get

$$\begin{aligned} \|p(x_\lambda) - \varpi\| &\leqslant \|\mathcal{V}(b,0)\mathbb{E}\mathcal{G}(x_\lambda) - x_g\| \\ &\quad + \mathcal{M}\int_0^b \|\mathcal{F}(s, x_\lambda(s)) - \mathcal{F}(s)\|ds \\ &\to 0 \quad \text{as} \quad \lambda \to 0^+. \end{aligned} \tag{18}$$

From (17), (18), and $(H5)$, we obtain

$$\begin{aligned} \|x_\lambda(b) - x^b\| &\leqslant \|\lambda R(\lambda, \Gamma_0^b)p(x_\lambda)\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_0^b)\varpi\| \\ &\quad + \|\lambda R(\lambda, \Gamma_0^b)\|\|p(x_\lambda) - \varpi\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_0^b)\varpi\| + \|p(x_\lambda) - \varpi\| \\ &\to 0 \text{ as } \lambda \to 0^+. \end{aligned}$$

Hence, (1) is approximately controllable. □

4. Example

Consider a control system governed by the following partial differential equation :

$$\begin{aligned} \frac{\partial}{\partial t}[x(t,z) - x_{zz}(t,z)] + [a(t,z) + \frac{\partial^2}{\partial z^2}]x(t,z) \\ = \mu(t,z) + \sin x(t,z), \\ z \in (0,\pi), \;\; t \in (0,1]; \\ x(t,0) = x(t,\pi) = 0, \;\; t \in [0,1]; \\ x(0,z) + \frac{e^t}{c(1+e^t)}\cos x(t,z) = x_0(z), \\ z \in (0,\pi); \end{aligned} \tag{19}$$

where $X = \mathbb{U} = \mathcal{L}^2([0,1] \times [0,\pi], \mathbb{R})$, $x_0(z) \in D(\mathbb{E})$, $a(t,z) \in C^1([0,\pi] \times [0,1], \mathbb{R})$, $J = [0,1]$, i.e. $b = 1$, and c is positive constant. Define

$$\begin{aligned} \mathbf{A}(t)x(t,z) &= [a(t,z) + \frac{\partial^2}{\partial z^2}]x(t,z), \\ \mathbb{E}x &= x - x_{zz}, \end{aligned} \tag{20}$$

where $D(\mathbf{A}(t))$, $D(\mathbb{E})$ is given by $H^2(0,\pi) \cap H_0^1(0,\pi)$. Therefore, $-\mathbf{A}(t)$ generates a compact evolution system of bounded linear operators $W(t,s)$ on X and is given by (see [19])

$$W(t,s)x = T(t-s)e^{\int_s^t a(\tau)d\tau}x, \quad x \in D(\mathbf{A}(t)). \tag{21}$$

Here

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n,$$

with $e_n(z) = \sqrt{\frac{2}{\pi}} sin(nz)$, $0 \leqslant z \leqslant \pi$, $n = 1, 2, \ldots$.

The operator $\mathbb{E}$ *can be written as following (see [5])*

$$\mathbb{E}x = \sum_{n=1}^{\infty} (1+n^2) < x, e_n > e_n, \; x \in D(E). \tag{22}$$

Furthermore for $x \in X$, *we have*

$$\mathbb{E}^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1+n^2} < x, e_n > e_n, \tag{23}$$

which is compact. So, the operator $-\mathbf{A}(t)\mathbb{E}^{-1}$ *generates a compact evolution system of bounded linear operators that is given as*

$$\mathcal{S}(t,s)x = U(t-s)e^{\int_s^t a(\tau)d\tau}x, \tag{24}$$

where

$$U(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t} < x, e_n > e_n.$$

Hence assumptions $(H1)$, $(H4)$ *hold. By putting* $x(t) = x(t,\cdot)$ *which means* $x(t)(z) = x(t,z)$, $t \in [0,1]$, $z \in [0,\pi]$ *and* $u(t) = \mu(t,\cdot)$ *is continuous. Let the bounded linear operator* $\mathbb{B} : \mathbb{U} \to X$ *is defined as* $\mathbb{B}u(t)(z) = \mu(t,z)$. *Further*

$$\begin{aligned} \mathcal{F}(t,x(t))(z) &= \sin x(t,z), \\ \mathcal{G}(x) &= \frac{e^t}{c(1+e^t)} \cos x. \end{aligned}$$

So, the system (19) *can be formulated into the abstract form of* (1). *Note that* $\mathbb{E}\mathcal{G}(x) = \frac{2e^t}{c(1+e^t)} \cos x$. *Observe that the functions* $\mathcal{F}, \mathcal{G}$ *satisfies the assumptions* $(H2)$, $(H3)$, *and also* $\mathcal{F}, \mathbb{E}\mathcal{G}$ *are uniformly bounded. Now it is needed to check the approximately controllability of the associated linear system, for this we show that*

$$\mathbb{B}^*\mathcal{V}^*(b,s)x = 0, \; s \in [0,b) \Rightarrow x = 0, \tag{25}$$

where $\mathcal{V}(t,s) = \mathbb{E}^{-1}\mathcal{S}(t,s)$. *Notice that* $\mathcal{S}$ *and* $\mathbb{E}^{-1}$ *are self adjoint. Indeed,*

$$\begin{aligned} \mathbb{B}^*\mathcal{V}^*(b,s)x &= \mathcal{V}^*(b,s)x = \mathcal{S}^*(b,s)(\mathbb{E}^{-1})^*x \\ &= \mathcal{S}(b,s)\mathbb{E}^{-1}x \\ &= e^{\int_s^b a(\tau)d\tau} \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}(b-s)} \\ &\quad < \mathbb{E}^{-1}x, e_n > e_n \\ &= e^{\int_s^b a(\tau)d\tau} \sum_{n=1}^{\infty} \frac{1}{1+n^2} e^{\frac{-n^2}{1+n^2}(b-s)} \\ &\quad < x, e_n > e_n. \end{aligned} \tag{26}$$

This implies that the condition (25) *holds, and hence the assumption* $(H5)$. *Thus by Theorem* 5, *the system* (19) *is approximately controllable on* J.

5. Conclusion

In this work, we have obtained that the mild solutions for non-autonomous Sobolev differential equations with nonlocal condition exist mainly by the help of evolution system of bounded linear operators and Krasnoselskii fixed point technique. Also we have determined the sufficient conditions for approximate controllability by using the controllability of corresponding linear system. The results developed in this article can be extended to the study of existence of mild solutions and approximate controllability for neutral and impulsive differential systems. Moreover the obtained results also can be generalized for fractional Sobolev, neutral and impulsive differential systems.

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